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Resolvent and heat kernel properties for second order elliptic differential operators with general boundary conditions on L^p

Dedicated to Professor Dr. Herbert Gajewski on the occasion of his sixtieth birthday

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Abstract

Under general (including mixed) boundary conditions, nonsmooth coefficients and weak assumptions on the spatial domain, resolvent estimates for second order elliptic operators in divergence form are proved. The semigroups generated by them are analytic, map into Hölder spaces, are positivity improving, and their heat kernels are Hölder continuous in both arguments. We regard perturbations of the elliptic operator by nonnegative potentials, by first order differential operators and multiplicative perturbations. Finally the results provide that the solutions of the corresponding linear and semilinear parabolic equations are Hölder continuous in space and time.

Contents

1 Introduction

During the last years considerable progress has been made in the investigation of elliptic differential operators in connection with nonsmooth situations. This concerns results covering Lipschitz domains [22] as well as possibly jumping coefficients. In particular, in case of Dirichlet boundary conditions explicit ranges of p's are known, where $-\Delta$ provides an isomorphism between $W_0^{1,p}$ and the dual of $W_0^{1,p'}$. Little effort has been made, however, to tackle mixed boundary conditions, although they play an important role in applied problems, cf. Amann [2] or Gajewski and Gröger [20] and the references cited there.

The present work is motivated by the study of reaction–diffusion equations of the type

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(D(u)\operatorname{grad} u\big) = f(u, \operatorname{grad} u), \tag{1.1}$$

where u is a concentration, $D(u)$ a diffusion coefficient, $J = -D(u)\operatorname{grad} u$ the current, and f represents external sources and reactions, cf. also [2, 20]. In unsmooth situations equations of this type have usually been regarded in negatively indexed Sobolev spaces, cf. [20] and the references cited there. The serious disadvantage of this approach is that one does *not* know in the end that for any time point the divergence of the current is a function from L^p; one only obtains that it is a distribution.

However, it would be highly satisfactory to know that the normal flow over any part of the Dirichlet boundary is well defined by Gauss' theorem, because the continuity of the normal component of the current plays an essential role in connecting and embedding of potential flow systems (1.1), not least in electronic device simulation, cf. Gajewski [13].

In order to deal with equation (1.1) in a function space we investigate elliptic differential operators in divergence form on L^p. Inspecting existing theories which can be possibly applied, cf. [1, 3, 26, 28] and the references cited there, one recognizes that a cornerstone are always resolvent estimates uniform on the left complex half plane which imply the generation property of an analytic semigroup in the appropriate space. In fact, the generator property of an analytic semigroup on L^p for operators $\operatorname{div} a \operatorname{grad}$ under general boundary conditions has already been proved by Arendt and ter Elst [4, Sect. 4]. However, the approach in [4] rests on a Nash–Moser type iteration by Fabes and Stroock [12], and explicit resolvent estimates are not available. We take a different approach to the problem: By means of recently obtained C^α regularity results of Griepentrog and Recke [17], operating in the conceptual framework of regular sets in the sense of Gröger [19], and an old estimation technique taken from Pazy [28, 7.3 Th.3.6], we are able to give explicit resolvent estimates in terms of the coefficient function. Moreover, we prove that a finite power of the resolvent maps L^2 into C^α. A fortiori the semigroup operators map L^2 continuously into C^α, are nuclear and the corresponding heat kernel is not only essentially bounded but Hölder continuous in both arguments. This provides the persistence of the spectral properties of the elliptic operator on the scale of L^p–spaces, cf. Davies [6]. Moreover, the semigroup is positivity improving.

The reader will notice that one of the main results, Theorem 4.2, is not only formulated for operators $\operatorname{div} a \operatorname{grad}$ but for operators $U \operatorname{div} a \operatorname{grad}$, where U is an L^∞ function, bounded from below by a strictly positive constant, cf. also Ouhabaz [27]. This is motivated as follows: In material heterostructures the concentration u in (1.1) may be given by a function relative to another, $u = \tilde{u}/U$, where U is a fixed function representing material properties, cf. e.g. [20]. Multiplying (1.1) by the reference density U leeds to an operator $U \operatorname{div} a \operatorname{grad}$. There are other settings of the problem dealing with reference functions U from L^∞, cf. Griepentrog [16, 2]; however, on L^p spaces they canonically act as multipliers.

Using, that the operator $U \operatorname{div} a \operatorname{grad}$ generates an analytic semigroup on L^p and the continuous embedding of the elliptic operators domain into a C^α–space, we prove that the solutions of corresponding linear and semilinear parabolic equations are Hölder continuous in space and time.

2 Notations, definitions, prerequisites

In the sequel Ω will always be a bounded domain in $\mathbb{R}^d$ and Γ a part of its boundary, which may be empty. If p is from $[1,\infty[$, then $L^p = L^p(\Omega)$ is the space of complex, Lebesgue measurable, p–integrable functions on Ω, and $W^{1,p} = W^{1,p}(\Omega)$ is the usual Sobolev space on Ω. The L^p–$L^{p'}$ duality shall be given by the extended L^2 duality

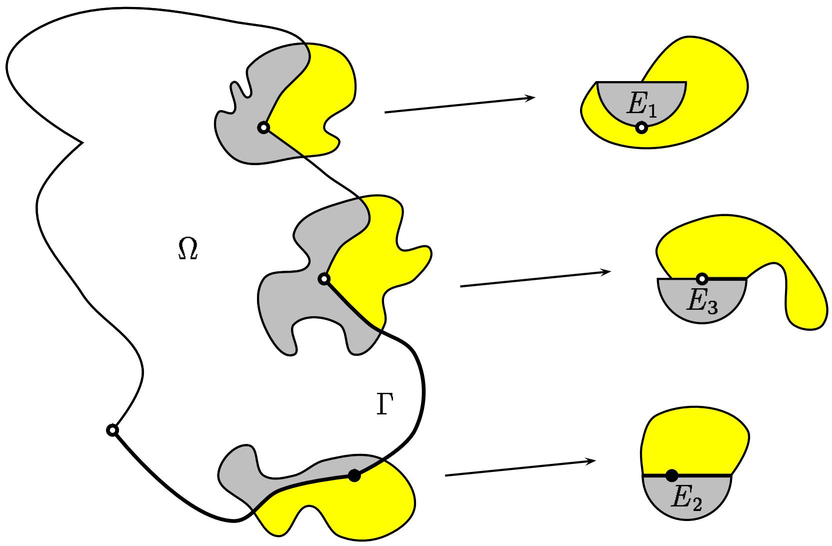
$$\langle \psi_1, \psi_2 \rangle = \int_\Omega \psi_1(x)\, \overline{\psi_2(x)}\, dx. \tag{2.1}$$

$L^\infty = L^\infty(\Omega)$ is the space of Lebesgue measurable, essentially bounded functions on Ω, and $C^\alpha = C^\alpha(\overline{\Omega})$ the space of up to the boundary α–Hölder continuous functions on Ω.

We assume that $\Omega \cup \Gamma$ is a regular set in the following sense:

2.1. Definition. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\Gamma \subset \partial\Omega$ be a part of its boundary. $\Omega \cup \Gamma$ is a regular set if for every point $\tilde{x} \in \partial\Omega$ there exist two open sets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$ and a bi–Lipschitz transformation L from $\mathcal{U}$ onto $\mathcal{V}$ such that, $\tilde{x} \in \mathcal{U}$, and $L\big(\mathcal{U} \cap (\Omega \cup \Gamma)\big)$ coincides with one of the three model sets

$$\begin{aligned} E_1 &= \{x \in \mathbb{R}^d : |x| < 1,\ x_d < 0\}, \\ E_2 &= \{x \in \mathbb{R}^d : |x| < 1,\ x_d \le 0\}, \\ E_3 &= \{x \in E_2 : x_d < 0 \text{ or } x_1 > 0\}. \end{aligned} \tag{2.2}$$



2.2. Definition. We define $W_0^{1,p}$ as the closure in $W^{1,p}$ of the set

$$C_0^\infty(\Omega \cup \Gamma) \stackrel{\text{def}}{=} \Big\{ u|_\Omega : u \in C_0^\infty(\mathbb{R}^d),\ \mathrm{supp}(u) \cap \big(\overline{\Omega} \setminus (\Omega \cup \Gamma)\big) = \emptyset \Big\}, \tag{2.3}$$

and $W^{-1,p}$ as the dual space to $W_0^{1,p'}$, where $1/p + 1/p' = 1$.

2.3. Remark. (Cf. [16, 1.1] or [17].) The above concept coincides exactly with Gröger's definition of regular sets, cf. [19], which seems well adjusted to mixed boundary value problems. N.B. from the definition of the regular set follows that Γ is relatively open in $\partial\Omega$. We can identify Γ with the Neumann and $\partial\Omega \setminus \Gamma$ with the Dirichlet part of the boundary $\partial\Omega$. Please note, that every bounded open set $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary is regular, but the converse statement is not true, cf. Grisvard [18, 1.2.1.4]. Nevertheless, it is easy to prove the $W^{1,p}$ extension domain property of Ω in $\mathbb{R}^d$, by means of the localization, transformation and reflection principles, cf. e.g. [16, 1.1]. Thus one obtains the usual embedding theorems $W^{1,p} \hookrightarrow L^q$. Futhermore, an adequate concept of surface measure σ on the boundary can be established by passing the boundary measure from the three model sets (2.2) via the bi–Lipschitz transformation L to the boundary of Ω. In particular, the embedding

$$W^{1,2} \hookrightarrow L^2(\partial\Omega, \sigma) \qquad \text{is compact,} \tag{2.4}$$

cf. e.g. Goldstein and Reshetnjak [15] or Griepentrog and Recke [17], and there is a constant M such that

$$\int_{\partial\Omega} |\psi|^2 \, d\sigma \le M \, \|\psi\|_{L^2} \sqrt{\int_\Omega \|\operatorname{grad}\psi\|^2_{\mathbb{C}^d} + |\psi|^2 \, dx} \quad \text{for } \psi \in W^{1,2}, \tag{2.5}$$

cf. Griepentrog [16, 1.1].

Throughout this paper $\mathcal{B}(X;Y)$ denotes the space of bounded linear operators from X to Y, X and Y being Banach spaces.

2.4. Definition. Let $a : \Omega \longrightarrow \mathcal{B}(\mathbb{R}^d;\mathbb{R}^d)$,

$$a : \Omega \ni x \longmapsto \begin{pmatrix} a_{1,1}(x) & a_{1,2}(x) & \dots & a_{1,d}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \dots & a_{2,d}(x) \\ \dots & \dots & \dots & \dots \\ a_{d,1}(x) & a_{d,2}(x) & \dots & a_{d,d}(x) \end{pmatrix} \tag{2.6}$$

be a measurable mapping into the set of real, symmetric $d \times d$ matrices, satisfying the relations

$$\operatorname*{vraimax}_{x\in\Omega} \|a(x)\|_{\mathcal{B}(\mathbb{R}^d;\mathbb{R}^d)} \le a^\bullet \tag{2.7a}$$

and

$$\sum_{k,l=1}^{d} a_{k,l}(x)\, \xi_k \, \xi_l \ge a_\bullet \sum_{k=1}^{d} \xi_k^2 \tag{2.7b}$$

for all $x \in \Omega$, all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ and two strictly positive constants $a_\bullet$ and $a^\bullet$. Further, let β be a nonnegative function from $L^\infty(\Gamma, d\sigma)$.

$\mathfrak{t}_\Omega$ and $\mathfrak{t}_\Gamma$ are the following two sesquilinear forms on $W_0^{1,2} \times W_0^{1,2}$:

$$\mathfrak{t}_\Omega[\psi_1, \psi_2] \stackrel{\text{def}}{=} \int_\Omega \langle a \operatorname{grad}\psi_1, \operatorname{grad}\psi_2\rangle_{\mathbb{C}^d} \, dx \tag{2.8a}$$

$$\mathfrak{t}_\Gamma[\psi_1, \psi_2] \stackrel{\text{def}}{=} \int_\Gamma \beta \, \psi_1 \, \overline{\psi_2} \, d\sigma \tag{2.8b}$$

$\mathfrak{t}$ is defined as the sum of the forms $\mathfrak{t}_\Omega$ and $\mathfrak{t}_\Gamma$.

We intend to define an operator on L^2 which corresponds to the form $\mathfrak{t}$ by the representation theorem of forms. Before doing this, we show

2.5. Lemma. *Let $\mathfrak{t}_\Omega$, $\mathfrak{t}_\Gamma$, and $\mathfrak{t}$ are according to Definition 2.4.*

i) The forms $\mathfrak{t}_\Omega$, $\mathfrak{t}_\Gamma$, and $\mathfrak{t}$ are well defined and symmetric on $W_0^{1,2}$.

ii) The quadratic forms associated to $\mathfrak{t}_\Omega$, $\mathfrak{t}_\Gamma$, and $\mathfrak{t}$ are nonnegative.

iii) The forms $\mathfrak{t}_\Omega$ and $\mathfrak{t}$ are densely defined on L^2 and closed.

Proof. Ad i. For $\mathfrak{t}_\Omega$ the statement follows from the boundedness of the function a; for $\mathfrak{t}_\Gamma$ it is implied by (2.5). Indeed, there is

$$\mathfrak{t}_\Gamma[\psi,\psi] \le M\|\beta\|_{L^\infty(\Gamma,\sigma)}\,\|\psi\|_{L^2}\,\|\psi\|_{W^{1,2}}.$$

Ad ii. The symmetry follows immediately from the coefficient matrix a being real symmetric, and from the real valuedness of the function β. The nonnegativity of $\mathfrak{t}_\Omega$ follows from (2.7b), that of $\mathfrak{t}_\Gamma$ from the nonnegativity of β.

