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Numerical methods for integral equations of Mellin type

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Abstract

We present a survey of numerical methods (based on piecewise polynomial approximation) for integral equations of Mellin type, including examples arising in boundary integral methods for partial differential equations on polygonal domains.

1 Introduction

In the last 30 years or so a great deal of interest has focused on the numerical analysis of boundary integral equations arising from PDEs on non-smooth domains (see [49] for one of the pioneering papers in this field). Here the chief difficulties are not only the loss of smoothness of the solution near non-smooth boundary points, but also (and more crucially) the singularity induced in the integral operator itself. The development of a proper understanding of these singularities has a huge practical motivation due to the large range of applications - particularly in engineering - and even the geometrically simple case of a polygonal domain still contains open problems of considerable mathematical subtlety. This survey concentrates on the numerical analysis of a class of equations which arises generically in such problems, namely the equations of *Mellin type*. The simplest case of such an equation contains the operator

$$Kv(s) = \int_0^1 \kappa\left(\frac{s}{\sigma}\right) v(\sigma)\frac{d\sigma}{\sigma}, \quad s \in [0,1] \ , \tag{1.1}$$

where the *kernel* κ is a given function on $\mathbb{R}^+ := [0,\infty)$. Often κ is a smooth function on $(0,\infty)$ satisfying certain asymptotic estimates at 0 and ∞, in which case $\kappa(s/\sigma)\sigma^{-1}$ is smooth at $s = \sigma > 0$ but blows up with $O(\sigma^{-1})$ when $s = \sigma \to 0$ (i.e. the operator (1.1) has a *fixed singularity* at the origin). Note that the upper limit of integration in (1.1) is to some extent arbitrary, since the operator $K_\epsilon v(s) := \int_0^\epsilon \kappa(s/\sigma)v(\sigma)d\sigma/\sigma,\ s \in [0,\epsilon]$ can easily be reduced to (1.1) via the transformation $\sigma \to \epsilon\sigma$.

The operator K (or more generally K_ϵ, for $\epsilon > 0$) can be considered as a localised version of the operator: $\mathcal{K}v(s) := \int_0^\infty \kappa(s/\sigma)d\sigma/\sigma,\ s \in \mathbb{R}^+$, which is normally treated using the *Mellin transform*: $\tilde{v}(z) = \int_0^\infty s^{z-1}v(s)ds$, for $z \in \mathbb{C}$. The convolution theorem then states that (for suitably well-behaved κ and v) we have $\widetilde{\mathcal{K}v} = \tilde{\kappa}\tilde{v}$, and from this it is easily shown that $\|K\|_2 \le \|\mathcal{K}\|_2 = \sup_{\mathrm{Re}(z)=1/2} |\tilde{\kappa}(z)|$. (Here $\|\cdot\|_2$ denotes the operator norm on the space L_2 of square-integrable functions.) Moreover, K is also bounded on L_∞ and on

C (the continuous functions on $[0,1]$), with $\|K\|_\infty = \int_0^\infty |\kappa(s)|ds/s < \infty$ (provided this integral exists), in which case

$$\lim_{s\to 0} Kv(s) = \tilde{\kappa}(0)v(0) \quad \text{for } v \in C \ . \tag{1.2}$$

Using (1.2) the following simple argument ([3]) shows that K is non-compact in C: For each $n \in \mathbb{N} := \{1,2,\ldots\}$, let $v_n : [0,1] \to \mathbb{R}$ denote a continuous function with $v_n(0) = 1 = \|v_n\|_\infty$ and $\operatorname{supp} v_n \subset [0, 1/n]$. If K were compact on C then the sequence $\{Kv_n\}$ would contain a convergent subsequence, $\left\{Kv_{n_j}\right\}$ in C. However, (1.2) implies that $Kv_{n_j}(0) = \tilde{\kappa}(0)$ for all j. Moreover for $s > 0$ we can employ the change of variable $x = s/\sigma$ to obtain $|Kv_{n_j}(s)| \leq \int_{n_j s}^\infty |\kappa(x)|dx/x \to 0$ as $j \to \infty$, demonstrating that $\left\{Kv_{n_j}\right\}$ cannot have a continuous limit when $\tilde{\kappa}(0) \neq 0$. In fact the spectrum of K contains all the values of $\tilde{\kappa}(z)$ for $\mathrm{Re}(z) = 0$, and K is not compact on any L_p space either (see §3 for a discussion of this).

All the problems which we shall consider in this paper have as their heart the solution of second-kind equations of the form

$$(I - K)u = f \tag{1.3}$$

with K as defined in (1.1). An important rôle in the theory of these equations is played by the *finite section operator* KT^τ, where T^τ is the truncation operator satisfying $T^\tau v(s) = 0$, for $s < \tau$ and $T^\tau v(s) = v(s)$ for $s \geq \tau$. Then, for $\tau \in (0,1]$, we have $KT^\tau v(s) = \int_\tau^1 \kappa(s/\sigma)v(\sigma)d\sigma/\sigma$. At various points in this review we will require assumptions on (i) the well-posedness of (1.3) and (ii) the stability of the corresponding finite section operators, i.e.

$$\text{(i)} \ \ \|(I-K)^{-1}\| \leq C \ , \quad \text{and} \quad \text{(ii)} \ \ \|(I - KT^\tau)^{-1}\| \leq C \quad \text{as } \tau \to \ 0, \tag{1.4}$$

for some norm $\|\cdot\|$. Throughout the paper we let $C, C_1, C_2, \ldots$ denote generic constants in the usual way.

To analyse (1.3), we introduce for $\alpha \in \mathbb{R}$ and $r \in \mathbb{N}$, the space $C^{r,\alpha}$ comprising the completion of the infinitely smooth functions on $(0,1]$ with respect to the norm $\|v\|_{r,\alpha} := \sup_{s\in(0,1],l=0,\ldots r} |s^{[l-\alpha]}D^l v(s)|$, where $[\beta] = \beta$ for $\beta \geq 0$ and $[\beta] = 0$ for $\beta < 0$. In general the solution u of (1.3) (or perhaps the higher derivatives of u) will have a singularity at $s = 0$, and thus will lie in $C^{r,\alpha}$ with the size of α depending on the zeros of the symbol $1 - \tilde{\kappa}(z)$, for $z \in \mathbb{C}$ (see, for example, [12] or [38, p.172-174]).

