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NUMERICAL SOLUTION OF DIRICHLET PROBLEMS FOR NONLINEAR PARABOLIC EQUATIONS BY PROBABILITY APPROACH

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Abstract. A number of new layer methods solving Dirichlet problems for semilinear parabolic equations is constructed by using probabilistic representations of their solutions. The methods exploit the ideas of weak sense numerical integration of stochastic differential equations in bounded domain. In spite of the probabilistic nature these methods are nevertheless deterministic. Some convergence theorems are proved. Numerical tests are presented.

1. Introduction

Numerical analysis of nonlinear partial differential equations (nonlinear PDE) is generally based on deterministic approaches (see, e.g., [1, 2, 3, 4]). A probability approach to constructing new layer methods for solving nonlinear PDE of parabolic type is proposed in [5] (see [6] as well). It is based on making use of the well-known probabilistic representations of solutions to linear parabolic equations (see, e.g., [7, 8]) and the ideas of weak sense numerical integration of SDE [9, 10, 11]. In spite of their probabilistic nature these methods are nevertheless deterministic. The probability approach takes into account a coefficient dependence on the space variables and a relationship between diffusion and advection in an intrinsic manner. In particular, the layer methods allow us to avoid difficulties stemming from essentially changing coefficients and strong advection. Other probabilistic applications to numerical solving nonlinear PDE are available, e.g., in [12, 13].

The papers [5, 6] are devoted to layer methods for the nonlinear Cauchy problem. The aim of this paper is to develop such methods for nonlinear problems with Dirichlet boundary conditions. Some probability methods solving boundary value problems for linear parabolic equations are proposed in [14, 15, 16].

Let G be a bounded domain in $\mathbf{R}^d$, $Q = [t_0, T) \times G$ be a cylinder in $\mathbf{R}^{d+1}$, $\Gamma = \bar{Q} \setminus Q$. The set Γ is a part of the boundary of the cylinder Q consisting of the upper base and the lateral surface.

Consider the Dirichlet problem for the semilinear parabolic equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(t,x,u)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{i=1}^{d} b^i(t,x,u)\frac{\partial u}{\partial x^i} + g(t,x,u) = 0, \ (t,x) \in Q, \tag{1.1}$$

$$u(t,x)|_{\Gamma} = \varphi(t,x). \tag{1.2}$$

The form of equation (1.1) is relevant to a probabilistic approach, i.e., the equation is considered under $t < T$, and the "initial" conditions are prescribed at $t = T$. Using the well known probabilistic representation of the solution to (1.1)-(1.2) (see [7, 8]), we get

$$u(t,x) = \mathbf{E}(\varphi(\tau, X_{t,x}(\tau)) + Z_{t,x,0}(\tau)). \tag{1.3}$$

In (1.3) $X_{t,x}(s),\ Z_{t,x,z}(s),\ (t,x) \in Q,\ s \geq t$, is the solution of the Cauchy problem for the Ito system of stochastic differential equations

$$dX = b(s,X,u(s,X))ds + \sigma(s,X,u(s,X))dw(s),\ X(t) = x,$$

$$dZ = g(s,X,u(s,X))ds,\ Z(t) = z, \tag{1.4}$$

where $w(s) = (w^1(s), ..., w^d(s))^\top$ is a standard Wiener process, $b(s,x,u) = (b^1(s,x,u), ..., b^d(s,x,u))^\top$ is the column vector, the matrix $\sigma = \sigma(s,x,u)$ is obtained from the equation

$$\sigma\sigma^\top = a,\ \sigma = \{\sigma^{ij}(s,x,u)\},\ a = \{a^{ij}(s,x,u)\},$$

and $\tau = \tau_{t,x}$ is the first exit time of the trajectory $(s, X_{t,x}(s))$ from the domain Q.

If the equation (1.1) is linear, system (1.4) does not contain the unknown function $u(s,x)$ and therefore one can use weak approximation schemes for solving (1.4) with the Monte Carlo realization of the representation (1.3). The representation involves the point $(\tau, X_{t,x}(\tau))$. To get a sufficiently effective approximation of this point is rather a hard problem. Some constructive schemes solving this problem in the case of linear parabolic equation are presented in [14, 15]. The procedures of [14, 15] together with the Monte Carlo approach allow us to find a value $u(t,x)$ at a single point even under a big dimension of the domain G.

Of course, the nonlinear case is much more complicated. But we are aimed to construct layer methods and due to this fact it becomes possible to use a one-step (local) version of the representation (1.3) (see formula (2.3) below). Introduce a time discretization, for definiteness the equidistant one

$$T = t_N > t_{N-1} > \cdots > t_0, \ h := \frac{T - t_0}{N} .$$

The proposed here methods give an approximation $\bar{u}(t_k, x)$ of the solution $u(t_k, x)$, $k = N, ..., 0$, $x \in \overline{Q}$, i.e., step by step everywhere in the domain Q. It is feasible if the dimension of the domain G is comparatively small ($d \leq 3$). To construct the layer methods, we exploit the ideas of weak sense numerical integration of stochastic differential equations in bounded domain and obtain some approximate relations on the basis of (2.3), (1.4). The relations allow us to express $\bar{u}(t_k, x)$, $k = N-1, ..., 0$, recurrently in terms of $\bar{u}(t_{k+1}, x)$. Despite the probabilistic nature these methods turn out to be deterministic as in [5, 6].

In Sections 2 and 4, we derive a few methods for nonlinear parabolic equations relying on the numerical integration of ordinary stochastic differential equations. In Section 3 we prove a convergence theorem using deterministic type arguments. To realize layer methods in practice, we need a discretization in the variable x with some kind of interpolation at every step to turn an applied method into an algorithm. Such numerical algorithms are constructed in Section 5. A majority of ideas can be demonstrated at $d = 1$ though that we restrict ourselves to this case in Sections 2-5. The case $d \geq 2$ is shortly discussed in Section 6. Numerical tests are presented in the last section.

2. Construction of layer method with one-step error $O(h^2)$

The boundary value problem (1.1)-(1.2) in the one-dimensional case has the following form:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2 u}{\partial x^2} + b(t,x,u)\frac{\partial u}{\partial x} + g(t,x,u) = 0, \quad (t,x) \in Q \ , \tag{2.1}$$

$$u(t,x)|_\Gamma = \varphi(t,x) \ . \tag{2.2}$$

In this case Q is the partly open rectangle: $Q = [t_0, T) \times (\alpha, \beta)$, and Γ consists of the upper base $\{T\} \times [\alpha, \beta]$ and two vertical intervals: $[t_0, T) \times \{\alpha\}$ and $[t_0, T) \times \{\beta\}$. We assume that $\sigma(t,x,u) \geq \sigma_0 > 0$ for $(t,x) \in \overline{Q}$, $-\infty < u < \infty$.

Let $u = u(t,x)$ be the solution to problem (2.1)-(2.2), which is supposed to exist, to be unique, and to be sufficiently smooth. One can find many theoretical results on this topic in [17, 18, 19, 20, 21] (see also references therein).

Analogously to (1.3) we have

$$u(t_k, x) = \mathbf{E}(u(\vartheta_{t_k,x}, X_{t_k,x}(\vartheta_{t_k,x})) + Z_{t_k,x,0}(\vartheta_{t_k,x})) \ , \tag{2.3}$$

where $\vartheta_{t_k,x} = \vartheta_{t_k,x}(t_{k+1}) := \tau_{t_k,x} \wedge t_{k+1}$, and X, Z satisfy system (1.4).

Let us suppose for a while that it is possible to extend the coefficients of the equation (1.1) so that the new equation has a solution $u(t,x)$ on $[t_0,T)\times\mathbf{R}$ which is an extension of the solution to the boundary value problem (1.1)-(1.2). Then instead of (2.3), we obtain (we suppose the layer $u(t_{k+1},x)$ to be known)

$$u(t_k,x)=\mathbf{E}(u(t_{k+1},X_{t_k,x}(t_{k+1}))+Z_{t_k,x,0}(t_{k+1})). \tag{2.4}$$

Applying the explicit weak Euler scheme with the simplest simulation of noise to the system (1.4), we get

$$\bar{X}_{t_k,x}(t_{k+1})=x+b(t_k,x,u(t_k,x))h+\sigma(t_k,x,u(t_k,x))\sqrt{h}\xi\ , \tag{2.5}$$

$$\bar{Z}_{t_k,x,0}(t_{k+1})=g(t_k,x,u(t_k,x))h\ , \tag{2.6}$$

where the ξ is distributed by the law: $P(\xi=\pm 1)=\dfrac{1}{2}$.

Using (2.4), we get to within $O(h^2)$

$$u(t_k,x)\simeq\mathbf{E}(u(t_{k+1},\bar{X}_{t_k,x}(t_{k+1}))+\bar{Z}_{t_k,x,0}(t_{k+1}))$$

$$=\frac{1}{2}u(t_{k+1},x+b(t_k,x,u(t_k,x))h-\sigma(t_k,x,u(t_k,x))\sqrt{h})$$

$$+\frac{1}{2}u(t_{k+1},x+b(t_k,x,u(t_k,x))h+\sigma(t_k,x,u(t_k,x))\sqrt{h})+g(t_k,x,u(t_k,x))h. \tag{2.7}$$

Now we can obtain an implicit relation for an approximation of $u(t_k,x)$. Applying the method of simple iteration to the implicit relation and taking $u(t_{k+1},x)$ as a null iteration, we get the following explicit one-step approximation $v(t_k,x)$ of $u(t_k,x)$:

$$v(t_k,x)=\frac{1}{2}u(t_{k+1},x+b_k\cdot h-\sigma_k\cdot\sqrt{h})+\frac{1}{2}u(t_{k+1},x+b_k\cdot h+\sigma_k\cdot\sqrt{h})+g_k\cdot h\ , \tag{2.8}$$

where $b_k,\ \sigma_k,\ g_k$ are the coefficients $b(t,x,u),\ \sigma(t,x,u),\ g(t,x,u)$ calculated at the point $(t_k,x,u(t_{k+1},x))$.