Ad iii. For the closedness of $\mathfrak{t}_\Omega$ it is sufficient to prove the closedness of $W_0^{1,2}$ in $W^{1,2}$, cf. Kato [23, VI.§1 Th. 1.11], but this we have by the definition of $W_0^{1,2}$. Knowing this, for the closedness of $\mathfrak{t}$ it is sufficient to show that $\mathfrak{t}_\Gamma$ is relatively bounded with respect to $\mathfrak{t}_\Omega$ with relative bound less than 1, cf. Kato [23, VI.§1 Th. 1.33]. Indeed, according to (2.5) there is

$$\begin{aligned}
\mathfrak{t}_\Gamma[\psi,\psi] &\le M\|\beta\|_{L^\infty(\Gamma,\sigma)}\,\|\psi\|_{L^2}\sqrt{\int_\Omega \|\operatorname{grad}\psi\|^2_{\mathbb{C}^d} + |\psi|^2\,dx}\\
&\le M\|\beta\|_{L^\infty(\Gamma,\sigma)}\,\|\psi\|_{L^2}\sqrt{\frac{1}{a_\bullet}\int_\Omega \langle a\ \operatorname{grad}\psi,\operatorname{grad}\psi\rangle_{\mathbb{C}^d} + a_\bullet|\psi|^2\,dx}\\
&\le \frac{1}{2}\int_\Omega \langle a\ \operatorname{grad}\psi,\operatorname{grad}\psi\rangle_{\mathbb{C}^d}\,dx + \Big(\frac{a_\bullet}{2} + \frac{M^2\|\beta\|^2_{L^\infty(\Gamma,\sigma)}}{2a_\bullet}\Big)\|\psi\|^2_{L^2}.
\end{aligned}$$

□

2.6. Definition. A_2 is the selfadjoint, nonnegative operator on L^2 which corresponds to the form $\mathfrak{t}$ from Definition 2.4 by the first representation theorem of forms, cf. Kato [23, VI.§2 Th. 2.1 and Th. 2.6]. For $p > 2$, A_p is the restriction of A_2 to L^p.

We will now prove some basic results for A_2.

2.7. Lemma. *The resolvent of A_2 is compact. The semigroup generated by $-A_2$ is contractive. If $\sigma(\partial\Omega \setminus \Gamma) > 0$ or $\int_\Gamma \beta \, d\sigma > 0$, then the operator A_2, has a strictly positive lower bound and the semigroup generated by $-A_2$ is even strictly contractive.*

Proof. For the compactness of $(A_2+1)^{-1}$ it is sufficient to know the compactness of $(A_2+1)^{-\frac{1}{2}}$. The latter operator provides a topological isomorphism between L^2 and the form domain, $W_0^{1,2}$, which compactly embeds into L^2. Now, as A_2 is nonnegative $(A_2+\rho)^{-1}$ is compact for every $\rho > 0$, cf. [23, III.§6 Th.6.29]. Moreover, there is

$$\left\|(A_2+\rho)^{-1}\right\|_{\mathcal{B}(L^2;L^2)} \le \frac{1}{\rho} \quad \text{for all } \rho > 0,$$

cf. [23, V.§3.5].

Suppose that the lower bound of A_2 is not strictly positive. Then, by the compactness of the resolvent, 0 has to be an eigenvalue of A_2 with an eigenvector $\psi \not\equiv 0$. According to Definition 2.6, ψ must satisfy the equation

$$\begin{aligned} 0 = \mathfrak{t}[\psi,\psi] &= \int_\Omega \langle a \operatorname{grad}\psi, \operatorname{grad}\psi\rangle \, dx \; + \; \int_\Gamma \beta \, |\psi|^2 \, d\sigma \\ &\ge a_\bullet \int_\Omega \langle \operatorname{grad}\psi, \operatorname{grad}\psi\rangle \, dx \; + \; \int_\Gamma \beta \, |\psi|^2 \, d\sigma. \end{aligned}$$

Hence, the terms on the right hand side must vanish. This means that $\operatorname{grad}\psi$ is zero almost everywhere on Ω and, consequently, ψ must be from the equivalence class of a constant function γ over $\overline{\Omega}$, cf. Ziemer [36, Corollary 2.1.9]. N.B. Ω is connected.

If $\int_\Gamma \beta \, d\sigma > 0$, then $\int_\Gamma \beta |\psi|^2 \, d\sigma = \gamma^2 \int_\Gamma \beta \, d\sigma = 0$ implies $\psi \equiv \gamma = 0$, which is a contradiction to $\psi \not\equiv 0$.

If $\partial\Omega \setminus \Gamma$ has a strictly positive surface measure, then $\psi \equiv \gamma$ is equivalent to zero on this set, hence $\psi \equiv \gamma = 0$, which again is a contradiction to $\psi \not\equiv 0$.

The contraction properties of the semigroup follow immediately from the lower bounds of A_2 by means of the spectral theorem. □

The following regularity result for elliptic boundary value problems due to Griepentrog and Recke is the essential ingredient in our subsequent proofs.

2.8. Proposition. *(Cf. [17].) Let $\Omega \cup \Gamma$ be a regular set in the sense of Definition 2.1, W be a nonnegative L^∞ function, A_2 be according to Definition 2.6, and $0 < a_\bullet \le a^\bullet < \infty$ be the constants from (2.7). For every p with $p \ge 2$ and $p > d/2$ there exist two constants $c = c(p, a_\bullet, a^\bullet, \Omega, \Gamma) > 0$ and*

$\alpha = \alpha(p, a_\bullet, a^\bullet, \Omega, \Gamma) \in]0,1[$ *such that for every* $f \in L^p$ *the solution* $u \in W_0^{1,2}$ *of the elliptic boundary value problem* $(A_2 + W)u = f$ *is Hölder continuous up to the boundary, and there is*

$$\left\|(A_2 + W + 1)^{-1} f\right\|_{C^\alpha} \le c\,\|f\|_{L^p}. \tag{2.9}$$

2.9. Remark. This result corresponds to one being known since long for the Dirichlet problem, cf. Gilbarg/Trudinger [14, 8.10].

From Proposition 2.8 one easily deduces the following

2.10. Lemma. *There is a positive number* j *such that for any nonnegative* L^∞ *function* W *the mapping*

$$\big(A_2 + W + 1\big)^{-j} : L^2 \longrightarrow C^\alpha \hookrightarrow L^\infty \tag{2.10}$$

is well defined and continuous. If $d = 2, 3$*, then* $j = 1$ *does the job; if* $d = 4, 5$*, then* $j = 3/2$ *works.*

Proof. If $d \le 3$, then (2.10) holds with $j = 1$, according to Proposition 2.8. If $d \in \{4, 5\}$, then $\frac{2d}{d-2} > \frac{d}{2}$ and Proposition 2.8 yields

$$L^2 \xrightarrow{(A_2+W+1)^{-\frac{1}{2}}} \operatorname{dom}(\mathfrak{t}) = W_0^{1,2} \hookrightarrow L^{\frac{2d}{d-2}} \xrightarrow{(A_2+W+1)^{-1}} C^\alpha.$$

If $d > 5$, then by Definition 2.6 and Proposition 2.8 one has

$$\begin{aligned}
(A_2 + W + 1)^{-1} : \qquad & L^2 \longrightarrow \operatorname{dom}(A_2) \hookrightarrow \operatorname{dom}(\mathfrak{t}) = W_0^{1,2} \hookrightarrow L^{\frac{2d}{d-2}} \\
(A_2 + W + 1)^{-1} : \qquad & L^{\frac{d+1}{2}} \longrightarrow C^\alpha \hookrightarrow L^\infty.
\end{aligned}$$

Hence, by the Riesz–Thorin interpolation theorem the mapping

$$(A_2 + W + 1)^{-1} \,:\, L^{\frac{2d}{d-2}(\frac{d}{d-2})^k} \longrightarrow L^{q(k)} \hookrightarrow L^{\frac{2d}{d-2}(\frac{d}{d-2})^{k+1}}$$

is continuous, for all positive integers k such that

$$\frac{2d}{d-2}\Big(\frac{d}{d-2}\Big)^k \le \frac{d+1}{2}.$$

Thus, with a finite resolvent power $(A_2 + W + 1)^{-k}$ one ends up in $L^{\frac{d+1}{2}}$. Now, applying once more $(A_2 + W + 1)^{-1}$ one arrives at C^α, due to Proposition 2.8.

□

2.11. Remark. It is not hard to see that j may be taken as 2 for $d = 6$.

2.12. Remark. Lemma 2.10 can be equivalently formulated: For each of the spaces $X = C^\beta$, $0 \le \beta \le \alpha$, α according to Proposition 2.8, and $X = L^p$, $1 \le p \le \infty$, there is a constant γ_X such that

$$\|\psi\|_X \le \gamma_X \left\|(A_2 + W + 1)^j \psi\right\|_{L^2} \quad \text{for all } \psi \in \operatorname{dom}\left((A_2 + W + 1)^j\right). \quad (2.11)$$

We recall that A_2 provides an isomorphism between $W_0^{1,p}$ and $W^{-1,p}$, a result due to Gröger and Rehberg.

2.13. Proposition. *(Cf. [21].) Let $\Omega \cup \Gamma$ be a regular set in the sense of Definition 2.1, A_2 be according to Definition 2.6, and $0 < a_\bullet \le a^\bullet < \infty$ be the constants from (2.7). There is a real constant $\epsilon = \epsilon(\Omega, \Gamma, a_\bullet, a^\bullet) > 0$, such that $(A_2 + 1)^{-1}$ continuously extends to a topological isomorphism between $W^{-1,p}$ and $W_0^{1,p}$ for all $p \in [2, 2 + \epsilon[$. Denoting the inverse of this toplogical isomorphism by B_p, one has the following resolvent estimate:*

$$\left\|(B_p + \rho)^{-1}\right\|_{\mathcal{B}(W^{-1,p};W^{-1,p})} \le \frac{N_p}{|\rho|} \qquad \textit{for all } \rho \in \mathbb{C} \textit{ with } \Re\rho \ge 0, \qquad (2.12)$$

where the constant N_p depends on Ω, Γ, $a_\bullet$, and $a^\bullet$.

2.14. Remark. It should be noticed that in view of the example in Shamir [30, p.151] one cannot expect in general that ϵ becomes much greater than zero, even for a smooth domain and constant coefficients.

3 The operators A_p

In this section we will regard more closely the operators A_p from Definition 2.6 and operators $A_p + W$, where W is a multiplication operator, induced by a nonnegative function. W of this type frequently occur as potentials of Schrödinger operators, cf. e.g. Reed/Simon [29, vol. IV:ch.XIII].

3.a Basic properties of the operators A_p

3.1. Theorem. *Suppose $p \in]2, \infty[$. For any $\rho > 0$ the operator $\left(A_p + \rho\right)^{-1}$ exists and is compact, hence, A_p is closed.*

Proof. Let first p be greater than $d/2$. According to Proposition 2.8 $\left(A_p+\rho\right)^{-1}$ is a continuous mapping from L^p into a Hölder space, hence it is compact, viewed as a mapping from L^p into itself. By Lemma 2.7, $\left(A_2 + \rho\right)^{-1}$ also is compact. Thus, using a well known interpolation theorem for L^p spaces, cf.

Davies [6, Th. 1.6.1], one obtains that $\big(A_p+\rho\big)^{-1}$ exists for $p \in]2, \frac{d}{2}]$ and is compact.

$\big(A_p+\rho\big)^{-1}$ is continuous for any $\rho > 0$, and, consequently, closed. Thus, $A_p+\rho$ and A_p are also closed operators. $\square$

For any $p \in [2,\infty[$ let $J_p : L^p \longrightarrow L^{p'}$, $1/p + 1/p' = 1$ denote the duality mapping

$$J_p : \psi \longmapsto \frac{1}{\|\psi\|_{L^p}^{p-2}}\, \psi|\psi|^{p-2} \tag{3.1}$$

from L^p into $L^{p'}$. N.B. the duality was defined by (2.1) as the extended L^2 duality, i.e. antilinear in the second argument. The duality mapping (3.1) has the following properties.

