Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Surface waves at a free interface of a saturated porous medium

Inna Edelman[1,2], Krzysztof Wilmański[1], Evgeniy Radkevich[3]

submitted: 4th August 1999

[1] Weierstrass Institute of
Applied Analysis and Stochastics
Mohrenstrasse 39
D – 10117 Berlin
Germany
E-Mail: edelman@wias-berlin.de
E-Mail: wilmanski@wias-berlin.de

[2] Russian Academy of Sciences
Institute of Physics of the Earth
B. Gruzinskaya 10
Moscow 123810
Russia

[3] Moscow State University
Department of Differential Equations
119899 Vorobievy Gori
Moscow
Russia

Preprint No. 513
Berlin 1999

1991 Mathematics Subject Classification. 35C20, 35L50, 73D20.

Key words and phrases. Partial differential equations, waves in porous media.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

Surface waves at a free interface of a saturated porous medium are investigated. Existence and peculiarities of surface wave propagation were revealed. Two types of surface waves proved possible: true surface wave, propagating almost without dispersion, and generalized Rayleigh wave, which attenuates along the propagation direction.

1 Introduction

The study of fundamental laws of propagation and interaction of elastic waves on the interface of fluid saturated porous medium is related to numerous geophysical, geological, acoustical, and marine problems as well as engineering applications. The theory of propagation of elastic waves through an infinite saturated isotropic porous medium was developed by M.Biot [1,2] in mid 1950's using macroscopic continuum mechanics and subsequently it was investigated by other scientists [3-6]. M.Biot proposed a phenomenological theory in which the average motions of solid and liquid phases are different. In other words, it is assumed that a medium is a mixture of interpenetrating continua, namely solid phase, constituting the matrix of the medium, and the liquid phase, constituting the saturating fluid. This theory takes into account the energy dissipation due to relative motion between the pore fluid and solid matrix. The theory predicts an existence of two compressional waves and one shear wave in saturated porous medium. The shear wave and the compressional wave of the first kind (fast wave) are similar to the waves in an ordinary single phase, isotropic, solid elastic medium. A particular and most interesting feature of Biot's model arises due to the prediction of the compressional wave of the second kind, or slow wave, which is strongly attenuated. It is essential that the velocity of slow wave is always lower than both compressional wave velocities in the solid and liquid. Biot's theory was confirmed experimentally by Plona [7].

Another approach was developed in [4]. Here the porous medium is considered also as a mixture of compressible solid and liquid continua. The main mechanism of dissipation is described by the linear contribution of the relative velocity to the momentum balance equations. The new feature of this model is the balance equation for porosity. This equation describes the microscopic relaxation processes through the presence of the source term as well as the transport processes of porosity. This yields an additional dissipation. This model, similarily to the Biot model, predicts three types of waves: two lingitudinal waves and one shear wave.

It is clear, that in most geophysical problems media are of finite extent. Thus, boundary conditions have to be considered. The existence of boundaries and interfaces allows for a generation of surface waves.

It is well known that at an interface between an isotropic solid and vacuum, there is only one surface wave—the Rayleigh wave [8]. This wave is a nondispersive plane inhomogeneous wave, undamped in its direction of propagation along the surface, and damped normal to the boundary. Its phase velocity c_R is a single-valued function of parameters of an elastic half-space. It does not depend on its frequency and is close t but somewhat less than the velocity of shear wave in unbounded media. The Rayleigh wave is a coupled compressional-shear system, propagating with unique velocity c_R.

The Rayleigh waves are modified if the vacuum-bounding plane elastic half-space is replaced by a liquid or by another solid. Early studies on this subject are due to Love [9] and Stoneley [10]. The essential results are that a wave corresponding to Rayleigh wave on a free surface, due to seepage of energy into another medium becomes exponentially attenuated along its direction of propagation, while simultaneously other new modes of surface waves appear. Whereas for certain values of elastic parameters these generalized Rayleigh waves cannot exist on a plane surface between two solids, they are always possible on a liquid-solid interface [11].

At a solid-liquid interface the phase velocity of generalized Rayleigh wave, which is a system of three waves (one in the liquid and two in the solid), is higher than the wave velocity in the fluid. This surface wave radiates energy continuously into the liquid, forming therein an inhomogeneous wave departing from the boundary. Since the energy flows across the interface (leaky wave), the wave attenuates along the propagation direction. At the same time on a solid-liquid interface there exists a true Stonely surface wave (sometimes called Scholte wave) [11,12], consisting of an inhomogeneous wave in the liquid and two inhomogeneous waves in the solid, and propagating parallel to the boundary without attenuation and being exponentially damped in both directions perpendicular to the interface. Its velocity is lower than all the bulk velocities in the solid and in the liquid.

Due to the presence of a second compressional wave in a fluid-saturated porous media, the properties of surface waves at interfaces of fluid-filled porous solid in contrast to either free interface of elastic half-space or liquid-solid interface should be different.

There are only a few papers published on this subject. Surface waves on a free boundary of a porous medium were examined by Deresiewicz [13] within the framework of Biot's model. A numerical analysis of the dispersion equation revealed that there is always a complex root which corresponds to the velocity of generalized Rayleigh wave. Hence, contrary to the elastic medium, Rayleigh waves in saturated porous medium are dispersive. This generalized Rayleigh wave leaks its energy into the slow compressional wave. For low frequencies, the velocity of this wave tends to the velocity of Rayleigh wave in an elastic medium. Additionally, as it was discovered experimentally [14] for the case of completely closed surface pores and viscosity-free fluid, there exists a true surface wave with a velocity slightly below that of the slow wave. Theoretically this true surface wave is slightly leaky as well since its velocity is higher than the sound velocity in air, although, in fact, the energy loss is negligible because of the large density difference between the two fluids.

This paper is the first part of the general work devoted to surface waves at the interface between saturated porous solid and different media. It concerns the surface waves propagating along the free boundary of a porous medium.

2 Mathematical Model

2.1 Govering Equations

Consider two semi-infinite spaces Ω^I and Ω^{II} having a common interface Γ. Let the region Ω^I is occupied by a saturated porous medium. In dimensionless variables the set of field equations describing the porous medium has the form ($x \in R^3,\ t \in [0,T]$) [4]:

Mass conservation equations

$$\frac{\partial}{\partial t}\varrho_f + \operatorname{div}(\varrho_f \mathbf{v}_f) = 0,$$

$$\frac{\partial}{\partial t}\varrho_s + \operatorname{div}(\varrho_s \mathbf{v}_s) = 0. \tag{2.1}$$

Here ρ is the mass density, $\mathbf{v}$ is the velocity vector and indices f and s indicate a fluid or solid phases, respectively.

Momentum conservation equations

$$\varrho_f \left[\frac{\partial}{\partial t} + (v_{fj}, \frac{\partial}{\partial x_j})\right] v_{fi} - \frac{\partial}{\partial x_j} T^f_{ij} + \pi(v_{fi} - v_{si}) = 0,$$

$$\varrho_s \left[\frac{\partial}{\partial t} + (v_{sj}, \frac{\partial}{\partial x_j})\right] v_{si} - \frac{\partial}{\partial x_j} T^s_{ij} - \pi(v_{fi} - v_{si}) = 0. \tag{2.2}$$

Here T^f_{ij} and T^s_{ij} are the stress tensors, π is a positive constant. The stress tensor in the fluid is assumed to be given by the following linear law:

$$T^f_{ij} = -p_f\delta_{ij} - \beta\Delta_m\delta_{ij}, \quad p_f = p_{f_0} + \kappa(\varrho_f - \varrho_{f_0}), \tag{2.3}$$

where p_f is the pore pressure. p_{f_0} and ϱ_{f_0} are the initial values of pore pressure and fluid mass density, respectively. κ is the constant compressibility coefficient of the fluid depending only on equilibrium value of the porosity m_E. $\Delta_m = m - m_E$ is the change of the porosity. β denotes the coupling coefficient of the components.

The stress tensor in skeleton has the following form:

$$T^s_{ij} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} + \beta\Delta_m\delta_{ij}, \tag{2.4}$$

where λ and μ are the Lame constants of the skeleton, which depend only on m_E, and e_{ij} is the strain tensor of small deformations.

Equation for the change of porosity

$$\frac{\partial}{\partial t}\Delta_m + (v_{si}, \frac{\partial}{\partial x_i})\Delta_m + m_E \operatorname{div}(\mathbf{v}_f - \mathbf{v}_s) = -\frac{\Delta_m}{\tau}, \tag{2.5}$$

where τ is the relaxation time of porosity.

For the strain tensor one has:

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_{si}}{\partial x_j} + \frac{\partial u_{sj}}{\partial x_i}\right), \tag{2.6}$$

where $\mathbf{u}_s$ is the displacement vector for the solid phase with $\mathbf{v}_s = \partial\mathbf{u}_s/\partial t$.