But in reality we know the layer $u(t_{k+1},x)$ for $\alpha\le x\le\beta$ only. At the same time the argument $x+b_kh-\sigma_k\sqrt{h}$ for x close to α is less than α and the argument $x+b_kh+\sigma_k\sqrt{h}$ for x close to β is more than β. Thus we need to extend the layer $u(t_{k+1},x)$ in a constructive manner.

Using the explicit weak Euler scheme for the initial point (t,α) with $t_k\le t\le t_{k+1}$, we put (cf. (2.5)-(2.6))

$$\bar{X}_{t,\alpha}(t_{k+1})=x+b(t,\alpha,u(t,\alpha))\cdot(t_{k+1}-t)+\sigma(t,\alpha,u(t,\alpha))\cdot\sqrt{t_{k+1}-t}\cdot\xi\ ,$$

$$\bar{Z}_{t,\alpha,0}(t_{k+1})=g(t,\alpha,u(t,\alpha))\cdot(t_{k+1}-t)\ . \tag{2.9}$$

Analogously we define $\bar{X}_{t,\beta}(t_{k+1}),\ \bar{Z}_{t,\beta,0}(t_{k+1})$.

We have (see (2.7), (2.9)) for $t_k\le t\le t_{k+1}$

$$u(t,\alpha)\simeq\mathbf{E}(u(t_{k+1},\bar{X}_{t,\alpha}(t_{k+1}))+\bar{Z}_{t,\alpha,0}(t_{k+1}))$$

$$=\frac{1}{2}u(t_{k+1},\alpha+b(t,\alpha,u(t,\alpha))\cdot(t_{k+1}-t)-\sigma(t,\alpha,u(t,\alpha))\cdot\sqrt{t_{k+1}-t})$$

$$+\frac{1}{2}u(t_{k+1},\alpha+b(t,\alpha,u(t,\alpha))\cdot(t_{k+1}-t)+\sigma(t,\alpha,u(t,\alpha))\cdot\sqrt{t_{k+1}-t})$$

$$+g(t,\alpha,u(t,\alpha))\cdot(t_{k+1}-t)\ . \tag{2.10}$$

If we replace (remember, $u(t,\alpha)=\varphi(t,\alpha)$ due to (2.2)) the argument $(t,\alpha,u(t,\alpha))=(t,\alpha,\varphi(t,\alpha))$ by $(t_k,\alpha,\varphi(t_{k+1},\alpha))$, the right-hand side of (2.10) is changed by a quantity of the order $O(h^2)$. Since the approximation in (2.10) is also of the order $O(h^2)$, we get

$$\frac{1}{2}u(t_{k+1},\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{t_{k+1}-t})$$

$$=\varphi(t,\alpha)-\frac{1}{2}u(t_{k+1},\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)+\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{t_{k+1}-t})$$

$$-g(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)+O(h^2). \tag{2.11}$$

Introduce

$$\alpha_0:=\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot h-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{h}.$$

Clearly $\alpha_0<\alpha$ and $\alpha_0\le\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot(t_{k+1}-t)-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{t_{k+1}-t}\le\alpha$ for $t_k\le t\le t_{k+1}$ under a sufficiently small h.

Analogously

$$\frac{1}{2}u(t_{k+1},\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot(t_{k+1}-t)+\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{t_{k+1}-t})$$

$$=\varphi(t,\beta)-\frac{1}{2}u(t_{k+1},\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot(t_{k+1}-t)-\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{t_{k+1}-t})$$

$$-g(t_k,\beta,\varphi(t_{k+1},\beta))\cdot(t_{k+1}-t)+O(h^2), \tag{2.12}$$

$$\beta_0:=\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot h+\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{h}.$$

The relations (2.11)-(2.12) give the desired extension of the function $u(t_{k+1},x)$ on the interval $[\alpha_0,\beta_0]$.

Let us return to the formula (2.8) now. The arguments $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}$ and $x+b_k\cdot h+\sigma_k\cdot\sqrt{h}$ are monotone increasing functions in $x\in[\alpha,\beta]$ under a sufficiently small h, their values belong to $[\alpha_0,\beta_0]$, and $x+b_k\cdot h+\sigma_k\cdot\sqrt{h}$ is always (for $x\in[\alpha,\beta]$) more than α while $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}$ is less than β. Let $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}<\alpha$ (clearly it is possible for x close to α). Due to the stated above, there exists the only root $\gamma_k(x)$, $0<\gamma_k(x)\le 1$, of the quadratic equation

$$\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{\gamma_k h}=x+b_k\cdot h-\sigma_k\cdot\sqrt{h}. \tag{2.13}$$

Analogously, if $x+b_k\cdot h+\sigma_k\cdot\sqrt{h}>\beta$, then there exists the only root $\delta_k(x)$, $0<\delta_k(x)\le 1$, of the quadratic equation

$$\beta+b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\delta_k h+\sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{\delta_k h}=x+b_k\cdot h+\sigma_k\cdot\sqrt{h}. \tag{2.14}$$

If, for instance, $x+b_k\cdot h-\sigma_k\cdot\sqrt{h}<\alpha$, then one can replace the value $u(t_{k+1},x+b_k\cdot h-\sigma_k\cdot\sqrt{h})/2$ in the formula (2.8) by the value due to the formulas (2.13) and (2.11):

$$\frac{1}{2}u(t_{k+1},x+b_k\cdot h-\sigma_k\cdot\sqrt{h})=$$

$$=\frac{1}{2}u(t_{k+1},\alpha+b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h-\sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{\gamma_k h})$$

$$\approx \varphi(t_{k+1-\gamma_k},\alpha) - \frac{1}{2}u(t_{k+1},\alpha + b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h + \sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{\gamma_k h})$$

$$-g(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h,$$

where $t_{k+1-\gamma_k} = t_k + (1-\gamma_k)\cdot h.$

As a result we obtain the following one-step approximation $v(t_k,x)$ for $u(t_k,x)$

$$v(t_k,x) = \frac{1}{2}u(t_{k+1},x + b_k\cdot h - \sigma_k\cdot\sqrt{h}) + \frac{1}{2}u(t_{k+1},x + b_k\cdot h + \sigma_k\cdot\sqrt{h}) \tag{2.15}$$

$$+g_k\cdot h, \text{ if } x + b_k\cdot h \pm \sigma_k\cdot\sqrt{h} \in [\alpha,\beta];$$

$$v(t_k,x) = \varphi(t_{k+1-\gamma_k},\alpha) - g(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h$$

$$-\frac{1}{2}u(t_{k+1},\alpha + b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\gamma_k h + \sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{\gamma_k h})$$

$$+\frac{1}{2}u(t_{k+1},x + b_k\cdot h + \sigma_k\cdot\sqrt{h}) + g_k\cdot h, \text{ if } x + b_k\cdot h - \sigma_k\cdot\sqrt{h} < \alpha;$$

$$v(t_k,x) = \frac{1}{2}u(t_{k+1},x + b_k\cdot h - \sigma_k\cdot\sqrt{h}) + \varphi(t_{k+1-\delta_k},\beta) - g(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\delta_k h$$

$$-\frac{1}{2}u(t_{k+1},\beta + b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\delta_k h - \sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{\delta_k h})$$

$$+g_k\cdot h, \text{ if } x + b_k\cdot h + \sigma_k\cdot\sqrt{h} > \beta,\ k = N-1,\ldots,1,0,$$

where (let us recall) b_k, σ_k, g_k are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ calculated at the point $(t_k,x,u(t_{k+1},x))$ and γ_k, δ_k are the corresponding roots of the equations (2.13) and (2.14).

Thus the layer method acquires the form

$$\bar{u}(t_N,x) = \varphi(t_N,x),\ x \in [\alpha,\beta], \tag{2.16}$$

$$\bar{u}(t_k,x) = \frac{1}{2}\bar{u}(t_{k+1},x + \bar{b}_k\cdot h - \bar{\sigma}_k\cdot\sqrt{h}) + \frac{1}{2}\bar{u}(t_{k+1},x + \bar{b}_k\cdot h + \bar{\sigma}_k\cdot\sqrt{h})$$

$$+\bar{g}_k\cdot h, \text{ if } x + \bar{b}_k\cdot h \pm \bar{\sigma}_k\cdot\sqrt{h} \in [\alpha,\beta];$$

$$\bar{u}(t_k,x) = \varphi(t_{k+1-\bar{\gamma}_k},\alpha) - g(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\bar{\gamma}_k h$$

$$-\frac{1}{2}\bar{u}(t_{k+1},\alpha + b(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\bar{\gamma}_k h + \sigma(t_k,\alpha,\varphi(t_{k+1},\alpha))\cdot\sqrt{\bar{\gamma}_k h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1},x + \bar{b}_k\cdot h + \bar{\sigma}_k\cdot\sqrt{h}) + \bar{g}_k\cdot h, \text{ if } x + \bar{b}_k\cdot h - \bar{\sigma}_k\cdot\sqrt{h} < \alpha;$$

$$\bar{u}(t_k,x) = \frac{1}{2}\bar{u}(t_{k+1},x + \bar{b}_k\cdot h - \bar{\sigma}_k\cdot\sqrt{h}) + \varphi(t_{k+1-\bar{\delta}_k},\beta) - g(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\bar{\delta}_k h$$

$$-\frac{1}{2}\bar{u}(t_{k+1},\beta + b(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\bar{\delta}_k h - \sigma(t_k,\beta,\varphi(t_{k+1},\beta))\cdot\sqrt{\bar{\delta}_k h})$$

$$+\bar{g}_k\cdot h, \text{ if } x + \bar{b}_k\cdot h + \bar{\sigma}_k\cdot\sqrt{h} > \beta;$$

$$k = N-1, ..., 1, 0,$$

where $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$ and $\bar{\gamma}_k$, $\bar{\delta}_k$ are the corresponding roots of the equations (2.13) and (2.14) with the right sides $x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}$ and $x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}$.

The method (2.16) is an explicit layer method for solving the Dirichlet problem (2.1)-(2.2). This method is deterministic, even though the probabilistic approach is used for its constructing. It is of the first order of smallness with respect to h (see below Theorem 3.1).

3. Convergence theorem

We shall keep the following assumptions.