3.2. Lemma. *Suppose $p \geq \max\left\{4, \frac{d+1}{2}\right\}$. If $\psi \in \operatorname{dom}(A_p)$, then $J_p\psi \in W_0^{1,2}$ and the (generalized) partial derivatives of $J_p\psi$ may be calculated as*

$$\frac{\partial}{\partial x_l} J_p\psi = \frac{1}{\|\psi\|_{L^p}^{p-2}} \left(|\psi|^{p-2}\frac{\partial \psi}{\partial x_l} + \frac{p-2}{2}\,\psi|\psi|^{p-4}\left(\overline{\psi}\frac{\partial \psi}{\partial x_l} + \psi\frac{\partial \overline{\psi}}{\partial x_l}\right)\right). \tag{3.2}$$

Proof. As $p \geq \max\left\{4, \frac{d+1}{2}\right\}$ we have $\psi \in C(\overline{\Omega})$ due to Proposition 2.8. Moreover, one has

$$\operatorname{dom}(A_p) \subset \operatorname{dom}(A_2) \subset \operatorname{dom}(A_2^{\frac{1}{2}}) = dom(\mathfrak{t}) = W_0^{1,2}. \tag{3.3}$$

Hence, due to the product rule it is sufficient to prove that $|\psi|^{p-2}$ is from $W_0^{1,2}$ and its partial derivatives may be calculated as

$$\frac{\partial}{\partial x_l}\,|\psi|^{p-2} = \frac{p-2}{2}|\psi|^{p-4}\left(\overline{\psi}\frac{\partial \psi}{\partial x_l} + \psi\frac{\partial \overline{\psi}}{\partial x_l}\right). \tag{3.4}$$

For $p = 4$, what is permitted in the cases $d \leq 7$, the statement follows immediately by the product rule. Let now p be greater than 4 and not smaller than $\frac{d+1}{2}$. With $\varphi = |\psi|^2$ the left hand side of (3.4) can be written as $\frac{\partial}{\partial x_l}\left(\varphi^{(\frac{p}{2}-1)}\right)$. φ is a positive function from $W_0^{1,2} \cap C(\overline{\Omega})$; we denote its supremum by M, and define the function $g : \mathbb{R} \longrightarrow [0,\infty[$ by

$$g(x) = \begin{cases} 0 & \text{if } x \in]-\infty, 0[\\ x^{\frac{p}{2}-1} & \text{if } x \in [0, M+1] \\ (M+1)^{\frac{p}{2}-1} & \text{if } x \in]M+1, \infty[. \end{cases} \tag{3.5}$$

Because the function φ takes its values only in the interval $[0, M]$, we have

$$|\psi|^{p-2} = \varphi^{\frac{p}{2}-1} = g(\varphi).$$

By construction, g is a continuous and piecewise continuously differentiable function with $g' \in L^\infty(\mathbb{R})$; thus the weak partial derivatives of $g(\varphi)$ are

$$\frac{\partial}{\partial x_l} g(\varphi) = g'(\varphi) \frac{\partial \varphi}{\partial x_l},$$

cf. Gilbarg/Trudinger [14, 7.4 Th. 7.8]. □

Lemma 3.2 allows to characterize the numerical range of the operators A_p:

3.3. Theorem. *Suppose $p \geq \max\left\{4, \frac{d+1}{2}\right\}$. If $\psi \in \operatorname{dom}(A_p)$, then*

$$\left|\Im\langle A_p\psi, J_p\psi\rangle\right| \leq \frac{a^\bullet}{a_\bullet} \frac{p-2}{2\sqrt{p-1}} \Re\langle A_p\psi, J_p\psi\rangle. \tag{3.6}$$

In particular, $-A_p$ is dissipative, and $-A_p$ is the infinitesimal generator of a strongly continuous semigroup of contractions.

Proof. By Proposition 2.8 and our assumption on p, $\operatorname{dom}(A_p)$ is contained in L^∞. Hence, the L^p–$L^{p'}$ duality $\langle A_p\psi, J_p\psi\rangle$ is equal to the scalar product between $A_p\psi$ and $J_p\psi$ in L^2. Further, $\psi \in \operatorname{dom}(A_p)$ implies by (3.3) and Lemma 3.2 that ψ and $J_p\psi$ belong to $W_0^{1,2} = \operatorname{dom}(\mathfrak{t})$. Hence, due to (3.2) there is

$$\langle A_p\psi, J_p\psi\rangle = \mathfrak{t}[\psi, J_p\psi] =$$

$$\frac{1}{\|\psi\|_{L^p}^{p-2}}\left(\int_\Omega \sum_{k,l=1}^d a_{k,l} \frac{\partial \psi}{\partial x_k}\left(|\psi|^{p-2}\frac{\partial \overline{\psi}}{\partial x_l} + \frac{p-2}{2}\overline{\psi}|\psi|^{p-4}\left(\psi\frac{\partial\overline{\psi}}{\partial x_l} + \overline{\psi}\frac{\partial \psi}{\partial x_l}\right)\right) dx + \int_\Gamma \beta\, |\psi|^p \, d\sigma\right) \tag{3.7}$$

We notice:

i) As far as the relation between the real and imaginary part of $\mathfrak{t}[\psi, J_p\psi]$ is concerned the factor $1/\|\psi\|_{L^p}^{p-2}$ on the right hand side of (3.7) may be omitted, what we will do in the sequel.

ii) If one neglects the (positive) term $\int_\Gamma \beta\, |\psi|^p \, d\sigma / \|\psi\|_{L^p}^{p-2}$, then the real part of the right hand side of (3.7) decreases.

We split

$$|\psi|^{\frac{p-4}{2}}\overline{\psi}\frac{\partial\psi}{\partial x_k} \stackrel{\text{def}}{=} \varphi_k + i\phi_k$$

into the real and imaginary parts and write down what remains on the right hand side of (3.7), thereby observing that the coefficient matrix (2.6) is **real symmetric**:

$$\int_\Omega \sum_{k,l} a_{k,l}\big(\varphi_k + i\phi_k\big)\left(\overline{(\varphi_l + i\phi_l)} + \frac{p-2}{2}\Big(\overline{(\varphi_l + i\phi_l)} + \big(\varphi_l + i\phi_l\big)\Big)\right) dx$$
$$= \int_\Omega \sum_{k,l} a_{k,l}\Big((p-1)\varphi_k\,\varphi_l \;+\; \phi_k\,\phi_l \;+\; i(p-2)\varphi_k\,\phi_l\Big)\,dx. \quad (3.8)$$

By means of (2.7b), the real part of (3.8) may be estimated from below by

$$a_\bullet\Big((p-1)\sum_{k=1}^d \int_\Omega \varphi_k^2\,dx \;+\; \sum_{k=1}^d \int_\Omega \phi_k^2\,dx\Big) \;\geq\; 0, \quad (3.9)$$

while the absolute value of the imaginary part of (3.8) can be estimated due to (2.7a) and (3.9) as follows:

$$(p-2)\left|\int_\Omega \sum_{k,l=1}^d a_{k,l}\,\varphi_k\,\phi_l\,dx\right| \leq a^\bullet(p-2)\sqrt{\int_\Omega \sum_{k=1}^d \varphi_k^2\,dx}\sqrt{\int_\Omega \sum_{k=1}^d \phi_k^2\,dx}$$
$$\leq a^\bullet\,\frac{p-2}{2}\left(\sqrt{p-1}\int_\Omega \sum_{k=1}^d \varphi_k^2\,dx \;+\; \frac{1}{\sqrt{p-1}}\int_\Omega \sum_{k=1}^d \phi_k^2\,dx\right)$$
$$\leq a^\bullet\,\frac{p-2}{2\sqrt{p-1}}\Big((p-1)\sum_{k=1}^d \int_\Omega \varphi_k^2\,dx \;+\; \sum_{k=1}^d \int_\Omega \phi_k^2\,dx\Big),$$

what proves the assertion (3.6). Now (3.6) implies immediately the dissipativity of $-A_p$, cf. Pazy [28, 1.4 Def. 4.1], and this together with Theorem 3.1 ensures by the Lumer–Phillips theorem [28, 1.4 Th. 4.3] , that $-A_p$ is the infinitesimal generator of a strongly continuous semigroup of contractions. □

3.4. Remark. The proof of Theorem 3.3 follows exactly the proof of Pazy [28, 7.3 Th. 3.6] for the case of smooth domains, smooth coefficients and homogeneous Dirichlet boundary conditions. Indeed, the crucial part of the proof in our setting is to show that the duality mapping J_p maps the domain of the operator A_p into the form domain of $\mathfrak{t}$.

Theorem 3.3 permits an essential conclusion:

3.5. Theorem. *If p is any number from $[2, \infty[$ then $\mathrm{dom}(A_p)$ is dense in L^p.*

Proof. At first let p be not smaller than $\max\left\{4, \frac{d+1}{2}\right\}$. According to a well known theorem, cf. Pazy [28, 1.4 Th. 4.6] it suffices to show that $1 + A_p$ has the whole space L^p as its range. Indeed, for any $\rho > 0$ the operator $A_p + \rho$ is surjective, because, due to the compactness of the resolvent, cf. Theorem 3.1, in the opposite case ρ would be an eigenvalue of $-A_p$. But, this is impossible because $-A_p$ is dissipative.

Thus, the assertion is proved for $p \geq p_0 = \max\left\{4, \frac{d+1}{2}\right\}$. Let now p be from $[2, p_0[$. There is

$$\mathrm{dom}(A_{p_0}) \subset \mathrm{dom}(A_p) \quad \text{for all } p \in [2, p_0[.$$

Hence, as $\mathrm{dom}(A_{p_0})$ is dense in L^{p_0} and L^{p_0} is dense in L^p, $\mathrm{dom}(A_p)$ must be dense in L^p. □

Theorem 3.5 justifies the following definition, supplementing Definition 2.6:

3.6. Definition. For $p < 2$, A_p is the adjoint of $A_{p'}$, where $p = p'/(p'-1)$.

We will now reproduce the statements on A_p for the case $p \in]1, 2[$.

3.7. Theorem. *Suppose $p \in]1, 2[$. A_p is closed and densely defined. The restriction of A_p to L^2 is equal to A_2. For any $\rho > 0$ the operator $\left(A_p + \rho\right)^{-1}$ exists and is compact, hence, its spectrum is discrete.*

Proof. In a reflexive space an operator is densely defined and closed if its adjoint is, cf. Kato [23, III.§5 Th. 5.29]. Thus the first assertion follows from Theorem 3.1 and the fact that L^p is dense in L^2 for all $p > 2$.

The second assertion follows from the selfadjointness of A_2.

The compactness of the resolvent of A_p follows from Theorem 3.1 and [23, III.§5 Th. 5.30], and there is

$$\left(A_p + \rho\right)^{-1} = \left(\left(A_{\frac{p}{p-1}} + \rho\right)^{-1}\right)^* \tag{3.10}$$

for any $\rho > 0$. □

3.8. Remark. According to Theorem 3.1 and Theorem 3.7, the operator $\left(A_p + 1\right)^{-1}$ is compact. Hence, $\mathrm{dom}(A_p)$ equipped with the graph norm $\|\psi\|_{\mathrm{dom}(A_p)} = \|(A_p + 1)\psi\|_{L^p}$ embeds compactly into L^p for all $p \in]1, \infty[$.

Now we may conclude the dissipativity of all the operators $-A_p$:

3.9. Theorem. *Suppose $p \in]1,\infty[$ and $\rho > 0$. There is*

$$\big\| \big(A_p + \rho\big)^{-1} \big\|_{\mathcal{B}(L^p;L^p)} \le \frac{1}{\rho}, \tag{3.11}$$

hence, $-A_p$ is dissipative.

Proof. In view of (3.10) it suffices to prove (3.11) for $p \in [2,\infty[$. For $p = 2$ and $p \ge \max\left\{4, \frac{d+1}{2}\right\}$ the inequality follows from the dissipativity of $-A_p$ and the surjectivity of $A_p + \rho$ and a well known theorem, cf. Pazy [28, Th. 1.4.2]; for $p \in]2, \max\left\{4, \frac{d+1}{2}\right\}[$, (3.11) follows by interpolation and the dissipativity of $-A_p$ follows again from [28, Th. 1.4.2]. □

3.b A_p: Perturbations by nonnegative potentials W

For $A_p + W$ to generate a strongly continuous semigroup of contractions and to allow resolvent estimates it is sufficient to know that the multiplication operator induced by the function W is relatively compact with respect to the operator A_p. First we prove a general lemma about relatively bounded perturbations of $A_p + 1$.

3.10. Lemma. *Suppose $p \in]1,\infty[$ and let $T : \mathrm{dom}(A_p) \longrightarrow L^p$ be relatively bounded with respect to $A_p + 1$:*

$$\|T\psi\|_{L^p} \le a\|\psi\|_{L^p} + b\|(A_p+1)\psi\|_{L^p} \quad \textit{for all } \psi \in \mathrm{dom}(A_p). \tag{3.12}$$

If $\rho > 1$, then

$$\|T\psi\|_{L^p} \le a\|\psi\|_{L^p} + 2b\|(A_p+\rho)\psi\|_{L^p} \quad \textit{for all } \psi \in \mathrm{dom}(A_p). \tag{3.13}$$

Moreover, if $b < 1/2$, then the operators A_p and $A_p + T$ have the same domain $\mathrm{dom}(A_p)$*, are closed and the resolvent of $A_p + T$ is compact.*

Proof. (3.13) results by means of (3.11) from the inequality

$$\begin{aligned}
\big\|(A_p+1)\psi\big\|_{L^p} &\le \big\|(A_p+1)(A_p+\rho)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \big\|(A_p+\rho)\psi\big\|_{L^p} \\
&\le \Big(1 + (\rho-1)\big\|(A_p+\rho)^{-1}\big\|_{\mathcal{B}(L^p;L^p)}\Big) \big\|(A_p+\rho)\psi\big\|_{L^p} \\
&\le 2\,\big\|(A_p+\rho)\psi\big\|_{L^p}.
\end{aligned}$$

Due to Theorem 3.1 and Theorem 3.7, A_p is closed. If $b < 1$, then the operators A_p and $A_p + T$ have the same domain $\mathrm{dom}(A_p)$ and are closed, cf. [23, IV.§1 Th. 1.1],

According to Theorem 3.1 and Theorem 3.7 $A_p + \rho$ is is compactly invertible for all $\rho > 0$, and there is (3.13). If $b < 1/2$, then by (3.11)

$$a \left\|(A_p + \rho)^{-1}\right\|_{\mathcal{B}(L^p;L^p)} + 2b \leq \frac{a}{\rho} + 2b < 1,$$

for ρ sufficiently great. Hence, the resolvent of $A_p + T$ is compact, cf. [23, IV.§2 Th. 1.16]. □

3.11. Theorem. *Suppose $p \in]1, \infty[$, and let W be a nonnegative, measurable function on Ω with lower bound $W_\bullet$. If the multiplication operator, induced by W on L^p is relatively compact with respect to A_p, then* $\mathrm{dom}(A_p + W)$ *equals* $\mathrm{dom}(A_p)$, *the operator $A_p + W$ is closed, its resolvent is compact, and*

$$\left\| \left(A_p + W + \rho\right)^{-1} \right\|_{\mathcal{B}(L^p;L^p)} \leq \frac{1}{\rho + W_\bullet} \quad \text{for all } \rho > -W_\bullet. \tag{3.14}$$

Moreover, the operator $-(A_p + W)$ generates a strongly continuous semigroup of contractions on L^p. If $\sigma(\partial\Omega \setminus \Gamma) > 0$, or $\int_\Gamma \beta \, d\sigma > 0$, or $W_\bullet > 0$, then this semigroup is even strictly contractive.

