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Adaptive detection of high-dimensional signal

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Abstract

Let n-dimensional Gaussian random vector $x = \xi + v$ be observed where ξ is a standard n-dimensional Gaussian vector and $v \in R^n$ is the unknown mean. In the papers [3, 5] there were studied minimax hypothesis testing problems: to test null - hypothesis $H_0 : v = 0$ against two types of alternatives $H_1 = H_1(\theta_n) : v \in V_n(\theta_n)$. The first one corresponds to multi-channels signal detection problem for given value b of a signal and number k of channels containing a signal, $\theta_n = (b, k)$. The second one corresponds to l_q^n-ball of radius $R_{1,n}$ with the l_p^n-ball of radius $R_{2,n}$ removed, $\theta_n = (R_{1,n}, R_{2,n}, p, q) \in R_+^4$. It was shown in [3, 5] that often there are essential dependences of the structure of asymptotically minimax tests and of the asymptotics of the minimax second kind errors on parameters θ_n. These imply the problem: to construct adaptive tests having good minimax property for large enough regions Θ_n of parameters θ_n.

This problem is studied here. We describe the sets Θ_n such that adaptation is possible without loss of efficiency. For other sets we present wide enough class of asymptotically exact bounds of adaptive efficiency and construct asymptotically minimax test procedures.

1 Statement of the problems

Let n-dimensional Gaussian random vector $x = \xi + v$ be observed where ξ is a standard n-dimensional Gaussian vector with zero mean and identity covariance matrix and $v \in R^n$ is the unknown mean. In the papers [3, 5] the asymptotically minimax hypothesis testing problems have been studied: to test null-hypothesis $H_0 : v = 0$ against two type of alternatives $H_1 = H_1(\theta_n) : v \in V_n(\theta_n) \subset R_n$.

The first type we call as *multi-channels problem* (MCP). Here the set $V_n(\theta_n)$ is finite collection of vectors

$$v = (v_1, ..., v_n) : \ v_i = 0 \text{ or } v_i = \pm b; \ \sum_i |v_i| = kb; \ k = k_n, \ b = b_n, \ \theta_n = (b, k).$$

The component v_i is an signal in i-th channel, the case $v_i = \pm b$ corresponds to a signal of level b, the value k, $1 \leq k \leq n$ is the number of channels containing a signal. Denote

$$\Theta_n^* = \{(b, k) : \ 0 < b, \ 1 \leq k \leq n, \ k \text{ is an integer}\}.$$

The second type we call as *balls-problem* (BP). Here the set $V_n(\theta_n)$ is the l_q^n-ball of radius $R_{1,n}$ with the l_p^n-ball of radius $R_{2,n}$ removed:

$$V_n(\theta_n) = \{v = (v_1, \dots, v_n) \in R^n : \sum_{i=1}^n |v_i|^p \geq R_{1,n}^p, \sum_{i=1}^n |v_i|^q \leq R_{2,n}^q\},$$

where $\theta_n = (R_{1,n}, R_{2,n}, p, q)$; the values p, q are define a "shape" of the balls, values $R_{1,n} > 0, R_{2,n} > 0$ are the radii. It is assumed

$$R_{1,n} \leq R_{2,n} \quad \text{for} \quad p > q; \quad R_{1,n} n^{-1/p} \leq R_{2,n} n^{-1/q} \quad \text{for} \quad p \leq q, \tag{1.1}$$

which imply that the sets V_n are nonempty. Sometimes it is more convenient to assume some stronger constraints: for some $c \in (0, 1)$

$$R_{1,n}/R_{2,n} \leq c \quad \text{for} \quad p > q; \quad R_{1,n}/R_{2,n} \leq c n^{1/p - 1/q} \quad \text{for} \quad p \leq q. \tag{1.2}$$

Denote as Θ_n^* the set $\theta_n \in R_+^4$ satisfying to (1.1) or (1.2). [1]

There are some points that motivate our interest in BP.

1. There are many practical problems, where data and unknown parameters are of large dimension. The problem under consideration seems to be the most natural minimax hypothesis testing problem of increasing dimension.

2. This problem is related to infinite-dimensional hypothesis testing problems about a signal in a white Gaussian noise or about the mean of an infinite-dimensional Gaussian random vector.

We deal with asymptotically minimax hypothesis testing problem. Let $\Psi_{n,\alpha}$ be the set of level α tests, $\alpha \in (0, 1)$, i.e. the set of measurable functions $\psi : R^n \to [0, 1]$ such that $\alpha(\psi) \leq \alpha$, where $\alpha(\psi) = E_{n,0}\psi$ is the first kind error. Here and below $E_{n,v}$ means the expectation with respect to the Gaussian measure $P_{n,v}$ with mean v and identity covariance matrix.

Let $\beta_n(\psi, v) = E_{n,v}(1 - \psi)$ be the second kind error and let $\beta_n(\psi, V_n) = \sup_{v \in V_n} \beta_n(\psi, v)$ be its maximum value for test ψ. Let

$$\beta_n(\alpha) = \beta_n(\alpha, V_n) = \inf_{\psi \in \Psi_{n,\alpha}} \beta_n(\psi, V_n) \tag{1.3}$$

be the minimax second kind error. It is clear that $0 \leq \beta_n(\alpha) \leq 1 - \alpha$.

The problem is to study sharp asymptotics of $\beta_n(\alpha, V_n(\theta_n))$ on θ_n [2] for any $\alpha \in (0, 1)$ and the structure of asymptotically minimax tests $\psi_{n,\theta_n,\alpha}$ such that

$$\alpha_n(\psi_{n,\theta_n,\alpha}) \leq \alpha + o(1), \quad \beta_n(\psi_{n,\theta_n,\alpha}) \leq \beta_n(\alpha, V_n(\theta_n)) + o(1).$$

[1] We use the same notations for both problems. Below it will be clear from context what problem is considered.

[2] It was assumed in [5] that p, q are fixed. However the results of [5] are uniform on any compacts. $K = \{(p, q)\} \subset R_+^2$ In fact, the main point using any specific relationship between p, q is the proofs of Lemmas 5.2, 6.1 – 6.3 where $p > q$ is assumed. However small modification of the proofs given in the proof of Lemmas 3.1 and 5.3 below provides estimations which are uniform on $0 < q < p \leq p_n$ for any $p_n = o(\log n)$.

Also it is of interest to study distinguishability conditions: $\beta_n(\alpha, V_n(\theta_n)) \to 1-\alpha$ or $\beta_n(\alpha, V_n(\theta_n)) \to 0;\ n \to \infty$ and to construct minimax consistent tests $\psi_{n,\theta_n,\alpha}$ such that $\beta_n(\psi_{n,\theta_n,\alpha}) \to 0$, if it is possible.

These problems were studied in [3, 5]. Following to [3] consider Bayesian hypothesis testing problem $H_0 : P = P_0$ versus $H_{\pi^n} : P = P_{\pi^n}$ on a distribution P of observed random vector $x \in R^n$. Let $\beta_n(\alpha, P_{\pi^n})$ is minimum second kind error in Bayesian problem. Consider product-priors $\pi^n = \pi^n(b, h)$ on $(R^n, \mathcal{B})$:

$$\pi^n = \pi^n(dv; b, h) = \prod_{i=1}^{n} \pi(dv_i; b, h). \tag{1.4}$$

Here

$$\pi(b, h) = \pi(dt; b, h) = (1-h)\delta_0(dt) + \frac{h}{2}(\delta_b(dt) + \delta_{-b}(dt)) \tag{1.5}$$

where δ_t is Dirac mass at the point $t \in R^1$. These priors correspond to random binomially distributed number k of channels containing a signal in MCP.

It was shown in [3] that the priors $\pi^n(b, h),\ h = k/n$ are asymptotically least favorable in MCP with $k \to \infty$:

$$\beta_n(\alpha, V_n(\theta_n)) = \beta_n(\alpha, P_{\pi^n}) + o(1) \tag{1.6}$$

Moreover, it was shown in [5] that by the special choose of the sequences $b = b_n = b_n(\theta_n) > 0,\ h = h_n = h_n(\theta_n)$ we obtain the priors of the type (1.4), (1.5) which are asymptotically least favorable for BP (it means that (1.6) holds as $h_n \to 0$). The choose is the following. Define the sequences

$$\lambda = \lambda_n = \lambda_n(\theta_n) = R_{n,1}/n^{1/p},\ \nu = \nu_n = \nu_n(\theta_n) = R_{n,2}/n^{1/q}$$

and the sequences $b = b_n = b_n(\theta_n) > 0,\ h = h_n = h_n(\theta_n)$. In view of (1.1) we have: either $\lambda \geq \nu$ or $p > q$. There are three possible equalities:

(i) $h = 1, \quad b = \lambda;$
(ii) $h = (\lambda/b(p))^p, \quad b = b(p) \quad$ (for $\quad p > 2$);
(iii) $hb^p = \lambda^p, \quad hb^q = \nu^q.$

Here $b(p) > 0$ for $p > 2$ is the root of equation: $p \tanh(b^2/2) = b^2$. The value $b(p)$ minimizes the function $b^{-p}\sinh(b^2/2)$.

The relations between p, q, λ, ν and (i), (ii), (iii) are described by the following.

Lemma 1.1 *([10]).*

1. *Let $p \leq 2$ and $p \leq q$. Then the relation* (i) *holds.*
2. *Let $p \leq 2$ and $p > q$. Then* (i) *holds if $\lambda < \nu$ and* (iii) *holds if $\lambda \geq \nu$.*

3. *Let* $\infty > p > 2$ *and* $p \le q$. *Then* (i) *holds if* $\lambda > b(p)$, (ii) *holds if* $\lambda \le b(p) \le (\nu^q/\lambda^p)^{1/(q-p)}$ *for* $p < q$ *or* $\lambda \le b(p)$ *for* $p = q$, *and* (iii) *holds if* $\lambda \le b(p)$, $p < q$, $(\nu^q/\lambda^p)^{1/(q-p)} \le b(p)$.

4. *Let* $\infty > p > 2$ *and* $p > q$. *Then* (i) *holds if* $b(p) < \lambda < \nu$, (ii) *holds if* $\lambda \ge \nu$ *and* $(\lambda/b(p))^p \le (\nu/b(p))^q$ *or* $\nu > \lambda$ *and* $\lambda < b(p)$, *and* (iii) *holds if* $\lambda \ge \nu$ *and* $(\lambda/b(p))^p > (\nu/b(p))^q$.

The asymptotics of the values $\beta_n(\alpha, P_{\pi^n})$ have been studied in [3]. Particularly, if b_n are bounded or $b_n^* - b_n \to \infty$ (here and below we denote $b_n^* = \sqrt{(\log n)/2}$), then the log-likelihood ratio are asymptotically Gaussian: $\log(dP_{\pi^n(b_n,h_n)}/dP_{n,0}) \to N(-u_n^2/2, u_n^2)$ under $P_{n,0}$-probability, and $\log(dP_{\pi^n(b_n,h_n)}/dP_{n,0}) \to N(u_n^2/2, u_n^2)$ under P_{π^n}-probability; here the function $u_n = u_n(b_n, h_n)$ is defined by

$$u_n^2 = nE_{1,0}\left(\frac{dP_{\pi(b_n,h_n)}}{dP_{1,0}} - 1\right)^2 = 2nh_n^2 \sinh^2(b_n^2/2). \tag{1.7}$$

These yield the asymptotics

$$\beta_n(\alpha, P_{\pi^n}) = \Phi(t_\alpha - u_n) + o(1). \tag{1.8}$$

Here and below Φ stands for the standard normal distribution function and t_α for its $(1-\alpha)$-quantile. The relations (1.6) and (1.8) imply the rates:

$$\beta_n(\alpha, V_n(\theta_n)) \to 1 - \alpha \ \text{ iff } u_n \to 0; \ \beta_n(\alpha, V_n(\theta_n)) \to 0 \ \text{ iff } u_n \to \infty. \tag{1.9}$$

However if b_n are closed to b_n^* or more than b_n^*, then either we have Gaussian asymptotics, but with differ u_n, or we have asymptotics of differ types defined by special functions $c_n = c_n(b_n, h_n)$ or $\lambda_n = \lambda_n(b_n, h_n)$ (see the next section). The asymptotically minimax tests were constructed in [5] for BP. Usually these depend on θ.

The object of interest in the paper is unknown θ_n in the problems. In fact, often the number of channels containing a signal and a level of signal are unknown in MCP. It is prefer to have tests which are good for wide enough sets of these values. Also in many practical problems an statistician can not choose justified enough constraints on an alternatives. He would like to have test procedures which have good properties for wide enough collections of constraints.

From mathematical point of view it means that we need to study an alternatives type

$$V_n(\Theta_n) = \bigcup_{\theta_n \in \Theta_n} V_n(\theta_n)$$

for wide enough sets $\Theta_n = \{\theta_n\} \subset \Theta_n^*$.

Following to Spokoiny [8] we call this problem as *adaptive*. We will consider "sharp" and "rate" adaptive problems. Sharp problem is to study sharp asymptotics of the values $\beta_n(\alpha, V_n(\Theta_n))$ and to construct asymptotically minimax tests. Rate problem is to study distinguishability conditions in the problem and to construct minimax consistent tests.

The first question in adaptive problem is following: is it possible to construct tests without loss of asymptotic efficiency (with the same asymptotics of second kind errors of with the same distinguishability conditions)? It was shown in [8] that it is impossible for some infinite-dimensional adaptive hypothesis testing problems: the losses are of $\log\log$-type. For wide class of infinite-dimensional adaptive hypothesis testing problems analogous results were obtain in [4, 6, 9]; the rate and sharp adaptive asymptotics have been studied in these papers.

In problems under consideration here we show that adaptation without loss of efficiency is possible for "small" or "large" enough b by using simple enough tests. In the most interesting cases we obtain rate and sharp adaptive asymptotics under general enough conditions. More difficult test procedures are required in these cases.

In the section 2 we remind the main results of [3, 5] which are the basis to study adaptive problems. In the section 3 we formulate main results. In the section 3.4 we consider the example BP with $R_{n,2} = An^a,\ a > 0$ and we are interesting: what is $R_{n,1}$ to obtain distinguishability ? The results show that distinguishability conditions may be very accurate for $p > q,\ aq < 1/2$.

The proofs and test procedures are given Sections 4 – 5. Adaptive lower bounds are proved in Section 4. Adaptive test procedures are constructed and proved in Section 5. The most technical elements of proofs are replaced in Appendix.

We denote as B positive values which do not depend on n and, may be, different.

2 Previous results

Denote

$$\xi(t,b) = \exp(-b^2/2)\cosh bt - 1,\ t \in R^1.$$

The likelihood ratio $L_n = dP_{\pi^n}/dP_{n,0}$ and log-likelihood ratios $l_n = \log L_n$ are of the form:

$$L_n(x) = \prod_{i=1}^{n}(1 + Z_n(x_i)),\ l_n(x) = l_n(x; b_n, h_n) = \sum_{i=1}^{n} W_n(x_i) \tag{2.1}$$

where

$$Z_n(x) = h_n\xi(x, b_n),\ W_n(x) = W_n(x; b_n, h_n) = \log(1+Z_n(x));\ x = (x_1, ..., x_n) \in R^n.$$

It was shown in [3], that there are three different types of asymptotics of the second kind error probabilities $\beta_n(\alpha, P_{\pi^n}) = \beta_n(\alpha, V_n) + o(1)$. They are defined by different types of limit distributions of log-likelihood ratio (2.1). To describe these types define the following sequences.

If $b_n \to \infty$ and $h_n \to 0$, let $T_n = T_n(b_n, h_n) > 0$ be such that

$$h_n \xi(T_n, b_n) = 1. \tag{2.2}$$

Note that

$$T_n \to \infty, \; h_n \sim 2 \exp\left(\frac{b_n^2}{2} - T_n b_n\right) = \tilde{h}_n \tag{2.3}$$

(more exactly, $h_n = \tilde{h}_n(1 + O(\tilde{h}_n))$ and

$$T_n = \frac{b_n}{2} + \frac{\log 2h_n^{-1}}{b_n} + O(h_n b_n^{-1}).$$

Put

$$\tau_n = \tau_n(\theta_n) = \frac{T_n}{b_n} = \frac{1}{2} + \frac{\log 2h_n^{-1}}{b_n^2} + O(h_n b_n^{-2})$$

and assume without loss of generality that $\tau_n \to \tau \in [1/2, \infty]$. If $b_n = O(1)$ or $h_n \asymp 1$, then put $\tau = \infty$.

Let $\tilde{l}_n$ be the sum of T_n-truncated items:

$$\tilde{Z}_n = Z_n \mathbf{1}_{Z_n < 1} = Z_n \mathbf{1}_{|x| < T_n}, \; \tilde{W}_n(x) = \log(1 + \tilde{Z}_n), \tag{2.4}$$

$$\tilde{l}_n(x) = \tilde{l}_n(x; b_n, h_n) = \sum_{i=1}^{n} \tilde{W}_n(x_i). \tag{2.5}$$

Different types of asymptotics correspond to the intervals: $\tau \in [2, \infty]$ (Gaussian type), $\tau \in (1, 2)$ (infinitely-divisible type) and $\tau \in [1/2, 1]$ (degenerate type).

2.1 Gaussian case: $\tau \in [2, \infty]$

Put

$$u_n^2(\theta_n) = u_n^2(b_n, h_n) = n E_{1,0} \tilde{Z}_n^2. \tag{2.6}$$

Denote $d_n = 2b_n - T_n$ and observe (see [3, 5] and sec. 6.2 below) that

$$u_n^2 = 2nh_n^2(\sinh(b_n^2/2))^2 \Phi(-d_n) + o(1) \sim \begin{cases} nh_n^2 b_n^4/2, & \text{if } b_n \to 0 \;, \\ 2nh_n^2(\sinh(b_n^2/2))^2, & \text{if } \tau \in (2, \infty] \;, \\ \frac{1}{2} nh_n^2 e^{b_n^2} \Phi(-d_n), & \text{if } b_n \to \infty \;. \end{cases} \tag{2.7}$$

Then (1.6), (1.8) hold with u_n defined by (2.7). Asymptotically minimax sequence of tests is of the form

$$\psi_{n,\alpha}^{(h_n, b_n)} = \mathbf{1}_{\{l_n > T_{n,\alpha}\} \cup \{\max_i |x_i| > H_n\}}. \tag{2.8}$$

Here H_n is a sequence such that $n\Phi(-H_n) \to 0$, $T_{n,\alpha} = t_\alpha u_n - u_n^2/2$; l_n are statistics of the form (2.1) with, possible, small modification of parameters b_n, h_n. It follows from the proof in [5] that one can replace test procedures (2.8) onto

$$\psi_{n,\alpha}(b_n, h_n) = \mathbf{1}_{\{\lambda_n(b_n,h_n)>t_\alpha\}\cup\{\max_i |x_i|>H_n\}}$$

which is based on the statistics

$$\lambda_n(b_n, h_n) = \frac{h_n}{u_n}\sum_{i=1}^n \xi(x_i, b_n) = (2n\Phi(T_n - 2b_n)\sinh^2(b_n^2/2))^{-1/2}\sum_{i=1}^n \xi(x_i, b_n). \quad (2.9)$$

These test procedures do not depend on h_n for $\tau > 2$.

Moreover, for BP, if $p \le 2$, $p \le q$, then we can replace the values u_n^2 onto $\tilde{u}_n^2 = nb_n^4/2$ and distinguishability are defined by b_n:

$$\beta_n(\alpha, V_n(\theta_n)) \to 1 - \alpha \quad \text{iff} \quad b_n n^{1/4} \to 0;$$
$$\beta_n(\alpha, V_n(\theta_n)) \to 0 \quad \text{iff} \quad b_n n^{1/4} \to \infty.$$

In this case we can construct asymptotically minimax tests which are based on χ^2 statistics:

$$\psi_{n,\alpha}^{(2)} = \mathbf{1}_{\{\lambda_{n,2}>t_\alpha\}}, \text{ where } \lambda_{n,2} = (2n)^{-1/2}\sum_{i=1}^n (x_i^2 - 1). \quad (2.10)$$

In BP denote $\kappa_n = (R_{2,n}, p, q)$; $\theta_n = (R_{1,n}, \kappa_n)$. Analogously with [1], we can define *critical radii* $R_{1,n}^* = R_{1,n}^*(\kappa_n)$ by the relation: $u_n(R_{1,n}^*, \kappa_n) \asymp 1$.

Example 1: bounded b_n (see [5].)

Let $R_{n,2} = An^a$, $a > 0$, $A > 0$. Assume (1.1) and let (1.2) holds if $p > q$ or $2 < p \le q$. Note that $\beta_n(\alpha) \to 0$ for $a < 1/q - 1/4$, $p \le q$, $p \le 2$ or $a < 1/2q$, $2 < p \le q$. Therefore assume $a \ge 1/q - 1/4$ if $p < q$ and $a \ge 1/2q$ if $p > q$. Denote $\eta = (p, q, a, A, C)$ and define the sets

$$\begin{aligned}
\Xi_1 &= \{\eta:\ p \le 2,\ a > \frac{1}{q} - \frac{1}{4} \text{ or } q \ge p,\ p \le 2 \text{ or } q < p \le 2,\ a = \frac{1}{q} - \frac{1}{4},\ A > C\},\\
\Xi_2 &= \{\eta:\ p > 2,\ a > \frac{1}{2q} \text{ or } p > 2,\ a = \frac{1}{2q},\ C^p \le A^q b(p)^{p-q}\},\\
\Xi_3 &= \{\eta:\ q > p > 2,\ \frac{1}{q} - \frac{1}{4} < a < \frac{1}{2q} \text{ or } 2 \ge p > q,\ \frac{1}{2q} \le a < \frac{1}{q} - \frac{1}{4}\\
&\qquad \text{or } q > p > 2,\ a = \frac{1}{2q},\ C^p > A^q b(p)^{p-q} \text{ or } 2 \ge p > q,\ a = \frac{1}{q} - \frac{1}{4},\ A \le C\}.
\end{aligned}$$

Let $R_{n,1} = C_n n^d, C_n \to C \in (0, \infty)$, where

$$d = \begin{cases} 1/p - 1/4, & \text{if } \eta \in \Xi_1,\\ 1/2p, & \text{if } \eta \in \Xi_2,\\ (p - q + 2aq(2-p))/2p(2-q), & \text{if } \eta \in \Xi_3. \end{cases}$$

Then, using the Lemma 1.1 we obtain

$$\begin{cases} b_n = C_n n^{-1/4}, \ h_n = 1, & \text{if } \eta \in \Xi_1 \\ b_n = b(p), \ h_n = (C_n/b(p))^p n^{-1/2}, & \text{if } \eta \in \Xi_2 \\ b_n = n^{-\delta}(C_n^p/A^q)^{1/(p-q)}, \ h_n = n^{\gamma-1}(A/C_n)^{pq/(p-q)}, & \text{if } \eta \in \Xi_3 \end{cases}$$

where for $\eta \in \Xi_3$

$$\gamma = \frac{q(4a-1)}{2(2-q)} \in [0,1]\,, \ \delta = \frac{2aq-1}{2(2-q)} \geq 0,$$

and we get the asymptotics (1.8) with

$$u_n^2 \sim \begin{cases} C^4/2, & \text{if } \ \eta \in \Xi_1, \\ 2(C/b(p))^{2p}\sinh^2(b(p)^2/2), & \text{if } \ \eta \in \Xi_2, \\ C^{2p(2-q)/(p-q)}A^{-2q(2-p)/(p-q)}/2, & \text{if } \ \eta \in \Xi_3\,, \ a \neq 1/2q, \\ 2(A/C)^{2pq/(p-q)}\sinh^2(C^p/A^q)^{2/(p-q)}/2) & \text{if } \ \eta \in \Xi_3\,, \ a = 1/2q. \end{cases}$$

Here critical radii are of the form $R^*_{1,n} = n^d$. Furthermore, we get the distinguishability conditions of the rate form:

$$\beta_n(\alpha) \to 1-\alpha \text{ iff } R_{1,n}/R^*_{1,n} = C_n \to 0,\ \beta_n(\alpha) \to 0 \text{ iff } R_{1,n}/R^*_{1,n} = C_n \to \infty.$$

Example 2: $\tau > 2$ [3]

Let $R_{n,2} = An^a$ and $p > q$, $1/4 < aq < 1/2$. More exactly, denote $\delta_n = 1-2aq$, $\eta_n = 1-4aq$ and assume $V_n = \delta_n \log n \to \infty$, $U_n = \eta_n\sqrt{\log n} \to -\infty$. We would like to obtain $R_{n,1}$ such that $u_n(\theta_n) = H_n + o(1)$. We are interesting in the case $H_n \asymp 1$, however for examples below we assume H_n is bounded away from 0 and $H_n = o(V_n/\log V_n)$, $\log H_n < cU_n^2$, $0 < c < 1/4$.