(i) There exists the only solution $u(t,x)$ to the problem (2.1)-(2.2) such that

$$u_\circ < u_* \le u(t,x) \le u^* < \overset{\circ}{u},\ t_0 \le t \le T,\ x \in [\alpha, \beta], \tag{3.1}$$

where $u_\circ$, u_*, u^*, $\overset{\circ}{u}$ are some constants, and there exist the uniformly bounded derivatives:

$$|\frac{\partial^{i+j} u}{\partial t^i \partial x^j}| \le K,\ i=0,\ j=1,2,3,4;\ i=1,\ j=0,1,2;\ i=2,\ j=0;\ t_0 \le t \le T,\ x \in [\alpha,\beta]. \tag{3.2}$$

(ii) The coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ and their first and second derivatives in x and u are uniformly bounded:

$$|\frac{\partial^{i+j} b}{\partial x^i \partial u^j}| \le K,\ |\frac{\partial^{i+j} \sigma}{\partial x^i \partial u^j}| \le K,\ |\frac{\partial^{i+j} g}{\partial x^i \partial u^j}| \le K,\ 0 \le i+j \le 2,$$

$$t_0 \le t \le T,\ x \in [\alpha,\beta],\ u_\circ \le u \le \overset{\circ}{u}. \tag{3.3}$$

Below we use the letters K and C without any index for various constants which do not depend on h, k, x.

First of all let us evaluate the one-step error $\rho(t_k, x)$ of the method (2.16).

Lemma 3.1. *Under the assumptions (i) and (ii) the one-step error $\rho(t_k,x)$ of the method* (2.16) *has the second order of smallness with respect to h, i.e.,*

$$|\rho(t_k,x)| = |v(t_k,x) - u(t_k,x)| \le Ch^2,$$

where $v(t_k,x)$ is defined by (2.15), *C does not depend on h, k, x.*

Proof. If both points $x + b_k \cdot h \pm \sigma_k \cdot \sqrt{h}$ belong to $[\alpha,\beta]$, the statement of this lemma follows directly from Lemma 4.1 of [5].

Let us consider the case when the point $x + b_k \cdot h - \sigma_k \cdot \sqrt{h} < \alpha$. Introduce the notation b_α, σ_α, g_α for the coefficients b, σ, g calculated at the point $(t_k, \alpha, \varphi(t_{k+1}, \alpha))$. We get from (2.13)

$$\alpha - x = b_k h - \sigma_k \sqrt{h} - b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h} = (\sigma_\alpha \sqrt{\gamma_k} - \sigma_k)\sqrt{h} + O(h) = O(\sqrt{h}). \tag{3.4}$$

Expand the terms of (2.15) at the point (t_k, x):

$$\varphi(t_{k+1-\gamma_k}, \alpha) = u(t_k + (1-\gamma_k)h, x + (\alpha - x)) = u + \frac{\partial u}{\partial t} \cdot (1-\gamma_k)h \tag{3.5}$$

$$+\frac{\partial u}{\partial x} \cdot (\alpha - x) + \frac{\partial^2 u}{\partial t \partial x} \cdot (1-\gamma_k)(\alpha - x)h + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \cdot (\alpha - x)^2 + \frac{1}{6}\frac{\partial^3 u}{\partial x^3} \cdot (\alpha - x)^3 + O(h^2),$$

$$u(t_{k+1}, \alpha + b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h}) = u(t_k + h, x + (\alpha - x + b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h})) \tag{3.6}$$

$$= u + \frac{\partial u}{\partial t} h + \frac{\partial u}{\partial x} \cdot (\alpha - x + b_\alpha \gamma_k h + \sigma_\alpha \sqrt{\gamma_k h}) + \frac{\partial^2 u}{\partial t \partial x} \cdot (\alpha - x + \sigma_\alpha \sqrt{\gamma_k h}) h$$

$$+ \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \cdot ((\alpha - x + \sigma_\alpha \sqrt{\gamma_k h})^2 + 2(\alpha - x + \sigma_\alpha \sqrt{\gamma_k h}) b_\alpha \gamma_k h)$$

$$+ \frac{1}{6} \frac{\partial^3 u}{\partial x^3} \cdot (\alpha - x + \sigma_\alpha \sqrt{\gamma_k h})^3 + O(h^2),$$

and

$$u(t_{k+1}, x + b_k h + \sigma_k \sqrt{h}) = u + \frac{\partial u}{\partial t} h + \frac{\partial u}{\partial x} \cdot (b_k h + \sigma_k \sqrt{h}) + \frac{\partial^2 u}{\partial t \partial x} \cdot \sigma_k h^{3/2} \tag{3.7}$$

$$+ \frac{1}{2} \frac{\partial^2 u}{\partial x^2} \cdot (\sigma_k^2 h + 2 b_k \sigma_k h^{3/2}) + \frac{1}{6} \frac{\partial^3 u}{\partial x^3} \cdot \sigma_k^3 h^{3/2} + O(h^2).$$

Here the function u and its derivatives are calculated at the point (t_k, x).

Substituting (3.5)-(3.7) in the corresponding expression for $v(t_k, x)$ of (2.15) and using (3.4), we obtain

$$v(t_k, x) = u + h(1 - \gamma_k) \cdot (\frac{\partial u}{\partial t} + \frac{\sigma_k^2}{2} \frac{\partial^2 u}{\partial x^2} + b_k \frac{\partial u}{\partial x} + g_k) + \frac{\partial u}{\partial x} \cdot (b_k - b_\alpha) \gamma_k h \tag{3.8}$$

$$+ \frac{\partial^2 u}{\partial t \partial x} \cdot (\sigma_k - \sigma_\alpha \sqrt{\gamma_k}) \gamma_k h^{3/2} + \frac{\partial^2 u}{\partial x^2} \cdot (\frac{1}{2} (\sigma_k^2 - \sigma_\alpha^2) \gamma_k h + b_\alpha (\sigma_k - \sigma_\alpha \sqrt{\gamma_k}) \gamma_k h^{3/2})$$

$$+ \frac{1}{2} \frac{\partial^3 u}{\partial x^3} \cdot \sigma_\alpha^2 (\sigma_k - \sigma_\alpha \sqrt{\gamma_k}) \gamma_k h^{3/2} + (g_k - g_\alpha) \gamma_k h + O(h^2).$$

Due to the assumptions (i) and (ii) and (3.4), we get

$$\sigma_\alpha = \sigma(t_k, \alpha, u(t_{k+1}, \alpha)) = \sigma(t_k, x, u(t_k, x)) + \frac{\partial \sigma}{\partial x} \cdot (\alpha - x) \tag{3.9}$$

$$+ \frac{\partial \sigma}{\partial u} \cdot (u(t_{k+1}, \alpha) - u(t_k, x)) + O(h)$$

$$= \sigma + \frac{\partial \sigma}{\partial x} \cdot (\sigma_\alpha \sqrt{\gamma_k} - \sigma_k) \sqrt{h} + \frac{\partial \sigma}{\partial u} \frac{\partial u}{\partial x} \cdot (\sigma_\alpha \sqrt{\gamma_k} - \sigma_k) \sqrt{h} + O(h)$$

$$= \sigma + (\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial u} \frac{\partial u}{\partial x}) \cdot \sigma_k (\sqrt{\gamma_k} - 1) \sqrt{h} + O(h),$$

$$b_\alpha = b(t_k, \alpha, u(t_{k+1}, \alpha)) = b + (\frac{\partial b}{\partial x} + \frac{\partial b}{\partial u} \frac{\partial u}{\partial x}) \cdot \sigma_k (\sqrt{\gamma_k} - 1) \sqrt{h} + O(h),$$

$$g_\alpha = g(t_k, \alpha, u(t_{k+1}, \alpha)) = g + (\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u} \frac{\partial u}{\partial x}) \cdot \sigma_k (\sqrt{\gamma_k} - 1) \sqrt{h} + O(h),$$

and

$$b_k = b(t_k, x, u(t_{k+1}, x)) = b + O(h), \ \ \sigma_k = \sigma + O(h), \ \ g_k = g + O(h),$$

where b, σ, g (without any indexes) and their derivatives are calculated at the point $(t_k, x, u(t_k, x))$.

Using (3.9) we obtain from (3.8):

$$v(t_k, x) = u(t_k, x) + h(1-\gamma_k)\cdot[\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + g] \tag{3.10}$$

$$+h^{3/2}\gamma_k\sigma(1-\sqrt{\gamma_k})\cdot[\frac{\partial^2 u}{\partial t\partial x} + \frac{\sigma^2}{2}\frac{\partial^3 u}{\partial x^3} + \sigma\cdot(\frac{\partial \sigma}{\partial x} + \frac{\partial \sigma}{\partial u}\frac{\partial u}{\partial x})\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x^2}$$

$$+(\frac{\partial b}{\partial x} + \frac{\partial b}{\partial u}\frac{\partial u}{\partial x})\frac{\partial u}{\partial x} + \frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial x}] + O(h^2)$$

$$= u(t_k, x) + (h(1-\gamma_k) + h^{3/2}\gamma_k\sigma(1-\sqrt{\gamma_k})\frac{\partial}{\partial x})[\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + g] + O(h^2).$$

Taking into account that $u(t, x)$ is the solution to the problem (2.1)-(2.2), the relation (3.10) implies

$$v(t_k, x) = u(t_k, x) + O(h^2).$$

The case $x + b_k\cdot h + \sigma_k\cdot\sqrt{h} > \beta$ can be considered analogously. Lemma 3.1 is proved.

Let us prove the following theorem on global convergence.

Theorem 3.1. *Under the assumptions* (*i*) *and* (*ii*) *the method* (2.16) *has the first order of smallness with respect to* h, *i.e.,*

$$|\bar{u}(t_k, x) - u(t_k, x)| \le Kh,$$

where K *does not depend on* h, k, x.