Proof. The multiplication operator induced by W maps $\mathrm{dom}(A_p)$ compactly into L^p. N.B. $\mathrm{dom}(A_p)$ equipped with the graph norm compactly embeds into L^p, cf. Remark 3.8. Further, the multiplication operator induced on L^p by the function $1/(1+W)$ is continuous and injective, because W is nonnegative, i.e.

$$\mathrm{dom}(A_p) \xrightarrow[\text{compact}]{W} L^p \xrightarrow[\text{continuous, injective}]{\frac{1}{1+W}} L^p.$$

According to Ehrling's lemma, cf. e.g. [35, I.§7 Satz 7.3], for every $b > 0$ there is an $a > 0$, such that

$$\|W\psi\|_{L^p} \leq a \left\| \frac{W}{1+W}\psi \right\|_{L^p} + b \left\|\psi\right\|_{\mathrm{dom}(A_p)} \leq a \left\|\psi\right\|_{L^p} + b \left\|(A_p + 1)\psi\right\|_{L^p}.$$

Thus, the multiplication operator, induced by W on L^p is relatively bounded by $A_p + 1$ with bound zero, and the assertions about $A_p + W$ follow from Lemma 3.10.

The multiplication operator, induced by $W - W_\bullet$ on L^p is dissipative. Indeed, one easily checks the definition [28, 1.4 Def. 4.1] by means of (3.1) thereby observing the nonnegativity of $W - W_\bullet$.

According to Theorem 3.3 $-A_p$ is the infinitesimal generator of a strongly continuous semigroup of contractions, and with the up to now obtained properties of $W - W_\bullet$ a perturbation theorem for such generators, cf. Pazy [28, 3.3 Cor. 3.3], applies. Hence, $-\big(A_p + W - W_\bullet\big)$ is the infinitesimal generator of a strongly continuous semigroup of contractions, and in particular dissipative.

Now the criterion [28, 1.4 Th. 4.2] for dissipativity provides

$$\big\|\big(A_p + W - W_\bullet + \rho\big)\psi\big\| \geq \rho\,\|\psi\| \qquad \text{for all } \rho > 0 \text{ and } \psi \in \operatorname{dom}(A_p + W - W_\bullet), \tag{3.15}$$

i.e. the operator $A_p + W - W_\bullet + \rho$ is injective. Due to the compactness of the resolvent $A_p + W - W_\bullet + \rho$ is also surjective. Consequently, (3.15) implies (3.14).

If $\sigma(\partial\Omega \setminus \Gamma) > 0$, or $\int_\Gamma \beta\, d\sigma > 0$, then the semigroup generated by $-(A_2 + W - W_\bullet)$ on L^2 is strictly contractive, cf. Lemma 2.7. Because the semigroups generated by $-(A_p + W - W_\bullet)$ on L^p are at least contractive, it follows by interpolation that the semigroups must be strictly contractive for all $p \in]1, \infty[$. If $W_\bullet > 0$, then the strict contractiveness follows from [28, 1.3 Cor. 3.8]. □

3.12. Remark. The question, whether $-(A_p + W)$ generates a strongly continuous semigroup of contractions on L^1 and L^∞ will be answered in Theorem 3.17.

3.c $A_p + W$: spectral properties

The question arises whether the spectra of the operators $A_p + W$ may differ for different p from each other or not. Later on we will establish ultracontractivity of the generated semigroup, which allows to answer this question, cf. Davies [6, Th. 1.6.3]. Here we will give a direct proof of the invariance of the spectrum. For the sake of technical simplicity we allow bounded potentials W only.

3.13. Theorem. *Suppose $p \in]1, \infty[$, and let W be a nonnegative L^∞ function. The spectrum of $A_p + W$ coincides with the spectrum of $A_2 + W$ and the geometric multiplicities are the same. For every eigenvalue λ of $A_p + W$ the algebraic multiplicity equals the geometric multiplicity, or, in other words, there are no nontrivial Jordan chains.*

Proof. According to Theorem 3.11 $A_p + W$ has a compact resolvent, hence its spectrum consist only of eigenvalues. We prove that the sets of eigenvalues are identical for all $p \in]1, \infty[$. Let p be firstly from $]1, 2[$. From Theorem 3.7

follows that all eigenvalues of $A_2 + W$ are also eigenvalues of $A_p + W$. Conversely, let λ be an eigenvalue of $A_p + W$. Then $\bar{\lambda}$ has to be an eigenvalue of $A_{\frac{p}{p-1}} + W$, hence, of $A_2 + W$, and due to the selfadjointness of $A_2 + W$, there is $\bar{\lambda} = \lambda$. This argument also applies to the geometric multiplicities, because the geometric multiplicities for the eigenvalue λ for $A_p + W$ is identical with the geometric multiplicity of $\bar{\lambda}$ as an eigenvalue of $A_{\frac{p}{p-1}} + W$, cf Kato [23, III.§6 Rem. 6.23]. The case $p > 2$ is proved by the inversed duality argument.

Now we show the second assertion. If $p > 2$, then no eigenvalue λ can have a nontrivial Jordan chain, because a Jordan chain in L^p also would be a Jordan chain in L^2 and this is impossible due to the selfadjointness of A_2+W. Hence, for $p > 2$ the dimension of the eigenprojection for an eigenvalue λ must be equal to the geometric multiplicity of λ. Let now p be smaller than 2 and λ be an eigenvalue of $A_p + W$. The dimension of the eigenprojection of λ in L^p equals the dimension of the corresponding eigenprojection for $A_{\frac{p}{p-1}} + W$, which is equal to the geometric multiplicity of λ. $\square$

3.14. Theorem. *Let W be a nonnegative L^∞ function. For any eigenvalue λ the corresponding eigenspaces of A_p+W are identical for all $p \in]1, \infty[$. The set of eigenvectors of the operator $A_p + W$ is total in L^p for all $p \in]1, \infty[$.*

Proof. If $p < q$, then any eigenspace of A_q is included in the corresponding eigenspace for A_p, and by Theorem 3.13 the eigenspaces have the same dimension.

The second statement is clear for L^2, due to the selfadjointness of $A_2 + W$. As the sets of eigenvectors for A_p and A_2 are identical this set is also total in L^p for $p \in]1, 2[$, because in that case L^2 is dense in L^p.

Let now p be from $]2, \infty[$ and let j be the resolvent power exponent from Lemma 2.10. Because $-(A_p + W + 1)$ generates a strongly continuous semigroup of contractions on L^p, cf. Theorem 3.11, $\operatorname{dom}\big((A_p + W + 1)^j\big)$ is dense in L^p, cf. Pazy [28, 1.2 Th.2.7]. Hence, it suffices to show that any element from the space $\operatorname{dom}\big((A_p + W + 1)^j\big) \subset L^p$ may be approximated by linear combinations of eigenvectors of $A_p + W$. Let for this purpose γ_p be the embedding constant

$$\gamma_p = \sup_{0 \neq \varphi \in \operatorname{dom}\left((A_p+W+1)^j\right)} \frac{\|\varphi\|_{L^p}}{\|(A_p + W + 1)^j \varphi\|_{L^2}} \tag{3.16}$$

of $\mathbb{1} : \operatorname{dom}\big((A_p + W + 1)^j\big) \to L^p$, which is finite due to Remark 2.12. Further, let $\psi \in \operatorname{dom}\big((A_p + W + 1)^j\big)$ and $\epsilon > 0$ be given. Because the system of eigenvectors of $A_2 + W$ is total in L^2, there is a finite sequence

$\{\psi_r\}_r$ of eigenvectors of A_2+W, λ_r being the corresponding eigenvalues, and a finite sequence $\{\mu_r\}_r$ of complex numbers such that

$$\Big\| \sum_r \mu_r \,\psi_r \;-\; (A_p + W + 1)^j \psi \Big\|_{L^2} < \frac{\epsilon}{\gamma_p}. \tag{3.17}$$

As

$$\sum_r \mu_r \,\psi_r \;-\; (A_p + W + 1)^j \psi = (A_p + W + 1)^j \left(\sum_r \frac{\mu_r}{(\lambda_r + 1)^j}\, \psi_r - \psi \right)$$

(3.17) implies by definition (3.16) of γ_p:

$$\Big\| \sum_r \frac{\mu_r}{(\lambda_r + 1)^j}\, \psi_r \;-\; \psi \Big\|_{L_p} \;<\; \epsilon,$$

i.e. the eigenvectors of $A_p + W$ form a total set in dom $\big((A_p + W + 1)^j\big)$. □

3.d Integral kernel properties

The next results affect kernel properties for resolvent powers of $A_2 + W$ and the semigroup operators generated by $-(A_2 + W)$. We will regard bounded potentials W only.

3.15. Theorem. *Let W be a nonnegative L^∞ function, α be the Hölder exponent from Proposition 2.8, and j the resolvent power exponent from Lemma 2.10.*

i) $(A_2 + W + 1)^{-j} : L^2 \to L^2$ *is a Hilbert–Schmidt operator.*

ii) $(A_2 + W + 1)^{-2j} : L^2 \to L^2$ *extends to a continuous mapping from L^1 into $C^\alpha \hookrightarrow L^\infty$.*

iii) $(A_2 + W + 1)^{-2j}$ *is an integral operator, the kernel $\mathcal{K}$ of which is essentially bounded. If $\hat\alpha$ is smaller than the Hölder exponent α from Proposition 2.8, then*

$$\|\mathcal{K}\|_{L^\infty(\Omega; C^{\hat\alpha})} = \big\| (A_2 + W + 1)^{-2j} \big\|_{\mathcal{B}(L^1; C^{\hat\alpha})}.$$

Proof. Ad i. The Hilbert–Schmidt property follows from the Pietsch factorization theorem, cf. Diestel/Jarchow/Tonge [8, 2.13 item iv], and Lemma 2.10. Indeed, there is a factorization over L^∞:

$$(A_2 + W + 1)^{-j} \;:\; L^2 \xrightarrow{(A_2 + W + 1)^{-j}} C^\alpha \hookrightarrow L^\infty \hookrightarrow L^2.$$

Ad ii. From Lemma 2.10 follows by duality that $(A_2 + W + 1)^{-j} : L^2 \to C^\alpha \hookrightarrow L^\infty$ extends to a continuous operator from L^1 into L^2. N.B. A_2+W+1 is selfadjoint.

Ad iii. This assertion follows from the abstract representation theorem of compact operators on L^1, cf. Diestel/Uhl [9, III.2 Th. 2]. Indeed, $(A_2 + W + 1)^{-2j} : L^1 \to C^\alpha \hookrightarrow C^{\hat\alpha}$ is compact, hence representable. N.B. the embedding $C^\alpha \hookrightarrow C^{\hat\alpha}$ is compact for $\hat\alpha < \alpha$. $\square$

3.16. Theorem. *Let W be a nonnegative L^∞ function, α be the Hölder exponent from Proposition 2.8.*

i) Each semigroup operator $e^{-t(A_2+W)}$, $t > 0$, maps L^2 continuously into $C^\alpha \hookrightarrow L^\infty$, i.e. in particular the semigroup $\{e^{-t(A_2+W)}\}_{t>0}$ is ultracontractive, cf. Davies [6, 2.1].

ii) Each semigroup operator $e^{-t(A_2+W)} : L^2 \to L^2$, $t > 0$, is nuclear and, consequently, an integral operator. The corresponding kernels are even from $C^\alpha(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$.

Proof. Let j be the resolvent power exponent from Lemma 2.10. There is

$$\begin{aligned}&\left\|e^{-t(A_2+W)}\right\|_{\mathcal{B}(L^2;C^\alpha)} \\ &\qquad \le \left\|(A_2 + W + 1)^{-j}\right\|_{\mathcal{B}(L^2;C^\alpha)} \left\|(A_2 + W + 1)^j e^{-t(A_2+W)}\right\|_{\mathcal{B}(L^2;L^2)}.\end{aligned}$$

The first factor on the right hand side is finite according to Lemma 2.10; the second one is finite due to the spectral theorem. Thus, there is a factorization of $e^{-t(A_2+W)}$ over L^∞

$$e^{-t(A_2+W)} \; : \; L^2 \xrightarrow{e^{-t(A_2+W)}} C^\alpha \hookrightarrow L^\infty \hookrightarrow L^2$$

for every $t > 0$, and according to the Pietsch factorization theorem, cf. Diestel/Jarchow/Tonge [8, 2.13 item iv], all the semigroup operators $e^{-t(A_2+W)} : L^2 \to L^2$, are Hilbert–Schmidt. Splitting up

$$e^{-t(A_2+W)} = \left(e^{-\frac{t}{2}(A_2+W)}\right)^2$$

this yields the nuclearity of $e^{-t(A_2+W)}$.