To approximate (1.3), we introduce piecewise polynomial spaces on $[0,1]$ as follows. For any integer $n \geq 1$, introduce a mesh $0 = x_0 < x_1 < \ldots < x_n = 1$.

Then for $r > d+1 \geq 0$, $S_n^{r,d}$ denotes the functions which reduce to polynomials of degree $r-1$ on each interval $I_i = (x_{i-1}, x_i)$ and have d continuous derivatives globally on $[0,1]$. Thus, for $r > 0$, $S_n^{r,-1}$ denotes the piecewise polynomials of degree $r-1$ which may be discontinuous at each x_i, $i = 1, \ldots, n-1$, whereas $S_n^{r,r-2}$ denotes the smoothest splines on $[0,1]$ (without any end-point conditions). We shall also need the 2π-periodic smoothest splines of degree $r-1$ (and C^{r-2} continuity), which we denote $S_{n,p}^r$. There is a well-worked literature on approximation in these spaces (see, e.g. [46,38,18]).

To deal with the singularity in u, one approach is to consider graded meshes constructed (either analytically or adaptively) to satisfy the inequalities

$$h_i \leq C_1(1/n)(i/n)^{q-1} \quad \text{and} \quad x_i \geq C_2(i/n)^q \ , \quad i = 1, \ldots, n \ , \qquad (1.5)$$

for some *grading exponent* $q \geq 1$, where $h_i = x_i - x_{i-1}$. These inequalities imply that near $x = 0$ mesh subintervals are of length $O((1/n)^q)$ whereas near $x = 1$ they are of length $O(1/n)$ as $n \to \infty$. We call meshes which satisfy (1.5) "q-*graded* at 0". A standard example of such a mesh is ([42]) $x_i := (i/n)^q$, $i = 0, \ldots, n$, which satisfies (1.5) with $C_1 = q$ and $C_2 = 1$.

To illustrate the properties of such meshes, consider approximating a function $u \in C^{r,\alpha}$ by $S_n^{r,-1}$ (where $r \geq 1$), and suppose for convenience that $\alpha \in (0,1]$. Then standard Taylor series estimates show that there exists a function $\phi_n \in S_n^{r,-1}$ such that $\|u - \phi_n\|_{\infty,I_i} \leq h_i^r \|D^r u\|_{\infty,I_i}$, provided the norm on the right-hand side is finite. Thus, for $i \geq 2$ making use of (1.5), we have $\|u - \phi_n\|_{\infty,I_i} \leq C h_i^r x_{i-1}^{\alpha - r} \|u\|_{r,\alpha} \leq C(1/n)^r((i-1)/n)^{q\alpha - r}\|u\|_{r,\alpha} \leq C(1/n)^r \|u\|_{r,\alpha}$, where the final inequality requires that the grading exponent q should be sufficiently large, namely $q \geq r/\alpha$. On the other hand, for $s \in I_1$, elementary arguments show that $|u(s) - u(0)| \leq C s^\alpha \|u\|_{r,\alpha} \leq C(1/n)^r \|u\|_{r,\alpha}$, again provided $q \geq r/\alpha$. So, setting $\phi_n \equiv u(0)$ on I_1 we see that $\|u - \phi_n\|_\infty$ is of optimal order $O(1/n)^r$. In some examples the solution u of (1.3) is not continuous but instead has an infinite singularity of order $s^{\alpha-1}$ (as $s \to 0$) for some $\alpha \in (1/2, 1)$. Then analogous arguments to those given above (but in the L_2 context) [17] show that there exists $\phi_n \in S_n^{r,-1}$ with $\phi_n \equiv 0$ on I_1 such that $\|u - \phi_n\|_2 = O(n^{-r})$ provided $q > r/(\alpha - 1/2)$. Both the L_2 and uniform estimates also extend to the case of approximation by splines of arbitrary smoothness [18].

An alternative way of dealing with a singularity in the solution $u(s)$ of (1.3) at $s = 0$ (and a method which we shall consider in more detail below) involves a change of variable $s = \gamma(x)$, where $\gamma : [0,1] \to [0,1]$ is an increasing function, with $\gamma(0) = 0$, $\gamma(1) = 1$ and $\gamma(x)$ having a zero of an appropriately high order at $x = 0$. For example if $u \in C^{r,\alpha}$ where $\alpha \in (0,1]$ and if γ has r continuous derivatives on $[0,1]$ with $(D^j\gamma)(x) = O(x^{q-j})$ for $j = 0, \ldots, r$, then it is easily shown that the function $u \circ \gamma$ has r continuous derivatives, provided $q \geq r/\alpha$. This function can then be approximated by a piecewise polynomial

ϕ_n of desired smoothness with respect to the *uniform mesh* $x_i = i/n$, yielding (after inverse transformation) an optimal order approximation $\phi_n(\gamma^{-1}(s))$ to $u(s)$. If (1.3) has a solution which blows up at $s = 0$ (for example the function $u(s) = s^{\alpha-1}$ with $\alpha \in (1/2, 1)$), then the straight substitution $s = \gamma(x)$ with $\gamma(x)$ given above makes it worse rather than better-behaved. This difficulty can be circumvented by considering instead the function $w(x) = (u \circ \gamma)(x)|\gamma'(x)|$, with γ as above, which arises naturally when u appears inside an integral. Then it is easily shown that $w(s)$ has r continuous derivatives provided $q \geq (r+1)/\alpha$ (see, e.g. [21]). Such nonlinear change of variables techniques can be found for example in [37,32,5,16,41,36],

A third method of obtaining optimal convergence for singular solutions (which we shall not discuss at length here) is to augment the approximating spaces with some of the singular terms occurring in the expansion of the solution (e.g. [51,12,33,34]).