Take a sequence $b_n^2 \to \infty$, $b_n^2 = O(\log n)$, which is concreted below and put

$$R_{n,1} = A^{q/p} b_n^{1-q/p} n^{aq/p}. \tag{2.11}$$

Then $b = b_n$ corresponds to the relation (iii) and

$$h_n = n^{aq-1}A^q b_n^{-q}, \tag{2.12}$$

Let b_n satisfy

$$b_n^2 = V_n + X_n = \frac{1}{2}(\log n + U_n\sqrt{\log n}) + X_n, \tag{2.13}$$

where

$$X_n = x + y,\ x = q\log b_n^2 - 2q\log A,\ y = \log(2H_n^2) + o(H_n^{-1}), \tag{2.14}$$

which corresponds to the relation

$$nh_n^2 e^{b_n^2} = 2H_n^2 + o(H_n).$$

[3]Examples 2 - 3 contain in [5] for fixed a or U_n. Uniform versions presented here have been studied by I. Suslina.

Assume below $2q \log A = o(\log V_n)$, as $V_n \to \infty$. Then the solution of (2.13) is of the form

$$b_n^2 = V_n + \tilde{X}_n = \tfrac{1}{2}(\log n + U_n \sqrt{\log n}) + \tilde{X}_n, \tag{2.15}$$

$$\tilde{X}_n = \tilde{x} + y + o(H_n^{-1}), \ \tilde{x} = q \log V_n - 2q \log A \tag{2.16}$$

and

$$T_n - 2b_n = -\frac{U_n + Y_n/\sqrt{\log n}}{(2 + 2U_n/\sqrt{\log n} + 4X_n/\log n)^{1/2}} \geq Z_n/\sqrt{2} \to \infty \tag{2.17}$$

where $Y_n = 2x + 3y - \log 4 + o(1), \ Z_n = -U_n - Y_n/\sqrt{\log n}$. These imply

$$\Phi(T_n - 2b_n) = 1 - o(\exp(-Z_n^2/4)) = 1 - o(H_n^{-1}),$$

by $U_n \to -\infty, \ X_n, Y_n = o(\sqrt{\log n}), \ Z_n^2 \sim U_n^2 > (\log H_n)/c$. Here and below we use the well known relations: as $x \to \infty, \ \delta = o(1)$

$$\Phi(-x) = \frac{\exp(-x^2/2)}{x\sqrt{2\pi}}(1 + O(x^{-2})), \ \Phi(-x + \delta) = \Phi(-x)e^{x\delta}(1 + O((x^{-1} + \delta)^2)). \tag{2.18}$$

Therefore we get:

$$2u_n^2 = nh_n^2 e^{b_n^2} \Phi(T_n - 2b_n) + o(1) = 2H_n^2 + o(H_n); \tag{2.19}$$

Let $H_n \asymp 1$. Then we obtain the critical radii of the form (2.11) with

$$b_n^2 = \delta_n \log n + q(\log\log\ n + \log \delta_n - 2\log A) + B_n; \ B_n = O(1). \tag{2.20}$$

If $B_n \to -\infty$, then $\beta_n(\alpha) \to 1 - \alpha$, if $B_n \to \infty$, then $\beta_n(\alpha) \to 0$.

Example 3: $\tau = 2$. .

Consider the case a is close to $1/4q, \ p > q$. More exactly, assume $\eta_n = 1 - 4aq = o((\log\log\ n)^{-2}), \ H_n = o(\log\log\ n)$. Denote, as above, $U_n = \eta_n\sqrt{\log n}$. Take a sequence $b_n, \ 2b_n^2 \sim \log n$ which is concreted below and put

$$R_{n,2} = An^a = An^{1/4q - U_n/4q\sqrt{\log n}}, \ R_{n,1} = A^{q/p} b_n^{1-q/p} n^{1/4p - U_n/4p\sqrt{\log n}}, \tag{2.21}$$

which correspond to (2.11) and imply (2.12).

Assume $U_n = o((\log n)^{1/6})$. Let b_n is defined by (2.13), (2.14) with

$$x = q(\log\log\ n - \log 2A^2) - \log \Phi(-U_n/\sqrt{2}) + o(1/H_n). \tag{2.22}$$

Then, using (2.17) with $\tilde{Y}_n = Y_n - \log \Phi(-U_n/\sqrt{2})$ in place of Y_n and (2.18), we get

$$T_n - 2b_n = -U_n/\sqrt{2} + \varepsilon_n, \ U_n\varepsilon_n = o(1/H_n) \tag{2.23}$$

which imply (2.19).

Assume $U_n \gg \sqrt{\log\log n}$. To choose b_n such that (2.19) hold we obtain the equation:

$$\log n + \log h_n^2 + b_n^2 + \log \Phi(T_n - 2b_n) = \log(2H_n^2) + o(H_n^{-1}). \tag{2.24}$$

By using (2.18) and in view of relation

$$T_n - 2b_n = -\frac{3}{2}b_n + b_n^{-1}\log(2b_n^{-1}) = -U_n/\sqrt{2} + O(U_n\eta_n)$$

we obtain bi-quadratic equation on b_n:

$$b_n^4 - 2Pb_n^2 + Q = 0,$$

where

$$\begin{aligned} P &= 4\log n + \log h_n^2 - 4\log U_n - 4\log 2H_n^2 - 2\log(\pi/8) + o(H_n^{-1}), \\ Q &= (1 - 4\log 2)\log h_n^2 + (\log 4)^2 + O(h_n \log n). \end{aligned}$$

Therefore

$$\begin{aligned} b_n^2 &= \frac{1}{2}(5 - \eta_n - 4\sqrt{1-\eta_n})\log n + q\log\log n - 2q\log A \\ &+ \log U_n + \log(2H_n^2) + \frac{1}{2}\log\pi - q\log 2 + o(H_n^{-1}). \end{aligned} \tag{2.25}$$

Let $H_n \asymp 1$. Then we obtain critical radii of the form (2.11), where if $U_n^2 = O((\log n)^{1/6}$, then:

$$b_n^2 = \frac{1}{2}(1+\eta_n)\log n + q\log\log\, n - 2q\log A - \log\Phi(-U_n/\sqrt{2}) + B_n; \tag{2.26}$$

and if $U_n \to \infty$, $\log\log\, n \ll U_n^2 = o(\log n(\log\log\, n)^{-4})$, then

$$b_n^2 = \frac{1}{2}(5-\eta_n-4\sqrt{1-\eta_n})\log n + (q+1/2)\log\log\, n - 2q\log A + \log\eta_n + B_n. \tag{2.27}$$

Here $B_n = O(1)$. If $B_n \to -\infty$, then $\beta_n(\alpha) \to 1-\alpha$, if $B_n \to \infty$, then $\beta_n(\alpha) \to 0$.

2.2 Infinitely divisible case: $\tau \in (1, 2)$

Put $c_n = c_n(\theta_n) = 2n\Phi(-T_n)$. It was shown in [3, 5] that

$$\beta_n(\alpha, V_n(\theta_n)) = \beta_n(\alpha, P_{\pi^n}) + o(1).$$

If $c_n(\theta_n) \to c \in (0,\infty)$, then the sharp asymptotics of these values are defined by special infinitely divisible distributions which depend on the (τ, c), see [3, 5]. However the rate are simple enough:

$$\begin{aligned} \beta_n(\alpha, V_n(\theta_n)) &\to 1-\alpha \text{ iff } c_n \to 0, \\ \beta_n(\alpha, V_n(\theta_n)) &\to 0 \text{ iff } c_n \to \infty. \end{aligned}$$

Moreover if $c_n(\theta_n) \to \infty$, then there exist minimax consistent test procedure which does not depend on θ_n and on $\tau_n \to \tau \in (1,2)$. This is based on thresholding:

$$\psi^{(\infty)}_{n,H_{n,\alpha}} = \mathbf{1}_{\{\max_i |x_i| > H_{n,\alpha}\}}, \tag{2.28}$$

here $H_{n,\alpha} = \sqrt{2\log n} + o(1)$ is such that $(1 - 2\Phi(-H_{n,\alpha}))^n = 1 - \alpha$ (this implies $\alpha(\psi_{n,\alpha}) = \alpha$).

Remark 2.1 *Distinguishability conditions are defined by the asymptotics of the values u_n not only for $\tau \geq 2$ but for $\tau > 1$ also.*

In fact, if $1 < \tau < 2$, then $T_n - 2b_n = b_n(\tau_n - 2) \to -\infty$ and by (2.3), (2.18) we get:

$$u_n^2 \sim \tau_n c_n/(2 - \tau_n) \asymp c_n.$$

Example 4: $\tau \in (1,2)$ (see [5].)

As in Example 2, put $R_{n,2} = An^a$. Consider $0 < a < 1/4q$ and let $R_{n,1}$ are defined by (2.11) with $b_n^2 = C_n \log n$. Denote $x = (aq)^{1/2},\ C = C(x) = 2(1-x)^2$,

$$C_n = C + y\frac{\log\log n}{\log n} + \frac{H_n}{\log n},\ y = \frac{(1-x)^2 + q(1-x)}{x}.$$

Then we get (2.12) and

$$T_n = D_n\sqrt{\log n} + \frac{q(\log\log n + \log(C_n/A^2)) + \log 4}{2\sqrt{C_n \log n}} + o\left(\frac{1}{\sqrt{\log n}}\right) \tag{2.29}$$

where

$$D_n = D(C_n, aq) = \left(\frac{C_n^{1/2}}{2} + \frac{1 - aq}{C_n^{1/2}}\right). \tag{2.30}$$

If $H_n = o(\log n)$, then using (2.29), (2.30) we get:

$$\tau_n = T_n/b_n \to \tau = (1-x)^{-1} \in (1,2).$$

If $H_n = O(1)$, then we obtain critical radii (2.11) and

$$c_n = 2n\Phi(-T_n) \sim \sqrt{2/\pi}(A\tau)^{q\tau}2^{-\tau(1+q/2)}\exp(H_n\tau(\tau-1)/2),$$

If $H_n \to -\infty$, then $\beta_n(\alpha) \to 1 - \alpha$, if $H_n \to \infty$, then $\beta_n(\alpha) \to 0$.

2.3 Degenerate case: $\tau \in [1/2, 1]$

Let $k = nh_n \to \infty, \tau = 1$. Put $\lambda_n = \lambda_n(\theta_n) = nh_n\Phi(b_n - T_n)$. Then (see [3, 5])

$$\beta_n(\alpha, V_n(\theta_n)) = \beta_n(\alpha, P_{\pi_n}) + o(1) = (1-\alpha)\exp(-\lambda_n(\theta_n)).$$

Let $k = nh_n = O(1)$, $\tau = 1$ and k is an integer. Then

$$\beta_n(\alpha, V_n(\theta_n)) = (1-\alpha)(\Phi(\sqrt{2\log n} - b_n))^{nh_n} + o(1).$$

Let $\tau < 1$. Then $\beta_n(\alpha, V_n(\theta_n)) \to 0$. Asymptotically minimax test procedure are based on the thresholding (2.28).

Example 5: $\tau = 1$ (see [5].)

Consider the case of small $a = d/\log n$. Let $R_{n,2} = A(\log n)^d$. Put

$$R_{n,1} = A^{q/p} C_n^{1/2-q/2p} (\log n)^{1/2-q/2p+dq/p}. \tag{2.31}$$

which correspond to (2.11). We get

$$\begin{aligned} b_n^2 &= C_n \log n,\ h_n = \frac{1}{n}(\log n)^{q(d-1/2)}(A^2/C_n)^{q/2},\ \tau_n = \frac{1}{2} + \frac{1}{C_n} + o(1); && (2.32)\\ T_n &= \left(\frac{\sqrt{C_n}}{2} + \frac{1}{\sqrt{C_n}}\right)\sqrt{\log n} + O\left(\frac{\log\log n}{\sqrt{\log n}}\right). && (2.33) \end{aligned}$$

Assume $d > 1/2 + 1/2q$ (this implies $nh_n/\sqrt{\log n \log\log n} \to \infty$). Put

$$C_n = 2 - 2z\sqrt{\frac{\log\log n}{\log n}} + \frac{2\log\log\log n}{z\sqrt{\log n \log\log n}} + \frac{H_n}{\sqrt{\log n \log\log n}},$$

where $z = \sqrt{2q(2d-1)}$. Let $H_n = o(\sqrt{\log n \log\log n})$. Then $\tau = 1$. Let $H_n = O(1)$. Then we obtain critical radii (2.31) and

$$\beta_n(\alpha) = (1-\alpha)e^{-\lambda_n} + o(1),\ \lambda_n \sim \frac{A^q}{z\sqrt{\pi}2^{q/2}} \exp\left(zH_n/4\right).$$

If $H_n \to -\infty$, then $\beta_n(\alpha) \to 1-\alpha$, if $H_n \to \infty$, then $\beta_n(\alpha) \to 0$.

Remark 2.2 *First, observe that if $T_n - b_n \to \infty$, then $c_n \sim (\tau_n - 1)\lambda_n/\tau_n$ which implies $c_n \asymp \lambda_n$ for $\tau > 1$ and $u_n^2 \asymp \lambda_n$ for $2 > \tau > 1$. Next, put*

$$b_n^* = \sqrt{(\log n)/2},\ \lambda_n^* = \lambda_n^*(\theta_n) = nh_n\Phi(b_n - H_n),$$

where $H_n = 2b_n^* + o(1)$ be such values that $n\Phi(-H_n) \asymp 1$. We can replace the values λ_n onto λ_n^* in the asymptotics of $\beta_n(\alpha, V_n(\theta_n))$ for the case $\tau_n \to 1$.

In fact, it was noted in [5], Lemma 5.4, that if $\lambda_n \asymp 1$, then $\lambda_n^* = \lambda_n + o(1)$ and if $\tau < \infty$ and $\lambda_n \to \infty$, then $\lambda_n^* \to \infty$. Analogously one can check that if $\lambda_n = o(1)$, then $\lambda_n^* = o(1)$.

Using Remarks 2.1, 2.2 we can express distinguishability conditions in the terms not of the values $\tau_n = \tau_n(\theta_n)$ and $u_n = u_n(\theta_n)$, $c_n = c_n(\theta_n), \lambda_n = \lambda_n(\theta_n)$ but in the terms of the values $b_n = b_n(\theta_n)$ and $u_n = u_n(\theta_n)$.

Proposition 2.1 *For any $\alpha \in (0,1)$:*

1) If $b_n - 2b_n^* \to \infty$, then $\beta_n(\alpha, V_n(\theta_n)) \to 0$; if $b_n \sim 2b_n^*$, $nh_n \to \infty$, then

$$\beta_n(\alpha, V_n(\theta_n)) = (1-\alpha)\exp(-\lambda_n^*) + o(1).$$

The tests (2.28) are asymptotically minimax in these cases.

2) Let $\liminf b_n(\theta_n)/b_n^* < 2$, $nh_n \to \infty$. If $u_n(\theta_n) \to \infty$, then $\beta_n(\alpha, V_n(\theta_n)) \to 0$. If $u_n(\theta_n) \to 0$, then $\beta_n(\alpha, V_n(\theta_n)) \to 1-\alpha$.

Proof. The statement 1) follows the properties of supremum-tests (2.28), see [5], Lemma 5.3. Consider the statement 2). If $\tau_n \to \tau \geq 2$, then it follows from the results for Gaussian case. If $\tau_n \to \tau < 2$, then $u_n^2 \asymp c_n = 2n\Phi(-T_n)$. If $u_n \to 0$, then $c_n \to 0$ which implies $T_n > 2b_n^* + o(1)$ and $\tau_n \to \tau > 1$; therefore it follows from the results for infinite-divisible case. Let $u_n \to \infty$. If $\tau_n \to \tau \neq 1$, then the statement is evident. If $\tau_n \to \tau = 1$, then either $\lambda_n \gg c_n \to \infty$ for $T_n - b_n \to \infty$, or, evidently, $\lambda_n \to \infty$ for $T_n - b_n \not\to \infty$. $\square$

3 Main results

To study adaptive setting observe evident inequality

$$\beta_n(\alpha, V_n(\Theta_n)) \geq \inf_{\theta_n \in \Theta_n} \beta_n(\alpha, V_n(\theta_n)) \tag{3.1}$$

In view of (3.1) necessary condition for adaptive distinguishability is uniform distinguishability on $\theta_n \in \Theta_n$:

$$\inf_{\theta_n \in \Theta_n} \beta_n(\alpha, V_n(\theta_n)) \to 0. \tag{3.2}$$

The first problem is the following: what are sets Θ_n such that asymptotic equality in (3.1) or the relation (3.2) are sufficient for adaptive distinguishability ?

The results above and the considerations below show that these hold for the sets Θ_n with small enough or with large enough values b_n.

3.1 The case of small b_n

For BP denote $\Theta_n^0 = \{\theta_n : p \leq 2,\ p \leq q\}$, in usual case put

$$\Theta_n^a = \{\theta_n : b_n(\theta_n) < a\};\ a > 0. \tag{3.3}$$

For subset $\Theta_n \subset \Theta_n^*$ put $u_n(\Theta_n) = \inf_{\theta_n \in \Theta_n} u_n(\theta_n)$. The results above show that for any $\Theta_n \subset \Theta_n^0$

$$\beta_n(\alpha, V_n(\Theta_n)) = \Phi(t_\alpha - u_n(\Theta_n)) + o(1)$$

and test procedures (2.10) are asymptotically minimax in BP:

$$\beta_n(\psi^{(2)}_{n,\alpha}, V_n(\Theta_n)) = \Phi(t_\alpha - u_n(\Theta_n)) + o(1).$$

This result is extended on the sets $\Theta_n \subset \Theta_n^{a_n}$ with $a_n = o(1)$ and $a_n = O(1)$ for MCP and BP.

Theorem 3.1 *1) Let $a_n = o(1)$. Then*

$$\beta_n(\alpha, V_n(\Theta_n^{a_n})) = \Phi(t_\alpha - u_n(\Theta_n^{a_n})) + o(1).$$

Asymptotically minimax tests $\psi_{n,\alpha}(a_n)$ are of the form:

$$\psi_{n,\alpha}(a_n) = \mathbf{1}_{\{\lambda_{n,2}>t_\alpha\}\cup\{\lambda_n(a_n)>t_\alpha\}};\ \lambda_n(a_n) = 2^{1/2}n^{-1/2}a_n^{-2}\sum_{i=1}^n \xi(x_i, a_n). \qquad (3.4)$$

2) Let $a_n = O(1)$. If $u_n(\Theta_n^{a_n}) \to \infty$, then $\beta_n(\alpha, V_n(\Theta_n^{a_n})) \to 0$, and if $u_n(\Theta_n^{a_n}) \to 0$, then $\beta_n(\alpha, V_n(\Theta_n^{a_n})) \to 1-\alpha$. Minimax consistence tests $\psi_n(a_n) = \psi_{n,\alpha_n}(a_n)$ are of the form (3.4) with some sequences $\alpha_n \to 0$.

The lower bound of the Theorem 3.1 follow directly from the results [3, 5] noted above. The properties of tests procedures provided upper bounds in Theorem 3.1 are studied in the proof of Theorem 3.4 below.

