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Self normal numbers

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1. Introduction

This paper is a continuation of [1] where we considered the link between expansions to a real base β and ergodic properties of the corresponding β–shift.
We found there a certain gap in the hierarchy of the sizes in the classification proposed by F. Blanchard. The aim of this paper is to fill in this gap. We will follow the notations in [1].

2. Statement of the main theorem

In [1] we have shown that the set C_5 of β–numbers whose orbit of 1 is dense in $[0,1)$ has full Lebesque measure and is residual. Therefore its complement C_4 has Lebesque measure zero and is meager. Here we want to define a set $\tilde{C}_5, C_4 \subset \tilde{C}_5 \subset C_5$, which still has full Lebesque measure and is meager.
Let $\pi : [0,1] \longrightarrow S_\beta$ denote the map assigning to each $x \in [0,1]$ its β expansion $\{i_n(x,\beta)\}$ and S_β the closure of the image of π (For definitions see [1].).

Definition 2.1. We say a real number $\beta \in (1,\infty)$ is self normal iff β belongs to the set

$$\tilde{C}_5 = \{\beta \in (1,\infty) \mid \begin{array}{l}\text{The orbit of 1 under } T_\beta \text{ is normally distributed} \\ \text{in } [0,1) \text{ w.r.t. } \nu_\beta\}\,.\end{array}$$

For an allowed word $B = [i_1, \ldots, i_m]$ and an β–expansion $\underline{x}$ of a real number in $[0,1)$ let $f_\ell(\underline{x}, B)$ denote the relative frequency

$$f_\ell(\underline{x}, B) = \frac{\#\{k \in [1,\ldots,\ell] | x_k = i_1 \ldots x_{k+m} = i_m\}}{\ell}$$

of the word B in $\underline{x}$. Then

$$\tilde{C}_5 = \{\beta \in (1,\infty) | \begin{array}{l}\lim_{\ell\to\infty} f_\ell(\{i_n(1,\beta)\}, B) \text{ exists for all allowed words } B \\ \text{and equals } \nu_\beta(\pi^{-1}B)\}\,.\end{array}$$

Now we can state the main

Theorem 2.1. *(1) $\tilde{C}_5$ has full Lebesque measure in $(1,\infty)$.*
(2) $\tilde{C}_5$ is meager.

3. Ergodic properties of shift spaces

Most of the results in this section are contained in [2]. Let $\sum$ be a shift space – that is a set of 1–sided infinite sequences over the alphabet $\{0,\dots,m\}$ which is invariant under the left–shift σ.

Definition 3.1. For $\underline{x} \in \sum$ we denote by $V_T(\underline{x})$ the set of accumulation points of

$$\left\{\frac{1}{N}\sum_{i=0}^{N-1}\mu\cdot\sigma^{-i}\right\}_{N=1}^{\infty}$$

Definition 3.2. For $\underline{x} \in \sum$ we define the lower entropy of $\underline{x}$ as

$$h(\underline{x}) = \inf\{h_\mu | \mu \in V_T(\underline{x})\}$$

and for $\beta \in (1,\infty)$

$$h(\beta) = h\left(\{i_n(1,\beta)\}\right).$$

Within these notations standard facts for shift spaces give:

Proposition 3.1. $h(\beta) = \inf\left\{-\frac{1}{N}\sum f_N(B) ln f_N(B)\right\}$
where the sum is taken over all allowed words of length N and
$f_N(B) = f_N(\{i_n(1,\beta)\},B)$.

4. Subshifts of finite type

We begin by recalling some facts about subshifts of fintite type (s.f.t.). A quite rich expository is contained in [2].
Let Λ be an irreducible s.f.t. that is given an alphabet $\{0,\dots,m-1\}$ and an irreducible and aperiodic matrix $T = (t_{ij})_{m\times m}$ whose entries are either 0 or 1 then Λ is the set

$$\Lambda = \left\{\underline{x} = [x_1\dots] \in \{0,\dots,m-1\}^{\mathbb{N}} \middle| t_{x_i x_{i+1}} = 1\right\}$$

Let σ be the shift to the left. A Markov measure is given by a stochastic matrix

$$P = (p_{ij})_{m\times m} \text{ with } p_{ij} > 0 \Leftrightarrow t_{ij} = 1.$$