Proof. Denote the error of the method (2.16) on the k-th layer ($(N-k)$-th step) as

$$R(t_k, x) := \bar{u}(t_k, x) - u(t_k, x). \tag{3.11}$$

If $x + \bar{b}_k\cdot h \pm \bar{\sigma}_k\cdot\sqrt{h} \in [\alpha, \beta]$, we have (see (2.16) and (3.11)):

$$u(t_k, x) + R(t_k, x) = \frac{1}{2}u(t_{k+1}, x + \bar{b}_k\cdot h - \bar{\sigma}_k\cdot\sqrt{h}) + \frac{1}{2}R(t_{k+1}, x + \bar{b}_k\cdot h - \bar{\sigma}_k\cdot\sqrt{h})$$

$$+\frac{1}{2}u(t_{k+1}, x + \bar{b}_k\cdot h + \bar{\sigma}_k\cdot\sqrt{h}) + \frac{1}{2}R(t_{k+1}, x + \bar{b}_k\cdot h + \bar{\sigma}_k\cdot\sqrt{h}) + \bar{g}_k\cdot h. \tag{3.12}$$

Expanding the functions $u(t_{k+1}, x + \bar{b}_k\cdot h \pm \bar{\sigma}_k\cdot\sqrt{h})$ at the point (t_k, x), we get

$$u(t_{k+1}, x + \bar{b}_k\cdot h \pm \bar{\sigma}_k\cdot\sqrt{h}) = u(t_k, x) + \frac{\partial u}{\partial t}h + (\bar{b}_k\cdot h \pm \bar{\sigma}_k\cdot\sqrt{h})\frac{\partial u}{\partial x}$$

$$+\frac{\bar{\sigma}_k^2}{2}\frac{\partial^2 u}{\partial x^2}\cdot h \pm \bar{b}_k\bar{\sigma}_k\frac{\partial^2 u}{\partial x^2}\cdot h^{3/2} \pm \bar{\sigma}_k\frac{\partial^2 u}{\partial t\partial x}\cdot h^{3/2} \pm \frac{\bar{\sigma}_k^3}{6}\frac{\partial^3 u}{\partial x^3}\cdot h^{3/2} + O(h^2), \tag{3.13}$$

where the derivatives are calculated at the point (t_k, x).

Here we have to suggest for a while that the value $u(t_{k+1}, x) + R(t_{k+1}, x)$ remains in the interval $(u_\circ, u^\circ)$ for a sufficiently small h (see the conditions (*ii*)). Clearly, $R(t_N, x) = 0$, and below we prove recurrently that $R(t_k, x)$ is sufficiently small under a sufficiently small h. Thereupon thanks to (3.1) this suggestion will be justified for such h.

Due to the assumptions (i) and (ii) and the notation (3.11), we obtain

$$\bar{b}_k = b(t_k, x, \bar{u}(t_{k+1}, x)) = b(t_k, x, u(t_{k+1}, x) + R(t_{k+1}, x)) = b(t_k, x, u(t_{k+1}, x)) + \Delta b$$

$$= b(t_k, x, u(t_k, x)) + \Delta b + O(h),$$

$$\bar{\sigma}_k = \sigma(t_k, x, u(t_k, x)) + \Delta\sigma + O(h),\ \bar{\sigma}_k^2 = \sigma^2(t_k, x, u(t_k, x)) + \Delta\sigma^2 + O(h),$$

$$\bar{g}_k = g(t_k, x, u(t_k, x)) + \Delta g + O(h), \tag{3.14}$$

where

$$|\Delta b|,\ |\Delta\sigma|,\ |\Delta\sigma^2|,\ |\Delta g| \leq K \cdot |R(t_{k+1}, x)|.$$

Substituting (3.13) in (3.12) and taking into account (3.14), we come to the relation

$$u(t_k, x) + R(t_k, x) = u(t_k, x) + h \cdot (\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + g) + r(t_k, x) + O(h^2)$$

$$+\frac{1}{2}R(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}) + \frac{1}{2}R(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h}), \tag{3.15}$$

where the derivatives are calculated at (t_k, x), b, σ, g are calculated at $(t_k, x, u(t_k, x))$, and

$$|r(t_k, x)| \leq Kh|R(t_{k+1}, x)|.$$

Since $u(t, x)$ is the solution to (2.1)-(2.2), the relation (3.15) implies

$$R(t_k, x) = \frac{1}{2}R(t_{k+1}, x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h}) + \frac{1}{2}R(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h})$$

$$+r(t_k, x) + O(h^2). \tag{3.16}$$

For x such that $x + \bar{b}_k \cdot h - \bar{\sigma}_k \cdot \sqrt{h} < \alpha$, we get (see (3.11) and (2.16))

$$u(t_k, x) + R(t_k, x) = \bar{u}(t_k, x) = \varphi(t_{k+1-\bar{\gamma}_k}, \alpha)$$

$$-\frac{1}{2}u(t_{k+1}, \alpha + b_\alpha\bar{\gamma}_k h + \sigma_\alpha\sqrt{\bar{\gamma}_k h})) + \frac{1}{2}u(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h})$$

$$-\frac{1}{2}R(t_{k+1}, \alpha + b_\alpha\bar{\gamma}_k h + \sigma_\alpha\sqrt{\bar{\gamma}_k h})) + \frac{1}{2}R(t_{k+1}, x + \bar{b}_k \cdot h + \bar{\sigma}_k \cdot \sqrt{h})$$

$$-g(t_k, \alpha, u(t_{k+1}, \alpha)) \cdot \bar{\gamma}_k h + \bar{g}_k \cdot h, \tag{3.17}$$

where $b_\alpha, \sigma_\alpha, g_\alpha$ are the corresponding coefficients calculated at the point $(t_k, \alpha, \varphi(t_{k+1}, \alpha))$.

In accordance with (2.16) and (2.13), we have (cf. (3.4))

$$\alpha - x = \bar{b}_k h - \bar{\sigma}_k\sqrt{h} - b_\alpha\bar{\gamma}_k h + \sigma_\alpha\sqrt{\bar{\gamma}_k h} = (\sigma_\alpha\sqrt{\bar{\gamma}_k} - \sigma_k)\sqrt{h} + O(h) = O(\sqrt{h}).$$

Recall that $\bar{\gamma}_k$ is the root of the equation (2.13) with the right side $x + \bar{b}_k h - \bar{\sigma}_k\sqrt{h}$.

Now we expand the first three terms in the right side of (3.17) in powers of h at the point (t_k, x) like it has been done in the proof of Lemma 3.1 (see (3.5)-(3.8)). The obtained new relation contains $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$, $\bar{\gamma}_k$ (instead of b_k, σ_k, g_k, γ_k in (3.8)) and b_α, σ_α, g_α. We present $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$ due to (3.14) and b_α, σ_α, g_α due to (3.9). As a result, we get (cf. (3.10))

$$u(t_k, x) + R(t_k, x) = u(t_k, x)$$

$$+(h(1 - \bar{\gamma}_k) + h^{3/2}\bar{\gamma}_k\sigma(1 - \sqrt{\bar{\gamma}_k})\frac{\partial}{\partial x})[\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + g]$$

$$+r_1(t_k,x)+O(h^2)-\frac{1}{2}R(t_{k+1},\alpha+b_\alpha\bar{\gamma}_k h+\sigma_\alpha\sqrt{\bar{\gamma}_k h}))+\frac{1}{2}R(t_{k+1},x+\bar{b}_k\cdot h+\bar{\sigma}_k\cdot\sqrt{h}),$$

where the derivatives of u are calculated at the point (t_k,x), the coefficients b, σ, g and their derivatives are calculated at the point $(t_k,x,u(t_k,x))$, and

$$|r_1(t_k,x)|\leq Kh|R(t_{k+1},x)|.$$

Since $u(t,x)$ is the solution to (2.1)-(2.2), finally we arrive at

$$R(t_k,x)=-\frac{1}{2}R(t_{k+1},\alpha+b_\alpha\bar{\gamma}_k h+\sigma_\alpha\sqrt{\bar{\gamma}_k h}))$$

$$+\frac{1}{2}R(t_{k+1},x+\bar{b}_k\cdot h+\bar{\sigma}_k\cdot\sqrt{h})+r_1(t_k,x)+O(h^2). \tag{3.18}$$

Clearly, for x such that $x+\bar{b}_k\cdot h+\bar{\sigma}_k\cdot\sqrt{h}>\beta$, we can obtain the relation similar to (3.18):

$$R(t_k,x)=-\frac{1}{2}R(t_{k+1},\beta+b_\beta\bar{\delta}_k h-\sigma_\beta\sqrt{\bar{\delta}_k h}))$$

$$+\frac{1}{2}R(t_{k+1},x+\bar{b}_k\cdot h-\bar{\sigma}_k\cdot\sqrt{h})+r_2(t_k,x)+O(h^2) \tag{3.19}$$

with

$$|r_2(t_k,x)|\leq Kh|R(t_{k+1},x)|.$$

Now introduce

$$R_k:=\max_{x\in[\alpha,\beta]}|R(t_k,x)|\ .$$

The relations (3.16), (3.18), and (3.19) imply (remember that $R(t_N,x)=0$)

$$R_N=0,\ R_k\leq R_{k+1}+KR_{k+1}h+Ch^2,\ k=N-1,...,1,0.$$

Therefore

$$R_k\leq\frac{C}{K}(e^{K(T-t_0)}-1)\cdot h,\ k=N,...,0.$$

Theorem 3.1 is proved.