Remark 3.1 *In BP assume $b = b_n > c > 0$. If $h = h_n = 1$ (this corresponds to the relation* (i) *in Lemma 1.1), then $u_n \to \infty$ and we can distinguish the hypothesis and alternative by using the combination of χ^2-tests and supremum-tests:*

$$\psi^{(2,\infty)}_{t_n,H_n} = \mathbf{1}_{\{\lambda_{n,2}>t_n\}\cup\{\max_i |x_i|>H_n\}};\ H_n \asymp \sqrt{\log n},\ n\Phi(-H_n) \to 0 \qquad (3.5)$$

with $t_n \to \infty$ but the rate of t_n is small enough. The same holds for

$$b^p h_n n^{p/4} \to \infty,\ \text{if } p \le 2;\quad (\log n)n^{1/2}h_n(b_n^2/\log n)^{p/2} \to \infty,\ \text{if } p > 2.$$

Therefore in BP we can exclude in consideration below cases (i) *with large enough $b = b_n > c > 0$ and cases with $b_n^2 \asymp \log n$, $\liminf nh_n^2 > 0$.*

Proof. One easily has (see [1], for example) that

$$\alpha(\psi^{2,\infty}_{n,\alpha}) \to 0,\ \beta(\psi^{2,\infty}_{n,\alpha}, v_n) \to 0$$

if

$$\max_{1\le i\le n} |v_i| > CH_n,\ C > 1\ \text{ or } n^{-1/2}\sum_{1\le i\le n} v_i^2 \gg t_n \to \infty.$$

It follows from Lemma 1.1 that $\sum_{1\le i\le n} |v_i|^p \ge R^p_{n,1} = nh_nb_n^p$. Let $0 < p \le 2$. Then

$$(n^{-1}\sum_{1\le i\le n} v_i^2)^{1/2} \ge (n^{-1}\sum_{1\le i\le n} |v_i|^p)^{1/p} \ge n^{-1/p}R_{n,1} = h_n^{1/p}b_n,$$

which implies

$$n^{-1/4}(\sum_{1\le i\le n} v_i^2)^{1/2} \ge n^{1/4}h_n^{1/p}b_n \gg t_n^{1/2},$$

if $b^p h_n n^{p/4} \gg t_n^{p/2} \to \infty$. Let $p > 2$ and $\max_{1\le i\le n}|v_i| \le CH_n$. Then

$$\sum_{1\le i\le n} v_i^2 = (CH_n)^2 \sum_{1\le i\le n}(v_i/CH_n)^2 \ge (CH_n)^2 \sum_{1\le i\le n}|v_i/CH_n|^p =$$
$$(CH_n)^{2-p}\sum_{1\le i\le n}|v_i|^p \ge (CH_n)^{2-p}nh_nb_n^p \gg n^{1/2}t_n,$$

if $(\log n)n^{1/2}h_n(b_n^2/\log n)^{p/2} \gg t_n \to \infty$. $\square$

3.2 The case of large b_n

Fix a positive sequence $\delta_n \to 0$ (in BP fix a sequence $p_n \to \infty$, $p_n = o(\log n)$ also) and a value $\delta > 0$. Assume below $\delta \le \min(p,q)$; $p \le p_n$ in BP. Denote

$$\Theta_n^\infty = \{\theta_n : \ b_n(\theta_n) > (2-\delta_n)b_n^*;\ nh_n(\theta_n) > \delta_n^{-1}\} \qquad (3.6)$$
$$\Theta_{n,1}^\infty = \{\theta_n : \ b_n(\theta_n) > (1+\delta)b_n^*;\ nh_n(\theta_n) > \delta_n^{-1}\}. \qquad (3.7)$$

Put

$$\lambda_n^*(\Theta_n) = \inf_{\theta_n\in\Theta_n}\lambda_n^*(\theta_n).$$

Theorem 3.2 *1) Let $\Theta_n \subset \Theta_n^\infty$. Then*

$$\beta_n(\alpha, V_n(\Theta_n)) = (1-\alpha)\exp(-\lambda_n^*(\Theta_n)) + o(1)$$

and tests (3.5) with $H_n = H_{n,\alpha}$ are asymptotically minimax:

$$\alpha(\psi^{(2,\infty)}_{t_n,H_{n,\alpha}}) \to \alpha,\ \beta_n(\psi^{(2,\infty)}_{t_n,H_{n,\alpha}}, V_n(\Theta_n)) = (1-\alpha)\exp(-\lambda_n^*(\Theta_n)) + o(1).$$

2) Let $\Theta_n \subset \Theta_{n,1}^\infty$. If $u_n(\Theta_n)) \to 0$, then $\beta_n(\alpha, V_n(\Theta_n)) \to 1-\alpha$. If $u_n(\Theta_n)) \to \infty$, then test procedure (2.28) is minimax consistence: $\beta_n(\psi^{(2,\infty)}_{t_n,H_{n,\alpha}}, V_n(\Theta_n)) \to 0$.

Proof. The lower bounds of Theorem 3.2 follow from Remarks 2.1, 2.2 and Proposition 2.1.

In view of Remark 3.1 to proof the upper bounds in BP we can exclude the cases $p \le q$ because $h_n = 1$ here (it follows from Lemma 1.1; it is the only point that we use χ^2-test in this proof). Next part of the proof is based on the study of minimax properties of the tests (2.28). It was shown in [3], sec. 6 (see also sec. 5.3 below) that uniformly on $v_n \in R^n$

$$\beta_n(\psi^{(\infty)}_{H_{n,\alpha}}, v_n) \to 0, \quad \text{as } \max_{1\le i\le n}(v_{n,i} - H_{n,\alpha}) \to \infty$$

and

$$\beta_n(\psi^{(\infty)}_{n,\alpha}, v_n) \le (1-\alpha)\exp(-\tilde F_n(v_n, H_{n,\alpha})) + o(1),$$

where

$$\tilde F_n(v,H) = \sum_{i=1}^n \tilde\phi(v_i, H);\ \ \tilde\phi(v,H) = \Phi(v-H) + \Phi(-v-H).$$

In MCP this implies

$$\beta_n(\psi^{(\infty)}_{n,\alpha}, V_n(b,k)) \le (1-\alpha)\exp(-k\tilde\phi(b, H_{n,\alpha})) + o(1) = (1-\alpha)\exp(-\lambda^*_n(\theta_n)) + o(1)$$

which implies the statement of Theorem. In BP denote

$$V(\theta_n, Q_n) = \{v \in V(\theta_n):\ \max_i |v_i| \le Q_n\},\ \ Q_n = Cb_n$$

and put

$$F_n(v,H) = \tilde F_n(v,H) - 2n\Phi(-H),\ \ \phi(v,H) = \tilde\phi(v,H) - 2\Phi(-H). \tag{3.8}$$

Lemma 3.1 *For any $C_1 > C_2 > 1$ there exist such $B > 0,\ \epsilon > 0$ (which do not depending on n, p, q) that if*

$$C_1 b > H \ge C_2 b > B,\ \ b < Q < C_1 b,\ \ p \le \epsilon b^2,\ \ p > q$$

(here $b = b_n(\theta_n)$), then

$$\inf_{v\in V(\theta_n, Q)} F_n(v,H) \ge nh\phi(b,H) \tag{3.9}$$

which imply

$$\inf_{v\in V(\theta_n, Q)} \tilde F_n(v,H) \ge nh\tilde\phi(b,H).$$

Proof of Lemma 3.1 is given in Appendix, sec. 6.5.2. It is modification of the proof of Lemma 6.1 in [5]. Using (3.9) we get

$$\beta_n(\psi^{(\infty)}_{n,\alpha}, V(\theta_n)) \le (1-\alpha)\exp(-nh_n\Phi(b_n - H_{n,\alpha})) + o(1)$$

uniformly on θ_n with $b_n \to \infty,\ p \le p_n = o(\log n),\ \delta > 0$. By Remarks 2.1, 2.2 these imply the upper bounds of Theorem 3.2. $\square$

The main results of the paper correspond to the case of moderate b_n: $b_n(\theta_n) \to \infty,\ b_n(\theta_n) < (2-\delta)b^*_n$.

3.3 The case of moderate b_n

Introduce semi-logarithmic scale. Namely, introduce variables $z = z_n(b_n) = z_n(\theta_n)$ and sets $\Theta_n(z_1, z_2)$:

$$z = z_n(b) = \begin{cases} b, & \text{if } b \le b^*_n + 1, \\ b^*_n + 1 + \log(b - b^*_n), & \text{if } b \ge b^*_n + 1, \end{cases} \tag{3.10}$$

$$\Theta_n(z_1, z_2) = \{\theta \in \Theta^*_n:\ z_1 \le z_n(\theta_n) \le z_2\}. \tag{3.11}$$

Here $z_l = z_{n,l},\ l = 1, 2,$

$$0 \le z_1 < z_2 \le z_n^* = b_n^* + 1 + \log(b_n^*). \tag{3.12}$$

These correspond to the inequalities on $b = b(z)$:

$$0 \le b(z_1) < b(z_2) \le 2b_n^*.$$

Here and below we denote as $b(z) = b_n(z)$ the inverse values: $z_n(b_n(z)) = z$. Observe that by the inequality $e^x \ge 1 + x$ one has

$$|z_n(b_{n,1}) - z_n(b_{n,2})| \le |b_{n,1} - b_{n,2}|. \tag{3.13}$$

For a set $\Theta_n \subset \Theta_n^*$ denote $Z_n(\Theta_n) = \{z(\theta_n) :\ \theta_n \in \Theta_n\}$.

Let us consider a sequence of functions $w_n(z) > 0,\ z \in [z_1,\ z_2]$ which will define possible adaptive bounds. Introduce the assumptions

W1. The functions $w_n^2(z),\ z \in [z_1,\ z_2]$ are uniform Lipschitzian: there exists $B > 0$

$$|w_n^2(z') - w_n^2(z'')| \le B|z' - z''| \ \text{ for all } z', z'' \in [z_1,\ z_2].$$

W2.

$$w_n = \inf_{z \in [z_1,\ z_2]} w_n(z) \to \infty.$$

W3.

$$\int_{z_1}^{z_2} \Phi(-w_n(z))dz \asymp 1.$$

Note that it follows from (3.12), W1 and W3

$$\sup_{z_1 \le z \le z_2} w_n(z) = O((\log n)^{1/4}). \tag{3.14}$$

Theorem 3.3 *Lower bounds. Assume W1, W2, W3. Let there exist $\tilde\Theta_n \subset \Theta_n$ such that $[z_1,\ z_2] \subset Z_n(\tilde\Theta_n)$ and*

$$\sup_{\theta_n \in \tilde\Theta_n} (u_n(\theta_n) - w_n(z_n(\theta_n)) \le R_n + o(1).$$

Then

$$\beta_n(\alpha, V_n(\Theta_n)) \ge (1 - \alpha)\Phi(-R_n) + o(1).$$

Theorem 3.4 *Upper bounds. Let $\Theta_n \subset \Theta_n(z_1, z_2),\ z_2 \le z_n^* - B,\ B > 0$ and*

$$\inf_{\theta_n \in \Theta_n} (u_n(\theta_n) - w_n(z_n(\theta_n)) \ge R_n.$$

Assume W1, W2, W3. Then

$$\beta_n(\alpha, V_n(\Theta_n)) \le (1 - \alpha)\Phi(-R_n) + o(1).$$

Corollary 3.1 *Let*

$$\Theta_n = \{\theta_n \in \Theta_n^* : z(\theta_n) \in [z_1,\ z_2];\ u_n(\theta_n) \ge w_n(z_n(\theta_n)) + R_n + o(1)\}.$$

Then

$$\beta_n(\alpha, V_n(\Theta_n)) = (1-\alpha)\Phi(-R_n) + o(1).$$

Remark 3.2 *First, note that the assumption $z_2 \le z_n^* - B,\ B > 0$ is equivalent to $b_n(\theta) \le (2-\delta)b_n^*,\ \delta > 0$. Also the assumption W2 in Theorem 3.4 may be replaced onto $R_n \to \infty$.*

Next, we can replace the assumption W3 *onto one of the following:*

W3a

$$\int_{z_1}^{z_2} \exp(-w_n^2(z)/2)dz \asymp 1;$$

W3b *There exist such $B_l \in R^1,\ l = 1,2,\ B_1 > B_2$ that*

$$\int_{z_1}^{z_2} \exp(-w_n^2(z)/2)w_n^{B_1}(z)dz \to \infty,\ \int_{z_1}^{z_2} \exp(-w_n^2(z)/2)w_n^{B_2}(z)dz \to 0.$$

In fact, under assumptions W2 and either W3a or W3b one can find such sequence $\delta_n \to 0$ that the sequence $\tilde{w}_n(z) = w_n(z) + \delta_n$ satisfies W3. This implies the upper and lower bounds with $\tilde{R}_n = R_n + \delta_n$ which are equivalent to original bounds.

In view of Remark 3.2 we can apply Theorems 3.3, 3.4 and Corollary 3.1, particularly, to the functions

$$w_n(z) = \sqrt{2\log z},\ z_1 \to \infty,\ z_2/z_1 \ge C > 1 \tag{3.15}$$

or

$$w_n(z) = \sqrt{2\log(z_n^* - z)},\ z_n^* - z_2 \to \infty,\ z_2/z_1 \ge C > 1. \tag{3.16}$$

Note the case of constant functions $w(z)$. Let

$$z_2 - z_1 \to \infty,\ w(z) = w_n = \sqrt{2\log(z_2 - z_1)}. \tag{3.17}$$

Corollary 3.2 *Let*

$$z_2 \le b_n^*,\ z_2 \asymp b_n^*,\ z_1 = o(b_n^*),\ w_n = \sqrt{2\log(z_2 - z_1)} = \sqrt{\log\log n} + o(1). \tag{3.18}$$

If $\Theta_n \subset \Theta_n(z_1, z_2))$, then

$$\beta_n(\alpha, V_n(\Theta_n)) \le (1-\alpha)\Phi(w_n - u_n(\Theta_n)) + o(1) \tag{3.19}$$

where, as above, $u_n(\Theta_n) = \inf_{\theta_n \in \Theta_n} u_n(\theta_n)$. If there exist $\tilde{\Theta}_n \subset \Theta_n$ such that $[z_1,\ z_2] \subset Z_n(\tilde{\Theta}_n)$, then

$$\beta_n(\alpha, V_n(\Theta_n)) \ge (1-\alpha)\Phi(w_n - u_n^+(\tilde{\Theta}_n)) + o(1). \tag{3.20}$$

where $u_n^+(\tilde{\Theta}_n) = \sup_{\theta_n \in \tilde{\Theta}_n} u_n(\theta_n)$.

3.4 Examples

In this section we consider adaptive version of examples 2 and 3 above. Let $R_{n,2} = An^a$, $a > 0$, $A > 0$. Assume A, p, q be fixed to simplicity and consider the case of unknown $a \in \mathcal{A}$. Denote $u_n(\theta_n) = u_n(a, R_{n,1})$, $b_n(\theta_n) = b_n(a, R_{n,1})$. Our arm here is to describe the asymptotics of adaptive critical radii $R_{n,1}(a) = R_{n,1}(a, A, p, q)$ such that uniformly on $a \in \mathcal{A}$

$$u_n(a, R_{n,1}) = w_n(z_n(a)) + C_n + o(1),\ C_n \asymp 1 \tag{3.21}$$

for wide enough sets $\mathcal{A}$ and for functions $w_n(z)$ type of (3.15) – (3.18); here $z_n(a) = z_n(b_n(a, R_{n,1}(a, A, p, q)))$.

Example 6: $\tau > 2$.

Analogously with Example 2, let $p > q$ and consider sets $\mathcal{A} = ((1+\eta_n^*)/4q,\ (1+\delta_n^*)/2q)$. Assume $\delta_n^* \log n \to \infty,\ c(\eta_n^*)^2 \geq \log\log\log n/\log n,\ 0 < c < 1/4$. Denote, as above, $V_n = (1-2aq)\log n,\ U_n = (1-4aq)\sqrt{\log n}$. Assuming

$$w_n(z_n(a)) = o(V_n/\log V_n),\ \log(w_n(z_n(a))) < cU_n^2, \tag{3.22}$$

we can use estimations of Example 2 for $H_n = w_n(z_n(a)) + C_n$ uniformly on $a \in \mathcal{A}$. By taken $b_n = b_n(a, A, p, q)$, define $R_{n,1} = R_{n,1}(a, A, p, q)$ according to (2.11). Then we get (2.12). Let b_n satisfy (2.13), (2.14) which implies (2.19) and correspond to (2.15), (2.16). Under assumptions above $\Phi(T_n - 2b_n) \geq 1 - o(1/\sqrt{\log b_n^2})$ and (3.21) holds. Note that $b_n = z_n = z_n(a)$ in this case. Thus we go to critical radii (2.11) with

$$b_n^2 = V_n + q(\log V_n - 2\log A) + \log(2w_n^2(b_n)) + C_n/w_n(b_n). \tag{3.23}$$

If $C_n \to -\infty$, then $\beta_n(\alpha) \to 1-\alpha$, if $C_n \to \infty$, then $\beta_n(\alpha) \to 0$.

Let $w_n^2(b) = \log b^2,\ b \leq b_n^*$. Then $w_n^2(b_n) \sim \log V_n$ and (3.22) hold. The relation (3.23) is of the form

$$b_n^2 = \delta_n \log n + q\log(\delta_n A^{-2}\log n) + \log(2\log V_n) + C_n/\sqrt{\log V_n}.$$

Let $w_n^2(b) = \log\log\ n$. Then (3.22) hold under additional assumption: $\delta_n \gg (\log\log\ n)^{1/2}\log\log\log n/\log n$. The relation (3.23) is of the form

$$b_n^2 = \delta_n \log n + q\log(\delta_n A^{-2}\log n) + \log(2\log\log n) + C_n/\sqrt{\log\log\ n}.$$

Example 7: $\tau = 2$.

Analogously with Example 3, consider the case $a = a_n$ is closed to $1/4q$. Let $\mathcal{A} = ((1-\eta_{n,2})/4q,\ (1-\eta_{n,1})/4q)$. Assume

$$0 < \eta_{n,1} < \eta_{n,2} = o((\log\log\ n)^{-2}),\ \eta_{n,1}\sqrt{\log n} \to \infty,\ \Omega_n = \log(\eta_{n,2}/\eta_{n,1}) \to \infty.$$

Put $R_{n,2},\ R_{n,1}$ according to (2.21) and let b_n is defined by (2.25) with $H_n = w_n(z_n(a)) + C_n$. Then we get (3.21). Particularly, one has the relations for $z_n = z_n(a_n)$:

$$z_n^* - z_n = -\log(\eta_n/2) + o(1/\log\log\, n),\ \ \eta_{n,1} < \eta_n = 1 - 4aq < \eta_{n,2};$$
$$z_1 = b_n^* + 1 - \log \eta_{n,2},\ \ z_2 = b_n^* + 1 - \log \eta_{n,1}.$$

We can rewrite (2.25) in the form:

$$\begin{aligned} b_n^2 &= \frac{1}{2}(5 - \eta_n - 4\sqrt{1-\eta_n})\log n + (q+1/2)\log\log\, n - 2q\log A \\ &+ \log \eta_n + \frac{1}{2}\log \pi - q\log 2 + \log(2w_n^2(z_n)) + C_n/w_n(z_n). \end{aligned} \tag{3.24}$$

Let $w_n^2(z) = \log \Omega_n^2$. Then we get: $\log(2w_n^2(z_n)) = \log 2 + \log\log \Omega_n^2$.

Let $w_n^2(z) = \log(z_n^* - z)^2,\ z > b_n^*$. Then

$$\log(2w_n^2(z_n)) = \log 2 + 2(\log\log \eta_n^{-1} + \log 2/\log \eta_n^{-1}).$$

As above, if $C_n \to -\infty$, then $\beta_n(\alpha) \to 1-\alpha$, if $C_n \to \infty$, then $\beta_n(\alpha) \to 0$.

4 Lower bounds

4.1 Methods of constructions

We use methods of [1, 6] based on Bayesian approach. It is enough to construct priors $\pi^n = \pi^n(dv)$

$$\pi^n(V_n(\tilde{\Theta}_n)) \to 1,\ \beta(\alpha, P_{\pi^n}) \geq (1-\alpha)\Phi(-R_n) + o(1);\ n \to \infty$$

We consider priors π^n which are finite mixtures of collections of particular priors $\pi_l^n,\ 1 \leq l \leq M$:

$$\pi^n = \sum_{l=1}^{M} p_{n,l}\pi_l^n,\ p_{n,l} \geq 0,\ \sum_{l=1}^{M} p_{n,l} = 1;\ M = M_n \to \infty. \tag{4.1}$$

Denote likelihood ratio statistics:

$$L_n = dP_{\pi^n}/dP_0 = \sum_{l=1}^{M} p_{n,l}\frac{dP_{\pi_l^n}}{dP_0} = \sum_{l=1}^{M} p_{n,l}L_{n,l};\ \ \ l_{n,l} = \log\left(dP_{\pi_l^n}/dP_0\right)$$

Let there are given sequences of collections $u_{n,l} > 0,\ w_{n,l} > 0,\ 1 \leq l \leq M_n$ and values R_n such that

$$\min_{1\leq l\leq M} w_{n,l} \to \infty;\ u_{n,l} \leq w_{n,l} + R_n + o(1);\ \sum_{l=1}^{M} \exp(-w_{n,l}^2/2) \asymp 1. \tag{4.2}$$

Put

$$p_{n,l} = \exp(-w_{n,l}^2/2)\left(\sum_{l=1}^{M}\exp(-w_{n,l}^2/2)\right)^{-1}. \qquad (4.3)$$

Let there are given statistics $\tilde{l}_{n,l}$ such that

$$\eta_n = P_{n,0}\{\ \exists\, l : l_{n,l} \neq \tilde{l}_{n,l},\ 1 \le l \le M\} = o(1). \qquad (4.4)$$

and uniformly on $1 \le l \le M$

$$m_{n,l} = E_{n,0}(\tilde{l}_{n,l}) = -u_{n,l}^2/2 + o(1),\ \sigma_{n,l}^2 = \mathrm{Var}_{n,0}(\tilde{l}_{n,l}) = u_{n,l}^2 + o(1). \qquad (4.5)$$

Consider centred and normalized statistics $\tilde{l}_{n,l}$:

$$\tilde{\lambda}_{n,l} = (\tilde{l}_{n,l} - m_{n,l})/\sigma_{n,l}.$$

Denote $\Delta_{n,l} = \sup_{x\in R^1}|P_{n,0}(\tilde{\lambda}_{n,l} < x) - \Phi(x)|$ and assume

$$\Delta_n = \sum_{1\le l\le M}\Delta_{n,l} = o(1). \qquad (4.6)$$

Let $\tilde{L}_{n,l,w} = \exp(\tilde{l}_{n,l})\mathbf{1}_{\tilde{\lambda}_{n,l}<w_{l,n}}$ be truncated statistics $\tilde{L}_{n,l} = \exp(\tilde{l}_{n,l})$. Assume

$$C_n = \max_{1\le l<k\le M}\mathrm{Cov}_{P_{n,0}}(\tilde{L}_{n,l,w}, \tilde{L}_{n,k,w}) \le o(1). \qquad (4.7)$$

The following Lemma is an extension of Theorem 4.2 in [1] and is analogous with Lemma 3.2 in [6].

Lemma 4.1 *Under the assumptions (4.2)–(4.7) for any* $\alpha \in (0,1)$

$$\beta(\alpha, P_{\pi^n}) \ge (1-\alpha)\Phi(-R_n) + o(1). \qquad (4.8)$$

Proof of the Lemma is given in Appendix, section 6.1.

Let as go to the proof of Theorem 3.3. First, note that under assumptions W1, W2, W3 we can assume that for any $b > 0$

$$z_1 \to \infty,\ z_n^* - z_2 \to \infty,\ \sup_{z_1\le z\le z_2} w_n^2(z)\exp(b(z - z_n^*)) = o(1). \qquad (4.9)$$

The first relations in (4.9) correspond to the inequalities on $b = b(z)$:

$$b(z_1) \to \infty,\ b(z_2) \le b_n^*(1 + o(1)). \qquad (4.10)$$

In fact, let us consider W3a to simplicity. Put

$$\tilde{z}_1 = z_1 + B_{n,1},\ \tilde{z}_2 = z_2 - B_{n,2},\ B_{n,l} \to \infty,\ B_{n,l} = o(\exp(w_n^2/2)),\ l = 1, 2.$$

Then

$$\int_{z_1}^{\tilde{z}_1} \exp(-w_n^2(z)/2))dz \le B_{n,1} \exp(-w_n^2/2) = o(1),$$
$$\int_{\tilde{z}_2}^{z_2} \exp(-w_n^2(z)/2))dz \le B_{n,2} \exp(-w_n^2/2) = o(1)$$

and we can consider $\tilde{z}_l$ in place of z_l. Moreover, put $\delta_n = B_{n,2}^{-1/2} \to 0$ and

$$\hat{z}_2 = \max\{z \in [\tilde{z}_1,\ \tilde{z}_2] : w_n^2(z) \le \delta_n \exp(b(z_n^* - z))\}.$$

If $z \in [\hat{z}_2,\ \tilde{z}_2]$, then $w_n^2(z) \ge \delta_n \exp(b(z^* - z)) > b\delta_n(z^* - z)$ and

$$\int_{\tilde{z}_2}^{\hat{z}_2} \exp(-w_n^2(z)/2))dz \le \int_{\tilde{z}_2}^{\hat{z}_2} \exp(b\delta_n(z - z_n^*)dz \le \exp(b\delta_n(\hat{z}_2 - z_n^*)/b\delta_n \to 0$$

Therefore we can replace $\tilde{z}_2$ onto $\hat{z}_2$ and if $z \in [\tilde{z}_1,\ \hat{z}_2]$, then by W1

$$w_n^2(z) \le w_n^2(\hat{z}_2) + B(\hat{z}_2 - z);\ w_n^2(\hat{z}_2) = \delta_n \exp(b(z_n^* - \hat{z}_2)) \le \delta_n \exp(b(z_n^* - z)),$$
$$B(\hat{z}_2 - z) \le (B/b)\exp(b(\hat{z}_2 - z)) = o(\exp(b(z_n^* - z)).$$

Thus (4.9) and W1 – W3 satisfies after replacing $z_1,\ z_2$ onto $\tilde{z}_1,\ \hat{z}_2$.