Let $(p_1,\dots,p_m)$ be the only eigenvector with eigenvalue 1. Then the measure of a cylinder $[x_1,\dots,x_n]$ is defined by

$$\mu_P([x_1,\dots,x_n]) = p_{x_1}p_{x_1x_2}\cdots p_{x_{n-1}x_n}.$$

This measure is shift–invariant and mixing.
It is well known that under these conditions the limit

$$\bar{P} = (\bar{p}_{ij}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P^n \qquad \text{exists}$$

and moreover

$$\overline{p_{ij}} = p_j.$$

Then these p_j will be called the stationary distributions. Also there exists the stationary distribution of any block $\overline{\mu}(B) = \mu_{\overline{P}}(B)$. Let $\{B_i\}$ be the blocks of length m. We omit the proof of the following easy lemma:

Lemma 4.1. *For $\underline{x} \in \Lambda$ and $\varepsilon > 0$ there is a $m \in \mathbb{N}$ and a Markov measure μ s.t.*

(1) $\lim_{k\to\infty} f_{n_k}(\underline{x}, B_i) - \varepsilon \leq \mu(B_i) \leq \lim_{k\to\infty} f_{n_k}(\underline{x}, B_i) + \varepsilon$

(2) $\lim_{k\to\infty} \frac{1}{n_k} \ln \mu(C_{n_k}(\underline{x})) \leq \frac{1}{m} \ln \prod \mu(B_i)^{\mu(B_i)} + \varepsilon$

with $C_n(\underline{x}) = \{\underline{y} \in \Lambda | x_1 = y_1 \dots x_n = y_n\}$ the n–cylinder of $\underline{x}$ in Λ and $\{n_k\}$ a subsequence of the naturals such that the limits of the relative frequencies exist.

A s.f.t. has topological entropy $\log \lambda$ where λ is the largest positive eigenvalue of the tran–sition matrix T. There exists a unique measure of maximal entropy $\mu^{(\lambda)}$. This measure is Markov and has the property , that for each cylinder C of length n

$$K^{-1}\lambda^{-n} \leq \mu^{(\lambda)}(C) \leq K\lambda^n, \quad K > 0.$$

The eigenvector of the eigenvalue λ for the corresponding stochastic matrix $P^{(\lambda)}$ we denote by $(p_1^{(\lambda)}, \dots, p_n^{(\lambda)})$.

5. Dimensions like values

We introduce a dimension–like value $\dim_e$ on the s.f.t. Λ (e from entropy).
For $X \subset \Lambda$, $s \in \mathbb{R}^+$ we define in Hausdorff measure style s–dimensional e–measure by

$$e_s(X) = \lim_{N\to\infty} \inf_{\mathcal{C}_N} \sum_{C\in\mathcal{C}_N} \lambda^{-sn(c)}$$

where the infimum is taken over all covers $\mathcal{C}_N$ of X consisting of cylinders C with length $n(c) \geq N$. One easily can check that there is a unique $s_0 = \dim_e X$ being the border between infinite and zero e_s–measure of X.

In [3] Ya. Pesin introduced the following notions of topological entropy: for a shift space $\sum, s \in \mathbb{R}^+, Y \subset \sum$ we set

$$\overline{m}_s(Y) = \lim_{N\to\infty} \inf_{\mathcal{C}_N} \sum_{C\in\mathcal{C}_N} e^{-sn(c)}$$

and

$$\widetilde{m}_s(Y) = \lim_{N\to\infty} \inf_{\mathcal{C}_N^0} \sum_{c\in\mathcal{C}_N^0} e^{-sN}$$

where the infimum in $\overline{m}_s$ is taken over all covers $\mathcal{C}_N$ of Y consisting of cylinders of length $n(C) \geq N$ and in $\widetilde{m}_s$ over covers consisting of cylinders of length N. The two numbers $\overline{h}$ and $\widetilde{h}$ defining the transition from ∞ to 0 for $\overline{m}_s$ and $\widetilde{m}_s$ respectively are the Pesin entropy and the Pesin entropy–capacity.
He proved that for compact and invariant sets these two notions coincide with the usual topological entropy h introduced by Bowen.