4. Layer method with one-step error $O(h^{3/2})$

Without exploiting the used above idea of involving the points outside the interval $[\alpha,\beta]$ while constructing a layer method, it is possible to get a layer method being more simple but less accurate than (2.16). To this end we approximate the solution $u(t_k,x)$, when the point x is close to α (or β), using values of the solution at a point $(t_{k+\lambda_k},\alpha)$ with some $\lambda_k\in(0,1)$ (or $(t_{k+\mu_k},\beta)$ with $\mu_k\in(0,1)$) and at the point $(t_{k+1},x+b_k\cdot h+\sigma_k\cdot\sqrt{h})$ (or $(t_{k+1},x+b_k\cdot h-\sigma_k\cdot\sqrt{h})$) with some (positive) weights. These two weights may be interpreted as probabilities of reaching and not reaching of α (or β). The method obtained on this way has the form

$$\bar{u}(t_N,x)=\varphi(t_N,x),\ x\in[\alpha,\beta], \tag{4.1}$$

$$\bar{u}(t_k,x)=\frac{1}{2}\bar{u}(t_{k+1},x+\bar{b}_k\cdot h-\bar{\sigma}_k\cdot\sqrt{h})+\frac{1}{2}\bar{u}(t_{k+1},x+\bar{b}_k\cdot h+\bar{\sigma}_k\cdot\sqrt{h})$$

$$+\bar{g}_k\cdot h,\ \text{if}\ x+\bar{b}_k\cdot h\pm\bar{\sigma}_k\cdot\sqrt{h}\in[\alpha,\beta];$$

$$\bar{u}(t_k,x)=\frac{1}{1+\sqrt{\bar{\lambda}_k}}\varphi(t_{k+\bar{\lambda}_k},\alpha)+\frac{\sqrt{\bar{\lambda}_k}}{1+\sqrt{\bar{\lambda}_k}}\bar{u}(t_{k+1},x+\bar{b}_k\cdot h+\bar{\sigma}_k\cdot\sqrt{h})$$

$$+\bar{g}_k\cdot\sqrt{\bar{\lambda}_k}h,\ \text{if } x+\bar{b}_k\cdot h-\bar{\sigma}_k\cdot\sqrt{h}<\alpha;$$

$$\bar{u}(t_k,x)=\frac{1}{1+\sqrt{\bar{\mu}_k}}\varphi(t_{k+\bar{\mu}_k},\beta)+\frac{\sqrt{\bar{\mu}_k}}{1+\sqrt{\bar{\mu}_k}}\bar{u}(t_{k+1},x+\bar{b}_k\cdot h-\bar{\sigma}_k\cdot\sqrt{h})$$

$$+\bar{g}_k\cdot\sqrt{\bar{\mu}_k}h,\ \text{if } x+\bar{b}_k\cdot h+\bar{\sigma}_k\cdot\sqrt{h}>\beta;$$

$$k=N-1,...,1,0,$$

where $\bar{b}_k$, $\bar{\sigma}_k$, $\bar{g}_k$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ calculated at the point $(t_k,x,\bar{u}(t_{k+1},x))$ and $0<\bar{\lambda}_k,\ \bar{\mu}_k<1$ are the roots of the quadratic equations (it is not difficult to verify that the roots exist and are unique)

$$\alpha=x+\bar{b}_k\cdot\bar{\lambda}_kh-\bar{\sigma}_k\cdot\sqrt{\bar{\lambda}_kh},$$

$$\beta=x+\bar{b}_k\cdot\bar{\mu}_kh+\bar{\sigma}_k\cdot\sqrt{\bar{\mu}_kh}.$$

This method involves one value of the function $\varphi(t,x)$ and one value of the approximate solution $\bar{u}(t_{k+1},y)$ on the previous layer in contrast to the method (2.16) which requires evaluating one value of the function $\varphi(t,x)$ and two values of the approximate solution $\bar{u}(t_{k+1},y)$ on the previous layer.

Lemma 4.1. *Under the assumptions* (*i*) *and* (*ii*) *the one-step error* $\rho(t_k,x)$ *of the method* (4.1) *is estimated as*

$$|\rho(t_k,x)|\le Ch^2 \text{ if } x+b_k\cdot h\pm\sigma_k\cdot\sqrt{h}\in[\alpha,\beta];$$

$$|\rho(t_k,x)|\le Ch^{3/2} \text{ if } x+b_k\cdot h-\sigma_k\cdot\sqrt{h}<\alpha \text{ or } x+b_k\cdot h+\sigma_k\cdot\sqrt{h}>\beta.$$

The proof is very similar (even more simply) to that of Lemma 3.1 and we do not give it here. The following convergence theorem for the method (4.1) takes place.

Theorem 4.1. *Under the assumptions* (*i*) *and* (*ii*) *the method* (4.1) *has the global error estimated as*

$$|\bar{u}(t_k,x)-u(t_k,x)|\le K\sqrt{h}, \tag{4.2}$$

where K does not depend on h, k, x.

The proof of the estimate (4.2) coincides, in general, with the proof of Theorem 3.1.

Remark 4.1. The assertions of Lemma 4.1 and Theorem 4.1 are also valid if we take weaker assumptions on the coefficients than (*ii*), namely:

$$|b|\le K,\ |\sigma|\le K,\ |g|\le K,$$

$$|b(t,x_2,u_2)-b(t,x_1,u_1)|+|\sigma(t,x_2,u_2)-\sigma(t,x_1,u_1)|+|g(t,x_2,u_2)-g(t,x_1,u_1)|$$

$$\le K(|x_2-x_1|+|u_2-u_1|),\ t_0\le t\le T,\ x\in[\alpha,\beta],\ u_\circ\le u\le \overset{\circ}{u}.$$

Remark 4.2. The layer methods of Sections 2 and 4 can be applied to solving the Dirichlet problem for linear parabolic equations. But if the dimension d of the linear problem is high ($d \geq 3$ in practice) and it is enough to find the solution in a few points only, the Monte Carlo approach is preferable [14, 15].

The used here weak approximations of SDE generate other random walk methods for solving the linear Dirichlet problem than the random walk proposed in [14, 15]. These and some other new random walks will be considered in a separate paper.

Remark 4.3. In the case of the linear Dirichlet problem one can prove using probabilistic arguments that the method (4.1) has the first order of smallness with respect to h. Apparently this is so in the nonlinear case as well and our numerical tests approve that (see Section 7). But we do not succeed in proving such a theorem.

Remark 4.4. Using other weak approximations for SDE, some new layer methods can be constructed (cf. [5, 6]). In particular, there are special methods of numerical integration in the weak sense for stochastic differential equations with small noise which are more effective than general ones [22]. In [6] they were used for constructing special layer methods for the Cauchy problem for semilinear parabolic equations with small parameter at higher derivatives. It is also possible to get some special layer methods in the case of the Dirichlet problem for semilinear parabolic equations with small parameter.

5. Numerical algorithms

To have become a numerical algorithm, the method (2.16) (just as other layer methods) needs a discretization in the variable x. Consider the equidistant space discretization with space step h_x (recall that the notation for time step is h): $x_j = \alpha + jh_x,\ j = 0, 1, 2, ..., M$, $h_x = (\beta - \alpha)/M$. Using, for example, linear interpolation, we construct the following algorithm (we denote it as $\bar{u}(t_k, x)$ again, since this does not cause any confusion):

$$\bar{u}(t_N, x) = \varphi(t_N, x),\ x \in [\alpha, \beta], \tag{5.1}$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}) + \frac{1}{2}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h})$$

$$+\bar{g}_{k,j} \cdot h,\ \text{if } x_j + \bar{b}_{k,j} \cdot h \pm \bar{\sigma}_{k,j} \cdot \sqrt{h} \in [\alpha, \beta];$$

$$\bar{u}(t_k, x_j) = \varphi(t_{k+1-\bar{\gamma}_{k,j}}, \alpha) - g(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \bar{\gamma}_{k,j} h$$

$$-\frac{1}{2}\bar{u}(t_{k+1}, \alpha + b(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \bar{\gamma}_{k,j} h + \sigma(t_k, \alpha, \varphi(t_{k+1}, \alpha)) \cdot \sqrt{\bar{\gamma}_{k,j} h})$$

$$+\frac{1}{2}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h}) + \bar{g}_{k,j} \cdot h,\ \text{if } x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h} < \alpha;$$

$$\bar{u}(t_k, x_j) = \frac{1}{2}\bar{u}(t_{k+1}, x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}) + \varphi(t_{k+1-\bar{\delta}_{k,j}}, \beta) - g(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \bar{\delta}_{k,j} h$$

$$-\frac{1}{2}\bar{u}(t_{k+1}, \beta + b(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \bar{\delta}_{k,j} h - \sigma(t_k, \beta, \varphi(t_{k+1}, \beta)) \cdot \sqrt{\bar{\delta}_{k,j} h})$$

$$+\bar{g}_{k,j} \cdot h,\ \text{if } x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h} > \beta;\ j = 1, 2, ..., M-1\ ,$$

$$\bar{u}(t_k, x) = \frac{x_{j+1} - x}{h_x}\ \bar{u}(t_k, x_j) + \frac{x - x_j}{h_x}\ \bar{u}(t_k, x_{j+1}),\ x_j \leq x \leq x_{j+1}, \tag{5.2}$$

$$j = 0, 1, 2, ..., M-1\ ,\ k = N-1, ..., 1, 0,$$

where $\bar{b}_{k,j}$, $\bar{\sigma}_{k,j}$, $\bar{g}_{k,j}$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$, $g(t,x,u)$ calculated at the point $(t_k, x_j, \bar{u}(t_{k+1}, x_j))$ and $0 < \bar{\gamma}_{k,j}$, $\bar{\delta}_{k,j} \leq 1$ are the roots of the equations (2.13) and (2.14) with the right sides $x_j + \bar{b}_{k,j} \cdot h - \bar{\sigma}_{k,j} \cdot \sqrt{h}$ and $x_j + \bar{b}_{k,j} \cdot h + \bar{\sigma}_{k,j} \cdot \sqrt{h}$ respectively.

Theorem 5.1. *If the value of h_x is taken equal to $\varkappa h$, $\varkappa$ is a positive constant, then under the assumptions (i) and (ii) the algorithm (5.1)-(5.2) has the first order of convergence, i.e., the approximation $\bar{u}(t_k, x)$ from the formula (5.2) satisfies the relation*

$$|\bar{u}(t_k, x) - u(t_k, x)| \leq Kh, \tag{5.3}$$

where K does not depend on x, h, k.

The proof of Theorem 5.1 differs only little from the proof of the corresponding theorem in [5] and is therefore omitted.

Remark 5.1. It is not difficult to prove that the algorithm based on the method (4.1) and linear interpolation has the global error $O(h^{1/2})$ if we choose the space step $h_x = \varkappa h^{3/4}$.