2.2 Boundary Conditions

For the general case, when the region Ω^{II} is occupied by a saturated porous medium as well, the boundary conditions at the interface Γ are:

1) the continuity of total stresses

$$(T^s_{ij} + T^f_{ij})n_j \mid^I = (T^s_{ij} + T^f_{ij})n_j \mid^{II}, \tag{2.7}$$

where $\mathbf{n}$ is a unit vector normal to the interface

2) the continuity of the displacements of the solid phases (i.e. the boundary Γ is material with respect to the skeleton)

$$\mathbf{u}_s \mid^I = \mathbf{u}_s \mid^{II} \tag{2.8}$$

3) the continuity of the mass flux across the interface

$$\varrho_f(\mathbf{v}_f - \mathbf{v}_s)\mathbf{n} \mid^I = \varrho_f(\mathbf{v}_f - \mathbf{v}_s)\mathbf{n} \mid^{II} \tag{2.9}$$

4) proportionality between discontinuity in pressure and relative velocity of the fluid with respect to solid phase [16]

$$\varrho_f(\mathbf{v}_f - \mathbf{v}_s)\mathbf{n} \mid^I = \alpha(p_f \mid^I - \frac{m^I}{m^{II}} p_f \mid^{II}). \tag{2.10}$$

Condition (10) reflects in a phenomenological way the existence of a boundary layer and relates the rate at which saturating fluid flows relative to the solid at the interface due to the pressure drop across the surface. Thus experimental constant α is a kind of surface porosity and the case $\alpha = 0$ corresponds to completely closed surface pores (impermeable boundaries), while the case $\alpha = \infty$ corresponds to the dynamical compatibility condition for partial tractions used in composits.

2.3 Linearized Equation System

Let us linearize the equation system (1)-(5) about some equilibrium state. The simplest case arises when the equilibrium state is taken to have constant values, namely $\varrho_f = \varrho_{f0}$, $\varrho_s = \varrho_{s0}$, $\mathbf{v}_f = 0$, $\mathbf{v}_s = 0$ and $\Delta_m = 0$. After the introduction of displacement vector for the fluid phase $\mathbf{u}_f$ [1] and linearization, the system (1)-(5) takes the following form:

$$\frac{\partial}{\partial t}\varrho_f + \varrho_{f0}\frac{\partial}{\partial x_i}\Big(\frac{\partial u_{fi}}{\partial t}\Big) = 0, \tag{2.11}$$

$$\frac{\partial}{\partial t}\varrho_s + \varrho_{f0}\frac{\partial}{\partial x_i}\Big(\frac{\partial u_{si}}{\partial t}\Big) = 0, \tag{2.12}$$

$$\varrho_{f0}\frac{\partial^2 u_{fi}}{\partial t^2} + \frac{\partial}{\partial x_j}p_f\delta_{ij} + \frac{\partial}{\partial x_j}\beta\Delta_m\delta_{ij} + \pi\frac{\partial}{\partial t}(u_{fi} - u_{si}) = 0, \tag{2.13}$$

$$\varrho_{s0}\frac{\partial^2 u_{si}}{\partial t^2} - \mu\Delta u_{si} - (\lambda+\mu)\nabla\,\mathrm{div}\,u_s - \frac{\partial}{\partial x_j}\beta\Delta_m\delta_{ij} - \pi\frac{\partial}{\partial t}(u_{fi} - u_{si}) = 0, \tag{2.14}$$

$$\frac{\partial}{\partial t}\Delta_m + m_E\,\mathrm{div}\,\frac{\partial}{\partial t}(\mathbf{u}_f - \mathbf{u}_s) = -\frac{\Delta_m}{\tau}. \tag{2.15}$$

Here the unknown variables are ϱ_f, ϱ_s, $\mathbf{u}_f$, $\mathbf{u}_s$ and Δ_m.

The general problem on propagation of elastic waves through the bounded space is rather complicated. We confine ourselves to the consideration of 2D problem ($i, j = 1, 2$, xy plane). Let us consider the propagation of elastic waves through the porous medium which occupies the semi-infinite space $y > 0$ and bounded by the

[1] Let us note that this is common in the fluid dynamics.

vacuum $y < 0$. On the interface $y = 0$ the following boundary conditions, which are the consequence of the general conditions (2.7)-(2.10), have to be satisfied:

a) the total stresses vanish

$$\mu(\frac{\partial u_{s1}}{\partial y} + \frac{\partial u_{s2}}{\partial x})|_{y=0} = 0, \tag{2.16}$$

$$\Big(\lambda \text{ div } \mathbf{u}_s + 2\mu \frac{\partial u_{s2}}{\partial y} - \kappa(\varrho_f - \varrho_{f0})\Big)|_{y=0} = 0, \tag{2.17}$$

b) relative velocity is equal to zero, i.e. $\alpha = 0$. Latter means that the pores at the interface are completely closed

$$\frac{\partial(u_{f2} - u_{s2})}{\partial t}|_{y=0} = 0. \tag{2.18}$$

We will show that for the boundary value problem (2.11)-(2.18), which we will call SWP (surface wave problem), there exist solutions in the form of surface waves. For this purpose we will investigate the propagation of harmonic wave along the positive axis $x > 0$ whose frequency is ω, wave number is k and amplitude depends on y. We will seek solutions which decrease rapidly with increasing distance from the interface, i.e. for $y \to \infty$.

3 Construction of Solution

Solution of SWP is sought in the following form:

$$\mathbf{u}_f = \nabla\varphi_f + ((\psi_f)_y, -(\psi_f)_x), \qquad \mathbf{u}_s = \nabla\varphi_s + ((\psi_s)_y, -(\psi_s)_x),$$

$$\varphi_f = A_f(y)\exp(i(kx - \omega t)), \quad \varphi_s = A_s(y)\exp(i(kx - \omega t)), \tag{3.1}$$

$$\psi_f = B_f(y)\exp(i(kx - \omega t)), \quad \psi_s = B_s(y)\exp(i(kx - \omega t)),$$

$$\varrho_f - \varrho_{f0} = A_{\varrho,f}(y)\exp(i(kx - \omega t)),$$

$$\varrho_s - \varrho_{s0} = A_{\varrho,s}(y)\exp(i(kx - \omega t)),$$

$$\Delta_m = A_{\Delta_m}(y)\exp(i(kx - \omega t)).$$

It should be noted here that in this paper we consider the solutions of (2.1)-(2.5) in the absence of external forces, which are defined uniquely by Cauchy data. In this case it is natural to derive ω as a function with respect to real wave number $k \in R^1$. Thus, $\mathrm{Re}\omega/|k|$ defines the phase velocity of the waves, while $\mathrm{Im}\omega$ defines attenuation.

Substituting the solution (3.1) into linearized equation system (2.11)-(2.15), one gets from the mass balance equations:

$$A_{\varrho,f}(y) + \varrho_{f0}(\frac{d^2}{d\,y^2} - k^2)A_f(y) = 0, \tag{3.2}$$

$$A_{\varrho,s}(y) + \varrho_{f0}(\frac{d^2}{d\,y^2} - k^2)A_s(y) = 0. \tag{3.3}$$

Under the assumptions $(3.1)_{1-3}$ we obtain from the momentum balance equation for the fluid phase:

$$\Delta\Big[\varrho_{f0}\frac{\partial^2\varphi_f}{\partial t^2} + \pi\frac{\partial}{\partial t}(\varphi_f - \varphi_s) + \beta\Delta_m + \kappa(\varrho_f - \varrho_{f0})\Big] = 0, \tag{3.4}$$

$$\Delta\Big[\varrho_{f0}\frac{\partial^2\psi_f}{\partial t^2} + \pi\frac{\partial}{\partial t}(\psi_f - \psi_s)\Big] = 0. \tag{3.5}$$

Consequently,

$$A_{\varrho,f} = k_{f1}(k_{f1}\varrho_{f0} + i\frac{\pi}{a_{f1}})A_f - i\frac{\pi}{a_{f1}}k_{f1}\,A_s - \frac{\beta}{a_{f1}^2}A_{\Delta_m}, \tag{3.6}$$

$$B_f = \frac{i\pi}{\omega\,\varrho_{f0} + i\pi}B_s. \tag{3.7}$$

Here $a_{f1}^2 = \kappa$ and $k_{f1}^2 = \omega^2/a_{f1}^2$.

From the momentum balance equation for the solid phase one has:

$$\Delta\Big[\varrho_{s0}\frac{\partial^2}{\partial t^2}\varphi_s - (\lambda + 2\mu)\Delta\varphi_s - \pi\frac{\partial}{\partial t}(\varphi_f - \varphi_s) - \beta\Delta_m\Big] = 0, \tag{3.8}$$

$$\Delta\Big[\varrho_{s0}\frac{\partial^2}{\partial t^2}\psi_s - \mu\Delta\psi_s - \pi\frac{\partial}{\partial t}(\psi_f - \psi_s)\Big] = 0, \tag{3.9}$$

which yields

$$\frac{d^2}{d\,y^2}A_s + (k_{s1}^2 - k^2)A_s - \frac{i\pi}{\varrho_{s0}a_{s1}}k_{s1}(A_f - A_s) + \frac{\beta}{\varrho_{s0}a_{s1}^2}A_{\Delta_m} = 0, \tag{3.10}$$

$$\frac{d^2}{d\,y^2}B_s + (k_{s2}^2 - k^2)B_s + \frac{i\pi}{\varrho_{s0}a_{s2}}k_{s2}(B_f - B_s) = 0, \tag{3.11}$$

where $k_{s1}^2 = \omega^2/a_{s1}^2$, $a_{s1}^2 = (\lambda + 2\mu)/\varrho_{s0}$, $k_{s2}^2 = \omega^2/a_{s2}^2$, $a_{s2}^2 = \mu/\varrho_{s0}$.