Let now $\{\lambda_r\}_{r=1}^\infty$ be the sequence of eigenvalues of $A_2 + W$, counting multiplicity, and $\{\psi_r\}_{r=1}^\infty$ a corresponding complete, orthonormal system of real eigenfunctions. Such a system may always be found because $A_2 + W$ commutes with the complex conjugation on L^2. We prove that the series

$$\sum_{r=1}^\infty e^{-t\lambda_r}\, \psi_r \otimes \psi_r \tag{3.18}$$

converges absolutely in $C^\alpha(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$. This implies that the series represents the integral kernel of $e^{-(A_2+W)t}$, because for any eigenfunction ψ_r one obtains the correct image under $e^{-(A_2+W)t}$. First, it follows from Lemma 2.10 that all eigenfunctions ψ_r belong to C^α because they belong to $\cap_{l=1}^\infty \operatorname{dom}\big((A_2 + W)^l\big)$, thus in particular to $\operatorname{dom}\big((A_2 + W)^j\big)$. Further, it is easy to check the inequality

$$\|\psi \otimes \varphi\|_{C^\alpha(\overline{\Omega}\times\overline{\Omega})} \leq 2\|\psi\|_{C^\alpha}\|\varphi\|_{C^\alpha} \quad \text{for all } \psi, \varphi \in C^\alpha. \tag{3.19}$$

Now one can estimate the terms of the sum (3.18) by means of (3.19) and (2.11) as follows:

$$\|\psi_r \otimes \psi_r\|_{C^\alpha(\overline{\Omega}\times\overline{\Omega};\mathbb{R})} \leq 2\gamma_{C^\alpha}^2 \left\|(A_2 + W + 1)^j \psi_r\right\|_{L^2}^2 \leq 2\gamma_{C^\alpha}^2 (\lambda_r + 1)^{2j}. \tag{3.20}$$

N.B. the ψ_r are L^2–normalized. According to Theorem 3.15 $(A_2 + W + 1)^{-j}$ is a Hilbert–Schmidt operator, hence $(A_2 + W + 1)^{-2j}$ must be nuclear. Thus, the series

$$\sum_{r=1}^{\infty} (\lambda_r + 1)^{-2j}$$

is convergent and due to (3.20) the series (3.18) converges absolutely in $C^\alpha(\overline{\Omega} \times \overline{\Omega}; \mathbb{R})$. □

3.e $A_p + W$: Semigroups on L^∞ and L^1

Next we will regard the semigroup $\{e^{-t(A_2+W)}\}_{t>0}$ with a nonnegative, bounded potential W on the spaces L^1 and L^∞.

3.17. Theorem. *Let W be a nonnegative L^∞ function. Then the semigroup $e^{-t(A_2+W)}$, $t > 0$ induces semigroups of contractions on L^∞ and L^1. The latter semigroup is strongly continuous, while the first is not.*

Proof. According to Theorem 3.16 there is $e^{-t(A_2+W)} \in \mathcal{B}(L^\infty, L^\infty)$, and $\{e^{-t(A_2+W)}\}_{t>0}$ obviously forms a semigroup on L^∞. It remains to show the contractivity of $e^{-t(A_2+W)}$ on L^∞. Indeed, due to the contractivity of $e^{-t(A_2+W)}$ for all $p \in [2, \infty[$, there is for all $\psi \in L^\infty$

$$\left\|e^{-t(A_2+W)}\psi\right\|_{L^\infty} \xleftarrow{\infty \leftarrow p} \left\|e^{-t(A_2+W)}\psi\right\|_{L^p} \leq \|\psi\|_{L^p} \xrightarrow{p\to\infty} \|\psi\|_{L^\infty}.$$

N.B. if $\psi \in L^\infty$, then $\|\psi\|_{L^\infty} = \lim_{p\to\infty}\|\psi\|_{L^p}$.

Now we regard the semigroup operators $e^{-t(A_2+W)}$ on L^2, but L^2 equipped with the L^1 norm. We state that they also have a norm not greater than 1. If this were not so, then there would exist a $t > 0$ and an element $\psi \in L^2$ with $\|\psi\|_{L^1} = 1$ such that $\left\|e^{-t(A_2+W)}\psi\right\|_{L^1} > 1$. By the Hahn–Banach theorem there would be a $\hat{\psi} \in L^\infty$ with $\|\hat{\psi}\|_{L^\infty} = 1$ such that

$$\begin{aligned} 1 \;<\; \left\|e^{-t(A_2+W)}\psi\right\|_{L^1} \;&=\; \langle e^{-t(A_2+W)}\psi, \hat{\psi}\rangle \;=\; \langle \psi, e^{-t(A_2+W)}\hat{\psi}\rangle \\ &\le\; \left\|e^{-t(A_2+W)}\hat{\psi}\right\|_{L^\infty} \|\psi\|_{L^1} \;\le\; \|\psi\|_{L^1}\,\|\hat{\psi}\|_{L^\infty} \;=\; 1. \end{aligned}$$

N.B. $\psi, \hat{\psi} \in L^2$, and $e^{-t(A_2+W)}$ is selfadjoint on L^2 and contractive on L^∞. This is a contradiction to $\left\|e^{-t(A_2+W)}\psi\right\|_{L^1} > 1$,

The operators $e^{-t(A_2+W)}$ extend by continuity from $L^1|_{L^2}$ to the whole space L^1; they are there contractive and satisfy the semigroup property. We now prove the strong continuity: for $\psi \in L^2$ the continuity of the mapping

$$\mathbb{R}_+ \ni t \longmapsto e^{-t(A_2+W)}\psi \in L^1$$

follows immediately from the strong continuity of the semigroup in L^2. For $\psi \in L^2$, $\phi \in L^1$, and $s, t \in [0, \infty[$ there is, due to the contractivity of the semigroup on L^1

$$\begin{aligned} &\left\|e^{-t(A_2+W)}\phi - e^{-s(A_2+W)}\phi\right\|_{L^1} \\ &\qquad\qquad \le 2\,\|\phi - \psi\|_{L^1} + \left\|e^{-t(A_2+W)}\psi - e^{-s(A_2+W)}\psi\right\|_{L^1}. \end{aligned}$$

The expression on the left hand side becomes arbitrarily small provided ψ is sufficiently close to ϕ with respect to the L^1 norm and provided s is sufficiently close to t.

Why is the semigroup not strongly continuous on L^∞? If it were, its generator would be densely defined in L^∞ according to the Hille–Yosida theorem, cf. Pazy [28, 1.3 Th. 3.1]. On the other hand, due to Proposition 2.8, $\mathrm{dom}(A_\infty + W)$ is contained in a Hölder space C^α, never being dense in L^∞. □

3.18. Remark. In particular, Theorem 3.17 together with Theorem 3.15 implies that the semigroup generated by $-(A_2 + W)$ is hypercontractive, cf. [29, vol. II:X.9].

3.19. Remark. With respect to the dual semigroup, cf. Pazy [28, ch. 1 Th. 10.4], it is an interesting question what the closure of $\mathrm{dom}(A_1^*)$ in L^∞ is. The authors believe that if $d = 2$, then

$$\overline{\mathrm{dom}(A_1^*)}^{L^\infty} = C(\overline{\Omega}) \cap \bigl\{\psi \,:\, \psi|_{\overline{\partial\Omega\setminus\Gamma}} = 0\bigr\},$$

but have no idea for the higher dimensional cases.

3.f $A_p + W$: positivity preservation

We now turn to the question: Do the semigroups generated by $A_p + W$ preserve positivity? Before we can give the — affirmative — answer, we have to prove a technical lemma:

3.20. Lemma. *If ψ is a real–valued function from $W_0^{1,2}$, then the positive part ψ^+ of ψ also belongs to $W_0^{1,2}$.*

Proof. It is well known, cf. Evans/Gariepy [11, 4.2.2 Th. 4.iii], that for any real–valued function $\psi \in W^{1,2}$ the positive part ψ^+ also belongs to $W^{1,2}$ and

$$\frac{\partial \psi^+}{\partial x_k} = \begin{cases} \frac{\partial \psi}{\partial x_k} & \text{a.e. on } \{\psi > 0\}, \\ 0 & \text{a.e. on } \{\psi \leq 0\}. \end{cases} \tag{3.21}$$

Hence, the nontrivial part of Lemma 3.20 is to prove that the functions ψ^+ also have the stated boundary behaviour. To that end, we show firstly:

$$\text{If } \psi \in C_0^\infty(\Omega \cup \Gamma; \mathbb{R}), \text{ then } \psi^+ \in W_0^{1,2}. \tag{3.22}$$

By definition (2.3) of $C_0^\infty(\Omega \cup \Gamma)$ the support of ψ, and a fortiori that of ψ^+, does not intersect $\overline{\Omega} \setminus (\Omega \cup \Gamma)$, hence, it has a positive distance δ to the latter set, which is closed, cf. Remark 2.3. Passing to mollifications of ψ^+, cf. e.g. [11, 4.2.1], one observes that the support of the regularized functions also does not intersect $\overline{\Omega} \setminus (\Omega \cup \Gamma)$, provided the mollification parameter being smaller than δ. Additionally, the regularized functions converge towards ψ^+ in $W^{1,2}(\mathbb{R}^d)$, hence in $W^{1,2}$.

Assume now ψ to be an arbitrary, real–valued function from $W_0^{1,2}$. By Definition 2.2 of $W_0^{1,2}$ there is a sequence $\{\psi_r\}_{r=1}^\infty$ from $C_0^\infty(\Omega \cup \Gamma)$ which converges to ψ in the $W_0^{1,2}$ topology. Obviously, the functions ψ_r may be taken real–valued, and without loss of generality, possibly passing to a subsequence, we may assume that this sequence converges pointwise to ψ almost everywhere. It is easy to see that the sequence $\{\psi_r^+\}_{r=1}^\infty$ is bounded in $W^{1,2}$ and converges pointwise almost everywhere to ψ^+. This implies that $\{\psi_r^+\}_{r=1}^\infty$ in fact weakly converges to ψ^+ in $W^{1,2}$. Because we already know that the elements ψ_r^+ belong to $W_0^{1,2}$, ψ^+ also does. □

3.21. Theorem. *If W is a nonnegative L^∞ function, then the semigroup operators $e^{-t(A_2+W)}$, $t > 0$ are positivity preserving.*

Proof. One has to show two things, cf. Liskevich/Semenow [25, Proposition 1.6], namely

i) Each operator $e^{-t(A_2+W)}$, $t>0$ maps real–valued functions from L^2 onto real–valued functions.

ii) The Phillips condition

$$\langle (A_2+W)\psi, \psi^+\rangle \;\geq\; 0 \quad \text{for all } \psi \in \operatorname{dom}(A_2) \cap L^2(\Omega;\mathbb{R}) \qquad (3.23)$$

holds, ψ^+ being the positive part of ψ.

By Theorem 3.16 the operators $e^{-t(A_2+W)}$ are integral operators with real–valued kernel; this implies the first property.

We show (3.23): for real–valued functions ψ from $\operatorname{dom}(A_2) \subset W_0^{1,2}$ the corresponding positive part ψ^+ belongs due to Lemma 3.20 to $W_0^{1,2}$, the form domain of A_2+W. Hence, one obtains by means of (3.21)

$$\begin{aligned}\langle (A_2+W)\psi, \psi^+\rangle &= \int_\Omega \langle a \operatorname{grad}\psi, \operatorname{grad}\psi^+\rangle_{\mathbb{R}^d} + W\psi\,\psi^+\,dx + \int_\Gamma \beta\psi\,\psi^+\,d\sigma \\ &= \int_\Omega \langle a \operatorname{grad}\psi^+, \operatorname{grad}\psi^+\rangle_{\mathbb{R}^d} + W\left(\psi^+\right)^2 dx + \int_\Gamma \beta\left(\psi^+\right)^2 d\sigma \;\geq\; 0.\end{aligned}$$

□

3.22. Remark. From Theorem 3.16 and Theorem 3.21 follows immediately that the operators A_2+W generate Markov semigroups.

3.23. Remark. The integral kernels $\mathcal{K}_t$ belonging to operators $e^{-t(A_2+W)}$, cf. Theorem 3.16, are nonnegative. This follows immediately from the Markov property and the ultracontractivity, cf. Davies [6, Lemma 2.1.2].

3.24. Remark. According to the first Beurling–Deny criterion, cf. Davies [6, Th. 1.3.2], Theorem 3.21 implies that the resolvent of A_2+W is positivity preserving.

3.g $A_p + W$: positivity improvement

It turns out that the semigroup operators $e^{-t(A_2+W)}$, $t>0$ and the resolvents of A_2+W are not only positivity preserving but positivity improving. In view of Reed/Simon [29, vol. IV:XIII.12 Th. XIII.44] we prove

3.25. Lemma. *Let W be a nonnegative L^∞ function, and let A_2 be the operator from Definition 2.6.*

$$\lambda_1 = \min\left\{\langle (A_2+W)\psi, \psi\rangle : \psi \in W_0^{1,2},\ \|\psi\|_{L^2} = 1\right\}$$

is the smallest eigenvalue of $A_2 + W$, and the minimum is attained for a function $\psi_1 \in W_0^{1,2}$, positive within Ω, which solves $(A_2 + W)\psi_1 = \lambda_1 \psi_1$.