Remark 5.2. It is natural to attract cubic interpolation instead of the linear one for constructing numerical algorithms. Exploitation of cubic interpolation allows us to take the space step $h_x = \varkappa\sqrt{h}$ (in contrast to $h_x = \varkappa h$ for the linear interpolation) and, thus, to reduce the volume of computations. Moreover, if we use cubic interpolation, we can avoid special formulas near the boundary choosing some appropriate $\varkappa$ (indeed, we can take, e.g., $\varkappa = 2 \max\limits_{t\in[t_0,T],\, x\in\overline{G},\, u\in[u_\circ,\, u^\circ]} \sigma(t,x,u)$, then for a sufficiently small h the points $x_j + \bar{b}_{k,j} \cdot h \pm \bar{\sigma}_{k,j} \cdot \sqrt{h}$ always belong to $[\alpha, \beta]$). Unfortunately, we do not succeed in proving a convergence theorem in the case of cubic interpolation. The way of proving Theorem 5.1 gives us some restriction on the type of interpolation procedure which we can use for constructing the numerical algorithm. The restriction is such that the sum of the absolute values of the coefficients staying at $\bar{u}(t_k, \cdot)$ in the interpolation procedure must be not greater than 1. Linear interpolation and B-splines of the order $O(h_x^2)$ satisfy this restriction. But cubic interpolation of the order $O(h_x^4)$ does not satisfy the restriction. In Section 7.1 we test an algorithm based on cubic interpolation. The tests give fairly good results. See also some theoretical explanations and numerical tests in [5, 6].

Remark 5.3. Clearly, the algorithms can be considered with variable time steps and space steps. An algorithm with variable space steps is used in our numerical tests (Section 7.1).

6. Extension to the multi-dimensional Dirichlet problem

In this section we generalize the layer method (4.1) to the multi-dimensional case ($d > 1$). A generalization of the layer method (2.16) to the multi-dimensional case is complicated and it is not considered here.

As it has been mentioned in Introduction, layer methods are feasible if the dimension d of the domain G is not more than 3. That is why, we restrict ourselves here to the cases $d = 2$ and $d = 3$. We mark only that it is not difficult to generalize the layer method (4.1) for an arbitrary d.

Consider the case $d = 2$. Introduce the notation ${}_iX_{k+1} := ({}_iX^1_{k+1}, {}_iX^2_{k+1})$,

$${}_iX^1_{k+1} = x^1 + \bar{b}^1_k h + \bar{\sigma}^{11}_k \sqrt{h} \cdot {}_i\xi^1 + \bar{\sigma}^{12}_k \sqrt{h} \cdot {}_i\xi^2,$$

$${}_iX^2_{k+1} = x^2 + \bar{b}^2_k h + \bar{\sigma}^{21}_k \sqrt{h} \cdot {}_i\xi^1 + \bar{\sigma}^{22}_k \sqrt{h} \cdot {}_i\xi^2,$$

$$i = 1,2,3,4,\ x = (x^1, x^2) \in G \subset R^2,$$

where ${}_1\xi = (-1,-1)$, ${}_2\xi = (-1,1)$, ${}_3\xi = -{}_1\xi$, ${}_4\xi = -{}_2\xi$ and $\bar{b}_k = (\bar{b}_k^1, \bar{b}_k^2)$, $\bar{\sigma}_k = \{\bar{\sigma}_k^{jl}\}$ are the coefficients $b(t,x,u)$, $\sigma(t,x,u)$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$.

If the point $x = (x^1, x^2) \in G$ is sufficiently far from the boundary ∂G (more precisely, if the points ${}_iX_{k+1}$, $i = 1,2,3,4$, belong to $\overline{G}$), the layer method has the form (cf. [5]):

$$\bar{u}(t_k, x^1, x^2) = \sum_{i=1}^{4} \frac{1}{4} \bar{u}(t_{k+1},\ {}_iX_{k+1}^1,\ {}_iX_{k+1}^2) + \bar{g}_k \cdot h, \tag{6.1}$$

where $\bar{g}_k$ is the coefficient $g(t,x,u)$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$.

If the point $x = (x^1, x^2) \in G$ is close to the boundary ∂G, then some of the points ${}_iX_{k+1} = ({}_iX_{k+1}^1,\ {}_iX_{k+1}^2)$, $i = 1,2,3,4$, may be outside of the domain $\overline{G}$. Let us connect the point x with the points ${}_{i^*}X_{k+1}$, which are outside of $\overline{G}$, by the curves $\psi_{i^*}(\lambda) = (\psi_{i^*}^1(\lambda), \psi_{i^*}^2(\lambda))$:

$$\psi_{i^*}^1(\lambda) = x^1 + \bar{b}_k^1 \lambda h + \bar{\sigma}_k^{11} \sqrt{\lambda h} \cdot {}_{i^*}\xi^1 + \bar{\sigma}_k^{12} \sqrt{\lambda h} \cdot {}_{i^*}\xi^2,$$

$$\psi_{i^*}^2(\lambda) = x^2 + \bar{b}_k^2 \lambda h + \bar{\sigma}_k^{21} \sqrt{\lambda h} \cdot {}_{i^*}\xi^1 + \bar{\sigma}_k^{22} \sqrt{\lambda h} \cdot {}_{i^*}\xi^2,\ 0 \le \lambda \le 1.$$

Due to the smoothness of the boundary ∂G, under a sufficiently small h there is a unique value of $\lambda = {}_{i^*}\bar{\lambda}_k$, $0 < {}_{i^*}\bar{\lambda}_k < 1$, such that the point ${}_{i^*}\eta_k = ({}_{i^*}\eta_k^1,\ {}_{i^*}\eta_k^2)$, where

$${}_{i^*}\eta_k^1 = x^1 + \bar{b}_k^1 \cdot {}_{i^*}\bar{\lambda}_k h + \bar{\sigma}_k^{11} \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^1 + \bar{\sigma}_k^{12} \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^2,$$

$${}_{i^*}\eta_k^2 = x^2 + \bar{b}_k^2 \cdot {}_{i^*}\bar{\lambda}_k h + \bar{\sigma}_k^{21} \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^1 + \bar{\sigma}_k^{22} \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^2,$$

belongs to the boundary ∂G (of course, ∂G is supposed to be sufficiently smooth).

Put ${}_j\bar{\lambda}_k = 1$ and ${}_j\eta_k = {}_jX_{k+1}$ for the points ${}_jX_{k+1}$ belonging to $\overline{G}$. Then the layer method takes the form

$$\begin{aligned}
\bar{u}(t_k, x^1, x^2) = {} & \frac{\sqrt{{}_2\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k}}{(\sqrt{{}_1\bar{\lambda}_k} + \sqrt{{}_3\bar{\lambda}_k})(\sqrt{{}_1\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k} + \sqrt{{}_2\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k})} \bar{u}(t_{k+{}_1\bar{\lambda}_k},\ {}_1\eta_k^1,\ {}_1\eta_k^2) \\
& + \frac{\sqrt{{}_1\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k}}{(\sqrt{{}_2\bar{\lambda}_k} + \sqrt{{}_4\bar{\lambda}_k})(\sqrt{{}_1\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k} + \sqrt{{}_2\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k})} \bar{u}(t_{k+{}_2\bar{\lambda}_k},\ {}_2\eta_k^1,\ {}_2\eta_k^2) \\
& + \frac{\sqrt{{}_1\bar{\lambda}_k \cdot {}_2\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k}}{(\sqrt{{}_1\bar{\lambda}_k} + \sqrt{{}_3\bar{\lambda}_k})(\sqrt{{}_1\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k} + \sqrt{{}_2\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k})} \bar{u}(t_{k+{}_3\bar{\lambda}_k},\ {}_3\eta_k^1,\ {}_3\eta_k^2) \\
& + \frac{\sqrt{{}_1\bar{\lambda}_k \cdot {}_2\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k}}{(\sqrt{{}_2\bar{\lambda}_k} + \sqrt{{}_4\bar{\lambda}_k})(\sqrt{{}_1\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k} + \sqrt{{}_2\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k})} \bar{u}(t_{k+{}_4\bar{\lambda}_k},\ {}_4\eta_k^1,\ {}_4\eta_k^2) \\
& + \bar{g}_k \cdot \frac{2\sqrt{{}_1\bar{\lambda}_k \cdot {}_2\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k}}{\sqrt{{}_1\bar{\lambda}_k \cdot {}_3\bar{\lambda}_k} + \sqrt{{}_2\bar{\lambda}_k \cdot {}_4\bar{\lambda}_k}} h.
\end{aligned} \tag{6.2}$$

Recall that if ${}_i\eta_k = ({}_i\eta_k^1,\ {}_i\eta_k^2) \in \partial G$ then $\bar{u}(t_{k+{}_i\bar{\lambda}_k},\ {}_i\eta_k^1,\ {}_i\eta_k^2) = \varphi(t_{k+{}_i\bar{\lambda}_k},\ {}_i\eta_k^1,\ {}_i\eta_k^2)$ (see (1.2)).

The error of the one-step approximation corresponding to (6.2) is of the order $O(h^{3/2})$, and the layer method (6.1)-(6.2) has the global error estimated by $O(h^{1/2})$ (see Remark

4.3 as well). These assertions can be checked directly without attracting some new ideas in comparison with Lemma 3.1 and Theorem 3.1.

Now consider the case $d = 3$. Introduce the notation ${}_iX_{k+1} = ({}_iX^1_{k+1},\ {}_iX^2_{k+1},\ {}_iX^3_{k+1})$, $i = 1, 2, \dots, 8$, where

$$ {}_iX^j_{k+1} := x^j + \bar{b}^j_k h + \bar{\sigma}^{j1}_k \sqrt{h} \cdot {}_i\xi^1 + \bar{\sigma}^{j2}_k \sqrt{h} \cdot {}_i\xi^2 + \bar{\sigma}^{j3}_k \sqrt{h} \cdot {}_i\xi^3,\ j = 1, 2, 3, $$

$$ x = (x^1, x^2, x^3) \in G \subset R^3. $$

Here $\bar{b}_k = \{\bar{b}^j_k\}$ and $\bar{\sigma}_k = \{\bar{\sigma}^{jl}_k\}$ are the coefficients $b(t, x, u)$ and $\sigma(t, x, u)$ calculated at the point $(t_k, x, \bar{u}(t_k, x))$ and ${}_i\xi = ({}_i\xi^1,\ {}_i\xi^2,\ {}_i\xi^3)$, $i = 1, \dots, 8$, are the following vectors:

$$ {}_1\xi = (-1, -1, -1),\ {}_2\xi = (-1, -1, 1),\ {}_3\xi = (-1, 1, -1),\ {}_4\xi = (1, -1, -1), $$

$$ {}_{i+4}\xi = -\,{}_i\xi,\ i = 1, 2, 3, 4. $$

If the points ${}_iX_{k+1}$, $i = 1, 2, \dots, 8$, belong to $\overline{G}$, the layer method has the form

$$ \bar{u}(t_k, x) = \sum_{i=1}^{8} \frac{1}{8} \bar{u}(t_{k+1},\ {}_iX_{k+1}) + \bar{g}_k \cdot h, \tag{6.3} $$

where $\bar{g}_k$ is the coefficient $g(t, x, u)$ calculated at the point $(t_k, x, \bar{u}(t_{k+1}, x))$.