Lemma 5.1. *Let β be a simple β–number – i.e. $\{i_n(1,\beta)\}$ is finite – and*

$$\widetilde{M}_N(\beta) = \left\{ \underline{x} \in s_\beta \left| \frac{h(\underline{x})}{\log\beta} < 1 - \frac{1}{N} \right.\right\}.$$

Then

$$\dim_e \widetilde{M}_N(\beta) \leq 1 - \frac{1}{N}.$$

Proof. We fix now the simple β–number β. It is well known that the topological entropy of S_β is $ln\ \beta$.
Let $\{\mu_n\}$ be a countable and dense – in the set of all Markov measures – set of Markov measures.
According to lemma 4.1. if $\underline{x} \in \widetilde{M}_N(\beta)$ then there is a measure μ_n with stationary distributions $\{\overline{\mu}_n(B_1), \ldots, \overline{\mu}_n(B_\ell)\}$ arbitrary close to the set $\{\ell im f_r(\underline{x}, B_i)\}_{i=1}^{\ell}$ of accumulation points of the sequences
$\{f_x(\underline{x}, B_i)\}$. Let us furthermore assume that supp $\mu_n = S_\beta$. Suppose that for a large enough m $\{B_i\}$ are the allowed words of length m. Then the following inequalities are fulfilled for a suitable choice of a subsequence r_k

$$\lim_{k\to\infty} -\tfrac{1}{r_k} ln\overline{\mu}_n\left(C_{r_k}(\underline{x})\right) = -\tfrac{1}{m} ln \prod \overline{\mu}_n(B_i)^{\overline{\mu}_n(B_i)} + \tfrac{\varepsilon}{2} =$$

$$= -\tfrac{1}{m} \sum \overline{\mu}_n(B_i) ln\overline{\mu}_n(B_i) + \tfrac{\varepsilon}{2} < h_{\overline{\mu}_n} + \varepsilon < \left(1 - \tfrac{1}{N}\right) ln\beta$$

If we consider the sets

$$\Delta_n = \left\{ \underline{x} \in S_\beta | \liminf_{m\to\infty} -\frac{1}{m} ln\overline{\mu}_n\left(C_m(\underline{x})\right) < \left(1 - \frac{1}{N}\right) ln\beta \right\}$$

then $\bigcup_{n=1}^{\infty} \Delta_n \supset \widetilde{M}_N(\beta)$.
Let $B_k = \{C_m(\underline{x}) | \underline{x} \in \widetilde{M}_N(\beta), \overline{\mu}_n(C_m(\underline{x})) > e^{-m(1-\frac{1}{N})ln\,\beta}\ , m \geq k\}$.
By the preceeding arguments B_k is a covering of $\widetilde{M}_N(\beta)$. Therefore we can proceed

$$1 \geq \lim_{k\to\infty} \inf_{B'\subset B_k} \sum_{C_m(\underline{x})\in B'} \overline{\mu}_n(C_m(\underline{x})) >$$

$$> \lim_{k\to\infty} \inf_{B'\subset B_k} \sum_{C_m(\underline{x})\in B'} e^{-m(1-\frac{1}{N})ln\beta} =$$

$$= \lim_{k\to\infty} \inf_{B'\subset B_k} \sum_{C_m(\underline{x})\in B'} \beta^{-(1-\frac{1}{N})m}$$

where the infimum is taken over all subcovers B' of B_k. But this means

$$\dim_e \widetilde{M}_N(\beta) < 1 - \frac{1}{N}.$$

□

6. Auxilary Lemma

For the prove of the main theorem the following lemma is useful:

Lemma 6.1. *Let β be a simple β–number. Then for $m \in \mathbb{N}$ the set $\Omega_m = \{x \in [0,1)|$ for some allowed word B of length m the relative frequencies of B in $\{i_m(x,\beta)\}$ do not converge to $\nu(B)\}$ where ν is the unique measure of maximal entropy is the union of sets M_k with $\dim_H M_k \leq 1 - \frac{1}{k}$.*

Proof. Let m be fixed. It is well known that for simple β–numbers the shift S_β is of finite type. By changing if neccessary the alphabet $\{0,\dots,[\beta]\}$ we obtain a s.f.t. Λ_m conjugate to S_β. The measure ν is the projection of the measure $\mu^{(\lambda)}$ on Λ_m under $\pi_m : \Lambda_m \to [0,1)$ the superposition of the conjugation of Λ_m and S_β and the map $\pi : S_\beta \to [0,1)$.
We consider the set A of all numbers $x \in [0,1)$ having the same limits of the relative frequencies of blocks or length m in their coding as correspond to the measure ν (such numbers are often called simple m–normal).
Since the measure ν has a bounded non–vanishing density [4]:

$$C\text{diam}C_n(x) \leq \nu(C_n(x)) \leq C^{-1}\text{diam}C_n(x)$$

for some $C > 0$ and $C_n(x) = \{y \in [0,1)|i_j(x,\beta) = i_j(y,\beta) \;\; j = 1,\dots,n\}$ and the length of the cylinder $C_n(x)$ is bounded inbetween

$$D^{-1}\lambda^{-n} \leq \text{diam}C_n(x) \leq D\lambda^{-n} \;\; (D > 0).$$

(see for instance [5])
the map π^{-1} is Lipschitz and consequently the set–functions $\dim_e(E)$ and $\dim_H(\pi^{-1}E)$ coincide. Hence, we can apply lemma 5.1.
Then the sets $\pi^{-1}\widetilde{M}_k = M_k$ fit the assertion of the lemma. □