If $\psi \in W_0^{1,2}$ is any solution of the equation $(A_2 + W)\psi = \lambda_1\psi$, then ψ is a multiple of ψ_1.

Proof. As $A_2 + W$ is a nonnegative, selfadjoint operator with compact resolvent, cf. Definition 2.7, λ_1 indeed is the smallest eigenvalue of $A_2 + W$. Moreover, one can choose a real valued eigenfunction to λ_1 and according to Theorem 3.14 and Proposition 2.8 any eigenfunction of $A_2 + W$ is Hölder continuous. If $\psi \in W_0^{1,2}$, then

$$(A_2 + W)\psi = \lambda_1\psi \qquad \Longleftrightarrow \qquad \langle (A_2 + W)\psi, \psi\rangle = \lambda_1 \|\psi\|_{L^2}^2.$$

Let us assume

$$(A_2 + W)\psi = \lambda_1\psi, \qquad \|\psi\|_{L^2} = 1, \qquad \psi \in W_0^{1,2}(\Omega; \mathbb{R}).$$

Since $\psi^+, \psi^- \in W_0^{1,2}$ and due to (3.21) we have the relation

$$\langle (A_2 + W)\psi^+, \psi^-\rangle = 0.$$

Accordingly, by the minimizer property of λ_1 we get

$$\begin{aligned} \lambda_1 = \langle (A_2 + W)\psi, \psi\rangle = \langle (A_2 + W)\psi^+, \psi^+\rangle + \langle (A_2 + W)\psi^-, \psi^-\rangle \\ \geq \lambda_1 \|\psi^+\|_{L^2}^2 + \lambda_1 \|\psi^-\|_{L^2}^2 = \lambda_1. \end{aligned}$$

Hence, this inequality must in fact be an equality, and again by the minimizer property it follows

$$\langle (A_2 + W)\psi^+, \psi^+\rangle = \lambda_1 \|\psi^+\|_{L^2}^2 \ \text{ and } \ \langle (A_2 + W)\psi^-, \psi^-\rangle = \lambda_1 \|\psi^-\|_{L^2}^2.$$

But this is equivalent to $(A_2 + W)\psi^+ = \lambda_1\psi^+$ and $(A_2 + W)\psi^- = \lambda_1\psi^-$. Hence, for all nonnegative $\varphi \in C_0^\infty(\Omega)$ there holds

$$\langle (A_2 + W)\psi^+, \varphi\rangle \geq 0 \ \text{ and } \ \langle (A_2 + W)\psi^-, \varphi\rangle \geq 0,$$

in other words, ψ^+, ψ^- are (weak) supersolutions of the equation $(A_2 + W)v = 0$. Now, we can distinguish three cases.

Case $\operatorname*{vraimin}_\Omega \psi^+ > 0$: Then, $\psi = \psi^+$ is positive within Ω.

Case $\operatorname*{vraimin}_\Omega \psi^- > 0$: In the same style $\psi = -\psi^-$ is negative within Ω.

Case $\operatorname*{vraimin}_\Omega \psi^+ = \operatorname*{vraimin}_\Omega \psi^- = 0$: Noticing, that Ω is an open, connected set, we conclude in the following way. Because of the property of ψ^+ and ψ^-

to be supersolutions of the equation $(A_2+W)v=0$ and the nonnegativity of $W\in L^\infty$ we can apply the strong maximum principle, cf. Gilbarg/Trudinger [14, 8.7], to get the validity of the following two alternatives:

$$\psi^+=0 \quad\text{or}\quad \operatorname*{vraimin}_K \psi^+ > \operatorname*{vraimin}_\Omega \psi^+ = 0 \quad\text{for every compact } K\subset\Omega$$

and

$$\psi^-=0 \quad\text{or}\quad \operatorname*{vraimin}_K \psi^- > \operatorname*{vraimin}_\Omega \psi^- = 0 \quad\text{for every compact } K\subset\Omega.$$

Because of $\psi\neq 0$ this is only possible if

$$\psi^+=0 \quad\text{and}\quad \operatorname*{vraimin}_K \psi^- > 0 \quad\text{for every compact subset } K\subset\Omega$$

or

$$\psi^-=0 \quad\text{and}\quad \operatorname*{vraimin}_K \psi^+ > 0 \quad\text{for every compact subset } K\subset\Omega.$$

Because ψ — as an eigenfunction — is from C^α, so are ψ^+ and ψ^-, hence the essential infima of ψ^+ and ψ^- over compact sets K are in fact minima.

Summing up the results of the considered cases we have proved, that if $\psi\in W_0^{1,2}$ is a solution of $(A_2+W)\psi=\lambda_1\psi$, then either

$$\psi \text{ is positive within } \Omega \quad\text{or}\quad \psi=0 \quad\text{or}\quad \psi \text{ is negative within } \Omega. \tag{3.24}$$

Two functions, each satisfying one of the conditions of (3.24), cannot be orthogonal to each other; thus, the eigenspace must be one dimensional. ☐

3.26. Remark. The proof of Lemma 3.25 follows the ideas of Evans [10, 2.6.5.1 Th. 2] for the case of smooth coefficients and homogeneous Dirichlet boundary conditions.

According to [29, vol. IV:XIII.12 Th. XIII.44] the statements of Lemma 2.7, Theorem 3.21, and Lemma 3.25 now imply immediately

3.27. Theorem. *Let W be a nonnegative L^∞ function, and A_2 be the operator from Definition 2.6.*

i) $(A_2+W+\rho)^{-1}$ is positivity improving, hence ergodic, at least for all $\rho > -\operatorname*{vraimin}_\Omega W$.

ii) $e^{-t(A_2+W)}$ is positivity improving, hence ergodic, for all $t>0$.

4 The operators UA_p

Now we turn to the investigation of operators $U \operatorname{div} a \operatorname{grad}$, where U is a positive L^∞ function, bounded from below by a strictly positive constant. These operators generate analytic semigroups on L^p. This property is stable with respect to perturbations by first order differential operators, at least for certain p.

4.a Resolvent estimates

This section is devoted to resolvent estimates for operators $UA_p + W$, where A_p is according to Definition 2.6 and Definition 3.6. W is a nonnegative measurable function, and U is a positive, essentially bounded function, which is bounded from below by a strictly positive constant, i.e. $U^{-1} \in L^\infty$. For the sake of technical simplicity we assume $W \in L^\infty$; this assures that Lemma 3.10 applies. By $W_\bullet$ we denote the essential infimum of W on Ω. In the sequel $\mathbb{H}$ will always be the closed complex right half plane. We abbreviate

$$\tau_p = \frac{a_\bullet}{a^\bullet}\frac{2\sqrt{p-1}}{p-2}, \quad \text{if } p \in \left[\max\left\{4, \tfrac{d+1}{2}\right\}, \infty\right[, \tag{4.1}$$

cf. Theorem 3.3.

4.1. Theorem. *Let U and W be nonnegative, essentially bounded functions. If $U^{-1} \in L^\infty$, then for any $p \in]1, \infty[$ there is a constant M_p such that*

$$\left\| \left(A_p + WU^{-1} + \rho U^{-1}\right)^{-1} \right\|_{\mathcal{B}(L^p;L^p)} \leq \frac{M_p \, \|U\|_{L^\infty}}{|\rho| + W_\bullet} \quad \textit{for all } \rho \in \mathbb{H} \setminus \{0\}.$$

The constant M_p can be specified as follows:

i) If $p = 2$, then $M_2 = \sqrt{2}$.

ii) If $p \in \left[\max\left\{4, \tfrac{d+1}{2}\right\}, \infty\right[$, then $M_p = \dfrac{\sqrt{1+\tau_p^2}}{\min\{1, \tau_p\}}$.

iii) If $p \in \left]2, \max\left\{4, \tfrac{d+1}{2}\right\}\right[$, then $M_p = 2^{\frac{1-\theta_p}{2}} \left(\dfrac{\sqrt{1+\tau^2}}{\min\{1,\tau\}}\right)^{\theta_p}$,

where $\tau = \tau_{\max\left\{4, \frac{d+1}{2}\right\}}$ and θ_p is defined by

$$\frac{1}{p} = \frac{\theta_p}{\max\left\{4, \frac{d+1}{2}\right\}} + \frac{1-\theta_p}{2}. \tag{4.2}$$

iv) If $p \in]1,2[$, *then* $M_p = M_{\frac{p}{p-1}}$.

Proof. We regard firstly the selfadjoint case $p = 2$. Let γ be the lower form bound for $\mathfrak{t}$, which is nonnegative, cf. Lemma 2.5.

$$\begin{aligned}\left\|\left(A_2 + WU^{-1} + \rho U^{-1}\right)\psi\right\|_{L^2}\|\psi\|_{L^2} &\geq \left|\mathfrak{t}_\Omega[\psi,\psi] + \int_\Omega (\rho + W)U^{-1}|\psi|^2\,dx\right| \\ &\geq \left|\gamma + \frac{\rho + W_\bullet}{\|U\|_{L^\infty}}\right| \|\psi\|_{L^2}^2 \geq \frac{|\rho + W_\bullet|}{\|U\|_{L^\infty}}\|\psi\|_{L^2}^2 \geq \frac{|\rho| + W_\bullet}{\sqrt{2}\|U\|_{L^\infty}}\|\psi\|_{L^2}^2, \quad (4.3)\end{aligned}$$

i.e. the operator $A_2 + WU^{-1} + \rho U^{-1}$ is injective. If one can show additionally the surjectivity of $A_2 + WU^{-1} + \rho U^{-1}$, then (4.3) implies the assertion. Indeed, $(W + \rho)U^{-1}$ is a bounded linear operator on L^2, hence it is A_2–bounded with bound equal zero. Thus Lemma 3.10 applies and provides that $A_2 + WU^{-1} + \rho U^{-1}$ has a compact resolvent, hence, it is surjective as it is injective.

Now, let p be from $[\max\left\{4, \frac{d+1}{2}\right\}, \infty[$, $p' = p/(p-1)$, and τ_p according to (4.1). For $\rho \in \mathbb{H} \setminus \{0\}$ we define

$$\hat{\rho} = \begin{cases} 1 + \tau_p i & \text{if } \rho > 0, \\ 1 - \tau_p \operatorname{sign}(\Im\rho) i & \text{if } \rho \in \mathbb{H} \setminus \mathbb{R}. \end{cases} \quad (4.4)$$

If $\psi \in \operatorname{dom}(A_p)$ and $\rho \in \mathbb{H} \setminus \{0\}$, then due to (3.1) and (3.6)

$$\begin{aligned}&\sqrt{1 + \tau_p^2}\,\left\|\left(A_p + (W + \rho)U^{-1}\right)\psi\right\|_{L_p}\|\psi\|_{L^p} \\ &= |\hat{\rho}|\,\left\|\left(A_p + (W + \rho)U^{-1}\right)\psi\right\|_{L_p}\|J_p\psi\|_{L^{p'}} \\ &\geq \left|\hat{\rho}\left\langle\left(A_p + (W + \rho)U^{-1}\right)\psi, J_p\psi\right\rangle\right| \\ &\geq \Re\left(\hat{\rho}\left\langle\left(A_p + (W + \rho)U^{-1}\right)\psi, J_p\psi\right\rangle\right) \\ &= \Re\left(\hat{\rho}\left(\Re\langle A_p\psi, J_p\psi\rangle + i\,\Im\langle A_p\psi, J_p\psi\rangle\right)\right) \\ &\qquad + \Re\left(\hat{\rho}\,\rho\right)\left\langle U^{-1}\psi, J_p\psi\right\rangle + \Re\left(\hat{\rho}\right)\left\langle WU^{-1}\psi, J_p\psi\right\rangle \\ &\geq \Re\langle A_p\psi, J_p\psi\rangle - \tau_p\left|\Im\langle A_p\psi, J_p\psi\rangle\right| + \frac{\Re\rho + \tau_p|\Im\rho| + W_\bullet}{\|U\|_{L^\infty}}\|\psi\|_{L^p}^2 \\ &\geq \min\{1, \tau_p\}\,\frac{|\rho| + W_\bullet}{\|U\|_{L^\infty}}\|\psi\|_{L^p}^2.\end{aligned}$$

Hence,

$$\|\psi\|_{L^p} \le \frac{\|U\|_{L^\infty}}{\big(|\rho| + W_\bullet\big)} \frac{\sqrt{1+\tau_p^2}}{\min\{1,\tau_p\}} \big\| \big(A_p + (W+\rho)U^{-1}\big)\psi \big\|_{L_p}$$

$$\text{for all } \psi \in \operatorname{dom}(A_p) \text{ and all } \rho \in \mathbb{H} \setminus \{0\}. \quad (4.5)$$

Now the assertion follows in the same way as in the case $p = 2$.

If $p \in \left]2, \max\left\{4, \frac{d+1}{2}\right\}\right[$, then interpolation between the cases $p = 2$ and $p = \max\left\{4, \frac{d+1}{2}\right\}$ with the Riesz–Thorin theorem provides the stated result.