From the equation for the change of porosity one gets:

$$A_{\Delta_m} = \frac{i\,m_E\omega}{\frac{1}{\tau} - i\omega}(\frac{d^2}{d\,y^2} - k^2)(A_f - A_s). \tag{3.12}$$

Thus we obtain three equations for $A_f(y)$, $A_s(y)$ and $B_s(y)$:

$$(\frac{d^2}{d\,y^2} - k^2)A_f + k_{f1}(k_{f1} + i\frac{\pi}{\varrho_{f0}a_{f1}})A_f - i\frac{\pi}{\varrho_{f0}a_{f1}}\kappa_{f1}\,A_s$$

$$+\frac{\beta}{\varrho_{f0}a_{f1}}\frac{m_E k_{f1}}{\frac{i}{\tau} + \omega}(\frac{d^2}{d\,y^2} - k^2)(A_f - A_s) = 0, \tag{3.13}$$

$$\frac{d^2}{d\,y^2}A_s + (k_{s1}^2 - k^2)A_s - \frac{i\pi}{\varrho_{s0}a_{s1}}k_{s1}(A_f - A_s)$$
$$-\frac{\beta\,m_E k_{s1}}{\varrho_{s0}a_{s1}(\frac{i}{\tau} + \omega)}(\frac{d^2}{d\,y^2} - k^2)(A_f - A_s) = 0, \tag{3.14}$$

$$\frac{d^2}{d\,y^2}B_s - \Big(k^2 - k_{s2}^2 + \frac{i\pi k_{s2}}{\varrho_{s0}a_{s2}}\frac{\omega\varrho_{f0}}{\omega\varrho_{f0} + i\pi}\Big)B_s = 0 \tag{3.15}$$

and four algebraic relations for $B_f(y)$, $A_{\Delta_m}(y)$, $A_{\varrho,s}(y)$, and $A_{\varrho,f}(y)$:

$$B_f = \frac{i\pi}{\omega\varrho_{f0} + i\pi}B_s, \tag{3.16}$$

$$A_{\Delta_m} = -\frac{m_E\omega}{\frac{i}{\tau} + \omega}(\frac{d^2}{d\,y^2} - k^2)(A_f - A_s), \tag{3.17}$$

$$A_{\varrho,s} = -\varrho_{s0}(\frac{d^2}{d\,y^2} - k^2)A_s, \tag{3.18}$$

$$A_{\varrho,f} = k_{f1}(k_{f1}\varrho_{f0} + \frac{i\pi}{a_{f1}})A_f - \frac{i\pi k_{f1}}{a_{f1}}A_s + \frac{\beta}{a_{f1}}\frac{m_E k_{f1}}{\frac{i}{\tau} + \omega}(\frac{d^2}{d\,y^2} - k^2)(A_f - A_s). \tag{3.19}$$

Next let us prove the existence of the solution for the system (3.13)-(3.14) and for the equation (3.15). First consider (3.15). The solution has the following form:

$$B_s = C_s(0)\exp(\pm\mu_s\, y) \tag{3.20}$$

with

$$\mu_s = \sqrt{k^2 - k_{s2}^2 + \frac{i\pi k_{s2}}{\varrho_{s0} a_{s2}} \frac{\omega \varrho_{f0}}{\omega \varrho_{f0} + i\pi}} \tag{3.21}$$

Let us define the following condition:

Condition 3.1

$$Re\Big[k^2 - k_{s2}^2 + \frac{i\pi k_{s2}}{\varrho_{s0} a_{s2}} \frac{\omega \varrho_{f0}}{\omega \varrho_{f0} + i\pi}\Big] > 0 \tag{3.22}$$

As we will show below, this condition can be indeed fulfilled by both surface waves which are proven to be possible. It is also quite natural. Namely, a similar condition in the classical elasticity theory yields the conclusion that phase velocity of the surface wave should be less than velocity of shear wave.

Then, the square root in (3.21) is defined as $\sqrt{1} = 1$ and in order to get bounded solution we choose

$$B_s = C_s(0)\exp(-\mu_s\, y). \tag{3.23}$$

We proceed to prove the existence of solution for (3.13)-(3.14). The solution is sought in the following form:

$$\begin{pmatrix} A_f \\ A_s \end{pmatrix} = C_j(0) \begin{pmatrix} R_{fj} \\ R_{sj} \end{pmatrix} \exp(\pm\gamma_j\, y). \tag{3.24}$$

Substituting (3.24) into (3.13),(3.14), one has:

$$d_{f1}(j) R_{fj} + d_{s1}(j) R_{sj} = 0,$$

$$d_{f2}(j) R_{fj} + d_{s2}(j) R_{sj} = 0. \tag{3.25}$$

Consequently,

$$d_{f1}(j) d_{s2}(j) - d_{f2}(j) d_{s1}(j) = 0. \tag{3.26}$$

In the above relations we have defined:

$$d_{f1}(j) = \Big(1 + \frac{\beta\, m_E k_{f1}}{\varrho_{f0} a_{f1}(\omega + \frac{i}{\tau})}\Big)\gamma_j^2 + k_{f1}(k_{f1} + i\frac{\pi}{\varrho_{f0} a_{f1}})$$
$$-\Big(\frac{\beta\, m_E k_{f1}}{\varrho_{f0} a_{f1}(\omega + \frac{i}{\tau})} + 1\Big)k^2,$$

$$d_{s1}(j) = -\frac{\beta\, m_E k_{f1}}{\varrho_{f0} a_{f1}(\omega + \frac{i}{\tau})}(\gamma_j^2 - k^2) + i\frac{\pi k_{f1}}{\varrho_{f0} a_{f1}},$$

$$d_{f2}(j) = -\frac{\beta\, m_E k_{s1}}{\varrho_{s0} a_{s1}(a_{s1} k_{s1} + \frac{i}{\tau})}(\gamma_j^2 - k^2) - i\frac{\pi k_{s1}}{\varrho_{s0} a_{s1}},$$

$$d_{s2}(j) = \Big(1 + \frac{\beta\, m_E k_{s1}}{\varrho_{s0} a_{s1}(a_{s1} k_{s1} + \frac{i}{\tau})}\Big)\gamma_j^2 + k_{s1}(k_{s1} + i\frac{\pi}{\varrho_{s0} a_{s1}})$$
$$-\Big(\frac{\beta\, m_E k_{s1}}{\varrho_{s0}^2 a_{s1}(a_{s1} k_{s1} + \frac{i}{\tau})} + 1\Big)k^2. \tag{3.27}$$

The constants γ_j follow from (3.26). For $\beta = 0$ one has:

$$\mathcal{A}_0 \gamma_j^4 - \mathcal{A}_1 \gamma_j^2 + \mathcal{A}_2 = 0,$$

where

$$\mathcal{A}_0 = 1,$$
$$\mathcal{A}_1 = 2k^2 - k_{f1}(k_{f1} + i\frac{\pi}{\varrho_{f0} a_{f1}}) - k_{s1}(k_{s1} + i\frac{\pi}{\varrho_{s0} a_{s1}}),$$
$$\mathcal{A}_2 = k_{f1}^2(k_{s1}^2 - k^2) + \frac{i\pi k_{s1} k_{f1}^2}{\varrho_{s0} a_{s1}} + \frac{i\pi k_{f1}(k_{s1}^2 - k^2)}{a_{f1}\varrho_{f0}}$$
$$-k^2(k_{s1}^2 - k^2) - \frac{i\pi k_{s1} k^2}{\varrho_{s0} a_{s1}}.$$

The assumption on the vanishing coefficient β means that we neglect a static coupling between components. This happens to be justified for the sufficiently small changes of porosity Δ_m even though these changes influence, however to much lesser extent than relative motion of the phases, both speeds of propagation and attenuation [15].