7. Proof of the main theorem

The proof of the theorem consists of several steps. First we subdivide $(1,\infty)$ into a countable union of closed intervals $\{I_n\}_{n=-\infty}^{\infty}$ to obtain uniform bounds on these compact sets I_n. Let $I = I_n = [\underline{\beta}, \overline{\beta}]$ $N, m \in \mathbb{N}$ be fixed in the further.
We will need a couple of observations.

(1) Suppose $\beta_0 \in (\underline{\beta}, \overline{\beta}]$. Then all expansions of 1 for $\underline{\beta} < \beta < \beta_0$ are at the same time expansions for some $x \in [0,1)$ in base β_0. We denote the set of such x by 1_{β_0}. If $\pi_0 : [0,1) \to S_{\beta_0}$ is the map assigning to each $x \in [0,1)$ its β_0–expansion we define a map $\rho_0 : 1_{\beta_0} \xrightarrow{\text{onto}} [\underline{\beta}, \beta_0]$ by $\rho_0(x)$ is the unique β having $\pi_0(x)$ as its expansion of 1.
A well–known fact is ([3],[4]) that if β_0 is a simple β–number S_{β_0} is of finite type and the Parry measure ν_0 is homogeneous in the sense of Bowen:

$$K^{-1}\beta_0^{-n} \leq \nu_0(\pi_0^{-1}C_n) \leq K\beta_0^n$$

for some $K > 0$ and any cylinder set C_n of length n.
From the explicit form of the density

$$h_0(x) = \sum_{x < T_{\beta_0}^m(1)} \frac{1}{\beta^m}$$

of the Parry measure we can derive that for $C = \frac{\beta}{\underline{\beta}-1}$ and any cylinder set C_n

$$C^{-1}\nu_0(\pi_0^{-1}(C_n)) \leq \text{ diam } (\pi_0^{-1}(C_n)) \leq C\nu_0(\pi_0^{-1}(C_n))$$

holds. Combining these inequalities we get:

$$\text{diam } (\pi_0^{-1}(C_n)) \geq C^{-1}K^{-1}\beta_0^{-n}.$$

On the other hand the maximal distance between β_1 and β_2 in I_n having both their expansion of 1 in the same cylinder set C_n is at most $\frac{1}{\beta_2^{n-1}}$ (See lemma 4.3. in [1]) which not depends on $\beta_0 \in (\underline{\beta}, \overline{\beta}]$. Therefore

$$|\rho(x) - \rho(y)| \leq C \cdot K \cdot L\, |x - y|^{\frac{\log \beta_1}{\log \beta_0}}$$

with $\beta_1 = \min(\rho(x), \rho(y))$ and $L > 0$.
Hence, there exist a δ depending on N, $\underline{\beta}$ and $\overline{\beta}$ but not on β_0 such that the restriction of ρ_0 to the set $\rho_0^{-1}(\beta_0 - \delta_1, \beta_0)$ is Hölder continuous with exponent $1 - \frac{1}{2N}$.

(2) For a simple β–number β_s let $M_N(\beta_s)$ be the set considered in the auxilary lemma 6.1. for the s.f.t. S_{β_s}. We write $\Xi_N(\beta_s)$ for the set

$$[\beta_s - \delta, \beta_s] \cap \rho(M_N(\beta_s)).$$

From observation 1. we know

$$\dim_H \Xi_N(\beta_s) \le \dim_H \rho(M_N(\beta_s)) \le \frac{\dim_H M_N(\beta_s)}{1-\frac{1}{2N}} < 1 - \frac{1}{2N-1}.$$

(3) There exists only a countable number of simple β–numbers in I_n. Therefore the set

$$\Xi_N = \bigcup_{\substack{\beta_s-\text{simple}\\ \beta_s \in I_n}} \Xi_N(\beta_s) =$$

$$= \{\beta \in I_n \,|\, \exists \text{ simple } \beta_s \ge \beta \ge \beta_s - \delta \text{ s.t. } \rho(M_N(\beta_s)) \ni \beta\}$$

has Hausdorff dimension at most $1-\frac{1}{2N-1}$ and hence zero Lebesque measure.