If $p \in]1,2[$, then one obtaines the assertion by duality. From the Definition 3.6 of A_p follows

$$A_p + (W+\rho)U^{-1} = \big(A_{\frac{p}{p-1}} + (W+\overline{\rho})U^{-1}\big)^*.$$

This yields, cf. Kato [23, III.§5 Th. 5.30]

$$\big(A_p + (W+\rho)U^{-1}\big)^{-1} = \Big(\big(A_{\frac{p}{p-1}} + (W+\overline{\rho}U^{-1})\big)^{-1}\Big)^*,$$

and then the already proved cases imply the assertion. □

4.2. Theorem. *Let U and W be nonnegative, essentially bounded functions. If $U^{-1} \in L^\infty$, then for any $p \in]1,\infty[$ the operator $UA_p + W$ is closed, has the same domain as A_p, and $-(UA_p + W)$ is the infinitesimal generator of an analytic semigroup on L^p. More precisely, for any $\rho \in \mathbb{H} \setminus \{0\}$ one has*

$$\big\|(UA_p + W + \rho)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \le M_p \|U\|_{L^\infty} \|U^{-1}\|_{L^\infty} \frac{1}{|\rho| + W_\bullet}, \quad (4.6)$$

where the constants M_p are those from Theorem 4.1.

Proof. The operator U is bounded and boundedly invertible on L^p. Thus, the operators $UA_p + W$ and $A_p + U^{-1}W$ have the same domain and are closed simultaneously, i.e. $UA_p + W$ is closed and has the domain $\operatorname{dom}(A_p)$ according to Theorem 3.1 and Theorem 3.7.

Let ρ be in $\mathbb{H} \setminus \{0\}$. According to Theorem 4.1 $A_p + (W+\rho)U^{-1}$ is continuously invertible, hence, $UA_p + W + \rho$ is continuously invertible and

$$\begin{aligned}
\big\|(UA_p + W + \rho)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} &\le \big\|\big(A_p + (W+\rho)U^{-1}\big)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \big\|U^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \\
&= \big\|\big(A_p + (W+\rho)U^{-1}\big)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} \big\|U^{-1}\big\|_{L^\infty}.
\end{aligned}$$

Thus, the asserted inequality (4.6) follows immediately from Theorem 4.1. (4.6) implies that $UA_p + W$ is the infinitesimal generator of an analytic semigroup on L^p, cf. e.g. Pazy [28, 2.5]. □

4.3. Remark. If $\sigma(\partial\Omega \setminus \Gamma) > 0$ or $\int_\Gamma \beta \, d\sigma > 0$, or $W_\bullet > 0$, then the singularity at $\rho = 0$ in (4.6) can be avoided.

4.b Perturbations by first order differential operators

Our next aim is to investigate the influence of perturbations upon an operator UA_p by first order differential operators. Again A_p is according to Definition 2.6 and Definition 3.6, and U is a positive, essentially bounded function on Ω with a strictly positive lower bound.

4.4. Theorem. *Let b_1, b_2, ... , b_d, and c be essentially bounded functions on Ω. We regard the first order differential operator:*

$$T_p : W_0^{1,p} \longrightarrow L^p \qquad\qquad T_p : \psi \longmapsto \sum_{k=1}^{d} b_k \frac{\partial \psi}{\partial x_k} + c\psi. \tag{4.7}$$

Let ϵ be the constant from Proposition 2.13; if $p \in]\frac{2d}{d+2}, 2+\epsilon[$, then

i) dom(UA_p) compactly embeds into dom(T_p) $= W_0^{1,p}$. T_p is relatively bounded with respect to UA_p and the bound is equal zero.

ii) $UA_p + T_p$ has the same domain as A_p, and is closed.

iii) $UA_p + T_p$ generates an analytic semigroup on L^p.

iv) The resolvent of $UA_p + T_p$ is compact.

Proof. Ad i. According to Theorem 4.2 there is $\mathrm{dom}(A_p) = \mathrm{dom}(UA_p)$. Let $\mathcal{M} \subset L^p$ be a set such that $(A_p + 1)\mathcal{M}$ is bounded in L^p. Thus, $(A_p + 1)\mathcal{M}$ is a precompact set in $W^{-1,p}$. If $p \in [2, 2+\epsilon[$, then Proposition 2.13 implies that $\mathcal{M}$ is precompact in $W_0^{1,p}$. If $p \in]\frac{2d}{d+2}, 2[$, then the compactness of the embedding $L^p \hookrightarrow W^{-1,2}$ provides the precompactness of $(A_p+1)\mathcal{M}$ in $W^{-1,2}$. Knowing this, the Lax–Milgram lemma implies the precompactness of $\mathcal{M}$ in $W_0^{1,2}$ and, by embedding, also in $W_0^{1,p}$. Thus, taking into account

$$\left\|T_p\psi\right\|_{L^p} \le \max\left\{\|c\|_{L^\infty}, \|b_1\|_{L^\infty}, \dots \|b_d\|_{L^\infty}\right\} \|\psi\|_{W^{1,p}} \quad \text{for all } \psi \in W_0^{1,p},$$

the first assertion is proved. The second one is implied by Ehrling's lemma, cf. [35, I.§7 Satz 7.3]: as the domain $\mathrm{dom}(A_p)$ of A_p equipped with the graph norm $\|\psi\|_{\mathrm{dom}(A_p)} = \|(A_p + 1)\psi\|_{L^p}$ compactly embeds into $W_0^{1,p}$, for any $p \in]\frac{2d}{d+2}, 2+\epsilon[$ we have

$$\mathrm{dom}(A_p) \xrightarrow[\text{compact}]{\mathbb{1}} W_0^{1,p} \xrightarrow[\text{continuous, injective}]{\mathbb{1}} L^p,$$

and for every $b > 0$ there is an $a > 0$, such that

$$\|\psi\|_{W^{1,p}} \leq a\,\|\psi\|_{L^p} + \frac{b}{\|U^{-1}\|_{L^\infty}}\,\|A_p\psi\|_{L^p} \leq a\,\|\psi\|_{L^p} + b\,\|UA_p\psi\|_{L^p}$$
$$\text{for all } \psi \in \operatorname{dom}(A_p). \quad (4.8)$$

Ad ii and iii. These claims follow from i), thereby observing Theorem 4.2, from abstract perturbation theorems, cf. Kato [23, IV.§1 Th. 1.1 and IX.§2 Th. 2.4] and Pazy [28, 3.2 Th. 2.1].

Ad iv. If a and $b < 1$ are two constants such that (4.8) applies, then according to (4.6) there is for some $\rho > 0$

$$a\big\|(UA_p + \rho)^{-1}\big\|_{\mathcal{B}(L^p;L^p)} + b \leq M_p\,\|U\|_{L^\infty}\|U^{-1}\|_{L^\infty}\,\frac{a}{\rho} + b < 1,$$

and this implies by the theorem on the stability of bounded (and compact) invertibility, cf. Kato [23, IV.§1 Th. 1.16], that the resolvent of $UA_p + T_p$ is compact as UA_p has a compact resolvent, cf. Theorem 4.2. □

4.5. Remark. Theorem 4.4 is primarily relevant in the low dimensional cases $d = 2, 3, 4$ where the permitted interval for p intersects the p–interval where $\operatorname{dom}(A_p)$ continuously embeds into a space C^α, cf. Proposition 2.8. Further, Theorem 4.4 is in correspondence to the results of Arendt and ter Elst [4], which also require restrictions on the first order differential operators.

5 A_2 on fractional Sobolev and Besov spaces

The operator $-A_2$ induces analytic semigroups on L^p spaces, cf. Theorem 3.3 and on certain spaces $W^{-1,q}$, cf. Proposition 2.13 and Gröger/Rehberg [21]. By interpolation it induces analytic semigroups also on fractional Sobolev spaces and on Besov spaces.

5.1. Theorem. *Let ϵ be the constant from Proposition 2.13; if*

$$q \in [2, 2+\epsilon[, \qquad p \in]1,\infty[, \qquad \theta \in]0,1[, \qquad s \in [1,\infty[, \qquad (5.1)$$

then the operator $-A_2$, cf. Definition 2.6, induces an analytic semigroup in any of the interpolation spaces

$$\big(W^{-1,q}, L^p\big)_{\theta,s} \qquad \text{and} \qquad \big[W^{-1,q}, L^p\big]_\theta. \qquad (5.2)$$

Proof. Let A_p be the operators on L^p from Definition 2.6 and Definition 3.6, let B_q be the operators on $W^{-1,q}$ from Proposition 2.13, and let $\mathbb{H}$ be again the closed complex right half plane. The resolvent estimates for the infinitesimal generators in the interpolation spaces, which imply the analyticity of the semigroups, cf. [28, 2.5], result by interpolation due to the following facts:

i) $W^{-1,q}$ and L^p are an interpolation couple, because both embed continuously into $W^{-1,r}$, $r = \min\{p, q\}$.

ii) For any $\rho \in \mathbb{H}$ the operators $(A_p + 1 + \rho)^{-1}$ and $(B_q + \rho)^{-1}$ coincide on $L^{\max\{2,p\}}$. This set is dense in L^p and $W^{-1,q}$; thus $(A_p + 1 + \rho)^{-1}$ and $(B_q + \rho)^{-1}$ may be viewed as the same operator.

iii) Real and complex interpolation are exact interpolation functors of type θ, cf. e.g. Triebel [32, 1.2.2].

It remains to show that the domain of the operator on the corresponding interpolation space is dense in this space: one knows, cf. [32, 1.6.2 and 1.9.3], that $W^{-1,q} \cap L^p$ is dense in $\left(W^{-1,q}, L^p\right)_{\theta,s}$ and in $\left[W^{-1,q}, L^p\right]_\theta$. (Because this is not necessarily true for the (real) interpolation index (θ, ∞), cf. [32, Rem. 1.6.2] one has to exclude this index in the assertion.) Further, the norm

$$\max\left\{\|\psi\|_{L^p},\ \|\psi\|_{W^{-1,q}}\right\} \tag{5.3}$$

on $W^{-1,q} \cap L^p$ is stronger than the induced norm from any of the interpolation spaces, cf. Triebel [32, 1.3.3 and 1.9.3].

Let now $p_0 \geq \max\{2, p\}$ be chosen, such that L^{p_0} continuously embeds into L^p and $W^{-1,q}$. Clearly, then one has $\mathrm{dom}(A_{p_0}) \subset \mathrm{dom}(A_p|_{W^{-1,q} \cap L^p})$ and the images under the embedding mappings $L^{p_0} \hookrightarrow W^{-1,p}$ and $L^{p_0} \hookrightarrow L^p$ are dense in both spaces. By Theorem 3.5 $\mathrm{dom}(A_{p_0})$ is dense in L^{p_0} and, consequently, is dense in $W^{-1,q} \cap L^p$ in the norm (5.3). $\square$

5.2. Remark. According to duality theorems from interpolation theory, cf. Triebel [32, 1.11], there is

$$\begin{aligned} \left(W^{-1,q}, L^p\right)_{\theta,s} &= \left(\left(W_0^{1,q'}, L^{p'}\right)_{\theta,s'}\right)^* \qquad \text{for } s \in]1, \infty[, \\ \left[W^{-1,q}, L^p\right]_\theta &= \left(\left[W_0^{1,q'}, L^{p'}\right]_\theta\right)^*, \end{aligned}$$

where $p' = p/(1+p)$, $q' = q/(1+q)$, $s' = s/(1+s)$.

6 Applications to parabolic equations

The property of $-UA_p$ to be an infinitesimal generator of an analytic semigroup on L^p paves the way for treating the corresponding parabolic equations on L^p, cf. Amann [3], Lunardi [26], Pazy [28]. With respect to the results obtained by Arendt and ter Elst [4] the basic finding in our context is the Hölder continuity of solutions to the parabolic equation in space and time, which rests on Proposition 2.8. Before we expatiate upon this, we prove a preparatory technical lemma:

6.1. Lemma. *Suppose $p > \frac{d}{2}$ and $p \geq 2$. Let A_p be an operator according to Definition 2.6, and let $U \in L^\infty$ be a positive function with a strictly positive lower bound. Further, let α be the Hölder exponent from Proposition 2.8. There are numbers $\Theta \in]0,1[$ and $0 < \tilde{\alpha} < \alpha$, such that*

$$\operatorname{dom}\big((UA_p + 1)^\Theta\big) \hookrightarrow C^{\tilde{\alpha}}. \tag{6.1}$$

Any function $u \in C^1\big(S; L^p\big) \cap C\big(S; \operatorname{dom}(UA_p)\big)$, where $S = [T_0, T]$ is an interval of the real (time) axis, is Hölder continuous in space and time, more precisely $u \in C^{1-\Theta}\big(S; C^{\tilde{\alpha}}\big) \hookrightarrow C^\beta(S \times \overline{\Omega})$.