Hence we have the solution:

$$\gamma_j^2 = \frac{1}{2}\Big[2k^2 - k_{f1}(k_{f1} + i\frac{\pi}{\varrho_{f0} a_{f1}}) - k_{s1}(k_{s1} + i\frac{\pi}{\varrho_{s0} a_{s1}})\Big]$$
$$\pm\frac{1}{2}\sqrt{\Big(2k^2 - \frac{1}{\varrho_{f0}}k_{f1}(k_{f1} + i\frac{\pi}{\varrho_{f0} a_{f1}}) - k_{s1}(k_{s1} + i\frac{\pi}{\varrho_{s0} a_{s1}})\Big)^2 - 4\mathcal{A}_2}. \tag{3.28}$$

or in a more convenient form:

$$\gamma_j^2 = \frac{1}{2}\Big[2k^2 - k_{s1}^2 - k_{f1}^2\Big] \pm \frac{1}{2}\mathrm{Re}\,\delta + \frac{i}{2}\Big[\pm \mathrm{Im}\,\delta - \frac{k_{f1}\pi}{\varrho_{f0}a_{f1}} - \frac{k_{s1}\pi}{\varrho_{s0}a_{s1}}\Big], \qquad (3.29)$$

where

$$\delta = \sqrt{\Big(2k^2 - k_{f1}(k_{f1} + i\frac{\pi}{\varrho_{f0}a_{f1}}) - k_{s1}(k_{s1} + i\frac{\pi}{\varrho_{s0}a_{s1}})\Big)^2 - 4\mathcal{A}_2}.$$

Similarly to the analysis of the equation (3.15) we assume that the following condition holds:

Condition 3.2

$$Re\Big[2k^2 - k_{s1}^2 - k_{f1}^2\Big] \pm Re\,\delta > 0. \qquad (3.30)$$

Then there exist two roots $\gamma_j,\ j = 1, 2$, such that

$$Re\,\gamma_j > 0,$$

$$\gamma_{1,2} = \frac{1}{2^{1/2}}\sqrt{\Big[2k^2 - k_{s1}^2 - k_{f1}^2\Big] \mp Re\,\delta + i\Big[\mp Im\,\delta - \frac{k_{f1}\pi}{\varrho_{f0}a_{f1}} - \frac{k_{s1}\pi}{\varrho_{s0}a_{s1}}\Big]}. \qquad (3.31)$$

(As before, square root is defined as $\sqrt{1} = 1$.) Physically condition (3.30) means that phase velocity of the surface wave should be less then velocities of longitudinal waves both in solid and liquid. We will show below that this condition holds for one of the surface waves, namely for the surface wave whose phase velocity $c_1 < a_{f1}$. For another surface wave (generalized Rayleigh wave) with phase velocity $c_{R'}$ such that $a_{f1} < c_{R'} < a_{s2}$, we prove that the following condition holds:

Condition 3.3

$$Re\Big[2k^2 - k_{s1}^2 - k_{f1}^2\Big] + Re\,\delta > 0, \quad Re\Big[2k^2 - k_{s1}^2 - k_{f1}^2\Big] - Re\,\delta < 0. \qquad (3.30')$$

Then $Re\gamma_2 > 0$ and γ_2 is defined as above. But square root in expression for γ_1 is defined in such a way that the so-called radiation condition [8], i.e. condition of boundness of solution, is satisfied:

$$\gamma_1 = -i\frac{1}{2^{1/2}}\sqrt{\Big[k_{s1}^2 + k_{f1}^2 - 2k^2\Big] + Re\,\delta + i\Big[Im\,\delta + \frac{k_{f1}\pi}{\varrho_{f0}a_{f1}} + \frac{k_{s1}\pi}{\varrho_{s0}a_{s1}}\Big]}. \qquad (3.31')$$

Thus, a bounded solution to (3.13)-(3.15) exists and it has the form:

$$\begin{pmatrix} A_f \\ A_s \end{pmatrix} = C_1(0)\begin{pmatrix} R_{f1} \\ R_{s1} \end{pmatrix} \exp(-\gamma_1\, y) + C_2(0)\begin{pmatrix} R_{f2} \\ R_{s2} \end{pmatrix} \exp(-\gamma_2\, y),$$

$$B_s = C_s(0)\exp(-\mu_s\, y). \tag{3.32}$$

In order to specify the constants $C_1(0)$, $C_2(0)$ and $C_s(0)$, one has to consider the boundary conditions (2.16)-(2.18). We proceed to do so.

4 Dispersion Relation and Surface Wave Velocity

4.1 Dispersion Relation

Substituting the solution (3.1), (3.32) into boundary conditions (2.16)-(2.18) for the case $\beta = 0$ one gets the following system of equations with respect to $C_1(0)$, $C_2(0)$ and $C_s(0)$:

$$\gamma_1 R_{f1} C_1 + \gamma_2 R_{f2} C_2 + \big(i\frac{\mu_s^2 - k^2}{2k} - \frac{k\pi}{\omega\varrho_{f0} + i\pi}\big) C_s = 0, \tag{4.1}$$

$$\gamma_1 R_{s1} C_1 + \gamma_2 R_{s2} C_2 + i\frac{\mu_s^2 + k^2}{2k} C_s = 0, \tag{4.2}$$

$$\begin{aligned} &(\lambda + 2\mu)\Big((\gamma_1^2 - k^2)R_{s1}C_1 + (\gamma_2^2 - k^2)R_{s2}C_2\Big) \\ &+2\mu\, k^2(R_{s1}C_1 + R_{s2}C_2) + 2i\mu k\mu_s C_s - \kappa\Big[k_{f1}(k_{f1}\varrho_{f0} + i\frac{\pi}{a_{f1}})(R_{f1}C_1 + R_{f2}C_2) \\ &-i\frac{\pi}{a_{f1}}k_{f1}(R_{s1}C_1 + R_{s2}C_2)\Big] = 0. \end{aligned} \tag{4.3}$$

For the sake of simplicity consider solely the case $|k| \gg 1$, i.e. short waves. Dividing (3.27) by k^2 one gets:

$$d_{f1}(j) = \frac{\gamma_j^2}{k^2} + \frac{1}{a_{f1}^2}\tilde{\omega}^2 - 1,$$

$$d_{s1}(j) = 0, \quad d_{f2}(j) = 0,$$

$$d_{s2}(j) = \frac{\gamma_j^2}{k^2} + \frac{1}{a_{s1}^2}\tilde{\omega}^2 - 1.$$

Here $\tilde{\omega} = \omega/|k|$. From (3.26)

$$\gamma_1 = |k|\sqrt{1 - \frac{1}{a_{f1}^2}\tilde{\omega}^2},$$

$$\gamma_2 = |k|\sqrt{1 - \frac{1}{a_{s1}^2}\tilde{\omega}^2}, \tag{4.4}$$

and, consequently, eigenvectors for (3.25) take the form:

$$R_1 = (R_{f1}, R_{s1}) = (1, 0), \quad R_2 = (R_{f2}, R_{s2}) = (0, 1). \tag{4.5}$$

Also, from (3.21) one has

$$\mu_s = |k|\sqrt{1 - \frac{\tilde{\omega}^2}{a_{s2}^2}}. \tag{4.6}$$

Solving (4.1)-(4.3) one gets:

$$C_1(0) = i\frac{k\tilde{\omega}^2}{2\gamma_1 a_{s2}^2} C_s(0),$$

$$C_2(0) = -i\frac{k(2 - \frac{\tilde{\omega}^2}{a_{s2}^2})}{2\gamma_2} C_s(0). \tag{4.7}$$

Substitution of (4.7) into (4.3) leads to the dispersion equation

$$-\frac{i\mu}{2\gamma_1\gamma_2}\Big[- \Big((\lambda + 2\mu)\frac{\tilde{\omega}^2}{\mu a_{s1}^2} - 2\Big)\gamma_1(2 - \frac{\tilde{\omega}^2}{a_{s2}^2}) + \varrho_{f0}\frac{\gamma_2\tilde{\omega}^4}{\mu a_{s2}^2} - 4\mu_s\frac{\gamma_1\gamma_2}{k^2}\Big] = 0. \tag{4.8}$$

Obviously, (4.8) includes radicals γ_1, γ_2, and μ_s, which are multi-valued functions. In order to make these functions single-valued, consider Riemann surface of $\tilde{\omega}$ with the cuts outgoing from the points $\pm a_{f1}$, $\pm a_{s2}$, $\pm a_{s1}$. In the following we will consider only the upper strip of the Riemann surface, where the signs of radicals on the real axis satisfy radiation condition [8]. The latter means that solutions (3.1) are bounded in the whole half-space.

Let either

Condition 4.1

$$1 > \max Re\Big(\frac{\tilde\omega^2}{a_{f1}^2}, \frac{\tilde\omega^2}{a_{s2}^2}, \frac{\tilde\omega^2}{a_{s1}^2}\Big), \tag{4.9}$$

and, consequently, γ_1, γ_2, *and* μ_s *are defined as in (4.4) and (4.6),*

or

Condition 4.2

$$Re\frac{\tilde\omega^2}{a_{f1}^2} > 1 > \max Re\Big(\frac{\tilde\omega^2}{a_{s2}}, \frac{\tilde\omega^2}{a_{s1}}\Big). \tag{4.9'}$$

Then γ_2 *and* μ_s *are defined as above and*

$$\frac{1}{|k|}\gamma_1 = -i\sqrt{\frac{1}{a_{f1}^2}\tilde\omega^2 - 1}.$$

Note, that condition (4.9) is similar to conditions (3.23), (3.30), i.e. phase velocity of one of the surface waves should be less then velocities of the body longitudinal and shear waves. Below we will show that dispersion equation (4.8) has two roots satisfying either (4.9) or (4.9').