If some points ${}_{i^*}X_{k+1} \notin \overline{G}$, we connect the point x with the points ${}_{i^*}X_{k+1}$ by the curves $\psi_{i^*}(\lambda) = (\psi^1_{i^*}(\lambda), \psi^2_{i^*}(\lambda), \psi^3_{i^*}(\lambda))$,

$$ \psi^j_{i^*}(\lambda) = x^j + \bar{b}^j_k\, \lambda h + \bar{\sigma}^{j1}_k \sqrt{\lambda h} \cdot {}_{i^*}\xi^1 + \bar{\sigma}^{j2}_k \sqrt{\lambda h} \cdot {}_{i^*}\xi^2 + \bar{\sigma}^{j3}_k \sqrt{\lambda h} \cdot {}_{i^*}\xi^3, $$

$$ j = 1, 2, 3,\ 0 \le \lambda \le 1. $$

Due to the smoothness of the boundary ∂G, under a sufficiently small h there is a unique value of $\lambda = {}_{i^*}\bar{\lambda}_k$, $0 < {}_{i^*}\bar{\lambda}_k < 1$, such that the point ${}_{i^*}\eta_k = ({}_{i^*}\eta^1_k,\ {}_{i^*}\eta^2_k,\ {}_{i^*}\eta^3_k)$, where

$$ {}_{i^*}\eta^j_k = x^j + \bar{b}^j_k \cdot {}_{i^*}\bar{\lambda}_k h + \bar{\sigma}^{j1}_k \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^1 + \bar{\sigma}^{j2}_k \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^2 + \bar{\sigma}^{j3}_k \sqrt{{}_{i^*}\bar{\lambda}_k h} \cdot {}_{i^*}\xi^3\,, $$

$$ j = 1, 2, 3, $$

belongs to the boundary ∂G.

Put ${}_j\bar{\lambda}_k = 1$ and ${}_j\eta_k = {}_jX_{k+1}$ for the points ${}_jX_{k+1}$ belonging to $\overline{G}$. Then the layer method takes the form

$$ \bar{u}(t_k, x) = \sum_{i=1}^{4} \frac{\gamma_k}{\sqrt{{}_i\bar{\lambda}_k} + \sqrt{{}_{i+4}\bar{\lambda}_k}} \Big(\frac{1}{\sqrt{{}_i\bar{\lambda}_k}} \bar{u}(t_{k+{}_i\bar{\lambda}_k},\ {}_i\eta_k) + \frac{1}{\sqrt{{}_{i+4}\bar{\lambda}_k}} \bar{u}(t_{k+{}_{i+4}\bar{\lambda}_k},\ {}_{i+4}\eta_k) \Big) \tag{6.4} $$

$$ + \bar{g}_k \cdot 4\gamma_k h, $$

where

$$ \gamma_k = \left(\sum_{j=1}^{4} \frac{1}{\sqrt{{}_i\bar{\lambda}_k \cdot {}_{i+4}\bar{\lambda}_k}} \right)^{-1}. $$

To construct the corresponding numerical algorithms, we attract linear interpolation as in the previous section. For example, consider the case $d = 2$. To this end put the domain $\overline{G}$ into a rectangle Π with corners (x^1_0, x^2_0), $(x^1_0, x^2_{M_2})$, $(x^1_{M_1}, x^2_0)$, $(x^1_{M_1}, x^2_{M_2})$ and introduce the equidistant space discretization of the rectangle Π:

$$ \Delta_{M_1,M_2} := \{(x^1_j, x^2_l) :\ x^1_j = x^1_0 + jh_{x^1},\ x^2_l = x^2_0 + lh_{x^2},\ j = 0, \dots, M_1,\ l = 0, \dots, M_2\}, $$

$$h_{x^1} = \frac{x^1_{M_1} - x^1_0}{M_1}, \ h_{x^2} = \frac{x^2_{M_1} - x^2_0}{M_2}.$$

The values of $\bar{u}(t_k, x^1_j, x^2_l)$ at the nodes of $\Delta_{M_1,M_2} \cap \overline{G}$ are found in accordance with (6.1)-(6.2). Let $(x^1, x^2) \in \overline{G}$ and $x^1_j \le x^1 \le x^1_{j+1}, \ x^2_l \le x^2 \le x^2_{l+1}$. If all the nodes $(x^1_j, x^2_l), \ (x^1_j, x^2_{l+1}), \ (x^1_{j+1}, x^2_l), \ (x^1_{j+1}, x^2_{l+1}) \in \overline{G}$, the value of $\bar{u}(t_k, x^1, x^2)$ is evaluated as

$$\bar{u}(t_k, x^1, x^2) = \frac{x^1_{j+1} - x^1}{h_{x^1}} \cdot \frac{x^2_{l+1} - x^2}{h_{x^2}} \bar{u}(t_k, x^1_j, x^2_l) + \frac{x^1_{j+1} - x^1}{h_{x^1}} \cdot \frac{x^2 - x^2_l}{h_{x^2}} \bar{u}(t_k, x^1_j, x^2_{l+1}) \tag{6.5}$$

$$+ \frac{x^1 - x^1_j}{h_{x^1}} \cdot \frac{x^2_{l+1} - x^2}{h_{x^2}} \bar{u}(t_k, x^1_{j+1}, x^2_l) + \frac{x^1 - x^1_j}{h_{x^1}} \cdot \frac{x^2 - x^2_l}{h_{x^2}} \bar{u}(t_k, x^1_{j+1}, x^2_{l+1}).$$

If the point $x = (x^1, x^2) : x^1_j \le x^1 \le x^1_{j+1}, \ x^2_l \le x^2 \le x^2_{l+1}$ is such that some of the nodes $(x^1_j, x^2_l), \ (x^1_j, x^2_{l+1}), \ (x^1_{j+1}, x^2_l), \ (x^1_{j+1}, x^2_{l+1})$ do not belong to $\overline{G}$, then we use some points on the boundary ∂G (due to (1.2) we know values of $u(t,x)$ for $x \in \partial G$) to find $\bar{u}(t_k, x^1, x^2)$ by linear interpolation.

If we take $h_{x^i} = \varkappa^i h^{3/4}$, $i = 1, 2$, $\varkappa^1, \varkappa^2 > 0$ are positive constants, the error of the proposed algorithm is estimated as $O(h^{1/2})$.

7. Numerical tests

In the previous sections we deal with semilinear parabolic equations with negative direction of time t : the equations are considered under $t < T$ and the "initial" conditions are given at $t = T$. This form of equations is suitable for the probabilistic approach which we use to construct numerical methods. Of course, the proposed methods are adaptable to semilinear parabolic equations with positive direction of time, and this adaptation is particularly easy in the autonomous case. In our numerical tests we use algorithms with positive direction of time (see, e.g., (7.13)-(7.14)).

7.1. **The Burgers equation.** Consider the Dirichlet problem for the one-dimensional Burgers equation:

$$\frac{\partial u}{\partial t} = \frac{\varepsilon^2}{2} \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \ \ t > 0, \ x \in (-1, 1), \tag{7.1}$$

$$u(0, x) = -A \sin \pi x, \ x \in [-1, 1], \tag{7.2}$$

$$u(t, \pm 1) = 0, \ t > 0. \tag{7.3}$$

This problem was used for testing various numerical methods in, e.g., [23, 24] (see also references therein). By means of the Cole-Hopf transformation, one can find the explicit solution of the problem (7.1)-(7.3) in the forms:

$$u(t, x) = -A \frac{\displaystyle\int_{-\infty}^{\infty} \sin \pi (x - y) \, \exp(-\tfrac{A}{\pi \varepsilon^2} \cos \pi (x - y) - \tfrac{y^2}{2\varepsilon^2 t}) \, dy}{\displaystyle\int_{-\infty}^{\infty} \exp(-\tfrac{A}{\pi \varepsilon^2} \cos \pi (x - y) - \tfrac{y^2}{2\varepsilon^2 t}) \, dy} \tag{7.4}$$

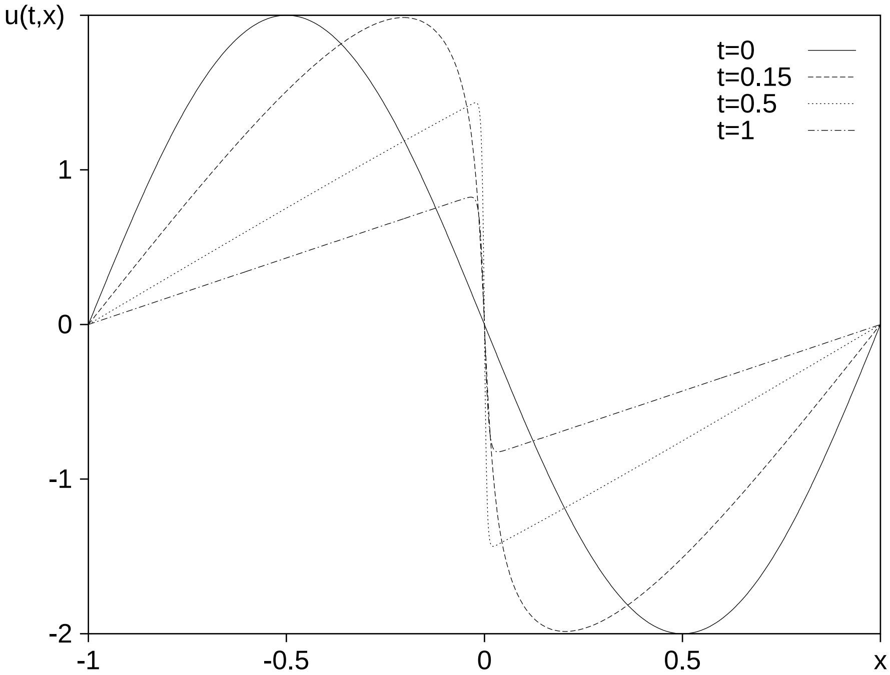

FIGURE 1. A typical solution $u(t,x)$ of the problem (7.1)-(7.3) for $\varepsilon = 0.1$, $A = 2$ and various time moments.