Let us choose such collections $\Theta_n^M = \{\theta_{n,1}, ..., \theta_{n,M}\} \subset \tilde{\Theta}_n \subset \Theta_n$ that the values

$$z_{n,l} = z(\theta_{n,l}) \in [z_1, z_2],\ w_{n,l} = w_n(z_{n,l}),\ u_{n,l} = u_n(\theta_{n,l})$$

satisfy (4.2). First, we choose $\tilde{z}_{n,l},\ l = 1, ..., 2M+1$ such that uniformly on l

$$\tilde{z}_{n,l+1} = \tilde{z}_{n,l} + \delta_{n,l},\ \delta_{n,l} \asymp w_n(\tilde{z}_{n,l}),\ l = 1, ..., 2M+1;\ \tilde{z}_{n,1} = z_1,\ \tilde{z}_{n,2M+1} = z_2$$

and consider intervals $I_{n,l} = (\tilde{z}_{n,l}, \tilde{z}_{n,l+1}),\ l = 1, ..., 2M$. Let

$$J_{n,l} = \int_{I_{n,l}} \Phi(-w_n(z))dz,\ A_n = \sum_{l=1}^{M} J_{n,2l-1},\ B_n = \sum_{l=1}^{M} J_{n,2l}.$$

Under assumption W3 we have: either $A_n \asymp 1$ or $B_n \asymp 1$. Let $A_n \asymp 1$. By using mean-value theorem we find $z_{n,l} \in I_{n,2l-1},\ l = 1, ..., M$ such that

$$J_{n,2l-1} = \Phi(-w_n(z_{n,l}))\delta_{n,l} \asymp \Phi(-w_n(z_{n,l}))w_n(\tilde{z}_{n,2l-1}).$$

Denote $a = \tilde{z}_{n,2l-1},\ b = z_{n,l}$ and observe that under the assumptions W1, W2 one has

$$|w_n(b) - w_n(a)| = \frac{|w_n^2(b) - w_n^2(a)|}{w_n(b) + w_n(a)} \le B\frac{|b-a|}{2w_n} = o(\delta_{n,l}) = o(w_n(a)).$$

Therefore

$$w_n(\tilde{z}_{n,2l-1}) \sim w_n(z_{n,l}),\ J_{n,2l-1} \asymp \Phi(-w_n(z_{n,l}))w_n(z_{n,l}) \asymp \exp(-w_n^2(z_{n,l})/2)$$

and we have:

$$\sum_{l=1}^{M} \exp(-w_n^2(z_{n,l})/2) \asymp 1; \;\; z_{n,l+1} - z_{n,l} \geq \delta_{n,l} \asymp w_n(z) \;\forall\; z \in [z_{n,l}, z_{n,l+1}]. \quad (4.11)$$

Other relations in (4.2) follow from the assumption of Theorem.

Denote $b_{n,l} = b_n(\theta_{n,l})$, analogously $h_{n,l}, T_{n,l}$ and so on.

We consider product priors of the type (1.4), (1.5) and put

$$\pi_l^n = \pi^n(b'_{n,l}, h'_{n,l}), \;\; b'_{n,l} = b_n(\theta'_{n,l}), \quad h'_{n,l} = h_n(\theta'_{n,l})$$

were primes correspond to small changing θ such that uniformly on l

$$u_n(b'_{n,l}, h'_{n,l}) = u_{n,l} + o(1); \;\; \pi_l^n(V_n(\theta_{n,l})) = 1 - o(1). \quad (4.12)$$

The constructions are described in [3, 5]. To simplicity of notation we omit primes below. In view of (4.12) one has

$$\pi^n(V_n(\Theta_n)) \geq \sum_l p_{n,l}\pi_l^n(V_n(\theta_{n,l})) \geq 1 - o(1)$$

which implies $\beta(\alpha, P_{\pi^n}) \leq \beta(\alpha, V_n(\Theta_n))$. Thus, it is enough to check the assumptions of Lemma 4.1.

The statistics $l_{n,l}$ and $\tilde{l}_{n,l}$ are defined by (2.1), (2.4), (2.5) in the case. Using (3.14) assume without loss of generality

$$R_n \leq B, \;\; b < u_{n,l} \leq B(\log n)^{1/4}, \;\; 1 \leq l \leq M \quad (4.13)$$

by in other case lower bounds of Theorem 3.3 are obvious. Put

$$r_{n,lk} = nE_{n,0}(\tilde{Z}_{n,l}\tilde{Z}_{n,k}); \;\; \rho_{n,lk} = \frac{r_{n,lk}}{u_{nl}u_{nk}}.$$

Denote $T_n^* = \min(T_{n,l}, T_{n,k})$. Then, using the relation (6.13) below one has: as $b_{n,l} \to \infty, \;\; b_{n,k} \to \infty, T_n^* - b_{n,l} - b_{n,k} \to \infty$

$$r_{n,lk} \sim nh_{n,l}h_{n,k} \exp b_{n,l}b_{n,k}\Phi(T_n^* - b_{n,l} - b_{n,k}), \quad (4.14)$$

$$\rho_{n,lk} \sim 2\exp(-(b_{n,l} - b_{n,k})^2/2)\frac{\Phi(T_n^* - b_{n,l} - b_{n,k})}{\sqrt{\Phi(-d_{n,l})\Phi(-d_{n,k})}} \quad (4.15)$$

Lemma 4.2 *Assume (4.13) and*

$$\max_{1\leq l\leq M} c_{n,l} = o(1), \;\; \sum_{1\leq l\leq M} c_{n,l}/u_{n,l}^3 = o(1). \quad (4.16)$$

where (remind) $c_{n,l} = 2n\Phi(-T_{n,l})$. *Then the relations (4.4), (4.5) and (4.6) hold. Moreover, assume*

$$\max_{1\leq l<k\leq M} r_{n,lk} = o(1). \quad (4.17)$$

Then (4.7) holds.

Proof of the Lemma is given in Appendix, section 6.3.

To obtain lower bounds we use estimations of $c_n = c_n(\theta_n)$ and $\rho_{n,lk} = \rho_n(\theta_l, \theta_k)$ in terms of $z_n = z_n(\theta_n)$ and $z_{n,l} = z_n(\theta_{n,l}), z_{n,k} = z_n(\theta_{n,k})$.

Proposition 4.1 *Let $b_n = b_n(\theta_n) \to \infty,\ u_n = u_n(\theta_n) = o(\log n)$. Then there exist $B > 0,\ b > 0$ such that for large enough n one has:*

$$c_n \quad \leq \quad Bu_n^2 \exp(b(z_n - z_n^*)). \tag{4.18}$$

Proof of Proposition is given in sec. 6.2

Proposition 4.2 *Let $b_{n,l}, b_{n,k} \to \infty$. Then there exist $B > 0,\ b > 0$ such that for large enough n one has:*

$$\rho_{n,lk} \quad \leq \quad B \exp(-b|z_{n,l} - z_{n,k}|). \tag{4.19}$$

Proof of Proposition is given in sec. 6.6.1

4.2 Proof of the lower bounds of Theorem 3.3

We need to check the assumptions (4.16) and (4.17). The first relation in (4.16) follows directly from (4.18) and (4.9). The second relation in (4.16) follows from (4.18), (4.11), (4.13) and from estimations:

$$\begin{aligned}
&z_n^* - z_{n,l} = z_n^* - z_2 + z_2 - z_{n,l} \geq (z_n^* - z_2) + b(M - l + 1)w_n,\\
&\sum_{1 \leq l \leq M} c_{n,l}/u_{n,l}^3 \leq B \sum_{1 \leq l \leq M} \exp(-b(z_n^* - z_{n,l})) \leq\\
&B \exp(-b(z_n^* - z_2) - b_1 w_n) = o(1).
\end{aligned}$$

The relation in (4.17) follows from (4.19), (4.11) by

$$\begin{aligned}
&|z_{n,l} - z_{n,k}| \geq b(w_{n,l} + w_{n,k});\\
&r_{n,lk} = u_{n,l}u_{n,k}\rho_{n,lk} \leq Bw_{n,l}w_{n,k} \exp(-b(w_{n,l} + w_{n,k})) = o(1).
\end{aligned}$$

$\square$

5 Upper bounds

5.1 Test procedures

We will use the assumption W3a in place of W3 without loss of generality.

Let us describe the test procedures which provide the upper bounds of Theorem 3.4. We need to provide such families of tests $\psi_n = \psi_{n,\alpha}$ that

$$\alpha(\psi_{n,\alpha}) \le \alpha + o(1);\ \beta(\psi_{n,\alpha}, V_n(\Theta_n)) \le (1-\alpha)\Phi(-R_n) + o(1) \tag{5.1}$$

It is enough to find such family ψ_n that

$$\alpha(\psi_n) \to 0;\ \beta(\psi_n, V_n(\Theta_n)) \le \Phi(-R_n) + o(1) \tag{5.2}$$

by the relations (5.2) implies (5.1) for tests $\psi_{n,\alpha} = \alpha + (1-\alpha)\psi_n$.

We assume below $R_n = R + o(1)$ by if $R_n \to -\infty$, then the upper bounds are trivial, and if $R_n \to \infty$, then we will obtain (5.2) for any $R_n^{'} = R + o(1)$ which imply the statement of Theorem.

The constructed families are based on collections of tests

$$\psi_n(x) = \max_{l \in L_n^*} \psi_{n,l}(x),\ L_n^* = \{0\} \cup L_n. \tag{5.3}$$

Let $\psi_{n,0} = \psi_{t_n,H_n}^{(2,\infty)}$ be tests defined by (3.5) with $t_n \to \infty,\ t_n = o(w_n)$ and

$$c_n^* = 2n\Phi(-H_n) \asymp (\log(w_n))^{-1} = o(1),\ H_n = \sqrt{2\log n} + o(1). \tag{5.4}$$

To describe tests $\psi_{n,l},\ l \in L_n$ let us consider the functions $u_n(b) = w_n(z_n(b)) + R_n$ and let $h(b) = h_n(b)$ be such values that $u_n(b) = u_n(b,h)$; the values $u_n^2(b,h)$ are defined by (2.6). Let us consider the collections

$$0 < z_1 = z_{n,1} < ... < z_{n,M} = z_2,\ z_{n,l+1} = z_{n,l} + \delta_{n,l};\ \delta_{n,l} \asymp (w_{n,l})^{-2/3},\ w_{n,l} = w_n(z_{n,l}).$$

Denote $b_{n,l} = b_n(z_{n,l}),\ c_{n,l} = c_n(b_{n,l}, h_{n,l}),\ u_{n,l} = u_n(b_{n,l})$ and so on.

Fix $\kappa \in (0, 2/3)$ and consider the set $L = L_n = \{l :\ c_{n,l} \le u_{n,l}^{\kappa}\}$. For any $l \in L$ we consider the tests $\psi_{n,l} = \psi_n(b_{n,l})$ of the type

$$\psi_n(b) = \mathbf{1}_{\{\lambda_{n,b} > w_n(b) + \omega_n\}};\ w_n(b) = w_n(z_n(b)), \tag{5.5}$$

where statistics $\lambda_{n,b}$ are defined by (2.9), values $\omega_n = o(1)$ are concreted below.

Note that if $b_n = o(n^{-1/4}(\log n)^{-1})$, then the test $\psi_n(b_n)$ is asymptotical equivalent to χ^2-test (2.10). This easily follows from Tailor expansion of statistics $\xi(x,b)$ for small b.

5.2 The first kind errors

For the tests (5.3) one has

$$\alpha(\psi_n) \le \sum_{l \in L_n^*} \alpha(\psi_{n,l}) \tag{5.6}$$

It is clear that

$$\alpha(\psi_{n,0}) \to 0, \tag{5.7}$$

and to obtain the first relation in (5.2) it is enough to show that

$$\sum_{l \in L_n} P_{n,0}(\lambda_{n,l} > w_{n,l}) = o(1). \tag{5.8}$$

Fix small $\delta > 0, \eta: \ \kappa < \eta < 2/3$ and put $b^*_{n,1} = \sqrt{\log n}/10, \ b^*_{n,2} = b^*_n(1-\delta)$. If $b \geq b^*_{n,1}$, then let us consider thresholds

$$\tilde{T}_n(b) = \begin{cases} T_n(b), & \text{if } b < b^*_{n,2} \\ T_n(b) + \xi_n(b), & \text{if } b \geq b^*_{n,2} \end{cases}, \text{ where } \zeta_n(b) = \eta \log(u_n(b))/T_n(b) = o(1).$$

Introduce $\tilde{T}_n(b)$-truncated statistics

$$\tilde{\lambda}_{n,b}(x) = \frac{h}{u_n(b)} \sum_{i=1}^{n} \tilde{\xi}_n(x_i, b); \ \tilde{\xi}(x,b) = \xi(x,b)\mathbf{1}_{|x| < \tilde{T}_n(b)}. \tag{5.9}$$

Denote $\tilde{\lambda}_{n,l} = \tilde{\lambda}_{n,b_{n,l}}, \ \tilde{T}_{n,l} = \tilde{T}_n(b_{n,l})$ and so on and observe that

$$P_{n,0}(\lambda_{n,l}(x) \neq \tilde{\lambda}_{n,l}(x) \text{ for any } l \in L) \leq 2n \max_{l \in L} \Phi(-\tilde{T}_{n,l}) = o(1) \tag{5.10}$$

because if $b < b^*_{n,2}$, then $d_n \to -\infty, \ \tau_n > 2$ and $2n\Phi(-\tilde{T}_{n,l}) = c_{n,l} \to 0$, (this follows from the relations (6.18) and (3.14)); if $b \geq b^*_{n,2}$, then by (2.18)

$$2n\Phi(-\tilde{T}_{n,l}) \sim c_{n,l} \exp(-T_{n,l}\zeta_{n,l}) \leq u_{n,l}^{(\kappa-\eta)} = o(1).$$

In view of (5.10) we can replace the statistics $\lambda_{n,l}$ onto $\tilde{\lambda}_{n,l}$ in (5.8). Put

$$\hat{\lambda}_{n,l} = (\tilde{\lambda}_{n,l} - m_{n,l})/\sigma_{n,l}, \ m_{n,l} = E_{n,0}\tilde{\lambda}_{n,l}, \ \sigma^2_{n,l} = \mathrm{Var}_{n,0}\tilde{\lambda}_{n,l}.$$

Denote, as in Lemma 4.1,

$$\Delta_n(b) = \sup_{x \in R^1} |P_{n,0}(\hat{\lambda}_{n,b} < x) - \Phi(x)|.$$

Proposition 5.1 *For some $\delta > 0, \ B > 0, \ 1 - 3\eta/2 < \varrho < 1 - 3\kappa/2$ and any $l \in L_n$ one has:*

$$\Delta_{n,l} \leq B \begin{cases} n^{-\delta}, & \text{if } b < b^*_{n,1}, \\ c_{n,l}u_{n,l}^{-3} + n^{-\delta}, & \text{if } b^*_{n,1} \leq b \leq b^*_{n,2}, \\ c_{n,l}(u_{n,l})^{-2-\varrho}, & \text{if } b > b^*_{n,2}. \end{cases}$$

Moreover, uniformly on $l \in L$

$$m_{n,l} \leq 0, \ m_{n,l} = o(1); \ \sigma_{n,l} \leq 1 + \omega_{n,l}, \ \omega_{n,l} = o(u_{n,l}^{-1}). \tag{5.11}$$

Proof of Proposition 5.1 is given in Appendix, section 6.3.

To estimate first find errors, first, let us show that

$$A_n = \sum_{l\in L} \Delta_n(b_{n,l}) = o(1). \tag{5.12}$$

In fact, because $M_n = O((\log n)^2)$, the sum of $n^{-\delta}$ is $o(1)$. It follows from (4.18), from the choose $\delta_{n,l}$ and from W1 that

$$\sum_{l:b_{n,l}\geq b^*_{n,1}} \Delta_n(b_{n,l}) \leq B\sum_{l=1}^{M_n} u_{n,l}^{-\xi} \exp(-b(z^*_n - z_{n,l})) \leq$$
$$B\exp(-bz^*_n)\int_{z_1}^{z_2} e^{bz} w_n^{2/3-\xi}(z)dz \leq \ Bw_n^{2/3-\xi}\exp(-b(z^*_n - z_2)) = o(1)$$

in view of W2. Next, let as show that

$$B_n = \sum_{l=1}^{M_n} \Phi(-w_{n,l}) = o(1). \tag{5.13}$$

In fact, using W1, W2, W3a one easily has

$$\sum_{l=1}^{M_n} \Phi(-w_{n,l}) \asymp \sum_{l=1}^{M_n} w_{n,l}^{-1}\exp(-w_{n,l}^2/2)$$
$$\asymp \int_{z_1}^{z_2} w_n^{-1/3}(z)\exp(-w_n^2(z)/2)dz = O(w_n^{-1/3}) = o(1)$$

Put $\omega_n = \max_{l\in L}\omega_{n,l}$. Using (5.7), (5.10), (5.11), (5.12) and (5.13) we get (5.8):

$$\sum_{l\in L_n} P_{n,0}(\lambda_{n,l} > w_{n,l}+\omega_n) \leq A_n + \sum_{l\in L_n} \Phi(-(w_{n,l}-m_{n,l}+\omega_n)/\sigma_{n,l}) \leq A_n+B_n = o(1).$$

Let us consider the tests $\psi_{n,\alpha}(a_n)$ from Theorem 3.1. Using (6.7) one can easily see that, as $a_n \to 0$,

$$E_{n,0}\lambda_{n,2} = E_{n,0}\lambda_n(a_n) = 0,\ E_{n,0}\lambda_{n,2}^2 = 1,\ E_{n,0}(\lambda_n(a_n))^2 = 4\sinh^2(a_n^2/2)/a_n^4 \sim 1,$$
$$E_{n,0}(\lambda_{n,2}\lambda_n(a_n)) = a_n^{-2}\left((E_{1,a_n}(x^2-1) + E_{1,-a_n}(x^2-1))/2 - E_{1,0}(x^2-1)\right) = 1$$

which imply

$$E_{n,0}(\lambda_{n,2} - \lambda_n(a_n))^2 \to 0. \tag{5.14}$$

It follows easily from Central Limit Theorem that the statistics $\lambda_{n,2},\ \lambda_n(a_n)$ are asymptotical $(0,1)$-Gaussian under $P_{n,0}$-probability and by (5.14) this implies $\alpha(\psi_{n,\alpha}^{a_n}) \to \alpha$. If $a_n = O(1)$, then one has:

$$\alpha(\psi_{n,\alpha_n}(a_n)) \leq 2\alpha \to 0,\ \text{as}\ \alpha \to 0.$$

5.3 The second kind errors

For the tests (5.3) one has

$$\beta(\psi_n, v) \le \min_{0\le l\le M} \beta(\psi_{n,l}, v), \ \ v \in l_2 \tag{5.15}$$

and to obtain the second relation in (5.2) it is enough to construct such collections of sets

$$\Theta_{n,l} \subset \Theta_n, \ \ \bigcup_{0\le l\le M} \Theta_{n,l} = \Theta_n$$

that uniformly on $0 \le l \le M$

$$\min_{0\le j\le M} \beta(\psi_{n,j}, V_n(\Theta_{n,l})) \le \Phi(-R) + o(1) \tag{5.16}$$

(we will consider no more than two tests $\psi_{n,l}, \psi_{n,l+1}$ in this minimum).

We will consider subsets $\Theta_{n,l} \subset \Theta_n$ such that $b_n(\theta_n) \in [b_{n,l-1}, \ b_{n,l}]$.

First, let us consider tests $\psi_{n,0}$. Observe

$$\begin{aligned}
\beta(\psi_{n,0}, v) \ &\le \ P_{n,v}(\max_i |x_i| \le H_n) = \prod_{i=1}^{n}(1 - \Phi(|v_i| - H_n) - \Phi(-|v_i| - H_n) \\
&\le \ \begin{cases} \exp\left(-\sum_{i=1}^{n}(\Phi(|v_i| - H_n) + \Phi(-|v_i| - H_n))\right) \\ \max_i \Phi(H_n - |v_i|) \end{cases} .
\end{aligned}$$

Therefore we need consider below only such $v = v_n \in V_n$ that for $B > -\log\Phi(-R)$

$$S_n(v) = \sum_{i=1}^{n}(\Phi(|v_i| - H_n) + \Phi(-|v_i| - H_n)) \le B, \ \max_i |v_i| \le H_n + R \tag{5.17}$$

Before to consider consider the tests $\psi_{n,l} = \psi_n(b_{n,l})$, $l \in L$, observe, that $\beta(\psi_{n,l}, v) \le \beta(\tilde\psi_{n,l}, v)$, where the tests $\tilde\psi_{n,l}$ are based on on $T_{n,l}$-truncated statistics $\tilde\lambda_{n,l}$ analogous with (5.9). Therefore we will used the tests $\tilde\psi_{n,l}$ for $b_{n,l} > b^*_{n,1}$ and omit tilde to simplicity of notation.

Lemma 5.1 *Uniformly on $v \in R^n$ under the constraints (5.17)*

$$\beta(\psi_{n,l}, v) \le \Phi(w_{n,l} - F^*_n(v, b_{n,l})) + o(1).$$

Here the functional $F^*_n(v, a)$ is of the form

$$F^*_n(v, a) = F_n(v, a)/\sqrt{F_n(a, a)}; \ \ F_n(v, a) = \sum_{i=1}^{n} \varphi_n(v_i, a) \tag{5.18}$$

where

$$\varphi_n(v, a) = \begin{cases} 2\sinh^2(av/2), & \textit{if} \ \ a < b^*_{n,1}, \\ 2\sinh^2(av/2) \ \Phi(T_n(a) - a - |v|), & \textit{if} \ \ a \ge b^*_{n,1}. \end{cases} \tag{5.19}$$

Proof of Lemma 5.1 is given in Appendix, section 6.4.