Expression in square brackets in (4.8) can be rewritten as

$$\begin{aligned}\mathcal{P}(\tilde\omega) = \tilde\omega^4\Big(\frac{\varrho_{f0}}{a_{s2}^2}\sqrt{1 - \frac{1}{a_{s1}^2}\tilde\omega^2} + \frac{\lambda + 2\mu}{a_{s1}^2 a_{s2}^2}\sqrt{1 - \frac{1}{a_{f1}^2}\tilde\omega^2}\Big)\\ -2\tilde\omega^2\sqrt{1 - \frac{1}{a_{f1}^2}\tilde\omega^2}\Big(\frac{\lambda + 2\mu}{a_{s1}^2} + \frac{\mu}{a_{s2}^2}\Big)\\ -4\mu\sqrt{1 - \frac{1}{a_{f1}^2}\tilde\omega^2}\Big(\sqrt{1 - \frac{\tilde\omega^2}{a_{s2}^2}}\sqrt{1 - \frac{1}{a_{s1}^2}\tilde\omega^2} - 1\Big) = 0.\end{aligned} \tag{4.10}$$

Due to condition (4.9) one of the roots of (4.10) is from the interval $[0, \Lambda_0)$, where $\Lambda_0 = \min(a_{f1}, a_{s2}, a_{s1}) = a_{f1} = \sqrt{\kappa}$.

Next let us prove that indeed there exists a root, satisfying (4.9). It is obvious that

$$\mathcal{P}(0) = 0. \tag{4.11}$$

Hence there exists the root

$$\tilde\omega_0 = 0.$$

It is easy to check that

$$\mathcal{P}(\Lambda_0) = \varrho_{f0} a_{f1}^4 \frac{1}{a_{s2}^2}\sqrt{1-\frac{a_{f1}^2}{a_{s1}^2}} > 0. \tag{4.12}$$

Next consider the derivative $\mathcal{P}'$ with respect to $\tilde{\omega}$. It is not difficult to show that

$$\mathcal{P}'(0) = -2\frac{\lambda+\mu}{a_{s1}^2} < 0. \tag{4.13}$$

Thus due to (4.11)-(4.13) there exists a root $\tilde{\omega}_1 \in (0, \sqrt{\kappa})$, satisfying Condition 4.1.

Now it is easy to prove that the following proposition holds true (see Appendix for some details):

Proposition 4.1 *For sufficiently small $\beta \le \beta_0$ and sufficiently large $|k| \ge k_0 > 0$ there exists a root of dispersion equation*

$$\tilde{\omega}_1 = O(\sqrt{\kappa}) + a_1 + ib_1, \quad a_1,\ b_1 = O(\beta + \frac{1}{\sqrt{|k|}}), \tag{4.14}$$

satisfying Condition 4.1.

4.2 Approximation

It is easy to see that dispersion equation (4.10) can be rewritten in the following form:

$$\frac{\varrho_{f0}}{\varrho_{s0}}\frac{1}{a_{s2}^4}\tilde{\omega}^4\sqrt{1-\frac{1}{a_{s1}^2}\tilde{\omega}^2} + \sqrt{1-\frac{1}{\kappa}\tilde{\omega}^2}\,\mathcal{P}_R = 0, \tag{4.15}$$

where

$$\mathcal{P}_R = \Big(2-\frac{1}{a_{s2}^2}\tilde{\omega}^2\Big)^2 - 4\sqrt{1-\frac{1}{a_{s1}^2}\tilde{\omega}^2}\sqrt{1-\frac{1}{a_{s2}^2}\tilde{\omega}^2} \tag{4.16}$$

is the classical Rayleigh equation [8]. Moreover, (4.15) corresponds exactly to the case of surface waves at the interface between liquid and solid half-spaces [8]. Equality (4.15), i.e. the condition of surface wave existence, is an equation for the definition of phase velocity of the surface waves and their attenuation. Let us show that (4.15) has a unique root, satisfying Condition 4.1 and defining the velocity of very slow surface wave propagating without dispersion. Indeed, as it was shown, the existence of such a root follows from general Proposition 4.1. Now we construct the asymptotics for this root.

Let us rewrite dispersion equation (4.15) as

$$\delta\, Z^2\sqrt{1-\nu\, Z} + \sqrt{1-\frac{1}{\sigma}Z}\Big((2-Z)^2 - 4\sqrt{1-Z}\sqrt{1-\nu\, Z}\Big) = 0, \tag{4.17}$$

where $Z = a_{s2}^2 \tilde{\omega}^2$, $\delta = \frac{\varrho_{f0}}{\varrho_{s0}}$, $\sigma = \frac{\kappa}{a_{s2}^2}$, $\nu = \frac{a_{s2}^2}{a_{s1}^2}$.

In order to satisfy Condition 4.1, the root of (4.17) should belong to the interval $(0, \sigma)$. After the change

$$Z = \sigma(1 - Y^2), \quad \text{where} \;\; Y \in (0,1), \;\; Y \sim 0, \tag{4.18}$$

one has:

$$\delta\sigma^2(1-Y^2)^2\sqrt{1-\nu\sigma(1-Y^2)}$$
$$+Y\Big((2-\sigma+\sigma Y^2)^2 - 4\sqrt{1-\sigma+\sigma Y^2}\sqrt{1-\nu\sigma+\nu\sigma Y^2}\Big) = 0. \tag{4.19}$$

Let

$$Y = \sigma Y_0 + \sigma^2 Y_1 + \dots \tag{4.20}$$

Substituting (4.20) into (4.19) one gets from the leading part that

$$Y_0 = \frac{\delta}{2(1-\nu)} \qquad \Longrightarrow \;\; Y = \sigma Y_0 < 1. \tag{4.21}$$

Finally, one obtains phase velocity

$$c_1^2 = \kappa\Big(1 - \frac{\varrho_{f0}^2}{4\mu^2(1-\nu)^2}\kappa^2 + O(\kappa^3)\Big) \tag{4.22}$$

It corresponds to very slow surface wave, propagating almost without dispersion. Its speed is less than the velocities of all bulk waves in porous medium and has order $O(\sqrt{\kappa})$.

Remark. Justification of asymptotics is presented in Appendix.

Next, let us show that dispersion equation (4.15) has also a complex root, satisfying Condition 4.2. It belongs to the interval (κ, a_{s2}^2) and corresponds to generalized Rayleigh wave whose phase velocity $c_{R'} \to c_R$ as $\varrho_{f0} \to 0$, where c_R is a velocity of the classical Rayleigh wave in elastic half-space. It should be noted here that in (4.15) for $\sqrt{1 - \frac{1}{\kappa}\tilde{\omega}^2}$ the following branch is taken (see Condition 4.2): $\sqrt{1 - \frac{1}{\kappa}\tilde{\omega}^2} = -i\sqrt{\frac{1}{\kappa}\tilde{\omega}^2 - 1}$. Also, since $c_{R'}$ should be close to a_{s2}, than $\sqrt{\frac{1}{\kappa}\tilde{\omega}^2 - 1} \approx \frac{\tilde{\omega}}{\sqrt{\kappa}}$. Thus (4.15) takes the form:

$$\sqrt{\kappa}\frac{\varrho_{f0}}{\varrho_{s0}}\frac{1}{a_{s2}^4}\tilde{\omega}^3\sqrt{1 - \frac{1}{a_{s1}^2}\tilde{\omega}^2} - i\mathcal{P}_R = 0. \tag{4.23}$$

Let

$$\tilde{\omega} = \Omega_0 + \sqrt{\kappa}\Omega_1 + \dots \tag{4.24}$$

It is easy to show that the leading part Ω_0 of (4.24) satisfies the Rayleigh equation $\mathcal{P}_R(\Omega_0) = 0$, i.e. $\Omega_0 = c_R$. For the next term Ω_1 of expansion (4.24) one gets the following equation:

$$\Big[\frac{4}{a_{s2}^4}\Omega_0^3 - 8\frac{1}{a_{s2}^2}\Omega_0 - 4\frac{d}{dc}\Big(\sqrt{1-\frac{1}{a_{s2}^2}\tilde{\omega}^2}\sqrt{1-\frac{1}{a_{s1}^2}\tilde{\omega}^2}\Big)\Big|_{c=\Omega_0}\Big]\Omega_1$$
$$= -i\Omega_0^3\frac{\varrho_{f0}}{\varrho_{s0}}\frac{1}{a_{s2}^4}\sqrt{1-\frac{1}{a_{s1}^2}\Omega_0^2}. \qquad (4.25)$$

Finally one has:

$$\tilde{\omega}_{R'} = c_R + \sqrt{\kappa}\Omega_1 + O(\kappa), \qquad (4.26)$$

where Ω_1 is imaginary and is determined by (4.25). Real part of (4.26), i.e. c_R, defines the phase velocity of generalized Rayleigh wave and $\mathrm{Im}\Omega_1$ corresponds to the attenuation of this wave. Thus the reradiation of the energy into the medium occures. Namely generalized Rayleigh wave radiates energy into the slow compressional wave and attenuates along the propagation direction. It is so-called leaky wave. But in contrast to the generalized Rayleigh wave at the interface between liquid and solid half-spaces, where energy is radiated from solid into liquid, leaky wave at the free interface of a porous medium radiates energy into the half-space, where the wave is localized. The first example of such type of surface waves was described at the concave cylindrical interface of elastic solid [8]. As it is clear from the research presented, such leaky waves, radiating energy into the medium where they are located, exist at the plane interface of a porous saturated medium.