However the chief difficulty in solving (1.3) is not the approximation of the singular solution u but rather proving the stability of the chosen numerical method, with the main theoretical barrier being the non-compactness of the operator K. This was emphasised in [10], where it was shown that there exist piecewise polynomial collocation methods which converge optimally when K is compact but which actually diverge for (1.3) when K is given by (1.1). In [10] a way around this barrier was found by considering a modified method (which excluded the counterexample but which was nevertheless very close to a standard collocation method) and proving stability and convergence for it. Subsequently this modification technique has been applied to a great variety of approximation methods for (1.3) (see [38] for an extensive review), and as far as we are aware it is still the standard way of proving stability and convergence for practical methods for integral equations of Mellin type. Examples of results which use this modification technique to prove stability (in conjunction with mesh grading) are [9,10,27,17,39,19,20,11,29,15,24], whereas the same technique is used in conjunction with a nonlinear change of parametrisation in [32,30,21,22,26,23,25,31,47,35]. The modification technique for proving stability later found a more practical use as a parameter for accelerating the convergence of multigrid-type algorithms ([4,40]).

It is important to point out that in the case of classical Galerkin methods for boundary integral equations on corner domains (where a variational formulation of the underlying integral equation is exploited and errors due to quadrature are not taken in to account), the stability analysis is not difficult provided one restricts to the energy norm. The numerical analysis then reduces to finding efficient ways of approximating the singular solution. In this context the literature is older and includes, for example, [51,12]. The papers [6,7] also concern the Galerkin method but analyse errors in the uniform norm and therefore require a more sophisticated stability analysis.

We begin this survey in §2 by illustrating the use of the modification technique in the (relatively simple) context of discontinuous piecewise polynomial collocation methods for (1.3). The modified method can be thought of as the discretization of the finite section approximation of K, and then a perturbation argument is the key to proving stability. In §3 we explain how this idea can be extended to a unified convergence theory of spline approximation methods for equation (1.3). §4 is devoted to some examples of second and first kind boundary integral equations for elliptic PDEs on corner domains leading to the model equation (1.3) (more precisely systems of such equations), with emphasis on Laplace's equation. In §5 we give a survey of results on Symm's integral equation and related first kind equations.

2 Introduction to Modification Techniques

To illustrate the technique of modification (mentioned in §1) in a simple setting, consider equation (1.3) and suppose that assumption (1.4) holds in the essential supremum norm. Assume also that κ satisfies the estimates

$$\int_0^\infty s^k |D^k \kappa(s)| ds/s < \infty, \quad \text{for all integers } \ k \geq 0. \tag{2.1}$$

To solve (1.3), we consider classical piecewise polynomial collocation methods in $S_n^{r,-1}$. To define the collocation procedure, choose r points $0 \leq \xi_1 \leq \ldots \leq \xi_r \leq 1$ in the reference domain $[0,1]$ and map these to each I_i with the formula: $x_{ij} = x_{i-1} + \xi_j h_i$, $i = 1, \ldots n$, $j = 1, \ldots r$. Defining the interpolatory projection P_n onto $S_n^{r,-1}$ by requiring that $P_n v(x_{ij}) = v(x_{ij})$ for all i, j it follows that P_n converges pointwise to the identity on C and has uniform norm bounded as $n \to \infty$. The classical collocation method for (1.3) seeks an approximate solution $u_n \in S_n^{r,-1}$ such that

$$(I - P_n K) u_n = P_n f \ . \tag{2.2}$$

To focus on the difficulty in analysing (2.2), recall that if $K : L_\infty \to C$ were compact, then $\|(I - K) - (I - P_n K)\|_\infty = \|(I - P_n)K\|_\infty \to 0$ as $n \to \infty$ (since pointwise convergence is uniform on compact sets). Hence, by the Banach perturbation lemma (applied in L_∞) and the well-posedness assumption (1.4)(i), $(I - P_n K)^{-1}$ exists for n sufficiently large and has uniform norm bounded as $n \to \infty$. In this case a unique collocation solution u_n exists, and $u - u_n = (I - P_n K)^{-1}(u - P_n u)$, from which we obtain the usual error estimate $\|u - u_n\|_\infty \leq C \|u - P_n u\|_\infty$.

As mentioned in §1, this argument fails for the non-compact operator K in (1.1), an observation which led in [10] to the introduction of the (slightly more general) *modified collocation method.* Here for simplicity we shall introduce this technique in the special case where the solution u of (1.3) satisfies $u(0) = 0$, although - as we indicate in §3 - the principle can be applied in the general case also. In its simplest form the modification technique involves choosing an integer $i* \geq 0$ and seeking $u_n \in S_n^{r,-1}$ satisfying $u_n \equiv 0$ on $[0, x_{i*}]$ and (instead of (2.2)) the modified collocation equations:

$$(I - K)u_n(x_{ij}) = f(x_{ij}), \quad j = 1, \dots r, \quad i = i*+1, \dots, n \ .$$

In operator form this can be written

$$u_n \in S_n^{r,-1} : \quad (I - P_n T^{x_{i*}} K)u_n = P_n T^{x_{i*}} f \ , \tag{2.3}$$

which is clearly equivalent to (1.3) when $i* = 0$.

To analyse (2.3), the first step is to recall the formal identity: $(I - T^{x_{i*}}K)^{-1} = I + T^{x_{i*}}(I - KT^{x_{i*}})^{-1}K$. Using this, together with the assumption (1.4)(ii) and the identity $\|T^\tau\|_\infty = 1$, it follows that, for fixed $i* \geq 0$, $\|(I - T^{x_{i*}}K)^{-1}\|_\infty$ is uniformly bounded as $n \to \infty$. Then, attempting to mimick the argument in the compact case, we can show that (2.3) is well-posed provided we show that $\|(I - T^{x_{i*}}K) - (I - P_n T^{x_{i*}}K)\|_\infty = \|(I - P_n)T^{x_{i*}}K\|_\infty$ is sufficiently small. Although this quantity does not approach zero as $n \to \infty$, we shall see in the next lemma that it can be made arbitrarily small independent of n by an appropriate choice of parameter $i*$.