In view of Lemma 5.1 we need to estimate the values

$$F_n(\theta_n, a) = \inf_{v \in V_n(\theta_n)} F_n(v, a), \text{ with } a = b_{n,l}.$$

First, let $b_{n,l} \le b^*_{n,1}$. Denote $b = b_n(\theta_n),\ h = h_n(\theta_n)$, and for BP $p = p(\theta_n),\ q = q(\theta_n)$. Remind that by Lemma 1.1 there are three possible relations (i), (ii) and (iii) for parameters $b,\ h$ in BP. We show that in BP the following inequalities are possible:

$$F_n(\theta_n, a) \ge F_{n,1}(\theta_n, a) = 2nh\sinh^2(ab/2), \tag{5.20}$$
$$F_n(\theta_n, a) \ge F_{n,2}(\theta_n, a) = 2nh\sinh^2(b^2(p)/2)(ab/b^2(p))^p, \tag{5.21}$$
$$F_n(\theta_n, a) \ge F_{n,3}(\theta_n, a) = 2n\sinh^2(abh^{1/p}/2). \tag{5.22}$$

Observe that in MCP the relation (5.20) holds. For BP introduce the assumptions (we assume $b(p) = 0$ for $p \le 2$):

$$A \ : \ \begin{cases} \text{or} \quad p > q,\ ab \ge b^2(p); \\ \text{or} \quad p < q,\ ab \le b^2(p); \end{cases} \tag{5.23}$$

$$B \ : \ \begin{cases} \text{or} \quad p \ge q,\ ab < b^2(p); \\ \text{or} \quad p \le q,\ ab > b^2(p) \ge h^{1/p}ab, \end{cases} \tag{5.24}$$

$$C \ : \quad h^{1/p}ab \le b^2(p), \tag{5.25}$$

$$D \ : \quad h^{1/p}ab > b^2(p). \tag{5.26}$$

Lemma 5.2

1. Assume: $\{$(iii) and $A\}$. Then (5.20) holds true.

2. Assume: either Bor $\{C$ and either (i) or (ii)$\}$. Then (5.21) holds true.

3. Assume: D and either $\{$(iii) and $p < q\}$, or $\{$either (i) or (ii)$\}$. Then (5.22) holds true.

Proof of Lemma 5.2 is given in Appendix, section 6.5.1.

Remark 5.1

First, observe that if $p \le 2$, then B, C are empty and D holds.

Next, if $h = 1$ (this means (i) *holds), then $F_{n,1}(\theta_n, a) = F_{n,3}(\theta_n, a)$. Therefore using Lemmas 1.1 and 5.2 one can easily see that if $p \le 2$, then (5.20) holds. Moreover, (5.20) does not holds for $p > 2$, if either $b \le b(p) \le a$, or $b \ge b(p) \ge a$.*

Denote $M^* = \max\{l : b_{n,l} \le b^*_{n,1}\}$. Using Lemmas 5.1, 5.2 we get for $1 \le l \le M^*$:

$$\beta_n(\psi_{n,l}, V_n(\theta_n)) \le \Phi(w_{n,l} - \rho_n(\theta_n, b_{n,l})u_n(\theta_n)) + o(1), \tag{5.27}$$

where under (5.20) (this holds for MCP)

$$\rho_n(\theta_n,a)=\rho_{n,1}(\theta_n,a)=\frac{\sinh^2(ab/2)}{\sinh(a^2/2)\sinh(b^2/2)};\tag{5.28}$$

under (5.21)

$$\rho_n(\theta_n,a)=\rho_{n,2}(\theta_n,a)=\frac{g_p^2(b(p))}{g_p(a)g_p(b)},\ g_p(b)=\frac{\sinh(b^2/2)}{b^p}\tag{5.29}$$

(remind that the value $b(p)$ minimizes $g_p(b)$ on $b>0$) and under (5.22)

$$\rho_n(\theta_n,a)=\rho_{n,3}(\theta_n,a)=\frac{\sinh^2(abh^{1/p}/2)}{h\sinh(b^2/2)\sinh(a^2/2)}.\tag{5.30}$$

Let us estimate the values $\rho_{n,1}(\theta_n,a)$.

Proposition 5.2 *1). Let $a=a_n\le b^*_{n,1},\ b=b_n(\theta_n)\le b^*_{n,1},\ |a-b|=O(w_n^{-2/3}(a))$. Then $\rho_{n,1}(\theta_n,a)\ge 1-O(w_n^{-4/3}(a))$, and if $b=b(p)\le a$, then $\rho_{n,l}(\theta_n,a)\ge 1-O(w_n^{-4/3}(a)),\ l=2,3$.*

2) If $a,b\to 0$, then $\rho_{n,1}(\theta_n,a)\to 1$, and if $a,b=O(1)$, then $\rho_{n,1}(\theta_n,a)\asymp 1$. If $b=b(p)\le a$, then analogous relations hold for $\rho_{n,l}(\theta_n,a),\ l=2,3$.

Proof of Proposition 5.2 is given in Appendix, section 6.6.2.

Return to estimations of $\beta(\psi_{n,l},V_n(\theta_n))$. Let $\theta\in\Theta_{n,l},\ b_{n,l+1}\le b^*_{n,1}$. It means $b_n(\theta_n)\in[b_{n,l},\ b_{n,l+1}]$. Observe that $l,l+1\in L$ because $c_{n,l},c_{n,l+1}=o(1)$ in view of (6.18). In BP let us divide the set $\Theta_{n,l}$ onto

$$\Theta^+_{n,l}=\{\theta\in\Theta_{n,l}:b_n(\theta_n)>b(p(\theta_n))\},\ \Theta^-_{n,l}=\{\theta\in\Theta_{n,l}:b_n(\theta_n)<b(p(\theta_n))\},$$
$$\Theta^0_{n,l}=\{\theta\in\Theta_{n,l}:b_n(\theta_n)=b(p(\theta_n))\}.$$

We use the test $\psi_{n,l}$ with $a=b_{n,l}$ for the sets $\Theta^-_{n,l}$ and the test $\psi_{n,l+1}$ with $a=b_{n,l+1}$ for the set $\Theta^+_{n,l},\Theta^0_{n,l}$. By using Remark 5.1 we see that the relation (5.20) holds for $\theta\in\Theta^{\pm}_{n,l}$ and $b=b(p)\le a$ for $\theta\in\Theta^0_{n,l}$. Using the choose of $z_{n,l}$ and Proposition 5.2, we obtain for $l\le M^*$

$$r_n=\rho_{n,1}(\theta_n,a)u_n(\theta_n)\ge u_n(\theta_n)-Bu_n(\theta_n)w_{n,l}^{-4/3},\ l=1,2,3.$$

If $u_n(\theta_n)<w_{n,l}^{7/6}$, then $r_n\ge u_n(\theta_n)+o(1)$, and if $u_n(\theta_n)\ge w_{n,l}^{7/6}$, then $r_n-w_{n,l}\to\infty$. This implies

$$\beta(\psi_{n,l},V_n(\theta_n))\le\Phi(w_{n,l}-u_n(\theta_n))+o(1)\le\Phi(-R)+o(1).$$

The study for MCP are analogous. Thus (5.16) is proved for $\Theta_{n,1}=\{\theta_n\in\Theta_n: b(\theta_n)\le b^*_{n,1}\}$.

Let us study the tests $\psi_{n,\alpha}(a_n)$ from Theorem 3.1. First, observe, that we need consider $b_n > a_{n,0}$ for any positive sequence $a_{n,0} \sim n^{-\delta},\ \delta > 1/4$ by $u_n(b_n, h_n) \to 0$, if $b_n = o(n^{-1/4})$. Using Tailor expansion one can easily replace the χ^2-statistics $\lambda_{n,2}$ onto $\lambda_n(a_{n,0})$. By Proposition 5.2 we get: $\rho_{n,l}(\theta_n, a) \sim 1$, if $l = 1,\ b_n(\theta_n) = o(1),\ a = o(1)$ or $b = b(p) \le a = o(1)$ and $\rho_{n,l}(\theta_n, a) \asymp 1$ if $l = 1, b_n(\theta_n) = 0(1),\ a = O(1)$ or $b = b(p) \le a = O(1)$. Therefore using similar arguments uniformly on $v \in V_n(\theta_n),\ a_{n,0} < b_n(\theta_n) \le a_n$ one has:

$$\beta_n(\psi_{n,\alpha}(a_n, v) \le \min\{P_{n,v}(\lambda_n(a_{n,0}) < t_\alpha),\ P_{n,v}(\lambda_n(a_n) < t_\alpha)\} \le \\ \Phi(t_\alpha - u_n(b_n)) + o(1).$$

The same estimations show that $\beta_n(\psi_{n,\alpha}(a_n), v) \to 0$ as $u_n(b_n) \to \infty$ uniformly on $v \in V_n(\theta_n),\ a_{n,0} < b_n(\theta_n) \le a_n = O(1)$.

Consider the cases $b_{n,l} > b^*_{n,1}$. Remind that we use $\tilde{T}_{n,l}$-truncated statistics $\tilde{\lambda}_{n,l}$ in these cases.

Proposition 5.3 *Let $v_n \in V_n(\theta_n) :\ \ b_n > b^*_{n,1},\ d_n \ge B w_n^{-c} b^*_n \to \infty;\ 0 < c < 2$. Then $S_n(v_n) \to \infty$ (the values $S_n(v_n)$ are defined by (5.17)).*

Proof of Proposition 5.3 is given in Appendix, section 6.2.

In view of Proposition 5.3 we can consider below alternatives $v \in V_n(\theta_n)$ with such $\theta_n \in \Theta_n$ that $d_n = o(b_n)$ by in opposite case they are rejected by the tests $\psi_{n,0}$. The considerations below follow to the scheme above. By Lemma 5.1 and Remark 3.1 we need to estimate the values

$$F_n(\theta_n, a) = \inf_{v \in V_n(\theta_n, Q_n)} F_n(v, a),\ V_n(\theta_n, Q_n) = \{v \in V_n(\theta_n) :\ \max_i |v_i| \le Q_n\}$$

assuming in BP: $p > q,\ a = b_{n,l} \to \infty,\ b_n = b_n(\theta_n) \to \infty$. In fact, if $p \le q,\ p \le p^*_n = o(\log n)$, then $b(p) \asymp p^{1/2} \ll b^*_{n,1} < b_n$. By Lemma 1.1 it is possible under the relation (i). This case was excluded before.

Lemma 5.3 . *In BP for any $0 < C_1 < 1 < C_2,\ 0 < C_3 < 1$ there exist such $B > 0,\ \delta > 0,\ \epsilon > 0$ that if $b = b_n(\theta_n) > B,\ C_1 b < a < C_2 b,\ d = d_n(a, b) = a + b - T < \delta a,\ p < \epsilon b^2,\ b < Q \le C_3(4a + 3b - 2T)$, where $T = T_n(a)$, then*

$$F_n(\theta_n, a) \ge 2nh_n \sinh^2(ab/2)\Phi(-d_n(a, b)).$$

Proof of Lemma 5.3 is given in Appendix, section 6.5.2.

Note that same relation holds for MCP. Using Lemma 5.1 and Lemma 5.3 we get for $M^* \le l \le M$:

$$\beta_n(\psi_{n,l}, V_n(\theta_n)) \le \Phi(w_{n,l} - \rho_n(\theta_n, b_{n,l})u_n(\theta_n)) + o(1), \tag{5.31}$$

where

$$\rho_n(\theta_n, a) = \rho_{n,4}(\theta_n, a) = \frac{\sinh^2(ab_n/2)}{\sinh(a^2/2)\sinh(b_n^2/2)} \frac{\Phi(-(d_n(a) + d_n(b_n))/2 + \delta_n/2)}{\sqrt{\Phi(-d_n(a))\Phi(-d_n(b_n))}},$$

and $d_n(b) = 2b_n - T_n(b), \delta_n = T_n(a) - T_n(b)$.

Proposition 5.4 . *Let* $b^*_{n,1} \le a, b = b_n(\theta_n),\ |z_n(a) - z_n(b)| = O((w_n(a))^{-2/3})$. *Then*

$$\rho_{n,4}(\theta_n, a) \ge 1 - O((w_n(a))^{-4/3}). \tag{5.32}$$

Proof of Proposition 5.4 is given in Appendix, section 6.6.3.

Consider subsets $\Theta_{n,l} \subset \Theta_n$ such that $b_n(\theta_n) \in [b_{n,l},\ b_{n,l+1}],\ b_{n,l} > b^*_{n,1}$. If $l \in L$, then we used the test $\psi_{n,l}$ here. Using (5.31), the choose of $z_{n,l}$ and Proposition 5.4 analogously with above we obtain the relation

$$\beta(\psi_{n,l}, V_n(\theta_n)) \le \Phi(w_{n,l} - u_n(\theta_n)) + o(1) \le \Phi(-R) + o(1).$$

Let $l \notin L$, it means $c_{n,l} > u^{\kappa}_{n,l} \to \infty$. We use the test $\psi_{n,0}$ here. It is enough to show $S_n(v) \to \infty$ for any $v = v_n \in V_n(\theta_n),\ \theta_n \in \Theta_{n,l}$.

To proof this, first, observe that $d_n = d_n(\theta_n) \ge d_{n,l} + o(1)$. In fact, it is clear, if $T_n = T_n(\theta_n) \le T_{n,l}$. If $T_n > T_{n,l}$, then it follows from $T_n - T_{n,l} = o(1)$ (the last is shown in the proof of Proposition 5.4). By $c_{n,l} > u^{\kappa}_{n,l}$, using (6.19) we get $d_{n,l}/b^*_n \asymp d_{n,l}/T_{n,l} > u^{\kappa-2}_{n,l} > w_n^{-c},\ c = 2-\kappa > 0$. This implies $d_n/b^*_n \ge B w_n^{-c}$. Using Proposition 5.3 we obtain required relation. The study for MCP are analogous.

The upper bounds of Theorem 3.4 are proved. $\square$

6 Appendix

6.1 Proof of Lemma 4.1

It is enough to show that under $P_{n,0}$-probability

$$L_n = \sum_{l=1}^{M} p_{n,l} L_{n,l} = D_n + o(1),\ \ D_n \ge \Phi(-R_n); \tag{6.1}$$

By the assumption (4.4) we can replace $L_{n,l}$ onto $\tilde{L}_{n,l,w}$. In fact, let

$$\tilde{L}_n = \sum_{l=1}^{M} p_{n,l} \tilde{L}_{n,l},\ \ \tilde{L}_{n,w} = \sum_{l=1}^{M} p_{n,l} \tilde{L}_{n,l,w}$$

(a tilde corresponds to the replacement $l_{n,l}$ onto $\tilde{l}_{n,l}$, an index w corresponds to $w_{n,l}$-truncation). Then we get

$$P_{n,0}(L_n \ne \tilde{L}_{n,w}) \le P_{n,0}(L_n \ne \tilde{L}_n) + P_{n,0}(\tilde{L}_n \ne \tilde{L}_{n,w});\ \ P_{n,0}(L_n \ne \tilde{L}_n) = \eta_n,$$
$$P_{n,0}(\tilde{L}_n \ne \tilde{L}_{n,w}) \le \sum_{l=1}^{M} P_{n,0}(\tilde{L}_{n,l} \ne \tilde{L}_{n,l,w}) \le \Delta_n + \sum_{l=1}^{M} \Phi(-w_{n,l}) = o(1)$$

by

$$\sum_{l=1}^{M} \Phi(-w_{n,l}) \asymp \sum_{l=1}^{M} \exp(-w^2_{n,l}/2)/w_{n,l} \le B w_n^{-1} \to 0.$$

To obtain (6.1) with this replacement it is enough to check that

$$\sum_{l=1}^{M} p_{n,l} E_{n,0} \tilde{L}_{n,l,w} \geq \Phi(-R_n) + o(1), \ \sum_{l=1}^{M} p_{n,l}^2 E_{n,0} \tilde{L}_{n,l,w}^2 = o(1). \tag{6.2}$$

In fact, using (6.2) and the assumption (4.7) we get:

$$E_{n,0} \tilde{L}_{n,w} \geq \Phi(-R_n) + o(1), \ \mathrm{Var}_{n,0}(\tilde{L}_{n,w}) \leq$$
$$\sum_{l=1}^{M} p_{n,l}^2 E_{n,0} \tilde{L}_{n,l,w}^2 + 2 \sum_{1 \leq l < k \leq M} p_{n,l} p_{n,k} \mathrm{Cov}_{n,0}(\tilde{L}_{n,l,w}, \tilde{L}_{n,k,w}) = o(1)$$

and using Chebyshev inequality we get (6.1).

To check (6.2) we use the equality for the moments of bounded random variables $0 \leq X \leq H$ with distribution functions $F(x) = P(X < x)$:

$$\int_0^H x dF(x) = \int_0^H (1 - F(x)) dx$$

which imply the inequalities for differences of moments of bounded random variables $0 \leq X_1, X_2 \leq H$ with distribution functions $F_1(x), \ F_2(x)$:

$$|EX_1 - EX_2| \ \leq \ H \sup_x |F_1(x) - F_2(x)|, \ k = 1, 2; \tag{6.3}$$

$$|EX_1^2 - EX_2^2| \ \leq \ H^2 \sup_x |F_1(x) - F_2(x)|. \tag{6.4}$$

Let ν be standard Gaussian random variable. Put

$$X_l \ = \ \exp(m_{n,l} + \sigma_{n,l}\nu) = e^{\delta_{n,l}} \exp(-\sigma_{n,l}^2/2 + \sigma_{n,l}\nu), \ X_{l,w} = X_l \mathbf{1}_{\{\nu < w_{n,l}\}},$$
$$H_{n,l} \ = \ \exp(m_{n,l} + \sigma_{n,l} w_{n,l}) \sim \exp(-\sigma_{n,l}^2/2 + \sigma_{n,l} w_{n,l}).$$

where $\delta_{n,l} \to 0$ uniformly on $1 \leq l \leq M$ and are nonrandom by (4.5). Using (6.3), (6.4) we get:

$$E_{n,0} \tilde{L}_{n,l,w} = EX_{l,w} + O(H_{n,l} \Delta_{n,l});$$
$$E_{n,0} \tilde{L}_{n,l,M}^2 = EX_{l,w}^2 + O(H_{n,l}^2 \Delta_{n,l}).$$

Note that

$$p_{n,l} H_{n,l} \asymp \exp(-(\sigma_{n,l} - w_{n,l})^2/2) \leq 1$$

which implies that sums of remainder terms are $O(\Delta_n) = o(1)$. At last, for Gaussian variable one has

$$EX_{l,w} \sim P(\nu + \sigma_{n,l} < w_{n,l}) = \Phi(w_{n,l} - u_{n,l} + o(1)) \geq \Phi(-R_n) + o(1),$$

which implies the first relation in (6.2). To obtain the second relation it is enough to show that uniformly on l

$$p_{n,l} EX_{l,w}^2 \asymp \exp(\sigma_{n,l}^2 - w_{n,l}^2/2) \Phi(w_{n,l} - 2\sigma_{n,l}) = o(1).$$

Consider differently the cases $w_{n,l} > 3\sigma_{n,l}/2;$ and $w_{n,l} \leq 3\sigma_{n,l}/2$. We get in these cases respectively:

$$\exp(\sigma_{n,l}^2 - w_{n,l}^2/2) \Phi(w_{n,l} - 2\sigma_{n,l}) \leq \begin{cases} \exp(\sigma_{n,l}^2 - w_{n,l}^2/2) \leq \exp(-w_{n,l}^2/18) & = \mathrm{o}(1), \\ B w_{n,l}^{-1} \exp(-(w_{n,l} - \sigma_{n,l})^2) & = \mathrm{o}(1), \end{cases}$$

□

6.2 Some properties of likelihood ratio

Remind some properties of the statistics $\xi(x,b)$ and truncated ones $\tilde{\xi}_n(x,b) = \xi(x,b)\mathbf{1}_{\{|x|\le T_n\}}$ under null-hypothesis.

If x is a standard Gaussian variable, then we have the representation:

$$\xi(x,b) = (\eta(x,b)+\eta(x,-b))/2-1;\ \eta(x,b) = e^{-b^2/2+xb}; \tag{6.5}$$
$$\eta(x,b_1)\eta(x,b_2) = e^{b_1b_2}\eta(x,b_1+b_2);\ E\eta(x,b)=1. \tag{6.6}$$

Using (6.5), (6.6) we get:

$$E\xi(x,b)=0,\ E\xi^2(x,b) = 2\sinh^2\frac{b^2}{2},\ E(\xi(x,b_1)\xi(x,b_2)) = 2\sinh^2\frac{b_1b_2}{2}, \tag{6.7}$$

Also for an integer $k>1$ one has

$$E\xi^{2k}(x,b) \le C_1(k)\exp(C_2(k)b^2)(E\xi^2(x,b))^k \tag{6.8}$$

where $C_1(k)>0,\ C_2(k)>0$ are constants (see the Lemma 1 in [2]). Particularly, one can check that $C_2(2)=4$.

Let there are sequences $b_n\to\infty,\ T_n-b_n\to\infty$. Denote $\hat{\xi}_n(x,b_n) = \xi_n(x,b_n)\mathbf{1}_{\{|x|<T_n\}}$. Direct calculation analogous to [6] gives:

$$E\hat{\xi}_n(x,b_n) \sim -\Phi(b_n-T_n), \tag{6.9}$$
$$E\hat{\xi}^2_n(x,b_n) = (2\sinh^2\frac{b_n^2}{2}+1)\Phi(T_n-2b_n)-1+o(1), \tag{6.10}$$
$$E(\hat{\xi}_n(x,b))^3 \sim \frac{1}{4}\exp(3b_n^2)\ \Phi(T_n-3b_n). \tag{6.11}$$

Observe that if $nh_n^2\to 0$ (this holds under assumption of Theorems 3.3, 3.4), then using (6.9), (6.10) we get:

$$nh_n^2\mathrm{Var}_{1,0}\hat{\xi}_n(x,b_n) = u_n^2+o(1). \tag{6.12}$$

If we have two sequences $b_{n,1}\to\infty, b_{n,2}\to\infty$ and a sequence T_n such that $T_n-b_{n,1}-b_{n,2}\to\infty$, then

$$E\hat{\xi}_n(x,b_{n,1})\hat{\xi}_n(x,b_{n,2}) = 2\sinh^2\frac{b_{n,1}b_{n,2}}{2}\ \Phi(T_n-b_{n,1}-b_{n,2})+O(1). \tag{6.13}$$

Assume

$$b_n\to\infty,\ \log u_n/\log n = o(1),\ \tau_n\ge 2-o(1). \tag{6.14}$$

Then using (2.3), (2.7), (2.18), one can easily see that

$$b_n^2 \sim \frac{\log n}{2(\tau_n-1)},\ d_n^2\sim\frac{(\tau_n-2)^2}{2(\tau_n-1)}\log n, \tag{6.15}$$
$$T_n^2 \sim \frac{\tau_n^2\log n}{2(\tau_n-1)} \ge 2\log n;\ \log(nh_n^2)\sim -\frac{\log n}{2(\tau_n-1)}. \tag{6.16}$$

Below we use the asymptotics which follow from the relations (2.3), (2.18):

$$c_n \sim \frac{u_n^2 \exp(-d_n^2/2)}{\sqrt{2\pi} T_n \Phi(-d_n)} \tag{6.17}$$

where (remind) $c_n = 2n\Phi(-T_n)$. This implies

$$\begin{aligned} c_n &\asymp u_n^2 \exp(-d_n^2/2)/T_n, \text{ if } d_n \le B; & (6.18) \\ c_n &= u_n^2 d_n/T_n(1 + O(d_n^{-2})), \text{ if } d_n \to \infty. & (6.19) \end{aligned}$$

Proof of Proposition 4.1. Let $\tau_n \to \tau > 2$. Then, by (6.14) holds under assumption (4.13), the relation (4.18) follows from (6.15), (6.18) because $b_n < b_n^*$, $d_n^2/2 \asymp (b_n - b_n^*)^2 > b_n^* - b_n - 1$. Therefore

$$\exp(-d_n^2/2)/T_n \le B \exp b(1 + b_n - b_n^*)/b_n^* = B \exp b(z_n - z_n^*).$$

Let $\tau_n \to 2$. Using the relations (6.15), (6.16) we get:

$$T_n = \sqrt{2\log n}\,(1 + o(1)),\ b_n = T_n/\tau_n \sim b_n^*. \tag{6.20}$$

Note that $\tau_n \le 2 + o(1)$ if and only if $b_n \le b_n^*(1 + o(1))$.