Similarly to Proposition 4.1 we can now prove that

Proposition 4.2 *For sufficiently small $\beta \le \beta_0$ and sufficiently large $|k| \ge k_0 > 0$ there exist two roots of the dispersion equation. The first one*

$$c_1 = \sqrt{Z} + a_1 + ib_1 \quad Z = \kappa\Big(1 - \frac{\kappa\varrho_{f0}^2}{4\mu^2(1-\nu)^2} + O(\kappa^4)\Big), \quad a_1,\ b_1 = O(\beta + \frac{1}{\sqrt{|k|}}),$$

defines phase velocity of the surface wave, which propagates without attenuation.

The second one

$$\tilde{\omega}_{R'} = c_R + \sqrt{\kappa}\Omega_1 + O(\kappa) + a_{R'} + ib_{R'}, \quad a_{R'}, b_{R'} = O(\beta + \frac{1}{\sqrt{|k|}}),$$

where Ω_1 is defined from (4.25), corresponds to the generalized Rayleigh wave, which propagates with phase velocity c_R and attenuates along the interface.

5 Conclusions

The results presented in the paper concern surface waves which propagate on free interfaces of saturated porous media. Such waves were not yet systematically investigated. The present research reveals new features of surface waves in porous media which do not appear for example in the case of classical liquid/elastic medium interface. They are connected with the presence of a slow compressional wave in a porous solid. In contrast to free interface of elastic half-space, where only the Rayleigh wave exists, in porous materials two types of surface waves are proven to be possible. They are due to the combination of three waves in porous medium: two longitudinal waves and one shear wave. Consequently, as in the case of interface between liquid and elastic half-spaces, there are two surface waves. However, qualitatively the properties of the surface waves in porous media differ significantly.

The first mode, which exists at the free interface of a saturated porous medium, is a true surface wave, which, as shown in (4.22), propagates almost without dispersion. Asymptotic analysis showed that its velocity is less than the velocities of all bulk waves in unbounded porous medium and is influenced primarily by the compressibility coefficient of the liquid phase. This surface wave is much slower than an analogous one at the interface of liquid and elastic half-spaces.

The second type of surface waves, which appear on interfaces of porous media are the so-called leaky waves. These leaky waves are generalized Rayleigh waves since their phase velocity is close to the velocity of the classical Rayleigh wave. In typical case of an interface between liquid and elastic half-spaces a generalized Rayleigh wave is carried mostly by the elastic half-space and radiates some of its energy into the liquid. This is not the case for porous materials in which the generalized Rayleigh wave is carried by the porous medium and radiates the energy into the porous medium itself. Most likely this energy is absorbed by a slow compressional wave. However, this statement has solely a physical nature and could not yet be proven. Such leaky modes are the intermediate waves between surfaces waves and bulk waves. It is obvious that due to energy radiation into the bulk of the medium, they can exist only in the limited domain. They are transformed into the bulk waves as soon as their surface component is transformed into the bulk one.

Appendix

Here we present justification of asymptotics for the roots of dispersion equation. In fact, the following asymptotic expansion

$$\tilde{\omega} = \tilde{\omega}_0 + \frac{1}{|k|}\tilde{\omega}_1 + \dots \qquad (A.1)$$

has been considered. Let us estimate now $\tilde{\omega}_1$.

Substituting (3.20), where

$$C_s(0) = C_{s0} + \frac{1}{|k|} C_{s1} + \dots, \quad \tilde{\mu}_s = \tilde{\mu}_{s0} + \frac{1}{|k|} \tilde{\mu}_{s1} + \dots, \tag{A.2}$$

into (3.15), one gets:

$$\tilde{\omega}_0 \varrho_{f0} \Big[\tilde{\mu}_{s0}^2 - 1 + \frac{1}{a_{s2}^2} \tilde{\omega}_0^2 \Big] C_{s0}$$
$$+ \frac{1}{|k|} \Big[\Big(i\pi \Big[\tilde{\mu}_{s0}^2 - 1 + \frac{1}{a_{s2}^2} \tilde{\omega}_0^2 \Big] - \varrho_{f0} \tilde{\omega}_0^2 \,\mathrm{sign}\,(k) \frac{i\pi}{\mu} + \tilde{\omega}_1 \varrho_{f0} \Big[\tilde{\mu}_{s0}^2 - 1 + \frac{1}{a_{s2}^2} \tilde{\omega}_0^2 \Big]$$
$$+ 2\tilde{\omega}_0 \varrho_{f0} [\tilde{\mu}_{s0} \tilde{\mu}_{s1} + \frac{1}{a_{s2}^2} \tilde{\omega}_0 \tilde{\omega}_1] \Big) C_{s0} + C_{s1} \tilde{\omega}_0 \varrho_{f0} \Big(\tilde{\mu}_{s0}^2 - 1 + \frac{1}{a_{s2}^2} \tilde{\omega}_0^2 \Big) \Big] = O(\frac{1}{|k|^2}). \tag{A.3}$$

Hence from the first approximation

$$\tilde{\mu}_{s0}^2 = 1 - \frac{1}{a_{s2}^2} \tilde{\omega}_0^2, \tag{A.4}$$

(see (4.4)) or

$$\tilde{\omega}_0 = 0,$$

and from the second approximation

$$- \mathrm{sign}\,(k) \frac{i\pi}{\mu} \tilde{\omega}_0 + 2(\tilde{\mu}_{s0} \tilde{\mu}_{s1} + \frac{1}{a_{s2}^2} \tilde{\omega}_0 \tilde{\omega}_1) = 0. \tag{A.5}$$

From (A.5) it is defined

$$\tilde{\mu}_{s1} = \frac{\tilde{\omega}_0}{\tilde{\mu}_{s0}} \Big(i \,\mathrm{sign}\,(k) \frac{\pi}{2\mu} - \frac{1}{a_{s2}^2} \tilde{\omega}_1 \Big). \tag{A.6}$$

Next dividing (3.27) by k^2 and keeping the terms $O(1/|k|)$, one has:

$$d_{f1}(j) = \tilde{\gamma}_j^2 - 1 + \frac{1}{\kappa} \tilde{\omega} (\tilde{\omega} + i \frac{\pi}{\varrho_{f0} k}),$$

$$d_{s1}(j) = -i \frac{\pi \tilde{\omega}}{\varrho_{f0} \kappa\, k},$$

$$d_{f2}(j) = -i \frac{\pi \tilde{\omega}}{(\lambda + 2\mu) k},$$

$$d_{s2}(j) = \tilde{\gamma}_j^2 - 1 + \frac{1}{a_{s1}^2} \tilde{\omega} (\tilde{\omega} + i \frac{\pi}{\varrho_{s0} k}), \tag{A.7}$$

where $\tilde{\gamma}_j = \frac{1}{|k|}\gamma_j$. Let us introduce the matrix

$$\mathcal{D}(j) = \mathcal{D}_0(j) + \frac{1}{k}\mathcal{D}_1. \qquad (A.8)$$

Here

$$\mathcal{D}_0(j) = \left\{ \begin{array}{ll} \tilde{\gamma}_j^2 - 1 + \frac{1}{\kappa}\tilde{\omega}^2 & 0 \\ 0 & \tilde{\gamma}_j^2 - 1 + \frac{1}{a_{s1}^2}\tilde{\omega}^2 \end{array} \right\},$$

and

$$\mathcal{D}_1 = i\tilde{\omega}\,\mathrm{sign}(k) \left\{ \begin{array}{ll} \frac{\pi}{\kappa\varrho_{f0}} & -\frac{\pi}{\kappa\varrho_{f0}} \\ -\frac{\pi}{(\lambda+2\mu)} & \frac{\pi}{(\lambda+2\mu)} \end{array} \right\}.$$

Solving (3.25) with

$$R_j = R_{j0} + \frac{1}{|k|}R_{j1} + \ldots, \quad \tilde{\gamma}_j = \tilde{\gamma}_{j0} + \frac{1}{|k|}\tilde{\gamma}_{j1} + \ldots\ , \qquad (A.9)$$

one gets from the first approximation

$$\mathcal{D}_{00}(j)R_{j0} = 0, \qquad (A.10)$$

where

$$\mathcal{D}_{00}(j) = \left\{ \begin{array}{ll} \tilde{\gamma}_{j0}^2 - 1 + \frac{1}{a_{f1}^2}\tilde{\omega}_0^2 & 0 \\ 0 & \tilde{\gamma}_{j0}^2 - 1 + \frac{1}{a_{s1}^2}\tilde{\omega}_0^2 \end{array} \right\}.$$

Hence $R_{10} = (1,0)$, $R_{20} = (0,1)$, $\tilde{\gamma}_{10} = \sqrt{1 - \frac{1}{a_{f1}^2}\tilde{\omega}^2}$, $\tilde{\gamma}_{20} = \sqrt{1 - \frac{1}{a_{s1}^2}\tilde{\omega}^2}$ (see (4.7), (4.5)).