Lemma 2.1 *There exists a constant C independent of n and $i*$ such that $\|(I - P_n)T^{x_{i*}}K\|_\infty \leq C(1/i*)^r$.*

Proof Let $v \in L_\infty$. For $i > i*$, we have, using (1.5),

$$\begin{aligned} \|(I - P_n)Kv\|_{\infty,I_i} &\leq Ch_i^r \|D^r Kv\|_{\infty,I_i} \leq Ch_i^r x_{i-1}^{-r} \|s^r(D^r Kv)(s)\|_{\infty,I_i} \\ &\leq C(1/i*)^r \|s^r(D^r Kv)(s)\|_{\infty,I_i} \ . \end{aligned} \tag{2.4}$$

Now, by assumption (2.1) the Mellin convolution operator $s^r D^r K$ (with kernel $s^r D^r \kappa$) is bounded on L_∞, and (2.4) proves the lemma. □

From this we can prove the stability of (2.3) using the Banach lemma by taking $i*$ sufficiently large:

Theorem 2.2 *There exists $i* \geq 0$ such that for all n sufficiently large, the modified collocation equations (2.3) have a unique solution u_n and satisfy the error estimate $\|u - u_n\|_\infty \leq C\|u - P_n T^{x_{i*}}u\|_\infty$.* □

If $u \in C^{r,\alpha}$ with $0 < \alpha \leq 1$ and $u(0) = 0$ then, as described in §1, the above error estimate implies convergence with optimal order $O(n^{-r})$ provided $q \geq r/\alpha$. Note the philosophy of the argument: Lemma 2.1 shows that there exists a modification parameter $i*$ (fixed with respect to n) which ensures stability. Then Theorem 2.2 shows that the resulting modified method converges optimally provided the mesh is appropriately graded. The choice of any fixed $i*$ for stability does not affect the rate of convergence as $n \to \infty$, although it does affect the asymptotic constant in the error estimate.

3 Further Results for Second-kind Equations

We now give a survey of results on piecewise polynomial collocation methods and their iterated and discrete versions for equation (1.3). Using graded meshes and modified spline spaces as described in the previous section, it is possible to obtain stability (provided $(I - K)$ is well-posed) and the same optimal orders of convergence as in the case of second-kind equations with smooth kernels. We present here general convergence results in the space $L_p = L_p(0,1)$, $1 \leq p \leq \infty$, for which we need the following assumptions:

(A1) For all $k \geq 0$, $\int_0^\infty s^{1/p+k} |D^k \kappa(s)| ds/s < \infty$.

(A2) The symbol $1 - \tilde{\kappa}(z)$ does not vanish on $\mathrm{Re}(z) = 1/p$, and the winding number of this function with respect to the origin is equal to 0.

(A3) For some $1 \geq \alpha > -1/p$, $u \in C^{k,\alpha}$ for all k.

Note that (A1) (with $k = 0$) ensures that K is bounded on L_p and $\tilde{\kappa}(z)$ is a continuous function on $\mathrm{Re}(z) = 1/p$ vanishing at infinity. It turns out that (A2) is then equivalent to each of the conditions (i) and (ii) in (1.4) for the L_p norm. This follows from known results on Wiener-Hopf integral equations (see [38]). The assumption (A3) holds if the right-hand side f of (1.3) is (infinitely) smooth on $[0,1]$ and $(1 - \tilde{\kappa}(z))^{-1}$ is analytic in the strip $-\alpha \leq \mathrm{Re}(z) \leq 1/p$; see [20] for precise regularity results. We first consider the modified collocation method (2.3) again and extend Theorem 2.2 to the L_p case [19].

Theorem 3.1 *Assume the mesh $\{x_i\}$ is q-graded at 0, and suppose $i*$ is sufficiently large. Then the collocation method (2.3) is stable in L_p. It converges in L_p with optimal order $O(n^{-r})$ provided $q > r/(\alpha + 1/p)$ (although when $\alpha \in [0,1]$, the additional assumption $u(0) = 0$ is required for convergence).*

The proof is analogous to that of Theorem 2.2. Note that the crucial estimate (2.4) (for the L_p norm) follows from the boundedness of the operators $s^r D^r K$ (ensured by (A1)) and the standard *local approximation property* of $S_n^{r,-1}$, i.e. $\|(I - P_n)v\|_{p,I_i} \leq C h_i^r \|D^r v\|_{p,I_i}$, with C independent of i, n and v (see, e.g.

[46]). To obtain consistency of the method in the case $\alpha \in [0,1], u(0) \neq 0$, more general modifications of the spline spaces instead of the simple cut-off by zero on $[0, x_{i*}]$ should be used; see [10] for a version including the piecewise constants on the first $i*$ subintervals and [20] for a method based on splines from $S_n^{r,0}$ which reduce to a (global) constant on $[0, x_{i*}]$. However, in general the stability of these methods cannot be obtained from (1.4) (ii) by small perturbation. To get around this problem, either an additional condition on the norm of K should be imposed ([10]) or another approach based on Wiener-Hopf factorization can be employed ([20]).

As mentioned in the introduction, in several important classical second kind boundary integral equations on corner domains (which have localisation of form (1.3) - see §4), the operator to be approximated turns out to be strongly elliptic and the Galerkin method is stable without modification in the energy norm. If one wants to prove convergence in other norms (e.g. the uniform norm) more delicate analyses are needed (e.g. [6,7,9]). More generally, for problem (1.3) under the assumptions (A1) - (A3) above, we must again consider modifications in order to prove stability even for the Galerkin method (see [18,20]). Indeed the unmodified method is in general unstable for operators satisfying only (A2) ([20]).