Put $d_n^* = 2(b_n - b_n^*) = 2b_n - t_n^*$ (it does not depend on $T_n,\ h_n$). Here and below $t_n^* = \sqrt{2\log(n)}$. Using the relation (6.19) we can see that if $d_n \to \infty,\ d_n(\theta) = o(t_n^*)$, then the difference

$$d_n^* - d_n = \delta_n = \delta_n(\theta) = T_n(\theta) - t_n^* = O(\log t_n^*/t_n^*). \tag{6.21}$$

In fact, using (2.18) and (6.19) we get:

$$\begin{aligned} &(t_n^*)^{-1} \asymp n\Phi(-t_n^*) \sim n\Phi(-T_n)\exp(t_n^*\delta_n + \delta_n^2/2) \asymp c_n \exp(t_n^*\delta_n + \delta_n^2/2); \\ &\exp(-t_n^*\delta_n - \delta_n^2/2) \asymp u_n^2 d_n,\ |\delta_n| \le \frac{2\log u_n + \log d_n + B}{t_n^*} = o(\log t_n^*/t_n^*). \end{aligned}$$

Here we use $\delta_n = o(t_n^*),\ u_n = o(\log n)$. Therefore we can replace d_n onto d_n^* in (6.19). Then (4.18) follows from (6.19) by $d_n^*/T_n \asymp \exp(z_n - z_n^*)$.

If $d_n \le B,\ b_n \le b_n^* + 1$, then by using (6.18) and arguments analogous with what given for $\tau > 2$, we get (4.18).

If $b_n > b_n^* + 1$, then $\exp(z_n - z_n^*) = (b_n - b_n^*)/b_n^* > 1/b_n^*$, and by using (6.18) we get: $c_n \le Bu_n^2/b_n^* \le Bu_n^2 \exp(z_n - z_n^*)$. □

Proof of Proposition 5.3. First, observe that $u_n(\theta_n) \ge w_n + R + o(1) \to \infty$. Denote

$$\delta_n = H_n - T_n,\ \lambda_n^* = nh_n\Phi(b_n - H_n),\ \lambda_n = nh_n\Phi(b_n - T_n),$$

where H_n is defined by (5.4). In view of Lemma 3.1 it is enough to show that $\lambda_n^* \to \infty$.

It follows from (6.19) that $c_n \geq Bw_n^{2-c}$ which implies $\delta_n > 0$. By $\lambda_n \sim c_n\tau_n/(\tau_n - 1) > c_n$, if $b_n - T_n \to \infty$ ($\tau_n \geq 1$) and $\lambda_n = nh_n\Phi(b_n - T_n) \geq n\exp(-T_n^2/2)\Phi(-B) \gg c_n$, if $b_n - T_n \geq -B$, then using (2.18) we get: if $T_n \geq H_n + 2(\log\log w_n)/b_n = H_n + o(1)$, then

$$\lambda_n = nh_n\Phi(b_n - H_n + \delta_n) \asymp \lambda_n^* \exp((H_n - b_n)\delta_n - \delta_n^2/2) > Bc_n \asymp$$
$$n\Phi(-H_n + \delta_n) \asymp c_n^* \exp(H_n\delta_n - \delta_n^2/2)$$

which imply (by $b_n \geq b_{n,1}^*$)

$$c_n/c_n^* \asymp \exp(H_n\delta_n - \delta_n^2/2) < \exp(H_n\delta_n); \quad \lambda_n^* > Bc_n^* \exp(b_n\delta_n) >$$
$$Bc_n^*(c_n/c_n^*)^{b_n/H_n} \geq Bc_n^{1/10\sqrt{2}+o(1)}(c_n^*)^{1-1/10\sqrt{2}+o(1)} >$$
$$Bw_n^{(2-c-o(1))/10\sqrt{2}}/\log(w_n) \to \infty.$$

by the choose of c. If $T_n < H_n + 2(\log\log w_n)/b_n$, then

$$\lambda_n^* \asymp c_n^* \exp(b_n(H_n - T_n)) \asymp \exp(b_n(H_n - T_n))/\log w_n > \log w_n \to \infty.$$

□

Remind some properties of the statistics $\xi(x, b)$ under alternatives.

If x is a standard Gaussian variable, then using (6.5), for any $v \in R^1$ we get:

$$\xi(x+v, b) = (e^{bv}\eta(x, b) + e^{-bv}\eta(x, -b))/2 - 1; \tag{6.22}$$
$$\xi(x+v, b) - E\xi(x+v, b) = \frac{1}{2}(e^{bv}(\eta(x, b) - 1) + e^{-bv}(\eta(x, -b) - 1)). \tag{6.23}$$

Using (6.6), (6.22), (6.23) we get:

$$E\xi(x+v, b) = 2\sinh^2\frac{bv}{2}, \tag{6.24}$$
$$Var\xi(x+v, b) = 2\sinh^2\frac{b^2}{2} + (e^{b^2} - 1)\sinh^2 bv; \tag{6.25}$$
$$E|\xi(x+v, b)|^3 \leq 2 + E(\xi(x+v, b))^3 \leq$$
$$2 + Be^{3b|v| + 3b^2}. \tag{6.26}$$

6.3 Proof of Lemma 4.2

To check (4.4) note, that

$$P_{n,0}\{\exists\, l : l_{n,l} \neq \tilde{l}_{n,l},\ 1 \leq l \leq M\} = P_{n,0}(\max_{1\leq i\leq n}|x_i| \geq \min_{1\leq l\leq M} T_{n,l}) \leq$$
$$nP_{1,0}(|x| \geq \min_{1\leq l\leq M} T_{n,l}) = 2n\Phi(-\min_{1\leq l\leq M} T_{n,l}) = \max_{1\leq l\leq M} c_{n,l} = o(1)$$

by the assumption (4.16).

Proposition 6.1 *Under (6.14) for some $\delta > 0,\ B > 0$ one has:*

$$nE_{1,0}|\tilde{Z}_n|^3 \le B(c_n + n^{-\delta}). \tag{6.27}$$

Proof of Proposition 6.1. First, let us establish analogous relation for non-absolute moments:

$$nE_{1,0}(\tilde{Z}_n^3) \le B(c_n + n^{-\delta}). \tag{6.28}$$

Note that by $-1 < \xi(x, b_n)$, then the difference between absolute and non-absolute moments of ξ_n is no more then 2. Therefore a difference between left-hand sides of (6.27) and (6.28) is bounded by the values of the rate

$$nh_n^3 = O(n^{-\eta_n}),\ \eta_n \sim \frac{2\tau_n + 1}{4(\tau_n - 1)} > 0$$

(we use (2.3), (6.16)). Therefore (6.28) implies (6.27).

Let us check (6.28). Let $\tau_n > 5/2$. Then using (2.3), (6.15), (6.16), (6.11) we get:

$$nE_{1,0}(\tilde{Z}_n^3) \le nh_n^3 e^{3b_n^2} \asymp n\exp(3b_n^2(3/2 - \tau_n)) \asymp n^{-\eta_n},\ \eta_n \sim \frac{2\tau_n - 5}{4(\tau_n - 1)} > 0.$$

Let $\tau_n \le 5/2$. Then $T_n - 3b_n \sim b_n(\tau_n - 3) \to -\infty$ and using (2.3), (2.18),(6.11) we get:

$$\begin{aligned} &nE_{1,0}(\tilde{Z}_n^3) = nh_n^3 e^{3b_n^2}\Phi(T_n - 3b_n) \asymp \\ &nT_n^{-1}\exp((9b_n^2 - 6b_nT_n - (T_n - 3b_n)^2)/2) = nT_n^{-1}\exp(-T_n^2/2) \asymp c_n. \end{aligned}$$

Thus (6.27) is proved. $\square$

Proof of (4.5), (4.6). First, estimate means and variances of statistics $\tilde{l}_{n,l}$ under $P_{n,0}$-probability. By $-h_{n,l} \le \tilde{Z}_{n,l} \le 1$ and $h_{n,l} \to 0$, one has

$$|\tilde{W}_{n,l} - \tilde{Z}_{n,l} + \tilde{Z}_{n,l}^2/2| \le B|\tilde{Z}_{n,l}|^3. \tag{6.29}$$

Using (6.9) we have:

$$nE_{1,0}\tilde{Z}_{n,l} \sim -nh_{n,l}\Phi(b_{n,l} - T_{n,l}) \sim -c_{n,l}\frac{\tau_{n,l}}{\tau_{n,l} - 1} \asymp -c_{n,l}. \tag{6.30}$$

The relations (6.27), (6.29), (6.30) imply

$$\begin{aligned} E_{n,0}\tilde{l}_{n,l} &= nE_{1,0}\tilde{W}_{n,l} = n(E_{1,0}\tilde{Z}_{n,l} - \frac{1}{2}E_{1,0}(\tilde{Z}_{n,l}^2) + O(E_{1,0}(|\tilde{Z}_{n,l}|^3) \\ &= -u_{n,l}^2/2 + O(c_{n,l} + n^{-\delta}), & (6.31) \\ \mathrm{Var}_{n,0}\tilde{l}_{n,l} &= n\mathrm{Var}_{1,0}\tilde{W}_{n,l} = nE_{1,0}(\tilde{W}_{n,l}^2) - n^{-1}(nE_{1,0}\tilde{W}_{n,l})^2 \\ &= u_{n,l}^2 + O(c_{n,l} + n^{-\delta}). & (6.32) \end{aligned}$$

The estimations above imply (4.5). Denote $\tilde{W}^0_{n,l} = \tilde{W}_{n,l} - E_{1,0}\tilde{W}_{n,l}$. Note that by (6.29) analogous to (6.27) relation holds:

$$nE_{1,0}|\tilde{W}_{n,l}|^3 \le B(c_{n,l} + n^{-\delta}). \tag{6.33}$$

The relation (4.6) follows directly from the assumption (4.16), from (6.33) and from Bahr-Essen inequality because $M = o((\log n)^{1/2})$ and

$$\Delta_{n,l} \le \frac{BnE_{1,0}|\tilde{W}^0_{n,l}|^3}{(nE_{1,0}(\tilde{W}^0_{n,l})^2)^{3/2}} = O((c_{n,l} + n^{-\delta})/u^3_{n,l}).$$

□

Proof of (4.7). Let us establish the inequality:

$$\mathrm{Cov}_{n,0}(\tilde{L}_{n,l,w}\tilde{L}_{n,k,w}) \le \exp(r_{n,lk}) - 1 + o(1) \tag{6.34}$$

which implies (4.7). Denote $r_{n,lk} = r_n,\ \rho_{n,lk} = \rho_n = r_n/u_{n,k}u_{n,l}$ to simplicity.

Proposition 6.2 *Under the assumptions of the Lemma*

$$E_{n,0}(\tilde{L}_{n,l,w}) = \Phi(w_{n,l} - u_{n,l}) + o(1), \tag{6.35}$$

$$E_{n,0}(\tilde{L}_{n,l,w}\tilde{L}_{n,k,w}) \le \exp(r_n)\Phi_{\rho_n}(w_{n,l} - u_{n,l} - \rho_n u_{n,k}, w_{n,k} - u_{n,k} - \rho_n u_{n,l})$$
$$(1 + o(1)) + o(1) \tag{6.36}$$

where $\Phi_\rho(x, y)$ stands for the joint distribution function of standard Gaussian random variables X, Y with $E(X, Y) = \rho$.

Proof of Proposition 6.2. We give the proof (6.36) only, because (6.35) is proved by analogous way (in particular, $E_{n,0}(\tilde{L}_{n,l}) = 1 + o(1)$ in view (6.30)).

First, note the equality

$$E_{n,0}(\tilde{L}_{n,l,w}\tilde{L}_{n,k,w}) = E_{n,0}(\tilde{L}_{n,l}\tilde{L}_{n,k})P^n(\tilde{L}_{n,l} \le H_{n,l}, \tilde{L}_{n,k} \le H_{n,k}), \tag{6.37}$$

where $H_{n,l} = \exp(-u^2_{n,l}/2 + w_{n,l}u_{n,l})$ and $P^n = P^n_{l,k}$ is the measure on $(R^n, \mathcal{B}_n)$ with likelihood ratio

$$\frac{dP^n}{dP_{n,0}}(x) = \frac{\tilde{L}_{n,l}(x)\tilde{L}_{n,k}(x)}{E_{n,0}(\tilde{L}_{n,l}\tilde{L}_{n,k})} = \prod_{i=1}^n \frac{(1 + \tilde{Z}_{n,l}(x_i))(1 + \tilde{Z}_{n,k}(x_i))}{E_{1,0}(1 + \tilde{Z}_{n,l})(1 + \tilde{Z}_{n,k})},\ x = (x_1, ..., x_n) \in R^n.$$

By the equality (6.37) it is enough to show

$$E_{n,0}(\tilde{L}_{n,l}\tilde{L}_{n,k}) = (E_{1,0}(1 + \tilde{Z}_{n,l})(1 + \tilde{Z}_{n,k}))^n \le e^{r_{n,l,k}}(1 + o(1)) \tag{6.38}$$

and that under P^n-distribution the statistics $(\tilde{\lambda}_{n,l}, \tilde{\lambda}_{n,k})$ are asymptotical Gaussian with

$$\begin{aligned}
E_{P^n}\tilde{\lambda}_{n,l} &= u_{n,l} + \rho_n u_{n,k} + o(1),\\
E_{P^n}\tilde{\lambda}_{n,k} &= u_{n,k} + \rho_n u_{n,l} + o(1),\\
\mathrm{Var}_{P^n}(\tilde{\lambda}_{n,l}) &= 1 + o(1),\\
\mathrm{Var}_{P^n}(\tilde{\lambda}_{n,k}) &= 1 + o(1),\\
\mathrm{Cov}_{P^n}(\tilde{\lambda}_{n,l}, \tilde{\lambda}_{n,k}) &= \rho_n + o(1).
\end{aligned}$$

The last follows from asymptotical normality of the statistics $(\tilde{l}_{n,l}, \tilde{l}_{n,k})$ under P^n-distribution with

$$\begin{aligned}
E_{P^n}\tilde{l}_{n,l} &= u_{n,l}^2/2 + r_n + o(1), \\
E_{P^n}\tilde{l}_{n,k} &= u_{n,k}^2/2 + r_n + o(1), \qquad (6.39)\\
\mathrm{Var}_{P^n}(\tilde{l}_{n,l}) &= u_{n,l}^2 + o(1), \\
\mathrm{Var}_{P^n}(\tilde{l}_{n,k}) &= u_{n,k}^2 + o(1), \qquad (6.40)\\
\mathrm{Cov}_{P^n}(\tilde{l}_{n,l}, \tilde{l}_{n,k}) &= r_n + o(1). \qquad (6.41)
\end{aligned}$$

To obtain (6.38) observe that

$$\begin{aligned}
&(E_{1,0}(1+\tilde{Z}_{n,l})(1+\tilde{Z}_{n,k}))^n = (1 + E_{1,0}\tilde{Z}_{n,l} + E_{1,0}\tilde{Z}_{n,k} + E_{1,0}\tilde{Z}_{n,l}\tilde{Z}_{n,k})^n \\
&\le \exp(n(E_{1,0}\tilde{Z}_{n,l} + E_{1,0}\tilde{Z}_{n,k} + E_{1,0}(\tilde{Z}_{n,l}\tilde{Z}_{n,k}))) = e^{r_n}(1+o(1))
\end{aligned}$$

because it follows from (4.16), (6.30) that

$$nE_{1,0}\tilde{Z}_{n,l} = o(1),\ nE_{1,0}\tilde{Z}_{n,k} = o(1),\ nE_{1,0}(\tilde{Z}_{n,l}\tilde{Z}_{n,k}) = r_n.$$

The items $\tilde{W}_{n,l}(x_i)$ in the sum $\tilde{l}_{n,l}$ are i.i.d. under P_n-distribution where $P_n = P_{n;l,k}$ is the measure on $(R^1, \mathcal{B}_1)$ with likelihood ratio

$$\frac{dP_n}{dP_{1,0}}(x) = (1 + \tilde{Z}_{n,l}(x) + \tilde{Z}_{n,k}(x) + \tilde{Z}_{n,l}(x)\tilde{Z}_{n,k}(x))(1 - \delta_n + O(\delta_n^2)),\ x \in R^1$$

and

$$\delta_n = E_{1,0}\tilde{Z}_{n,l} + E_{1,0}\tilde{Z}_{n,k} + E_{1,0}(\tilde{Z}_{n,l}\tilde{Z}_{n,k}) = o(1/n).$$

Therefore we have:

$$\begin{aligned}
E_{P^n}\tilde{l}_{n,l} &= nE_{1,0}(\tilde{W}_{n,l} + \tilde{W}_{n,l}\tilde{Z}_{n,l} + \tilde{W}_{n,l}\tilde{Z}_{n,k} + \tilde{W}_{n,l}\tilde{Z}_{n,l}\tilde{Z}_{n,k}) \\
&\quad (1+o(1/n)), \qquad (6.42)\\
\mathrm{Var}_{P^n}(\tilde{l}_{n,l}) &= n\mathrm{Var}_{P_n}(\tilde{W}_{n,l}) = nE_{P_n}(\tilde{W}_{n,l}^2) - n^{-1}(E_{P^n}\tilde{l}_{n,l})^2 \\
&= nE_{1,0}(\tilde{W}_{n,l}^2 + \tilde{W}_{n,l}^2\tilde{Z}_{n,l} + \tilde{W}_{n,l}^2\tilde{Z}_{n,k} + \tilde{W}_{n,l}^2\tilde{Z}_{n,l}\tilde{Z}_{n,k}) \\
&\quad (1+o(1/n)) - n^{-1}(E_{P^n}\tilde{l}_{n,l})^2, \qquad (6.43)\\
\mathrm{Cov}_{P^n}(\tilde{l}_{n,l}, \tilde{l}_{n,k}) &= n\mathrm{Cov}_{P_n}(\tilde{W}_{n,l}, \tilde{W}_{n,k}) = nE_{P_n}(\tilde{W}_{n,l}\tilde{W}_{n,k}) - n^{-1}E_{P^n}\tilde{l}_{n,l}\ E_{P^n}\tilde{l}_{n,k} \\
&= nE_{1,0}(\tilde{W}_{n,l}\tilde{W}_{n,k} + \tilde{W}_{n,l}\tilde{W}_{n,k}\tilde{Z}_{n,l} + \tilde{W}_{n,l}\tilde{W}_{n,k}\tilde{Z}_{n,k} + \\
&\quad \tilde{W}_{n,l}\tilde{W}_{n,k}\tilde{Z}_{n,l}\tilde{Z}_{n,k})(1+o(1/n)) - n^{-1}E_{P^n}\tilde{l}_{n,l}\ E_{P^n}\tilde{l}_{n,k}. \qquad (6.44)
\end{aligned}$$

The relation (6.29) and estimations analogous to (6.30) - (6.32) imply:

$$\begin{aligned}
nE_{1,0}(\tilde{W}_{n,l}) &= -u_{n,l}^2/2 + o(1),\quad nE_{1,0}(\tilde{W}_{n,l}\tilde{Z}_{n,l}) = u_{n,l}^2 + o(1), \qquad (6.45)\\
nE_{1,0}(\tilde{W}_{n,l}\tilde{Z}_{n,k}) &= r_n + o(1),\quad nE_{1,0}(\tilde{W}_{n,l}\tilde{W}_{n,k}) = r_n + o(1), \qquad (6.46)\\
nE_{1,0}(\tilde{W}_{n,l}^2) &= u_{n,l}^2 + o(1),\quad nE_{1,0}(\tilde{W}_{n,k}^2) = u_{n,k}^2 + o(1). \qquad (6.47)
\end{aligned}$$

Moreover, Proposition 6.1 and boundness of $\tilde{Z}_{n,l},\ \tilde{W}_{n,l}$ imply that the moments of the order 3, 4 are of the rate $o(1/n)$. These and the relations (6.42) - (6.47) imply (6.39) - (6.41).

The asymptotical normality of the statistics $(\tilde{l}_{n,l}, \tilde{l}_{n,k})$ follows from (6.39) - (6.41) and from two-dimensional Bahr-Essen inequality:

$$\begin{aligned}
&\sup_{x,y} |P^n(\tilde{l}_{n,l} + u_{n,l}^2/2 \le xu_{n,l}, \tilde{l}_{n,k} + u_{n,k}^2/2 \le yu_{n,k}) - \\
&\Phi_{\rho_n}(x - u_{n,l} - \rho_n u_{n,k}, y - u_{n,k} - \rho_n u_{n,l})| \\
&\le \frac{Bn(E_{1,0}|W_{n,l}|^3 + E_{1,0}|\tilde{W}_{n,k}|^3)}{(1-|\rho_n|)(n(E_{1,0}|\tilde{W}_{n,l}|^2 + E_{1,0}|\tilde{W}_{n,k}|^2))^{3/2}} \le \frac{B(c_{n,l} + c_{n,k} + n^{-\delta})}{u_{n,l}^3 + u_{n,k}^3} = o(1).
\end{aligned}$$

□

The relation (6.34) follows from Proposition 6.2 by (we omit index n here)

$$\begin{aligned}
&\mathrm{Cov}_{P_0}(\tilde{L}_{l,w}, \tilde{L}_{k,w}) = E_0(\tilde{L}_{l,w}\tilde{L}_{k,w}) - E_0(\tilde{L}_{l,w})E_0(\tilde{L}_{k,w}) \le \\
&(e^r - 1)\Phi_\rho(w_l - u_l - \rho u_k, w_k - u_k - \rho u_l)(1 + o(1)) + o(1) \\
&+\Phi_\rho(w_l - u_l - \rho u_k, w_k - u_k - \rho u_l) - E_0(\tilde{L}_{l,w})E_0(\tilde{L}_{k,w})); \\
&|\Phi_\rho(w_l - u_l - \rho u_k, w_k - u_k - \rho u_l) - \Phi(w_l - u_l)\Phi(w_l - u_k)| \\
&\le B(|\rho|(1 + u_l + u_k))
\end{aligned}$$

for some $B > 0$ (the last relation follows from estimation of Hellinger distance),

$$\max_{1\le k<l\le M} \rho_{n,lk}(1 + u_{n,l} + u_{n,k}) \le B \max_{1\le l<k\le M} r_{n,lk} = o(1)$$

and in view on (6.35), $E_{n,0}(\tilde{L}_{n,l,w}) = \Phi(w_{n,l} - u_{n,l}) + o(1)$.