Second approximation leads to the following system:

$$\mathcal{D}_{00}(j)R_{j1} + \mathcal{D}_{01}(j)R_{j0} + \mathcal{D}_{10}R_{j0} = 0, \qquad (A.11)$$

where

$$\mathcal{D}_{01}(j) = 2\left\{ \begin{array}{ll} \tilde{\gamma}_{j0}\tilde{\gamma}_{j1} + \frac{1}{\kappa}\tilde{\omega}_0\tilde{\omega}_1 & 0 \\ 0 & \tilde{\gamma}_{j0}\tilde{\gamma}_{j1} + \frac{1}{a_{s1}^2}\tilde{\omega}_0\tilde{\omega}_1 \end{array} \right\}$$

and $\mathcal{D}_{10} = \mathcal{D}_1|_{\tilde{\omega}=\tilde{\omega}_0}$.

Solvability condition to the system (A.11)

$$\left\langle \mathcal{D}_{01}(j)R_{j0} + \mathcal{D}_{10}R_{j0}, R_{j0} \right\rangle = 0, \qquad (A.12)$$

results in definition of $\tilde{\gamma}_{j1}$. Consequently, R_{j1} can be obtained from (A.11). Finally one gets:

$$\tilde{\gamma}_1 = \tilde{\gamma}_{10} - \frac{1}{|k|}\frac{\tilde{\omega}_0}{2a_{f1}^2\tilde{\gamma}_{10}}\Big[2\tilde{\omega}_1 + i\,\mathrm{sign}(k)\frac{\pi}{\varrho_{f0}}\Big],$$

$$\tilde{\gamma}_2 = \tilde{\gamma}_{20} - \frac{1}{|k|}\frac{\tilde{\omega}_0}{2a_{s1}^2\tilde{\gamma}_{20}}\Big[2\tilde{\omega}_1 + i\,\mathrm{sign}(k)\frac{\pi}{\varrho_{s0}}\Big], \qquad (A.13)$$

$$R_1 = (R_{f1}, R_{s1}) = (1,0) - i\,\mathrm{sign}(k)\frac{1}{|k|}\frac{\pi a_{f1}^2}{\varrho_{s0}\tilde{\omega}_0}\Big(a_{s1}^2 - a_{f1}^2\Big)^{-1}(0,1),$$

$$R_2 = (R_{f2}, R_{s2}) = (0,1) + i\,\mathrm{sign}(k)\frac{1}{|k|}\frac{\pi\, a_{s1}^2}{\tilde{\omega}_0\varrho_{f0}}\Big(a_{s1}^2 - a_{f1}^2\Big)^{-1}(1,0). \qquad (A.14)$$

In order to get a dispersion equation, consider boundary conditions (4.1)-(4.3). Let

$$C_j = C_{j0} + \frac{1}{|k|}C_{j1} + \dots \qquad (A.15)$$

Substitution of (A.1), (A.2), (A.9), and (A.15) into (4.1)-(4.3) yields:

$$\tilde{\omega}_0\varrho_{f0}\Big(\tilde{\gamma}_{10}C_{10} - \frac{1}{2a_{s2}^2}i\tilde{\omega}_0^2C_{s0}\Big)$$
$$+\frac{1}{|k|}\Big[i\pi\tilde{\gamma}_{10}C_{10} + \tilde{\omega}_0\varrho_{f0}\Big(\tilde{\gamma}_{11}C_{10} + \tilde{\gamma}_{10}C_{11} + \tilde{\gamma}_{20}C_{20}R_{f2,1}\Big)$$
$$+\pi\Big(-1 + \frac{1}{2a_{s2}^2}\tilde{\omega}_0^2\Big)C_{s0} - i\frac{1}{2a_{s2}^2}\tilde{\omega}_0^3\varrho_{f0}C_{s1} + \varrho_{f0}\tilde{\gamma}_{10}C_{10}\tilde{\omega}_1$$
$$-\frac{3}{2}i\frac{1}{a_{s2}^2}\tilde{\omega}_0^2\tilde{\omega}_1\varrho_{f0}C_{s0} + i\tilde{\mu}_{s0}\tilde{\mu}_{s1}C_{s0}\Big] + O(\frac{1}{|k|^2}) = 0, \qquad (A.16)$$

$$\tilde{\gamma}_{20}C_{20} + i\Big(1 - \frac{1}{2a_{s2}^2}\tilde{\omega}_0^2\Big)C_{s0}$$
$$+\frac{1}{|k|}\Big[\tilde{\gamma}_{10}C_{10}R_{s1,1} + \tilde{\gamma}_{21}C_{20} + \tilde{\gamma}_{20}C_{21} + i\Big(1 - \frac{1}{2a_{s2}^2}\tilde{\omega}_0^2\Big)C_{s1}$$
$$-i\frac{1}{a_{s2}^2}\tilde{\omega}_0\tilde{\omega}_1C_{s0} + i\tilde{\mu}_{s0}\tilde{\mu}_{s1}C_{s0}\Big] + O(\frac{1}{|k|^2}) = 0, \qquad (A.17)$$

$$-(\varrho_{s0}C_{20} + \varrho_{f0}C_{10})\tilde{\omega}_0^2 + 2\mu\, C_{20} + 2i\mu\tilde{\mu}_0^sC_{s0}$$

$$+\frac{1}{|k|}\Big[-2(\varrho_{s0}C_{20}+\varrho_{f0}C_{10})\tilde\omega_0\tilde\omega_1+i\pi\tilde\omega_0C_{20}-(\lambda+2\mu)\Big(\frac{1}{\kappa}R_{s1,1}C_{10}+\frac{1}{a_{s1}^2}C_{21}\Big)\tilde\omega_0^2$$

$$+2(\lambda+2\mu)\tilde\gamma_{20}\tilde\gamma_{21}C_{20}+2\mu\,(R_{s1,1}C_{10}+C_{21})+2i\mu(\tilde\mu_{s0}C_{s1}+\tilde\mu_{s1}C_{s0})$$

$$-\varrho_{f0}\tilde\omega_0^2(C_{11}+C_{20}R_{f2,1})-i\pi\tilde\omega_0C_{10}\Big]+O(\frac{1}{|k|^2})=0. \qquad (A.18)$$

Leading part of (A.16)-(A.18) results in the following system:

$$\tilde\gamma_{10}C_{10}-i\frac{1}{2a_{s2}^2}\tilde\omega_0^2C_{s0}=0, \qquad (A.19)$$

$$\tilde\gamma_{20}C_{20}+i\Big(1-\frac{1}{2a_{s2}^2}\tilde\omega_0^2\Big)C_{s0}=0, \qquad (A.20)$$

$$-(\varrho_{f0}C_{10}+\varrho_{s0}C_{20})\tilde\omega_0^2+2\mu\,C_{20}+2i\mu\tilde\mu_{s0}C_{s0}=0. \qquad (A.21)$$

The latter leads to the dispersion relation with respect to $\tilde\omega_0$ (see (4.15)):

$$\mathcal{P}_0(\tilde\omega_0)=-i\frac{\mu}{2\tilde\gamma_{10}\tilde\gamma_{20}}\Big[\tilde\gamma_{10}\mathcal{P}_R+\tilde\gamma_{20}\frac{\varrho_{f0}}{\varrho_{s0}}\frac{\tilde\omega_0^4}{a_{s2}^4}\Big]. \qquad (A.22)$$