The collocation method is of practical interest because its implementation requires less numerical integration than the Galerkin method. However even the collocation method generally requires quadrature for its implementation, and this should be included in an error anaylsis. Thus we now discuss a fully discrete version of the collocation method (2.3), which also turns out to be closely related to the classical Nyström method. To define this method, introduce an r point interpolatory quadrature rule on $[0,1] : \int_0^1 v \cong \sum_{j=1}^r \omega_j v(\xi_j)$ with weights ω_j and points $0 \leq \xi_1 < \ldots < \xi_r \leq 1$. Let R be the *order* of this rule so that $R \geq r$ and $R = 2r$ if and only if ξ_j are the r Gauss-Legendre points on $[0,1]$. Define x_{ij} as in §2 and set $Q = \{(i,j) : i = i*+1, \ldots, n;\ j = 1, \ldots, r\}$. Then the (modified) *composite quadrature rule* obtained by shifting the above rule on $[0,1]$ to each I_i, and summing over $i > i*$, is $\int_0^1 v \cong \sum_Q \omega_j v(x_{ij}) h_i$. The integral operator K in (1.3) will be approximated by

$$K_n v(s) = \sum_Q \omega_j \kappa(s/x_{ij}) v(x_{ij}) h_i / x_{ij} \ . \tag{3.1}$$

The (modified) *discrete collocation method* for (1.3) seeks an approximate solution $u_n \in S_n^{r,-1}$ satisfying $u_n \equiv 0$ on $[0, x_{i*}]$ such that

$$(I - P_n T^{x_{i*}} K_n) u_n = P_n T^{x_{i*}} f \ , \tag{3.2}$$

where P_n is the interpolatory projection defined in §2. The *Nyström* (or *discrete iterated collocation*) solution $u_n^*(s)$ to (1.3) is then defined by $u_n^* = f + K_n u_n$, and it satisfies

$$(I - K_n)u_n^* = f \ ; \tag{3.3}$$

note that $P_n T^{x_{i*}} u_n^* = u_n$. By collocation at $s = x_{ij}$, $(i,j) \in Q$, (3.3) is reduced to the linear system (3.2) for $u_n^*(x_{ij}) = u_n(x_{ij})$, the solution of which in turn gives $u_n^*(s)$ for all $s \in [0,1]$. The following result extends Theorem 3.1 to the discrete collocation method (3.2) and establishes *superconvergence* for the Nyström method (3.3).

Theorem 3.2 *Under the assumptions of the preceding theorem, the method (3.2) is stable and optimally convergent in L_p. Moreover, if the grading exponent satisfies the (possibly stronger) requirement $q > R/(\alpha + 1/p)$, then the Nyström solution converges with the error bound $\|u - u_n^*\|_p = O(n^{-R})$ as $n \to \infty$.*

For details of the proof of Theorem 3.2, we refer to [19,27]. To give a brief overview of the proof, we remark that the stability of (3.2) can be obtained from that of the collocation method by small perturbation in the operator norm, as described in [19]. It is also possible to approach (3.3) directly in the case $p = \infty$. In [27] it is shown that the operator K_n defined in (3.1) is uniformly bounded on C. This allows a more straightforward approach to stability by regarding $I - K_n$ as a small perturbation of the finite section operator $I - KT^{x_{i*}}$. The error bound for the Nyström solution follows from the estimate $\|u - u_n^*\|_p \leq C\|(K - K_n)u\|_p$, where the last term is of order $O(n^{-R})$, provided that $u \in C^{R,\alpha}$, for $0 \leq \alpha \leq 1$, $u(0) = 0$, and the grading exponent satisfies $q \geq R/\alpha$.

For the model problem (1.3) it is simple to extend all the above methods to the case when $u(0) \neq 0$. Using (1.3) together with (1.2), it follows that $(1 - \tilde{\kappa}(0))u(0) = f(0)$. Then it is easy to see that the function $v := u - u(0)$ satisfies $v(0) = 0$ and can be computed by solving (1.3) with the modified right-hand side $f(s) - f(0)(1 - K1(s))/(1 - \tilde{\kappa}(0))$ (where 1 is the unit function on $[0,1]$). In more general situations (such as the second-kind boundary integral equations described in §4), equation (1.3) appears only as a localised model problem in a coupled system and in this context it is not possible to compute $u(0)$ explicitly. Nevertheless, stable and consistent methods can be constructed by considering appropriate extended systems [27].

All the results mentioned in this section can be generalised to systems of equations of the form (1.3). In particular, the stability of the methods can be again obtained from the stability of the finite section operators by small perturbation. However, for matrix operators, condition (1.4)(ii) is no longer equivalent to the well–posedness of $(I - K)$ and requires the invertibility of an additional Mellin convolution operator; see [38] for a discussion of this in the case of Wiener–Hopf operators. Fortunately, there is an important special case where (1.4) (ii) is always satisfied in the L_2 norm, namely the case of a strongly elliptic (matrix) symbol, i.e. $\mathrm{Re}(I-\tilde{\kappa}(z))$ is uniformly positive definite for $\mathrm{Re}(z) = 1/2$. Together with (A1) and (A3) (for $p = 2$), this implies stability and optimal convergence for the modified collocation and quadrature methods, whereas Galerkin's method is of course stable with $i* = 0$. We indicate an important application of this technique in §5.

Finally we want to emphasise that the simple perturbation argument presented in Lemma 2.1 is restricted to the case of continuous symbols. The stability analysis of more general classes of convolution operators (containing singular integral operators of Cauchy type for example) requires more sophisticated methods combining Mellin and local Banach algebra techniques; see, e.g. [39,38,14,29].