Lemma 4.2 is proved. □

Proof of of Proposition 5.1 If $b_{n,l} \le b_{n,2}^*$, then estimations are the same as in the proof of Proposition 6.1 and (4.5), (4.6). If $b_{n,l} > b_{n,2}^*$, then using analogous estimations we get:

$$\begin{aligned}
&m_{n,l} \sim -nh_{n,l}\Phi(b_{n,l} - \tilde{T}_{n,l})/u_{n,l} < 0; \\
&-m_{n,l} \asymp c_{n,l}\exp(-(T_{n,l} - b_{n,l})\zeta_{n,l})/u_{n,l} < c_{n,l}u_{n,l}^{-1} = o(1); \\
&\sigma_{n,l}^2 = nh_{n,l}^2 e^{b_{n,l}^2}\Phi(\zeta_{n,l} - d_{n,l})/2u_{n,l}^2 + o(u_{n,l}^{-2}); \\
&\sigma_{n,l}^2 = 1 + o(u_{n,l}^{-2}) + \begin{cases} B\zeta_{n,l} = o(u_{n,l}^{-1}), & \text{if } d_{n,l} < B \\ e^{d_{n,l}\zeta_{n,l}} - 1 = o(u_{n,l}^{-1}), & \text{if } d_{n,l} > B \end{cases} = 1 + o(u_{n,l}^{-1}),
\end{aligned}$$

which imply (5.11). Here we use (6.19) and definition of the set L to obtain the relation: $\exp(d_{n,l}\zeta_{n,l}) = 1 + o(u_{n,l}^{-1})$. The estimation of $\Delta_{n,l}$ are based on the estimation of Lyupunov ratio. It is bounded by

$$\begin{aligned}
&nh_{n,l}^3 E_{1,n}\tilde{\xi}_n^3(x)/u_{n,l}^3 \asymp c_{n,l}\exp((3b_{n,l} - T_{n,l})\zeta_{n,l})/u_{n,l}^3 \\
&\le Bc_{n,l}\exp(\eta(3 + \tau_{n,l})/\tau_{n,l} - 3)\log u_{n,l}) \le Bu_{n,l}^{-2-\varrho}
\end{aligned}$$

because $2 + \delta > \tau_{n,l} \ge 2 - o(1)$ which follows from the constraints on $c_{n,l}$ and (6.19).

□

6.4 Proof of Lemma 5.1

To prove Lemma 5.1 for $b \le b^*_{n,1}$ observe that by (6.25), (6.26) for any $\delta > 0$ and $\max_i |v_i| \le Q = (1+\delta)t_n$ and using the equality $\sinh(b^2/2) = (e^{b^2}-1)/2e^{b^2/2}$ we get:

$$E_{n,v}\lambda_{n,b} = \frac{2}{\sqrt{2n}\sinh(b^2/2)}\sum_{i=1}^n \sinh^2(bv_i/2) = F^*_n(v,b),$$
$$\mathrm{Var}_{n,v}\lambda_{n,b} - 1 = \frac{e^{b^2}-1}{2n\sinh^2(b^2/2)}\sum_{i=1}^n \sinh^2 bv_i \le \frac{2e^{b^2/2+bQ}}{(n/2)^{1/2}}E_{n,v}\lambda_{n,b} \le n^{-\eta}E_{n,v}\lambda_{n,b}$$

(the last relation holds for small enough $\delta > 0$, $\eta > 0$ by $b \le b^*_{n,1} = \sqrt{\log n}/10$). Using Chebyshev inequality we get for $E_{n,v}\lambda_{n,b} \ge 2w_n(b) \ge 2w_n \to \infty$:

$$\beta(\psi_{n,l}(v)) \le P_{n,v}(\lambda_{n,b} < w_n(b)) \le P_{n,v}(|\lambda_{n,b} - E_{n,v}\lambda_{n,b}| > E_{n,v}\lambda_{n,b} - w_n(b)) \le$$
$$\mathrm{Var}_{n,v}\lambda_{n,b}/(E_{n,v}\lambda_{n,b} - w_n(b))^2 = o(1). \tag{6.48}$$

Let $E_{n,v}\lambda_{n,b} < 2w_n(b)$. Then by Bahr-Essen inequality and (6.26)

$$|P_{n,v}(\lambda_{n,b} < w_n(b)) - \Phi(w_n(b) - E_{n,v}\lambda_{n,b})| \le Be^{3(bQ+b^2)}n^{-1/2} + o(1) = o(1)$$

(the last relation holds for small enough $\delta > 0$, $\eta > 0$ by $b \le b^*_{n,1} = \sqrt{\log n}/10$). These imply the statement of Lemma for $l < M^*$.

Let us consider the case $b > b^*_{n,1}$. Here and below in the proof we denote

$$\hat{\xi}_n(x,b_n) = \tilde{\xi}_n(x,b_n) - E_{1,0}\tilde{\xi}_n(x,b_n),\ \tilde{\xi}_n(x,b_n) = \xi(x,b_n)\mathbf{1}_{|x|<T_n}.$$

Replace λ_n onto statistics

$$\hat{\lambda}_n = \tilde{\lambda}_n - E_{n,0}\tilde{\lambda}_n = \frac{h_n}{u_n(b)}\sum_{i=1}^n \hat{\xi}_n(x_i,b_n).$$

(it is possible by $E_{n,0}\tilde{\lambda}_n = o(1)$ for T_n-truncated statistics $\tilde{\lambda}_n$ and by remark before Lemma).

Proposition 6.3 *Let*

$$b_n \ge b^*_{n,1},\ T_n \ge b_n(1+\delta),\ |v| \le T_n(1+\delta);\ \delta > 0. \tag{6.49}$$

Then

$$m_n(v) = E_{1,v}\hat{\xi}_n(x,b_n) = 2\sinh^2(b_nv/2)\Phi(T_n - b_n - |v|)(1 + o(n^{-\delta})), \tag{6.50}$$
$$\Delta\sigma^2_n(v) = \mathrm{Var}_{1,v}\hat{\xi}_n(x,b_n) - \mathrm{Var}_{1,0}\hat{\xi}_n(x,b_n) + m^2_n(v) = O(R_n(v) + n^{-\delta}m_n(v));$$
$$R_n(v) = e^{b_n^2}\sinh^2(b_nv/2)e^{b_n|v|}\Phi(T_n - 2b_n - |v|). \tag{6.51}$$

Proof of Proposition 6.3. Direct calculation analogous with [3] gives (we omit index n):

$$\begin{aligned}
&m(v) = E_v\tilde{\xi}(x,b) - E_0\tilde{\xi}(x,b) = E_0\tilde{\xi}(x,b)\xi(x,v)\\
&= \frac{1}{2}e^{bv}(\Phi(T-b-v) - \Phi(-T-b-v))\\
&+\frac{1}{2}e^{-bv}(\Phi(T-b+v) - \Phi(-T-b+v))\\
&-\Phi(T-b) + \Phi(-T-b) + \Phi(-T-v) + \Phi(-T+v) - 2\Phi(-T). \quad (6.52)
\end{aligned}$$

If $b|v| = o(1)$ or $b|v| = O(1)$, then using Tailor expansion and (2.18) one can get

$$m(v) = 2\sinh^2(bv/2)\Phi(T-b-|v|)(1+o(n^{-\delta_1})),\ \delta_1 > 0;$$

if $b|v| \to \infty,\ |v| < \delta_2(T-b),\ 0 < \delta_2 < 1$, then using (2.18) we get

$$|m(v) - 2\sinh^2(bv/2)\Phi(T-b-|v|)| = o(n^{-\delta_3}),\ \delta_3 > 0,$$

and if $\delta_2(T-b) \le |v| \le T_n + \delta_4\sqrt{b(2T-b)},\ 0 < \delta_4 < 1$, then

$$\begin{aligned}
&|m(v) - 2\sinh^2(bv/2)\Phi(T-b-|v|)| = O(1),\\
&2\sinh^2(bv/2)\Phi(T-b-|v|) > n^{\delta_5},\ \delta_5 > 0,
\end{aligned}$$

which implies (6.50).

Analogously, using direct calculation we can get:

$$\begin{aligned}
&\mathrm{Var}_{1,v}\hat{\xi}_n(x,b_n) - \mathrm{Var}_{1,0}\hat{\xi}_n(x,b_n) = E_{1,v}(\tilde{\xi}^2(x,b_n) - E_{1,0}(\tilde{\xi}^2(x,b_n)) - m_n^2(v) -\\
&2m_n(v)E_{1,0}\tilde{\xi}_n(x,b_n);\ E_v(\tilde{\xi}^2(x,b) - E_0(\tilde{\xi}^2(x,b)) = E_0(\tilde{\xi}^2(x,b)\xi(x,v)) \le\\
&\le BR_n(v),\ E_{1,0}\tilde{\xi}_n(x,b_n) \sim -\Phi(b_n - T_n) = o(n^{-\delta_6}),\ \delta_6 > 0,
\end{aligned}$$

which implies (6.51). $\square$

Proposition 6.4 *Assume (6.49). Then*

$$h_n R_n(v) \le C_{n,1}(v)m_n(v) \qquad (6.53)$$

where $C_{n,1}(v) = O(1)$ *and if* $|v| \le T_n - b_n(1-u_n^{-1})$, *then* $C_{n,1}(v) = o(1)$. *Also*

$$h_n m_n(v) = o(1);\ h_n m_n(v) \le Bu_n(b_n)\Phi(|v| - T_n),\ \textit{if}\ |v| \ge T_n - b_n(1-u_n^{-1}). \quad (6.54)$$

Proof of Proposition 6.4. In view of (2.3) the relation (6.53) follows from

$$\exp(3b_n^2/2 - T_n b_n + b_n|v|)\Phi(T_n - 2b_n - |v|) \le C_{n,1}\Phi(T_n - b_n - |v|). \qquad (6.55)$$

To check (6.55) consider differently cases

$$\begin{aligned}
(a)\ &:\ |v| < T_n - 3b_n/2 - \delta,\\
(b)\ &:\ T_n - 3b_n/2 - \delta \le |v| \le T_n - b_n + B,\\
(c)\ &:\ T_n - b_n + B < |v| \le T_n - b_n(1-u_n^{-1}),\\
(d)\ &:\ |v| > T_n - b_n(1-u_n^{-1}).
\end{aligned}$$

In the cases (a) (6.55) holds by the argument of the function $\Phi(\cdot)$ in right-hand side tends to ∞ and the argument of exponent is bounded by $-b_n\delta \to -\infty$. In the case (b) the argument of the function $\Phi(\cdot)$ in right-hand side is bounded away from $-\infty$, the argument of the function $\Phi(\cdot)$ in left-hand side tends to $-\infty$ and using (2.18) we get that left-hand side is bounded by $B\exp(-(T_n - b_n - |v|)^2/2)/(B+b_n)$. This implies (6.55) with $C_{n,1} = o(1)$. In the case (c) and (d) using (2.18) we get that left-hand is of the rate $\exp(-(T_n - b_n - |v|)^2/2)/(B + b_n + x)$ and right-hand side is of the rate $\exp(-(T_n - b_n - |v|)^2/2)/(B+x)$, $x = |v| - T_n + b_n - B \geq 0$. Also in the case (c) we have: $(B + x)/(B + b_n + x) \leq (u_n(b_n))^{-1} = o(1)$. This implies (6.55) with $C_{n,1} = O(1)$.

Analogously in view of (2.3) the relations (6.54) follow from

$$
\begin{aligned}
&\exp(b_n^2/2 - (T_n - |v|)b_n) = o(1), \text{ if } |v| \leq T_n - b_n(1 - u_n^{-1});\\
&S(v) = \exp(b_n^2/2 - (T_n - |v|)b_n)\Phi(T_n - b_n - |v|) \leq\\
&\exp(-(T_n - |v|)^2/2)(|v| + b_n - T_n)^{-1} \leq u_n/b_n = o(1), \text{ if } |v|| \in I_{n,1} \cup I_{n,2};\\
&S(v) \geq B \begin{cases} \frac{T_n - |v|}{|v| + b_n - T_n}\Phi(|v| - T_n) \leq u_n(b_n)\Phi(|v| - T_n), & \text{if } |v| \in I_{n,1}\\ \frac{\Phi(|v| - T_n)}{|v| + b_n - T_n} \leq \Phi(|v| - T_n), & \text{if } |v| \in I_{n,2} \end{cases},
\end{aligned}
$$

where $I_{n,1} = [T_n - B, T_n - b_n(1 - u_n^{-1})]$, $I_{n,2} = [T_n - B, T_n(1 + \delta)]$, which are established by using (2.18). $\square$

It follows from Proposition 6.3 that

$$
\begin{aligned}
E_{n,v}\hat{\lambda}_n &= \frac{h_n}{u_n(b_n)}\sum_{i=1}^n m_n(v_i) = \frac{\sum_{i=1}^n \sinh(b_n v_i/2)\Phi(T_n - b_n - |v_i|)}{\sinh(b_n^2/2)\sqrt{n\Phi(T_n - 2b_n)/2}}(1 + o(n^{-\delta}))\\
&= F_n^*(v, b_n)(1 + o(n^{-\delta})).
\end{aligned}
$$

Proposition 6.5 *Under constraints (5.17), (6.49)*

$$
\mathrm{Var}_{n,v}\hat{\lambda}_n = 1 + o(F_n^*(v, b_n)/u_n(b_n) + 1). \tag{6.56}
$$

Proof of Proposition 6.5. Using (6.12) we get: $\mathrm{Var}_{n,0}\hat{\lambda}_n = 1 + o(1)$. Using Propositions 6.3, 6.4 we get:

$$
u_n^{-2}(b_n)\sum_{i=1}^n h_n^2 m_n^2(v_i) = o(u_n^{-2}\sum_{i=1}^n h_n m_n(v_i)) = o(F_n^*(v, b_n)/u_n(b_n)).
$$

Therefore

$$
\begin{aligned}
&\mathrm{Var}_{n,v}\hat{\lambda}_n - \mathrm{Var}_{n,0}\hat{\lambda}_n = \frac{h_n^2}{u_n^2(b_n)}\sum_{i=1}^n(\Delta\sigma_n^2(v_i) - m_n^2(v_i)) \leq\\
&B\frac{h_n^2}{u_n^2(b_n)}\sum_{i=1}^n R_n(v_i) + o(F_n^*(v, b_n)/u_n(b_n))
\end{aligned}
$$

and it is enough to check that

$$h_n^2 \sum_{i=1}^n R_n(v_i) = o\left(\sum_{i=1}^n h_n m_n(v) + u_n^2(b_n)\right). \qquad (6.57)$$

Denote $I_n(v) = \{i: \ |v_i| \geq T_n - b_n(1 - (u_n(b_n))^{-1})\}$. It follows from Proposition 6.4 that

$$\begin{aligned} h_n^2 \sum_{i \notin I_n(v)} R_n(v_i) &\leq \eta_n \sum_{i \notin I_n(v)} h_n m_n(v_i), \ \eta_n = o(1); \\ h_n^2 \sum_{i \in I_n(v)} R_n(v_i) &\leq B u_n \sum_{i \in I_n(v)} h_n m_n(v_i) \leq B \sum_{i \in I_n(v)} \Phi(|v_i| - T_n) = u_n \tilde{S}_n(v). \end{aligned}$$

Observe that if $T_n > H_n$, then $\tilde{S}_n(v) \leq S_n(v) = O(1)$ by (5.17) which implies (6.56). Let $T_n \leq H_n$; $\delta_n = H_n - T_n = o(1)$. If $|v_i| \geq T_n - B$, then $\Phi(|v_i| - T_n) \sim \Phi(|v_i| - H_n)$ Note that

$$c_n = 2n\Phi(-T_n) = 2n\Phi(-H_n + \delta_n) \asymp 2n\Phi(-H_n)e^{\delta_n T_n} = c_n^* e^{\delta_n T_n} \asymp e^{\delta_n T_n} / \log(w_n).$$

In view of (6.19) this implies $e^{\delta_n T_n} \asymp c_n \log(w_n) \leq u_n^\kappa(b_n) \log(w_n)$. Therefore if $T_n - b_n(1 - u_n^{-1}(b_n)) \leq |v_i| \leq T_n - B, \ T_n \geq b_n(1 + \delta)$, then:

$$\frac{\Phi(|v_i| - T_n)}{\Phi(|v_i| - H_n)} \sim e^{\delta_n(H_n - |v_i|)} \leq e^{\delta_n b_n(1 - u_n^{-1})} \leq B(u_n^\kappa \log(w_n))^{(1 - u_n^{-1})/(1+\delta)} = o(u_n^\kappa).$$

Therefore $u_n(b_n)\tilde{S}_n(v) \leq o(S_n(v) u_n^{1+\kappa}(b_n)) = o(u_n^2(b_n))$. This implies (6.56). □

The relation (6.54) and estimations above imply the statement of Lemma. In fact, using Proposition 6.5 and Chebyshev inequality analogously to (6.48) we get for $E_{n,v}\hat{\lambda}_{n,b} \geq 2u_n(b)$:

$$\beta(\psi_{n,l}(v)) \leq P_{n,v}(\hat{\lambda}_{n,b} < w_n(b)) \leq \frac{\mathrm{Var}_{n,v}\hat{\lambda}_{n,b}}{(E_{n,v}\hat{\lambda}_{n,b} - w_n(b))^2} = o(1).$$

If $E_{n,v}\hat{\lambda}_{n,b} < 2u_n(b)$, then by $C_n = \sup_{|x|<T_n} |\tilde{Z}_n(b)| = 1$ and using Bahr-Essen inequality, we get

$$\begin{aligned} &E_{n,v}\hat{\lambda}_{n,b} = F_n^*(v, b) + o(1), \ \mathrm{Var}_{n,v}\hat{\lambda}_{n,b} = 1 + o(1), \\ &|P_{n,v}(\hat{\lambda}_{n,b} < w_n(b)) - \Phi(w_n(b) - E_{n,v}\hat{\lambda}_{n,b})| \leq O(C_n/u_n(b)) + o(1) = o(1). \end{aligned}$$

□

6.5 Extreme problems

Let $\Xi_n = \Xi_n^{p,q}(R_1, R_2, Q)$ be the set of collections of probability measures $\bar{r} = (r_1, ..., r_n)$ on the real line supported on the interval $[-Q, Q]$ subject to constraints

$$F_1(\bar{r}) = \sum_{i=1}^n \int \phi_1(v) r_i(dv) \geq H_1, \ F_2(\bar{r}) = \sum_{i=1}^n \int \phi_2(v) r_i(dv) \leq H_2$$

where
$$\phi_1(v) = |v|^p,\ \phi_2(v) = |v|^q,\ H_1 = R^p_{n,1},\ H_2 = R^q_{n,2}.$$
We assume $Q = \infty$ under assumptions of Lemma 5.2. Let
$$\Phi(r) = \int \phi(v) r(dv),\ F(\bar{r}) = \sum_{i=1}^n \Phi(r_i)$$
where the functions $\phi(v) = \phi_n(v, a)$ is defined by (5.19) or $\phi(v) = \phi(v, H)$ is defined in (3.8). Consider linear convex minimization problems:
$$F = \inf_{\bar{r}\in\Xi_n} F(\bar{r});\quad F(\bar{r}) = \sum_{i=1}^n \int \phi(v) r_i(dv). \tag{6.58}$$
It is clear that $F \le F_n(\theta_n) = \inf F(v)$, where $F(v) = \sum_{i=1}^n \phi(v_i)$ and infimum is taken over $v \in V_n(\theta_n)$: $\max_i |v_i| \le Q$. It is enough to study the problem (6.58).

Note that for the types (i) and (ii) we can put $R_{n,2} = \infty$. This corresponds to widest case and does not affect on the values h, b and $u = u_n(\theta_n)$.

By symmetry of the problems the infimum is attained on collections $\bar{r}^*_n = (r^*, ..., r^*) \in \Xi_n$ of equal symmetric measures r^*. Furthermore, using the method of sub-differentials and the theorem by Kuhn and Tucker (see, for example, Ioffe and Tikhomirov [7], pp. 76-77) one gets sufficient conditions for infimum: there exist $\lambda = \lambda_n \ge 0,\ \mu = \mu_n \ge 0,\ \eta = \eta_n$ such that
$$\psi(v) = \phi(v) - \lambda\phi_1(v) + \mu\phi_2(v) \ge \eta \text{ for all } v \in [-Q, Q] \tag{6.59}$$
(note that $\eta \le 0$ by $\phi(0) = 0$) and
$$r^*(\{v :\ \phi(v) - \lambda\phi_1(v) + \mu\phi_2(v) = \eta\}) = 1. \tag{6.60}$$
Moreover, if $F_1(\bar{r}^*_n) > H_1$, then $\lambda = 0$, and if $F_2(\bar{r}^*_n) < H_2$, then $\mu = 0$ (this implies $\mu = 0$ for $H_2 = \infty$).

6.5.1 Proof of Lemma 5.2

Let $\phi(v) = 2\sinh^2(av/2),\ b = b_n(\theta_n),\ h = h_n(\theta_n)$. Consider the measures
$$r^*_1 = \pi(b, h);\ r^*_2 = \pi(b^*, h^*),\ b^* = \frac{b^2(p)}{a},\ h^* = h\left(\frac{ab}{b^2(p)}\right)^p;\ r^*_3 = \pi(\tilde{b}, 1),\ \tilde{b} = bh^{1/p}$$
(note that $h^* \le 1$ under assumptions of n. 2.). It is clear that $F(\bar{r}^*_{n,l}) = F_{n,l}(\theta, a),\ l = 1, 2, 3$ (see (5.20) - (5.22)). We show that measures r^*_1 – r^*_3 attain the minimum for extreme problem (6.58) under assumptions 1) – 3) of Lemma.

Let us observe that $\bar{r}^*_{n,l} \in \Xi_n$ under assumptions 1) – 3). Using Lemma 1.1 can easily check that $F_1(\bar{r}^*_{n,l}) = nhb^p = H_1,\ l = 1, 2, 3$. To check $F_2(\bar{r}^*_{n,l}) \le H_2$, if (iii), then observe $F_2(\bar{r}^*_{n,1}) = nhb^q = H_2$ by Lemma 1.1. Also
$$F_2(\bar{r}^*_{n,2}) = nhb^q(b^2(p)/ab)^{q-p} \le H_2,$$

if B or either (i) or (ii) (by $H_2 = \infty$). At last, if $q \geq p$ or $h = 1$ or either (i) or (ii), then

$$F_2(\bar{r}^*_{n,3}) = n\tilde{b}^q = nh^{q/p}b^q \leq nhb^q \leq H_2.$$

Let us consider $\bar{r}^*_{n,1}$ under assumptions A and (iii). By Lemma 1.1 this correspond the equality $F_2(\bar{r}^*_{n,1}) = H_2$ and either $p > q,\ p \leq 2$ or $p > 2,\ p > q,\ ab \geq b^2(p)$ or $p > 2,\ p < q,\ ab \leq b^2(p)$. Put $\eta = 0$ and

$$\lambda = \lambda(a,b) = \frac{b\phi'(b) - q\phi(b)}{b^p(p-q)},\ \mu = \mu(a,b) = \frac{b\phi'(b) - p\phi(b)}{b^q(p-q)}. \tag{6.61}$$

The relations (6.61) imply the required equalities for $v = 0$ and $v = b$ in (6.60). Moreover, these imply that the line $w = 0$ on the half-plane $\{(v,w);\ v > 0\}$ is tangent to $\psi(v)$ at the point $v = b$.