First approximation of (A.16)-(A.18) yields:

$$C_{11}-i\frac{1}{2a_{s2}^2\tilde\gamma_{10}}\tilde\omega_0^2C_{s1}=\frac{1}{\tilde\gamma_{10}\varrho_{f0}\tilde\omega_0}F_1C_{s0}, \qquad (A.23)$$

$$C_{21}+i\frac{1}{\tilde\gamma_{20}}(1-\frac{1}{2a_{s2}^2}\tilde\omega_0^2)C_{s1}=\frac{1}{\tilde\gamma_{20}}F_2C_{s0}, \qquad (A.24)$$

$$-(\varrho_{f0}C_{11}+\varrho_{s0}C_{21})\tilde\omega_0^2+2\mu\,C_{21}+2i\mu\tilde\mu_{s0}C_{s1}=F_3C_{s0}, \qquad (A.25)$$

where

$$-\frac{1}{\tilde\gamma_{10}\varrho_{f0}\tilde\omega_0}F_1=i\Big(\tilde\omega_0\frac{\tilde\gamma_{11}}{\tilde\gamma_{10}}+\tilde\omega_1\Big)\frac{1}{2a_{s2}^2\tilde\gamma_{10}}\tilde\omega_0$$

$$+\,\mathrm{sign}(k)\frac{2\pi\, a_{s1}^2}{\varrho_{f0}\tilde\omega_0\tilde\gamma_{10}}\Big(1-\frac{1}{2a_{s2}^2}\tilde\omega^2\Big)(a_{s1}^2-a_{f1}^2)^{-1}$$

$$-\frac{\pi}{\varrho_{f0}\tilde\omega_0\tilde\gamma_{10}}-i\frac{3}{2a_{s2}^2\tilde\gamma_{10}}\tilde\omega_0\tilde\omega_1+i\frac{\tilde\mu_{s0}\tilde\mu_{s1}}{\varrho_{f0}\tilde\omega_0\tilde\gamma_{10}},$$

$$-\frac{1}{\tilde\gamma_{20}}F_2=-i\frac{\tilde\omega_0\tilde\omega_1}{\tilde\gamma_{20}}\Big[-\frac{1}{a_{s1}^2\tilde\gamma_{20}^2}\Big(1-\frac{\tilde\omega_0^2}{2a_{s2}^2}\Big)+\frac{1}{a_{s2}^2}+\frac{1}{a_{s1}^2}\Big]$$

$$+\mathrm{sign}(k)\frac{\pi\tilde\omega_0}{2\mu\tilde\gamma_{20}}\Big(\kappa(a_{s1}^2-a_{f1}^2)^{-1}-1\Big)$$

$$-\mathrm{sign}(k)\frac{\pi\tilde\omega_0}{2a_{s1}^2\tilde\gamma_{20}^3\varrho_{s0}}\Big(1-\frac{\tilde\omega_0^2}{2a_{s2}^2}\Big),$$

$$-F_3=i\frac{\tilde\omega_0\,\varrho_{s0}}{\tilde\gamma_{20}}\Big(1-\frac{\tilde\omega_0^2}{2a_{s2}^2}\Big)(2\tilde\omega_1+i\,\mathrm{sign}(k)\frac{\pi}{\varrho_{s0}})+2i\mu\frac{\tilde\omega_0}{\tilde\mu_{s0}}(i\,\mathrm{sign}(k)\frac{\pi}{2\mu}-\frac{\tilde\omega_1}{a_{s1}^2})$$

$$+2i\frac{\tilde\omega_0\tilde\omega_1\,\varrho_{s0}}{\tilde\gamma_{20}}\Big(1-\frac{\tilde\omega_0^2}{2a_{s2}^2}\Big)-i\frac{\tilde\omega_1\,\varrho_{f0}}{\tilde\gamma_{10}}\frac{\tilde\omega_0^3}{a_{s2}^2}+\frac{\pi\tilde\omega_0}{\tilde\gamma_{20}}\Big(1-\frac{\tilde\omega_0^2}{2a_{s2}^2}\Big)+\frac{\pi}{\tilde\gamma_{10}}\frac{\tilde\omega_0^3}{2a_{s2}^2}$$

$$\mathrm{sign}(k)\pi(a_{s1}^2-a_{f1}^2)^{-1}\Big[-\frac{\tilde\omega_0^3}{\tilde\gamma_{10}}\frac{a_{s1}^2}{2a_{s2}^2}+\frac{\tilde\omega_0\kappa}{\tilde\gamma_{10}}-\frac{\tilde\omega_0}{\tilde\gamma_{20}}a_{s1}^2\Big(1-\frac{\tilde\omega_0^2}{2a_{s2}^2}\Big)\Big].$$

Obtaining C_{11} and C_{21} from (A.23), (A.24) and substituting these expressions into (A.25), one gets relation

$$i\frac{\mu}{2\tilde\gamma_{10}\tilde\gamma_{20}}\mathcal{P}_0(\tilde\omega_0)C_{s1}=\Big[\frac{2\mu}{\tilde\gamma_{20}}(1-\frac{\tilde\omega_0^2}{2a_{s2}^2})F_2-\frac{\tilde\omega_0}{\tilde\gamma_{10}}F_1-F_3\Big]C_{s0},\qquad (A.26)$$

which leads to the dispersion equation with respect to $\tilde\omega_1$:

$$\frac{2\mu}{\tilde\gamma_{20}}(1-\frac{\tilde\omega_0^2}{2a_{s2}^2})F_2-\frac{\tilde\omega_0}{\tilde\gamma_{10}}F_1-F_3=0.\qquad (A.27)$$

Taking into account that $\tilde\omega_0=c_1=\sqrt{\kappa}(1+O(\kappa^2))$ (see (4.22)) and $\tilde\gamma_{10}=O(\kappa)$, it is not difficult to estimate that

$$\frac{2\mu}{\tilde\gamma_{20}}(1-\frac{\tilde\omega_0^2}{2a_{s2}^2})F_2-\frac{\tilde\omega_0}{\tilde\gamma_{10}}F_1-F_3$$

$$= -i\frac{\tilde{\omega}_0^5 \varrho_{f0}}{4\kappa\tilde{\gamma}_{10}^3 a_{s1}^2}(2\tilde{\omega}_1 + i\,\mathrm{sign}(k)\frac{\pi}{\varrho_{f0}})(1+O(|\tilde{\omega}_0|))$$

$$+\frac{\pi\tilde{\omega}_0}{\tilde{\gamma}_{10}}\Big[\mathrm{sign}(k)a_{s1}^2(1-\frac{\tilde{\omega}_0^2}{2a_{s2}^2})(a_{s1}^2-a_{f1}^2)^{-1}-1\Big]\Big(1+O(|\tilde{\omega}_0|)\Big)=0. \qquad (A.28)$$

Hence

$$\tilde{\omega}_1 = -i\,\mathrm{sign}(k)\frac{\pi}{2\varrho_{f0}} - i\frac{2\pi\kappa\tilde{\gamma}_{10}^2 a_{s1}^2}{\tilde{\omega}_0^4\varrho_{f0}}\Big[\mathrm{sign}(k)a_{s1}^2(1-\frac{\tilde{\omega}_0^2}{2a_{s2}^2})(a_{s1}^2-a_{f1}^2)^{-1}-1\Big]+O(\kappa^{3/2})$$

$$= -i\,\mathrm{sign}(k)\frac{\pi}{2\varrho_{f0}} + O(\kappa). \qquad (A.29)$$

Thus the asymptotics is valid if

$$\frac{1}{|k|} = \kappa^{\frac{1}{2}+\epsilon}, \quad \epsilon > 0. \qquad (A.30)$$

References

[1] Biot, M.A.: 1956, Theory of propagation of elastic waves in a fluid - saturated porous solids I. Low frequency range & II. High frequency range, *J. Acoust. Soc. Amer.* **28**, 168-186.

[2] Biot, M.A.: 1962, Generalized theory of acoustic propagation in porous dissipative media, *J. Acoust. Soc. Amer.* **34(9)**, 1254-1264.

[3] Nikolaevskiy, V.N.: 1990, *Mechanics of Porous and Fractured Media*, World Scientific, Singapore.

[4] Wilmanski,K.: 1998, *Thermomechanics of Continua*, Springer-Verlag, Berlin.

[5] Maksimov, A., Radkevich, E., Edelman, I.: 1994, Mathematical model of modulated wave generation in a gas-saturated porous media, *Differential Equations* **30(3)**, 596-607.

[6] Edelman, I.Ya.: 1999, Wave dynamics of saturated porous media and evolutionary equations, *Transport in Porous Media* **34(1-3)**, 117-128.

[7] Plona, T.J.: 1980, Observation of a second bulk compressional wave in a porous medium at ultrasonic frequencies, *Appl. Phys. Let.* **36(4)**, 259-261.

[8] Viktorov, I.A.: 1967, *Rayleigh and Lamb waves*, Plenum, New York.

[9] Love, A.E.H.: 1926. *Some Problems in Geodynamics,* Cambridge University Press, London.

[10] Stoneley, R.: 1924, Elastic waves at the surface of separation of two solids, *Proc. Roy. Soc. London*, Ser. A 106, 416.

[11] Ewing, W.M., Jardetky, W.S., Press, F.: 1957, *Elastic waves in Layered Media,* McGraw–Hill, New York.

[12] Scholte, J.G.: 1947, The range of existence of Rayleigh and Stoneley waves, *Mon. Not. R. Astr. Soc., Geophys. Suppl.* **5**, 120-126.

[13] Deresiewicz, H.: 1962, The effect of boundaries on wave propagation in a liquid-filled porous solid. IV. Surface waves in a half-space. *Bull. Seism. Soc. Am.* 52(3), 627-638.

[14] Nagy, P.B.: 1992, Observation of a new surface mode on a fluid-saturated permeable solid. *Appl. Phys. Lett.* 60(22), 2734-2737.

[15] Wilmanski,K.: 1999, Sound and shoch waves in porous and granular materials, *Proc. of WASCOM'99, Italy.*

[16] Wilmanski,K.: 1999, On conditions on boundaries and interfaces of porous materials, *Mech. Res. Comm.* (in preparation).