4 Boundary Integral Equations on Corner Domains

Boundary value problems for linear elliptic PDE's can be reduced to boundary integral equations through the use of a fundamental solution. For Laplace's equation in $2D$ this is the function $G(\mathbf{x},\mathbf{y}) = (2\pi)^{-1}\log|\mathbf{x}-\mathbf{y}|^{-1}$. If U satisfies Laplace's equation in a bounded polygonal domain Ω with boundary Γ then the Cauchy data $u := U|_\Gamma$ and $v := \partial_n U|_\Gamma$ satisfy *Green's identity*

$$\mathcal{V}v(\mathbf{x}) - \mathcal{W}u(\mathbf{x}) = -(1/2)u(\mathbf{x}), \quad \mathbf{x} \in \Gamma ,$$

for all smooth points $\mathbf{x}$ of Γ, where $\mathcal{V}v(\mathbf{x}) = \int_\Gamma G(\mathbf{x},\mathbf{y})v(\mathbf{y})d\Gamma(\mathbf{y})$ is the single layer potential, $\mathcal{W}u(\mathbf{x}) = \int_\Gamma \partial_{n(\mathbf{y})}G(\mathbf{x},\mathbf{y})u(\mathbf{y})d\Gamma(\mathbf{y})$ is the double layer potential, and ∂_n denotes differentiation in the outward normal direction from Ω. This identity can be extended to all $\mathbf{x} \in \Gamma$ by taking appropriate limits. An analogous relation holds for exterior problems. For the Neumann problem, with v given, we have to solve the second-kind equation

$$u(\mathbf{x}) - 2\mathcal{W}u(\mathbf{x}) = g(\mathbf{x}) := -2\mathcal{V}u(\mathbf{x}) , \tag{4.1}$$

for the Dirichlet data u. For the Dirichlet problem with u given, we have to solve the first-kind equation

$$2\mathcal{V}v(\mathbf{x}) = g(\mathbf{x}) := -u(\mathbf{x}) + 2\mathcal{W}u(\mathbf{x}) \tag{4.2}$$

for the Neumann data, and for mixed Dirichlet-Neumann conditions a first-second kind system arises. Analogous equations arise from the classical indirect boundary integral method [2]. A rigorous justification of the underlying potential theory in non-smooth domains can be found in [12]. The method is of course applicable to much more general PDEs (e.g. [50]).

To see how the model problem (1.3) arises from these applications, consider the case that Γ is (infinitely) smooth with the exception of a corner, without loss of generality situated at the origin $\mathbf{0}$. We further assume that Γ in the neighbourhood of $\mathbf{0}$ consists of two straight lines intersecting with an interior angle $(1-\chi)\pi$, $0 < |\chi| < 1$. Consider a parametrisation $\boldsymbol{\gamma}(s) : [-\pi, \pi] \to \Gamma$, $|\boldsymbol{\gamma}'(s)| > 0$ for $s \in [\pi, \pi]$, which near $s = 0$ may be given by

$$\boldsymbol{\gamma}(s) = \begin{cases} (-\cos\chi\pi, \sin\chi\pi)|s| & , \quad s \in [-\epsilon, 0] \\ (1,0)|s| & , \quad s \in [0, \epsilon]. \end{cases} \tag{4.3}$$

Considering first of all the relatively straightforward case (4.1), let ψ be a smooth function on Γ with $\psi(\mathbf{x}) \equiv 1$ when $|\mathbf{x}| \leq \epsilon/2$ and $\psi(\mathbf{x}) \equiv 0$ when $|\mathbf{x}| > \epsilon$ and observe that $2\mathcal{W} - \psi 2\mathcal{W}\psi$ is an operator with smooth kernel. The behaviour of (4.1) is thus dominated by the localised operator, $I - \psi 2\mathcal{W}\psi$. A short calculation shows that

$$(\psi 2\mathcal{W}\psi)u(\boldsymbol{\gamma}(s)) = \begin{cases} \int\limits_0^\epsilon \kappa(\frac{s}{\sigma}) u(\gamma(\sigma)) \frac{d\sigma}{\sigma} & , \quad s \in [-\epsilon, 0] \\ -\int\limits_{-\epsilon}^0 \kappa(\frac{s}{\sigma}) u(\gamma(\sigma)) \frac{d\sigma}{\sigma} & , \quad s \in [0, \epsilon] \ , \end{cases}$$

where

$$\kappa(s) := \frac{\sin\chi\pi}{\pi} \left\{ \frac{s}{1 - 2s\cos\chi\pi + s^2} \right\} .$$

Thus $\psi 2\mathcal{W}\psi$ corresponds to a matrix of operators of form (1.1) and the analysis of (1.3) is the key to understanding (4.1). The above argument can be extended to the case of many corners in an obvious way.

Now let us turn to the first-kind equation (4.2). The connection to (1.3) here is much less obvious. With the parametrisation $\boldsymbol{\gamma} : [-\pi, \pi] \to \Gamma$ introduced above, we can write

$$(2\mathcal{V}u)(\boldsymbol{\gamma}(s)) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \log \frac{1}{|\boldsymbol{\gamma}(s) - \boldsymbol{\gamma}(\sigma)|} w(\sigma) d\sigma =: Vw(s) \ , \tag{4.4}$$

where $w(\sigma) = u(\gamma(\sigma))|\boldsymbol{\gamma}'(\sigma)|$. (Note that here we take the Jacobian into the unknown. As indicated in §1, this is useful when nonlinear parametrisations are used to treat corner singularities.) In the theory of V a special role is played by the operator

$$Aw(s) := \frac{1}{\pi}\int_{-\pi}^{\pi} \log\frac{1}{|2\sin(s-\sigma)/2|} w(\sigma)d\sigma + Jw \ , \tag{4.5}$$

where $Jw = (1/2\pi)\int_{-\pi}^{\pi} w(\sigma)d\sigma$. The first term in the expression for A is simply the operator V restricted to the unit circle $\boldsymbol{\gamma}(s) = (\cos s, \sin s)$. The additional compact perturbation J is added to make A invertible with the result that A is an isometry from H^k onto H^{k+1} for all k (where H^k denotes the usual $2\pi-$ periodic Sobolev space of order k). It is a special feature of V that it can in some sense be conveniently regularised by the operator A^{-1}. More precisely we can write

$$A^{-1}Vw = A^{-1}(A + (V-A))v =: (I+M)v \ , \tag{4.6}$$

where $M = A^{-1}(V-A)$ and

$$(V-A)w(s) = \frac{1}{\pi}\int_{\Gamma} \log\frac{|2\sin(s-\sigma)/2|}{|\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(\sigma)|} w(\sigma)d\sigma - Jw \ .$$