We need to check the inequalities $\lambda \geq 0,\ \mu \geq 0$ and (6.59). Note that

$$\lambda \;=\; \lambda(a,b) = \frac{4x\sinh x\cosh x - 2q\sinh^2 x}{(p-q)b^p}, \tag{6.62}$$

$$\mu \;=\; \mu(a,b) = \frac{4x\sinh x\cosh x - 2p\sinh^2 x}{(p-q)b^q},\ x = ab/2. \tag{6.63}$$

The relation $\mu \geq 0$ implies $\lambda \geq 0$. The inequality $\mu \geq 0$ is equivalent to

$$p\tanh x \;\leq\; 2x,\ \text{if } p > q, \tag{6.64}$$

$$p\tanh x \;\geq\; 2x,\ \text{if } p < q. \tag{6.65}$$

These relations are equivalent to: $2x = ab \geq b^2(p)$, if $p > q$ and $2x = ab \leq b^2(p)$, if $p < q$. These hold under assumptions A.

The inequality (6.59) is equivalent to

$$\psi_1(v) = \phi(v)/v^p - \lambda + \mu v^{q-p} \geq 0 \text{ for all } v > 0 \tag{6.66}$$

and the tangent property observed above holds for $\psi_1(v)$.

Observe the following convex property.

Proposition 6.6 *The function $f_p^\beta(u) = \sinh^2(u^\beta)/u^{\beta p}$ is convex on $R^+ = \{u > 0\}$ for $\beta \geq 1/2,\ p \geq 2$ and for $\beta < 0,\ p \leq 2$.*

Proof of Proposition 6.6 is given in [10].

Let $p > 2$ and either $p > q$ or $q \geq p+1$. Then the inequality (6.66) follows from convexity of the functions $\psi_p(v) = \phi(v)/v^p$ and v^{q-p} for $v > 0$.

Let either $p > 2,\ 1 > q - p > 0$, or $q < p \leq 2$. Put $u = v^{q-p},\ \beta = 1/(q-p)$. The inequality (6.66) is equivalent to

$$\psi_2(u) = \phi(u^\beta)/u^{\beta p} - \lambda + \mu u$$

from convexity of the function $\psi_2(u)$ by Proposition 6.6.

Assume C or B (remind that this is possible for $p > 2$ only). Consider the measure r_2^*. Put

$$\mu = 0,\ \lambda = \lambda(a, b^*) = \frac{b^*\phi'(b^*)}{p(b^*)^p},\ \eta = \eta(a, b^*) = \phi(b^*) - \lambda(b^*)^p = \phi(b^*) - b^*\phi'(b^*)/p \tag{6.67}$$

and note that $\lambda > 0$ and

$$p\eta = -4x \sinh x \cosh x + 2p \sinh^2 x = 0, \quad \text{by } 2x = ab^* = b(p)^2.$$

The inequality (6.59) follows from the line $w = 0$ is tangent to $\psi(v)$ at the point $v = b^*$ and from the convexity of the function $\psi_1(v) = \psi(v)/v^p - \lambda$.

Assume D. Consider the measure r_3^*. Put analogously with (6.67)

$$\mu = 0,\ \lambda = \lambda(a, b) = \frac{\tilde{b}\phi'(\tilde{b})}{p\tilde{b}^p},\ \eta = \eta(a, \tilde{b}) = \phi(\tilde{b}) - \lambda(\tilde{b})^p = \phi(\tilde{b}) - \tilde{b}\phi'(\tilde{b})/p. \tag{6.68}$$

Note that $\lambda > 0$. This implies

$$p\eta = -4x \sinh x \cosh x + 2p \sinh^2 x \le 0$$

by $2x = a\tilde{b} \ge b^2(p)$ which is D.

The relation (6.67) implies that the line $w = 0$ is tangent to $\tilde{\psi}(v) = \phi(v) - \lambda v^p + \xi,\ \xi = -\eta \ge 0$ at the point $v = \tilde{b}$. The inequality (6.59) is equivalent to

$$\tilde{\psi}_1(v) = \phi(v)/v^p - \lambda + \xi v^{-p} \ge 0$$

and follows from the convexity $\tilde{\psi}_1(v)$ for $p \ge 2$, and from convexity

$$\tilde{\psi}_2(u) = \phi(u^\beta)/u^{\beta p} - \lambda + \xi u,\ u = v^{-p},\ \beta = -1/p$$

by Proposition 6.6 for $p \le 2$. Lemma 5.2 is proved. $\square$

6.5.2 Proof of Lemmas 3.1 and 5.3

Lemmas 3.1 and 5.3 correspond to extreme problems (6.58) for the functions

$$\phi(v) = \phi(v, a, T) = 2\sinh^2(av/2)\Phi(T - a - |v|)),\ T = T_n(a), \tag{6.69}$$
$$\phi(v) = \phi(v, H) = \Phi(v - H) + \Phi(-v - H) - 2\Phi(-H),\ H = H_n. \tag{6.70}$$

We show that $r^* = \pi(b, h)$ is extreme measure in the problem (6.58): $F = F(\pi(b, h))$. It is enough to check that r^* satisfies (6.60) and (6.59) holds for some $\lambda \ge 0,\ \mu \ge 0,\ \eta = 0$.

Relation (6.60) implies the equality in (6.59) for $v = 0,\ v = b$ and $v = -b$. We choose $\lambda,\ \mu$ by (6.61) which implies (6.60) and the line $w = 0$ is tangent $\psi(v)$ at the point $v = b$. By $p > q$ the inequality $\lambda \geq 0$ follows from $\mu \geq 0$ which is the same that

$$b\phi^{'}(b)/\phi(b) - p \geq 0. \tag{6.71}$$

Denote below $\eta = b\phi^{'}(b)/\phi(b) - p$ and put

$$\phi_p(v) = \phi(v)/v^p, \ z = v/b, \ v > 0; \ \psi_p(v) = \psi(v)/v^p = \phi_p(v) - \lambda + \mu v^{q-p}.$$

The inequality (6.59) is equivalent: $\psi_p(v) \geq 0$ for $0 < v \leq Q$. This may be rewrite in the form

$$\frac{\phi_p(v)}{\phi_p(b)} - \frac{\eta(1 - z^{q-p})}{(p-q)} - 1 \geq 0, \ 0 < v \leq Q. \tag{6.72}$$

By $(1 - z^{q-p})/(p-q) \leq \log z$, the inequality (6.72) holds for $\eta > 1, \ z \leq 1 - 1/\eta$. Analogously, the inequality also (6.72) holds, if

$$\phi_p(v)/\phi_p(b) \geq 1 + \eta \log C; \ 1 < c \leq z \leq Q/b = C. \tag{6.73}$$

The inequality (6.73) follows from

$$\phi(v)/\phi(b)) \geq C^p(1 + \eta \log C), \ 1 < c \leq z \leq C. \tag{6.74}$$

At last, (6.72) holds, if

$$\phi_p^{''}(v) \geq 0, \ 1 - 1/\eta < z < c. \tag{6.75}$$

This follows from convexity $\psi_p(v)$ in this case.

Therefore we need to check the relations (6.71) and to choose such $c > 1$ that (6.74), (6.75) hold under assumptions of Lemmas for the functions (6.69), (6.70).

Under assumption of Lemma 3.1 the inequality (6.71) follows from (2.18):

$$\eta \sim b(H - b) \to \infty \text{ as } b \asymp (H - b) \to \infty, \ p = o(b^2).$$

Analogously, under assumption of Lemma 5.3, it follows from the limit relations: as $a, b \to \infty, \ p = o(ab), \ d = o(a)$

$$b\phi'(b) = ab\sinh(ab)\Phi(-d) - 2b\sinh^2(ab/2)\exp(-d^2/2)/\sqrt{2\pi} \sim$$
$$(ab/2)\exp(ab)\Phi(-d), \ \phi(b) \sim \exp(ab)\Phi(-d)/2, \ \eta(b) \sim ab.$$

To check (6.74) under assumption of Lemma 3.1 put $c = (b + H)/2b, \ 1 + C_2 < 2c < 1 + C_1$. Let $x = H - b, \ y = H - cb$. It is enough to show that if $x \asymp y \asymp x - y \asymp b \to \infty, \ p = o(b^2)$, then $b^{-2}\log(\phi(cb)/\phi(b)) \asymp 1$ which easily follow from (2.18).

To check (6.74) under assumption of Lemma 5.3 let $a \asymp b \to \infty, \ \ d \leq o(a), \ p = o(b^2)$. We have for $u = v - b > d + b^{1/2} = o(b)$:

$$\frac{\phi(v)}{\phi(b))} \sim \frac{\exp(u(a - d) - (u^2 + d^2)/2)}{\sqrt{2\pi}\Phi(-d)(u + d)} >$$
$$A\exp(-(u - a + d)^2/2 + (a - d)^2/2); \ A = \frac{\exp(-d^2/2)}{\sqrt{2\pi(d + b(C_1 - 1))}}.$$

Therefore the inequality (6.74) follows from

$$(u-a+d)^2 \le (a-d)^2 - 2\log(A/B);\ B = C^p(1+\eta\log C) = o(a^2). \qquad (6.76)$$

Because $\log(A/B) = o(a^2)$, the inequality (6.76) implies (6.74) for lager enough a, b, any fixed $\xi \in (0,1)$ and $c = 1+\xi,\ Q = C_1 b < (b+2(a-d))(1-\xi)$.

To check (6.75) it is enough to show that

$$f(v) = v^{p+2}\phi_p^{''}(v) = v^2\phi^{''}(v) - 2pv\phi^{'}(v) + p(p+1)\phi(v) > 0,\ 1-1/\eta \le z \le c.$$

Under assumption of Lemma 3.1 it follows from asymptitic relation: for any fixed $0 < c_1 < c_2 < 1$ as $b \asymp H - b \to \infty,\ c_1 b < v < c_1 H$

$$f(v) \sim \phi(v)\left((v(H-v))^2 - pv(H-v) + p(p+1) + o(1+p^2)\right) =$$
$$\phi(v)\left((v(H-v)-p)^2 + p + o(1+p^2)\right) \asymp \phi(v)b^2 > 0.$$

Under assumption of Lemma 5.3 the same relations hold with replacing H on T.

□

6.6 Correlations properties

6.6.1 Proof of Proposition 4.2

If $d_{n,k}, d_{n,l} \le B$, then, by $b_{n,l},\ b_{n,k} \to \infty$, we have:

$$\rho_{n,lk} \asymp \exp(-(b_{n,l} - b_{n,k})^2/2)$$

which implies (4.19). Let $\tilde d_{n,k}, d_{n,l} \ge B$. Let $T^*_n = T_{n,l} \le T_{n,k}$. Denote $\tilde d_{n,k} = 2b_{n,k} - T^*_n \ge d_{n,k}$. Using (4.15), (2.18), we get:

$$\rho_{n,lk} \sim \exp(-(b_{n,l} - b_{n,k})^2/2)\ \frac{\Phi(-(d_{n,l} + \tilde d_{n,k})/2)}{\sqrt{\Phi(-d_{n,l})\Phi(-\tilde d_{n,k})}}\ \sqrt{\Phi(-\tilde d_{n,k})/\Phi(-d_{n,k})}\ ;$$
$$\sqrt{\Phi(-\tilde d_{n,k})/\Phi(-d_{n,k})} \asymp \sqrt{d_{n,k}/\tilde d_{n,k}}\ \exp\left(d^2_{n,k}/4 - \tilde d^2_{n,k}/4\right) \le \sqrt{d_{n,k}/\tilde d_{n,k}}\ ;$$
$$\frac{\Phi(-(d_{n,l} + \tilde d_{n,k})/2)}{\sqrt{\Phi(-d_{n,l})\Phi(-\tilde d_{n,k})}} \asymp 2\exp(\Delta_n)\sqrt{d_{n,l}\tilde d_{n,k}}/(d_{n,l} + \tilde d_{n,k})$$

where, by definitions $d_{n,l}, \tilde d_{n,k}$,

$$\Delta_n = \frac{1}{4}\left((2b_{n,l} - T^*_n)^2 + (2b_{n,k} - T^*_n)^2 - 2(b_{n,l} + b_{n,k} - T^*_n)^2\right) = (b_{n,l} - b_{n,k})^2/2.$$

Therefore (assume $d_{n,k} \le d_{n,l}$)

$$\rho_{n,lk} \le \frac{B\sqrt{d_{n,l}\tilde d_{n,k}}}{d_{n,l} + \tilde d_{n,k}}\sqrt{\frac{d_{n,k}}{\tilde d_{n,k}}} = \frac{B\sqrt{d_{n,l}d_{n,k}}}{d_{n,l} + \tilde d_{n,k}} \le \frac{B\sqrt{d_{n,l}d_{n,k}}}{d_{n,l} + d_{n,k}} \le B\sqrt{d_{n,k}/d_{n,l}}$$

We can replace $d_n = 2b_n - T_n$ on $d^*_n = 2(b_n - b^*_n)$ in this relation which implies:

$$\rho_{n,lk} \le B\sqrt{d^*_{n,k}/d^*_{n,l}} = B\exp(-|z_{n,l} - z_{n,k}|/2).$$

The cases type of $\tilde d_{n,k} \le B,\ d_{n,l} \ge B$ are considered by similar way. □

6.6.2 Proof of Proposition 5.2

First, consider the case (5.28). Let $a, b \le B w_n^{-1/2}(a)$. Using the relation

$$\sinh(x) = x + x^3/6 + O(x^5) \text{ as } x \to 0 \tag{6.77}$$

we get:

$$1 - \rho(\theta_n, a) = O(a^4 + b^4 + (ab)^2) = O(w_n^{-2}(a)).$$

Let $a, b > B w_n^{-1/2}(a)$. Introduce the function $g(t) = \log\sinh(e^t/2)$. Put $x = 2\log a,\ y = 2\log b$ and note that

$$\log\rho(\theta_n, a) = -(g(x) + g(y) - 2g((x+y)/2) = -g''(t)(x-y)^2,\ t \in [x, y].$$

Observe

$$g''(t) = \frac{u(\sinh(u) - u)}{4\sinh^2(u/2)} \le Bu,\ u = e^t,$$

for some $B > 0$ and all $u > 0$. Then by $(b-a)/a = o(1)$ we get:

$$-\log\rho(\theta_n, a) \le B b_{n,l}^2 (\log(b_{n,l}/b_{n,l-1})^2 \le B(b-a)^2 = O(w_n^{-2}(a)).$$

Consider the cases (5.29), (5.30). By $b(p)$ minimizes $g_p(b) = \sinh(b^2/2)/b^p$,

$$\frac{\sinh^2((abh^{1/p})^2/2)}{h\sinh(b^2/2)\sinh(a^2/2)} = \frac{g_p^2(x)}{g_p(a)g_p(b)} \ge \frac{g_p^2(b(p))}{g_p(a)g_p(b)},\ x = abh^{1/p}.$$

Therefore it is enough to consider the case (5.29) with $b = b(p)$.

Using the relation (6.77) and by

$$\cosh(x) = 1 + x^2/3 + O(x^4),\ x \to 0;\ \tanh(x) = 1 + O(e^{-2x}),\ x \to \infty$$

one can easily get

$$b^4(p) \sim 6(p-2) \text{ as } p \to 2,\ p > 2;\ b^2(p) \sim p \text{ as } p \to \infty. \tag{6.78}$$

Let $b = b(p) \le a \le B w_n^{-2/3}(a)$. By (6.78), $(p-2) = O(b_{n,2}^4)$ and one easily get:

$$1 \le g_p(a)/g_p(b(p)) \le (b(p)/a)^{(p-2)}(1 + O(a^2)) \le 1 + O(a^2)$$

which imply the required relation.

Let $b = b(p) \in [a - w_n^{-2/3}(a), a],\ a > B w_n^{-2/3}(a), B > 2$. Introduce the function

$$f_p(u) = \log g_p(a) = \log\sinh(u) - (p/2)\log(2u),\ u = u(a) = a^2/2,\ u_p = u(b(p)).$$

Observe

$$\log(g_p(a)/g_p(b(p))) = f_p''(v)(u - u_p)^2/2,\ v \in [u, u_p]$$

and

$$f_p''(u) = p/(2u^2) - \sinh^{-2}(u) \le p/(2u^2);\ f_p''(u) \le B(p-2)u^{-2} + O(1),\ \text{if } u = O(1).$$

By (6.78) we get: for bounded p, $b(p)$, a:

$$\log(g_p(a)/g_p(b(p))) \le B(p-2)b^{-4}(p)(a^2 - b^2(p))^2 = O(w_n^{-4/3}(a)),$$

and if $p \to \infty$, then, analogously,

$$\log(g_p(a)/g_p(b(p))) \le Bpb^{-4}(p)(a^2 - b^2(p))^2 \asymp (a - b(p))^2 = O(w_n^{-4/3}(a)).$$

The statement 2) follows from estimation above. □

6.6.3 Proof of Proposition 5.4

Denote

$$R_n(a, b_n) = \frac{\Phi(-(d_n(a) + d_n(b_n))/2 + \delta_n/2)}{\sqrt{\Phi(-d_n(a))\Phi(-d_n(b_n))}},$$
$$R_n^*(a, b_n) = \frac{\Phi(-(d_n(a) + d_n(b_n))/2)}{\sqrt{\Phi(-d_n(a))\Phi(-d_n(b_n))}},$$
$$Q_n(a, b_n) = \frac{\sinh^2(ab_n/2)}{\sinh(a^2/2)\sinh(b_n^2/2)},\ Q_n^*(a, b_n) = \exp(-(a - b_n)^2/2).$$

By $a, b_n \ge b_{n,1}^*$ one easily see:

$$Q_n(a, b_n) = Q_n^*(a, b_n) + o(n^{-\eta}),\ \eta > 0.$$

It is well known that the function $\log\Phi(x)$ is concave. Therefore $R_n^*(a, b_n) \ge 1$. If $\delta_n = T_n(a) - T_n(b_n) \ge 0$, then $R_n(a, b_n) \ge R_n^*(a, b_n)$. This implies

$$\rho_{n,4} = Q_n(a, b_n)R_n(a, b_n) \ge \exp(-(a - b_n)^2/2) + o(n^{-\eta}) \ge 1 - (a - b_n)^2/2 + o(n^{-\eta})$$

which implies (5.32). Therefore we need to consider the case $T_n(b_n) > T_n(a)$ which implies $d_n(b) \le d_n(a) + |a - b_n|$.

First, observe, that if $d_n(a) \le -4\log w_n(a)$, then

$$H_n = -(d_n(a) + d_n(b_n))/2 + \delta_n/2) = -d_n(a) + a - b > 4\log w_n(a) + o(1)$$

and

$$R_n(a, b_n) \ge \Phi(H_n) = 1 - \Phi(-H_n) \ge 1 - w_n^{-2}(a)$$

which implies (5.32).

Let $-4\log w_n(a) \le d_n(a) \le B$. Then $d_n(b_n) \le B + o(1)$. Denote $\tilde{u}_n = u_n(\theta_n)$. Using (6.18) we get:

$$0 < T_n^2(b_n) - T_n^2(a) = 2\log(u_n^2(a)/\tilde{u}_n^2) + d_n^2(b_n) - d_n^2(a) + O(1).$$

By

$$u_n^2(a)/\tilde{u}_n^2 \le u_n^2(a)/u_n^2(b_n) \le 1 + |u_n^2(a) - u_n^2(b_n)|/u_n^2(b_n) \le$$
$$1 + B(|z_n(a) - z_n(b_n)|w_n^2(b_n) \le 1 + Bw_n^{-8/3}(b_n)$$

this yields

$$T_n(b_n) - T_n(a) = b_n^{-1}(2\log(u_n^2(a)/\tilde{u}_n^2) + (b-a)d_n(a) + (b-a)^2 + O(1))$$
$$= o(1);\ T_n(b_n) - T_n(a) = O(b_n^{-1}) = O(w_n^{-2}(a)),$$
$$\Phi(-(d_n(a) + d_n(b_n))/2 + \delta_n/2) = \Phi(-(d_n(a) + d_n(b_n))/2)(1 - O(w_n^{-2}(a))$$

which implies (5.32) in this case.

By using analogous estimation one can see that if $d_n(a) \to \infty$, then $d_n(b) \to \infty$. Let $d_n(a) \to \infty,\ d_n(b_n) \to \infty$. Using (6.19) we get:

$$0 < T_n^2(b_n) - T_n^2(a) = 2\log(u_n^2(a)/\tilde{u}_n^2) + \log(d_n(a)/d_n(b_n))$$
$$+O(d_n^{-1}(a) + d_n^{-1}(b_n));\ d_n(a) - d_n(b_n) = o(1);$$
$$T_n(b_n) - T_n(a) = \frac{2\log(u_n^2(a)/\tilde{u}_n^2) + \log(d_n(a)/d_n(b_n)) + O(d_n^{-1}(a))}{T_n(b_n) + T_n(a)}$$
$$= O((d_n(a)b_n^*)^{-1}),$$

which analogously implies

$$\Phi(-(d_n(a) + d_n(b_n))/2 + \delta_n/2) = \Phi(-(d_n(a) + d_n(b_n))/2)(1 - O(w_n^{-2}(a)).$$

Therefore we need to estimate

$$\tilde{\rho}_{n,4} = Q_n^*(a, b_n)R_n^*(a, b_n) = \exp(-\Delta) \ge 1 - \Delta,$$

where

$$\Delta = (\Psi(d_n(a)) + \Psi(d_n(b_n))/2 - \Psi((d_n(a) + d_n(b_n))/2),\ \Psi(x) = \log(e^{-x^2/2}\Phi(-x)).$$

Using the expansions $\Psi(x) = -\log x + a_1/x + a_2/x^2 + \ldots,\ x \to \infty$ we get $\Psi''(x) \sim x^{-2}$, and for some $\tilde{d} \in [d_n(a),\ d_n(b_n)]$ one has:

$$\Delta = \Psi''(\tilde{d})(d_n(a) - d_n(b_n))^2/2 \sim (1 - d_n(b_n)/d_n(a))^2/2.$$

Therefore by

$$T_n(a) = 2b_n^* + o(1),\ d_n(a) \sim 2(a - b_n^*),\ d_n(a) - d_n(b_n) = 2(a - b_n) + o(1/b_n^*)$$

one has:

$$1 - \tilde{\rho}_{n,4} \le B(1 - d_n(b_n)/d_n(a))^2 \asymp (\exp(z_n(a) - z_n(b_n) + o(1/b_n^*)) - 1)^2) \asymp$$
$$(z_n(a) - z_n(b_n) + o(1/b_n^*))^2 = O(w_n^{-2}(a)).$$

This implies (5.32). $\square$

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