Proof. Let Θ be in $]0,1[$. If $\theta < \Theta$, then the embeddings

$$\operatorname{dom}\big((UA_p + 1)^\Theta\big) \hookrightarrow \big(\operatorname{dom}(UA_p + 1), L^p\big)_{1-\Theta,\infty} \hookrightarrow \big(\operatorname{dom}(UA_p + 1), L^p\big)_{1-\theta,1}$$

are continuous, cf. Triebel [32, 1.15.2 and 1.3.3]. The chain of continuous embeddings may be continued by applying Proposition 2.8 and [32, 1.10.3]

$$\begin{aligned}\big(\operatorname{dom}(UA_p + 1), L^p\big)_{1-\theta,1} = \big(\operatorname{dom}(A_p), L^p\big)_{1-\theta,1} \\ \hookrightarrow \big(C^\alpha, L^p\big)_{1-\theta,1} \hookrightarrow \big[C^\alpha, L^p\big]_{1-\theta}.\end{aligned}$$

Thus, it remains to show that

$$\big[C^\alpha, L^p\big]_{1-\theta} = \big[L^p, C^\alpha\big]_\theta \hookrightarrow C^{\tilde{\alpha}}, \qquad \text{for some } \tilde{\alpha} > 0.. \tag{6.2}$$

By means of the localization, transformation and reflection principles, cf. e.g. [16, 1.1], one can construct a simultaneous extension operator for

$$C^\alpha \longrightarrow C^\alpha(\mathbb{R}^d) \qquad \text{and} \qquad L^p \longrightarrow L^p(\mathbb{R}^d),$$

cf. [34, 1.2.2 Th. 1.2]. Hence, it is sufficient to prove

$$\big[L^p(\mathbb{R}^d), C^\alpha(\mathbb{R}^d)\big]_\theta \hookrightarrow C^{\tilde{\alpha}}(\mathbb{R}^d),$$

cf. e.g. [32, 1.2.4] instead of (6.2). According to [33], [31, VI.2.2] the space $C^\alpha(\mathbb{R}^d)$ is identical with the Besov space $B^\alpha_{\infty,\infty}(\mathbb{R}^d)$. Further, the interpolation space with L^p coincides with a Lizorkin–Triebel space, and continuously embeds into a Hölder space. More precisely,

$$\Big[L^p(\mathbb{R}^d), B^\alpha_{\infty,\infty}(\mathbb{R}^d)\Big]_\theta = F^{\theta\alpha}_{\frac{p}{1-\theta},\frac{2}{1-\theta}}(\mathbb{R}^d) \hookrightarrow C^{\tilde\alpha}(\mathbb{R}^d),$$

$$\text{if } \tilde\alpha \overset{\text{def}}{=} \theta\alpha - (1-\theta)\frac{d}{p} > 0,$$

cf. [33] and [32, 2.8.1] respectively. By choosing θ and Θ sufficiently close to 1 one always finds a strictly positive $\tilde\alpha$.

Let now u be from the space $C^1\big(S; L^p\big) \cap C\big(S; \operatorname{dom}(UA_p)\big)$ and let s, t be different numbers from the interval S. We have by the first statement of this lemma

$$\frac{\|u(s)-u(t)\|_{C^{\tilde\alpha}}}{|s-t|^{1-\Theta}} \le \|\mathbb{1}\|_{\mathcal{B}(\operatorname{dom}((UA_p+1)^\Theta);C^{\tilde\alpha})} \frac{\big\|(UA_p+1)^\Theta\big(u(s)-u(t)\big)\big\|_{L^p}}{|s-t|^{1-\Theta}}.$$

There is a constant γ, such that this inequality may be prolonged, cf. Pazy [28, 2 Th. 6.10], as follows

$$\le \gamma \,\big\|(UA_p+1)\big(u(s)-u(t)\big)\big\|^\Theta_{L_p} \left(\frac{\|u(s)-u(t)\|_{L^p}}{|s-t|}\right)^{1-\Theta}.$$

Due to the supposition on u, the expression on the right hand side is uniformly bounded for all $s \ne t \in S$. □

We will now draw some conclusions for parabolic equations, starting with the linear case.

Let us regard an initial–boundary value problem

$$\frac{\partial u}{\partial t} - U \operatorname{div} a \operatorname{grad} u = f, \quad u(0) = u_0, \quad \text{and boundary conditions.}$$

If we regard this equation in L^p, then $-U \operatorname{div} a \operatorname{grad}$ gets the precise meaning of the operator UA_p, cf. §4, and the fully elaborated existence, uniqueness and regularity theory for parabolic equations

$$\frac{\partial u}{\partial t} + UA_p u = f, \qquad u(0) = u_0. \tag{6.3}$$

related to the infinitesimal generator of an analytic semigroup applies, cf. Amann [3], Lunardi [26], Pazy [28].

For convenience, we formulate the new and essential fact for operators with mixed boundary conditions as

6.2. Theorem. *Suppose $p > \frac{d}{2}$, $p \geq 2$, and $T > 0$. Let A_p be an operator according to Definition 2.6, and $U \in L^\infty$ be a positive function with a strictly positive lower bound. If the right hand side f of (6.3) is Hölder continuous as a mapping from $[0,T]$ into L^p, then for any $T_0 \in]0,T[$ the solution of (6.3) is Hölder continuous on the set $[T_0,T] \times \overline{\Omega}$. If, in addition, the initial value u_0 is from $\operatorname{dom}(UA_p) = \operatorname{dom}(A_p)$, then the solution is Hölder continuous on $[0,T] \times \overline{\Omega}$.*

Proof. The proof results from Lemma 6.1 and classical regularity results, cf. Pazy [28, 4.3]. □

6.3. Remark. The Hölder continuity of solutions of (6.3) on $[0,T] \times \overline{\Omega}$ also has been obtained by Griepentrog [16, 2.3] in a completely different way.

If $U \equiv 1$, then the suppositions on the right hand side may be considerably relaxed:

6.4. Theorem. *Let A_p be an operator according to Definition 2.6, W be a nonnegative essentially bounded function, and $T > 0$. Then for any $q \in]1,\infty[$ the operator $A_p + W$ satisfies q–regularity; in other words: for any $q \in]1,\infty[$ the operator $\frac{\partial}{\partial t} + A_p + W$ provides a topological isomorphism between*

$$L^q\big([0,T];\operatorname{dom}(A_p)\big) \cap \big\{v \in W^{1,q}\big([0,T];L^p\big) \,:\, v(0,\cdot) \equiv 0\big\} \quad \textit{and} \quad L^q\big([0,T];L^p\big).$$

Moreover, if $p > \frac{d}{2}$, $p \geq 2$, and f is from $L^\infty\big([0,T];L^p\big)$, then the solution u of the initial value problem

$$\frac{\partial u}{\partial t} + (A_p + W)u \,=\, f, \qquad u(0) = u_0 \tag{6.4}$$

is Hölder continuous in space and time.

Proof. The first statement follows from the positivity of the operator $A_2 + W$, Theorem 3.17 and a result of Lamberton, cf. [24]. Let α be the Hölder exponent from Proposition 2.8. From the trace method in interpolation theory, cf. Ashyralyev and Sobolevskii [5, 1.3] or Triebel [32, 1.8.2], follows for any $q \in]1,\infty[$ the existence of a continuous embedding

$$\begin{aligned} L^q\big([0,T];\operatorname{dom}(A_p)\big) \cap W^{1,q}\big([0,T];L^p\big) &\hookrightarrow C\big(S;(\operatorname{dom}(A_p),L^p)_{\frac{1}{q},q}\big) \\ &\hookrightarrow C\big(S;(C^\alpha,L^p)_{\frac{1}{q},q}\big). \end{aligned} \tag{6.5}$$

We choose q great enough, so that $\big(C^\alpha,L^p\big)_{\frac{1}{q},q}$ continuously embeds into a space C^β with some $\beta > 0$, and η small enough, so that we still have an embedding $\left(C^\beta, L^p\right)_{\eta,s} \hookrightarrow C^\gamma$ for some $\gamma > 0$; both is possible by Lemma 6.1. Defining Θ by

$$\Theta = \eta + \tfrac{1-\eta}{q}, \tag{6.6}$$

we have by the reiteration theorem for real interpolation

$$\left(C^\alpha, L^p\right)_{\Theta,s} = \left(\left(C^\alpha, L^p\right)_{\frac{1}{q},q}, L^p\right)_{\eta,s}$$

and by the suppositions on q, η the continuity of the embedding

$$\left(\left(C^\alpha, L^p\right)_{\frac{1}{q},q}, L^p\right)_{\eta,s} \hookrightarrow \left(C^\beta, L^p\right)_{\eta,s} \hookrightarrow C^\gamma,$$

cf. Triebel [32, 1.10.3]. Using the corresponding interpolation inequality, one can estimate for any $t_1, t_2 \in [0,T]$:

$$\begin{aligned}
\frac{\|u(t_1)-u(t_2)\|_{(C^\alpha,L^p)_{\Theta,s}}}{|t_1-t_2|^{\Theta-\frac{1}{q}}} &\le \delta\, \frac{\|u(t_1)-u(t_2)\|_{L^p}^{\eta}}{|t_1-t_2|^{\Theta-\frac{1}{q}}}\, \|u(t_1)-u(t_2)\|^{1-\eta}_{(C^\alpha,L^p)_{\frac{1}{q},q}} \\
\le\ \delta\, \frac{\left\| \int_{t_1}^{t_2} u'(\tau)\, d\tau \right\|_{L^p}^{\eta}}{|t_1-t_2|^{\Theta-\frac{1}{q}}} &\left(2 \sup_{\tau\in[0,T]} \|u(\tau)\|_{(C^\alpha,L^p)_{\frac{1}{q},q}}\right)^{1-\eta} \\
\le\ \delta\, \frac{\left(\int_{t_1}^{t_2} \|u'(\tau)\|_{L^p}^q\, d\tau\right)^{\frac{\eta}{q}} |t_1-t_2|^{\frac{\eta}{q'}}}{|t_1-t_2|^{\Theta-\frac{1}{q}}} &\left(2 \sup_{\tau\in[0,T]} \|u(\tau)\|_{(C^\alpha,L^p)_{\frac{1}{q},q}}\right)^{1-\eta}.
\end{aligned}$$

By the definition (6.6) of Θ we have $\frac{\eta}{q'} = \Theta - \frac{1}{q}$, what proves the boundedness of the right hand side, independently from $t_1, t_2 \in [0,T]$. □

Next we will regard the semilinear case.

6.5. Theorem. *Let $\mathcal{F} : [0,T] \times \mathbb{C} \longrightarrow \mathbb{C}$ be a function which is Hölder continuous in the first argument and locally Lipschitz continuous in the second. (For $t \in [0,T]$ we identify the function $\mathcal{F}(t,\cdot)$ with the induced Nemytzkij operator on L^∞.) We assume the existence of a uniform Hölder exponent for every bounded set of $z \in \mathbb{C}$, and that the local Lipschitz constants may be taken uniform over $[0,T]$. Suppose $p > \frac{d}{2}$ and $p \ge 2$. Let A_p be an operator according to Definition 2.6 and $\Theta \in]0,1[$ be a number such that* dom $\left((A_p+1)^\Theta\right) \hookrightarrow C^{\tilde\alpha}$ *for some $\tilde\alpha > 0$. (Such numbers Θ and $\tilde\alpha$ exist according to Lemma 6.1.) Then the equation*

$$\frac{\partial u}{\partial t} + U A_p u = \mathcal{F}(t,u), \qquad u(0) = u_0 \in \operatorname{dom}((A_p+1)^\Theta) \tag{6.7}$$

has a unique local solution

$$u \in C\big([0,T_1[;L^p\big) \cap C^1\big(]0,T_1[;L^p\big) \cap C\big(]0,T_1[;\mathrm{dom}(A_p)\big),$$

which, by Lemma 6.1, is Hölder continuous in space and time on any set $[T_0,T_2]\times\overline{\Omega}$, $0<T_0<T_2<T_1$.

Proof. The local existence, uniqueness and asserted regularity follow from standard results, cf. Pazy [28, 6.3 Th. 3.1 and 4.3 Th. 3.5], provided one can prove that

$$[0,T]\times\mathrm{dom}((A_p+1)^\Theta) \ni (t,\psi) \longmapsto \mathcal{F}(t,\psi) \in L^\infty \hookrightarrow L^p$$

is Hölder continuous in the first variable and Lipschitzian in the second. But this follows immediately from our supposition $\mathrm{dom}\big((A_p+1)^\Theta\big) \hookrightarrow C^{\tilde{\alpha}}$ and the suppositions on $\mathcal{F}$. □

6.6. Remark. Much more could be said about fine properties of solutions of (6.3), (6.4) and (6.7) in dependence of the initial values u_0 and $\mathcal{F}(0,u_0)$, respectively; for particulars we refer to Lunardi [26]. We do not expatiate this here because in our highly nonsmooth constellation it is impossible in general to determine $\mathrm{dom}(A_p)$ or $\mathrm{dom}\big((A_P+1)^\Theta\big)$ explicitely, or to say how regular $\mathcal{F}(0,u_0)$ is.

As mentioned in the introduction, we are interested primarily in reaction–diffusion equations, especially in semiconductor equations. This requires a solution theory for coupled evolution equations, where, among others, the following two problems, cf. e.g. Pazy [28, 5.6], arise:

6.7. Problem. Under what conditions on two $L^\infty(\Omega;\mathcal{B}(\mathbb{R}^d,\mathbb{R}^d))$ functions a and $\tilde{a}$ with strictly positive lower bounds $a_\bullet$ and $\tilde{a}_\bullet$ the domains of the corresponding operators A_p and $\tilde{A}_p$ coincide? Do, at least the domains of fractional powers of A_p and $\tilde{A}_p$ coincide?

6.8. Problem. Let $t \longmapsto a_t$ be a function from $[0,T]$ into $L^\infty(\Omega;\mathcal{B}(\mathbb{R}^d,\mathbb{R}^d))$ and let $A_{p,t}$ be the operator corresponding to a_t, according to Definition 2.6. What can be said about Hölder continuity, in an appropriate sense, of the function $t \longmapsto A_{p,t}$, cf. e.g. Pazy [28, 5.6].

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