When Γ is smooth (e.g. C^∞), the kernel of the first term in $V-A$ has a removable singularity and it can be shown that the operator $(V-A)$ maps L_2 to H^k for all $k \geq 0$ and hence that M is compact from L_2 to H^k for all k. Thus in the smooth case the first kind equation (4.2) is equivalent to the nonstandard second kind equation

$$(I+M)w = f := A^{-1}g \ . \tag{4.7}$$

When Γ is polygonal the regularization (4.6) can still be carried out, but the resulting operator M is no longer compact. In fact local to each corner of Γ, M turns out to be composed of Mellin convolution operators of the form (1.1). To see this we need some more detail about the operator A. We have the well–known relations (see e.g. [38]) $DA = H$ and $A^{-1} = -HD + J$, where H is the 2π-periodic Hilbert transform $Hv(s) = -(2\pi)^{-1}\int_{-\pi}^{\pi} \cot\left((s-\sigma)/2\right) v(\sigma)d\sigma$ (with the integral to be interpreted in the Cauchy Principle Value sense) and D is the 2π-periodic differentiation operator. Hence the essential behaviour of M near each corner can be found by studying $HD(V-A)$. To compute this, we observe that

$$DVw(s) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{(\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(\sigma))\cdot\boldsymbol{\gamma}'(s)}{|\boldsymbol{\gamma}(s)-\boldsymbol{\gamma}(\sigma)|^2} w(\sigma)d\sigma \ .$$

For w locally supported near $\mathbf{0}$, i.e. supp $w \subset [-\epsilon, \epsilon]$, we have the representation

$$DVw(s) = \begin{cases} -\dfrac{1}{\pi}\displaystyle\int_{-\epsilon}^{0} \dfrac{w(\sigma)}{s-\sigma} d\sigma + \displaystyle\int_{0}^{\epsilon} \kappa_1\left(\dfrac{s}{\sigma}\right) w(\sigma)\dfrac{d\sigma}{\sigma} \quad , \quad s \in [-\epsilon, 0] \\ \displaystyle\int_{-\epsilon}^{0} \kappa_1\left(\dfrac{s}{\sigma}\right) w(\sigma)\dfrac{d\sigma}{\sigma} - \dfrac{1}{\pi}\displaystyle\int_{0}^{\epsilon} \dfrac{w(\sigma)}{s-\sigma} d\sigma \quad , \quad s \in [0, \epsilon] \end{cases}$$

where

$$\kappa_1(s) = \frac{1}{\pi}\left\{\frac{\cos \chi\pi - s}{1 - 2s\cos\chi\pi + s^2}\right\} .$$

This calculation, which shows that $D(V - A) = DV - H$ can be represented as a matrix of operators of the form (1.1), was first given in [53] and shows that $A^{-1}(V - A)$ is represented (local to each corner) as a product of H with operators of the form (1.1). From this a numerical analysis of collocation methods followed [52,28]. However this analysis was somewhat restricted, mainly because $M = A^{-1}(V - A) = -HD(V - A)$ (modulo compact operators) and although the operator $D(V - A)$ was well-understood (as above) the important product $HD(V - A)$ was not. In [21] this product was computed using the symbolic calculus for Mellin operators. This is possible since (local to the corner $\mathbf{0}$) we can write (modulo a compact operator)

$$Hv(s) = \int_{-\epsilon}^{\epsilon} \frac{1}{s-\sigma} v(\sigma) d\sigma = \int_{-\epsilon}^{\epsilon} (s/\sigma - 1)^{-1} v(\sigma)\frac{d\sigma}{\sigma} , \quad s \in [-\epsilon, \epsilon]$$

which can be also treated using the Mellin transform. In fact in [21] more general results than this were obtained. Following the "parametrisation" method for handling singularities outlined in §1, [21] considered parametrisations of Γ which varied more slowly than arc length near each corner. An example is to parametrise Γ near $\mathbf{0}$ by replacing $|s|$ with $|s|^q$ in (4.3). The above calculation first of V in (4.4) (which now depends on q) and of $A^{-1}(V - A)$ can again be performed and yields again a representation of M near each corner involving operators of the form (1.1).

In this section we have shown that model problem (1.3) arises in both standard and non-standard ways from localisations of boundary integral equations on non-smooth domains. (Here we have restricted to the Laplace equation but similar local problems arise, for example, from the Helmholtz equation [11] and in linear elasticity [24].) For the classical second-kind boundary integral equations (such as (4.1)) on polygonal domains it is possible to give a complete error analysis of (modified) methods, using the knowledge of numerical methods for the local model problem (1.3) outlined in §2, 3 - see, for example

[27]. However for first kind equations such as (4.2) which are connected to the model problem (1.3) in less standard way, the numerical analysis is more complicated. In the final section we give a brief survey of this area with pointers to the literature where the reader can find more details.

5 Results for First-kind Equations

We first discuss the numerical solution of Symm's integral equation (4.2) on polygonal domains by high order spline collocation methods. To approximate the singularities of solutions at the corner points, the first idea that comes to one's mind is to attack this equation directly by using splines on graded meshes as in the case of the double layer potential equation (4.1). This approach was taken in [13] where stability and optimal convergence rates for piecewise linear break–point collocation were proved with respect to a weighted Sobolev norm. So far these results have not been generalized to higher order splines.

On the other hand, if Γ is smooth then the operator in (4.2) is a classical periodic pseudodifferential operator, and thus the full force of the general convergence theory developed in [1,44,45] for collocation methods with smooth splines (mostly) on uniform grids becomes available; see also the review in [48] for these and related methods and the detailed presentation in [38]. Although the piecewise constant mid–point collocation method was shown to converge for quite general meshes in [8], this analysis is restricted to smooth boundaries and there is still no general convergence analysis for (4.2) for general boundaries and general piecewise polynomial approximation schemes.

This situation essentially motivated the approach in [21] where the use of a nonlinear parametrisation (or mesh grading transformation) of the boundary curve together with a uniform mesh has allowed a first stability and convergence analysis of high order collocation methods in the presence of corners.