(4) Finally we define

$$\Xi = \bigcup_{N=1}^{\infty} \Xi_N \quad =$$

$$= \{\beta \in I_N \,|\, \exists\, N \in \mathbb{N} \text{ and a simple } \beta_s \ge \beta \ge \beta_s - \delta(N) \text{ s.t. } \rho(M_N(\beta_s)) \ni \beta\}.$$

Then we can estimate its Lebesque measure:

$$\mathcal{L}(\Xi) \le \sum_{N=1}^{\infty} \mathcal{L}(\Xi_N) = 0.$$

(5) For a given number $\beta \in I_n$ and any allowed word B of length m the set $\pi_\beta^{-1}(B)$ is a nonempty interval. Here π_β is the map $\pi : [0,1] \to S_\beta$. Moreover, the word B is also allowed for all $\beta' > \beta$ and the length of B changes continuously because it depends only on the first $(m-1)$ iterates of T_β. This ensures for each $\beta \in I_n$ the existence of a simple β–number $\beta_s = \beta_s(\beta)$ with the property:
For all (in S_β) allowed words B of length m

$$\frac{1}{2} \le \frac{\text{length } (\pi_\beta^{-1}(B))}{\text{length } (\pi_{\beta_s}^{-1}(B))} \le 2$$

holds. Using the bounds on the density h_β for $\beta \in I_n$ (see observation 1) we get

$$\frac{1}{2C^2} \le \frac{\nu_\beta(\pi_\beta^{-1}(B))}{\nu_{\beta_s}(\pi_{\beta_s}^{-1}(B))} \le 2C^2.$$

(6) Let $M > 0$ and suppose that for all allowed words $\{B\}$ in base β their relative frequencies $\{f_l(\underline{x}, \beta)\}$ in some expansion $\underline{x} = \{i_n(x,\beta)\}$ are bounded by the measure of B:

$$M^{-1}\nu_\beta(B) \le \{\lim f_l(\underline{x}, B)\} \le M\nu_\beta(B).$$

then all measures in $V_{T_\beta}(x)$ are absolutely continuous w.r.t. ν_β and therefore they all coincide with ν_β. But

$$V_{T_\beta}(x) = \{\nu_\beta\}$$

implies that x is normal. Here we denoted the set of accumulation points of the sequence $\{f_\ell(\underline{x}, B)\}$ by $\{\lim f_\ell(\underline{x}, B)\}$.

(7) Let β be a non–selfnormal number in I_n. By 6. there exists a sequence $\{B_{m_k}\}$ of allowed words in S_β with length m_k s.t.

$$\{\lim f_\ell(1, B_{m_k})\} \not\subset \left[\frac{1}{2C^2}\nu_\beta(\pi^{-1}B_{m_k}), 2C^2\nu_\beta(\pi^{-1}B_{m_k})\right]$$

Using 5. we then can find intervals $[\beta, \beta_k]$ where the relative frequencies $f_\ell(\{i_n(1,\beta)\}, B_{m_k})$ are not converging to $\nu_{\beta'}$ for any $\beta' \in [\beta, \beta_k]$. Because β is not selfnormal we know that

$$h(\beta) < \ln\beta = h_{\nu_\beta}(T_\beta) \leq \ln\beta' = h_{\nu_{\beta'}}(T_\beta).$$

Therefore there exist naturals N and m and a real $\tilde{\beta} > \beta$ with

$$1 - \frac{1}{N} \geq \frac{\lim_{k\to\infty} -\frac{1}{m}\sum_{B_m} f_{\ell_k}(B_m)\log f_{\ell_k}(B_m)}{\log\tilde{\beta}} \geq \frac{h(\beta)}{\log\tilde{\beta}}$$

for a suitable sequence $\ell_1 < \ell_2 < \ldots$ where the sum is over all allowed words B_m of length m.

Making $\tilde{\beta}$ a simple β–number in $[\beta, \beta + \delta(N)]$ we can apply the auxilary lemma and can conclude $\beta \in \Xi$. This together with 4. gives the first part of the theorem. The proof of the second part is included in proposition 6.1. of [1] by observing that the set G in that proposition is contained in $C_5 \setminus \tilde{C}_5$.

□

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