To illustrate this type of result, we retain the notation of the preceding section and parametrise the boundary Γ with one corner at $\mathbf{0}$ by $\boldsymbol{\gamma} : [-\pi, \pi] \to \Gamma$ such that $\boldsymbol{\gamma}(0) = \mathbf{0}$, and, near $s = 0$,

$$\boldsymbol{\gamma}(s) = \begin{cases} (-\cos \chi\pi, \sin \chi\pi)|s|^q \quad , & s \in [-\epsilon, 0] \\ (1,0)|s|^q \quad , & s \in [0, \epsilon] \ . \end{cases} \tag{5.1}$$

Here the grading exponent q is an integer ≥ 1. The equation (4.2) transforms to

$$Vw(s) = g(s) \, , \quad s \in [-\pi, \pi] \ , \tag{5.2}$$

where V and w are defined as in (4.4) (but using the nonlinear parametrisation (5.1)), and $g(s) := g(\boldsymbol{\gamma}(s))$. By appropriate choice of q, the solution w of (5.2) can be made smooth local to the corner (provided g is smooth), and hence w can be optimally approximated using splines from $S^r_{n,p}$ (the 2π-periodic smoothest splines of degree $r-1$ on the uniform mesh $x_i = ih$, $i = 0, \ldots, n$, with meshsize $h = 2\pi/n$). To discretise (5.2), introduce the interpolant $Q_n v \in S^r_{n,p}$ by requiring

(a) when r is odd $\quad Q_n v(t_i) = v(t_i), \;\; i = 1, \ldots, n;$
(b) when r is even $\quad Q_n v(x_i) = v(x_i), \; i = 0, \ldots, n-1,$

where t_i are the mid-points of subintervals. Then the collocation method for (5.2) seeks $w_n \in S^r_{n,p}$ such that

$$Q_n V w_n = Q_n g \; . \tag{5.3}$$

The approach to the analysis of (5.3) is analogous to that used in §4 where (4.2) is transformed to the non–standard second kind equation (4.7). In fact (5.3) can be rewritten as a non–standard projection method for (4.7) as follows. For any $v \in H^0$, let $P_n v \in S^r_{n,p}$ solve the collocation equations $Q_n A(P_n v) = Q_n A v$ for the circle operator A defined in (4.5). It is well–known (see [38, pp. 492-493] and the references listed at the beginning of this section) that this prescription defines a (uniformly) bounded projection operator $P_n : H^0 \to S^r_{n,p}$. It is then straightforward to see that (5.3) is equivalent to

$$(I + P_n M) w_n = P_n f \;, \quad \text{with} \quad M = A^{-1}(V - A)\,, \; f = A^{-1} g \; . \tag{5.4}$$

To overcome the difficulty in the stability analysis of (5.3), or equivalently (5.4), one may introduce an analogous cut-off procedure in the vicinity of the corner as in the case of the model second kind equation (1.3). To describe the modification, introduce the truncation $T^\tau v$ as $T^\tau v(s) = 0$, for $|s| < \tau$, and $T^\tau v(s) = v(s)$ for $\tau < |s| < \pi$. Then, for any fixed $i* \geq 0$, consider the method

$$Q_n (A + (V - A) T^{i*h}) w_n = Q_n g \;, \tag{5.5}$$

which coincides with (5.3) when $i* = 0$. By mimicking the derivation of (5.4) from (5.3), it is easily seen that (5.5) is equivalent to

$$(I + P_n M T^{i*h}) w_n = P_n f \; . \tag{5.6}$$

Applying the technique outlined in §2,3 to the projection method (5.6) and employing the (non–trivial) Mellin analysis of the operator M discussed in the previous section, one then can prove the following convergence result for the modified collocation method [21].

Theorem 5.1 *Suppose the grading exponent satisfies $q > (r+1/2)(1+|\chi|)$, where $(1-\chi)\pi$ is the interior angle at the corner. Then there exists $i*$ such that (5.5) has a unique solution for all n sufficiently large and is optimally convergent in the L_2 norm, i.e. $\|w - w_n\|_2 = O(n^{-r})$ as $n \to \infty$.*

A crucial prerequisite for this result is the strong ellipticity of the second kind operator $I + M$, i.e. $\mathrm{Re}(I + M)$ is positive definite in $H^0 = L_2$, modulo compact operators. Together with a uniqueness result for the transformed integral equation (5.2), this implies the analogue of (1.4) in this setting, i.e. the well–posedness of $I + M$ and the stability of the finite section operators $I + MT^{i*h}$ (as $h \to 0$) in H^0. The final step in the stability proof for (5.6) is again a perturbation argument similar to that of Lemma 2.1, which however requires a thorough study of the Mellin convolution kerenel of the operator M localised to the corner (see [21], with improvements given in [23]). The optimal error estimate then follows from standard spline approximation results since, as it was also shown in [21], the solution of (5.2) satisfies $w \in H^r$ and has appropriate decay as $s \to 0$ provided the grading exponent q is sufficiently large.

The above stability and convergence results may be extended to various related parametrisation methods and to other first kind equations on polygonal boundaries. In [23] it was shown that Theorem 5.1 remains true when the collocation integrals are approximated using singularity subtraction and a suitable composite quadrature rule. A fully discrete trigonometric collocation method is given in [26]. This method is based on the trapezoidal rule and is easier to implement than the quadrature–collocation scheme of [23]. More general results on discrete qualocation methods can be found in [31]. Parametrisation methods based on global algebraic polynomials have recently been applied to Symm's equation [35] and to the generalized airfoil equation for an airfoil with a flap [36,47]. [25] presents a convergence analysis of the trigonometric collocation method applied to mixed boundary value problems on corner domains.

In conclusion we remark that the numerical analysis of these 2D corner problems is still not as satisfactory as in the case of smooth boundaries where even fully discrete high order methods of almost linear computational complexity are known. However, fast solution methods for classical first kind integral equations on open arcs have recently been obtained applying the cosine transform and discrete trigonometric collocation (see [43] and the references there). The development of analogous methods for more general problems on corner domains remains a challenge for the future.

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