or

$$u(t,x) = \frac{\pi\varepsilon^2}{2} \frac{\sum_{n=1}^{\infty} n a_n \exp(-\frac{1}{8}\varepsilon^2\pi^2 n^2 t) \sin \frac{1}{2}\pi n(x+1)}{\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \exp(-\frac{1}{8}\varepsilon^2\pi^2 n^2 t) \cos \frac{1}{2}\pi n(x+1)} \tag{7.5}$$

with

$$a_n = \int_{-1}^{1} \exp(-\frac{A}{\pi\varepsilon^2} \cos \pi x) \cos \frac{1}{2}\pi n(x+1)\, dx.$$

We shall simulate the problem (7.1)-(7.3) on relatively small time intervals $[0,T]$, where the formula (7.4) is more convenient. For a small ε, there is a thin internal layer, where the solution to (7.1)-(7.3) has singular behavior (see, e.g., [25] and references therein). Derivatives of the solution go to infinity as $\varepsilon \to 0$. A typical behavior of the solution is demonstrated on Fig. 1.

Here we test the following three algorithms: the algorithm (5.1)-(5.2), the algorithm based on the layer method (4.1) and linear interpolation, and the algorithm based on cubic interpolation (see also Remark 5.2). In the algorithm (5.1)-(5.2) and the algorithm based on the layer method (4.1) and linear interpolation we take the space step h_x being equal to the time step h.

The algorithm based on cubic interpolation in the case of the problem (7.1)-(7.3) has the form (cf. [6])

$$\bar{u}(0,x) = -A \sin \pi x, \; x \in [-1,1], \tag{7.6}$$

$$\bar{u}(t_{k+1}, x_0) = \bar{u}(t_{k+1}, -1) = 0,$$

$$\bar{u}(t_{k+1}, x_M) = \bar{u}(t_{k+1}, 1) = 0,$$

TABLE 1. The Burgers equation. Dependence of the errors $err^c(t)$ and $err^l(t)$ in h for the algorithms (5.1)-(5.2) and (7.6) under $t = 0.5$, $\varepsilon = 0.1$, and $A = 2$.

h	algorithm (5.1)-(5.2)		algorithm (7.6)	
	$err^c(t)$	$err^l(t)$	$err^c(t)$	$err^l(t)$
0.01	$1.239 \cdot 10^{-1}$	$3.035 \cdot 10^{-2}$	$1.854 \cdot 10^{-1}$	$3.081 \cdot 10^{-2}$
0.0016	$4.574 \cdot 10^{-2}$	$5.311 \cdot 10^{-3}$	$5.855 \cdot 10^{-2}$	$5.481 \cdot 10^{-3}$
0.0001	$2.673 \cdot 10^{-3}$	$3.288 \cdot 10^{-4}$	$3.737 \cdot 10^{-3}$	$3.466 \cdot 10^{-3}$
0.000016	$4.261 \cdot 10^{-4}$	$5.259 \cdot 10^{-5}$	$5.919 \cdot 10^{-4}$	$5.527 \cdot 10^{-5}$

$$\bar{u}(t_{k+1}, x_j) = \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j) - \varepsilon h^{1/2}) + \frac{1}{2}\bar{u}(t_k, x_j - h\bar{u}(t_k, x_j) + \varepsilon h^{1/2}),$$

$$j = 1, \ldots, M-1,$$

$$\bar{u}(t_k, x) = \sum_{i=0}^{3} \Phi_{j,i}(x)\bar{u}(t_k, x_{j+i}), \quad x_j < x < x_{j+3},$$

$$\Phi_{j,i}(x) = \prod_{m=0, m\neq i}^{3} \frac{x - x_{j+m}}{x_{j+i} - x_{j+m}},$$

$$k = 0, \ldots, N-1.$$

Here we use a nonequidistant discretization of the interval $[-1, 1]$. In the thin internal layer (in a neighborhood of $x = 0$) we take $h_x := x_{j+1} - x_j = \varepsilon\sqrt{h}$ and outside the layer $h_x = \sqrt{h}$. Since $\varepsilon \ll 1$ in our experiments and $h_x = \sqrt{h}$ for nodes x_j close to the end of the interval $[-1, 1]$, it is clear that the points $x_j - h\bar{u}(t_k, x_j) \pm \varepsilon h^{1/2}$, $j = 1, \ldots, M-1$, always belong to the interval $(-1, 1)$. Thus, we avoid using special formulas near the boundary in (7.6) (see Remark 5.2 as well).

Table 1 gives numerical results obtained by using the algorithms (5.1)-(5.2) and (7.6). The algorithm based on the layer method (4.1) and linear interpolation gives results being practically identical to the ones for (5.1)-(5.2). We present the errors of the approximate solutions $\bar{u}$ in the discrete Chebyshov norm and in l^1-norm:

$$err^c(t) = \max_{x_i} |\bar{u}(t, x_i) - u(t, x_i)|,$$

$$err^l(t) = \sum_i |\bar{u}(t, x_i) - u(t, x_i)| \cdot h_x\,.$$

7.2. **Quasilinear equation with power law nonlinearities.** Consider the Dirichlet problem for quasilinear parabolic equation with power law nonlinearities [4, 19]

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}(u^q \frac{\partial u}{\partial x}), \ t \in (0, 1), \ x > 0, \ q > 0, \tag{7.7}$$

with the initial condition

$$u(0, x) = (1 - x/L)^{2/q}, \ x \in [0, L], \tag{7.8}$$

$$u(0, x) = 0, \ x > L,$$

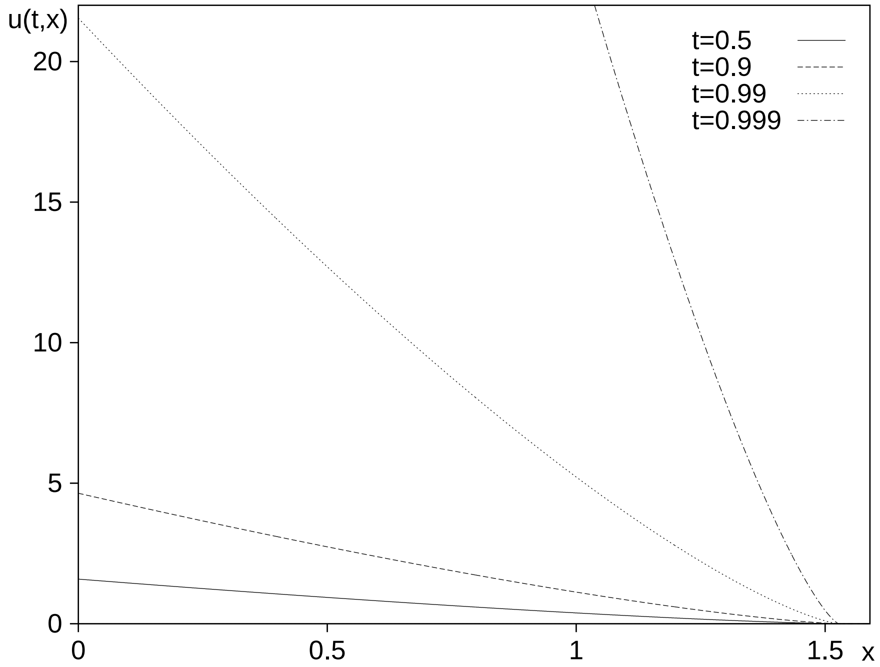

FIGURE 2. A typical solution $u(t,x)$ of the problem (7.7)-(7.9) for $q = 1.5$ and various time moments.

and the boundary regime

$$u(t,0) = (1-t)^{-1/q}, \ t \in [0,1), \tag{7.9}$$

where $L = \sqrt{(q+2)/q}$.

The exact solution to this problem has the form [4, 19]

$$u(t,x) = \left(\frac{1-x/L}{\sqrt{1-t}}\right)^{2/q} \text{ for } x \in [0,L]$$

and

$$u(t,x) = 0 \text{ for } x > L.$$

The temperature $u(t,x)$ grows infinitely as $t \to 1$. At the same time the heat remains being localized in the interval $[0,L)$. Figure 2 presents a typical behavior of the solution to (7.7)-(7.9).

The equation (7.7) is not of the form (2.1). The function

$$v = u^{q+1}$$

satisfies the problem

$$\frac{\partial v}{\partial t} = \frac{1}{2} v^{q/(q+1)} \frac{\partial^2 v}{\partial x^2}, \ t \in (0,1), \ x > 0, \tag{7.10}$$

$$v(0,x) = (1-x/L)^{2(q+1)/q}, \ x \in [0,L], \tag{7.11}$$

$$v(0,x) = 0, \ x > L,$$

$$v(t,0) = (1-t)^{-(q+1)/q}, \ t \in [0,1). \tag{7.12}$$

The equation (7.10) has the form (2.1).

TABLE 2. Quasilinear equation with power law nonlinearities. Dependence of errors $err_{\bar v}(t,h)$ (top position) and $err_{\bar u}(t,h)$ (lower position) in h and t for the algorithm (7.13)-(7.14) under $q = 1.5$.

	$h = 10^{-1}$	$h = 10^{-2}$	$h = 10^{-3}$	$h = 10^{-4}$
$t = 0.5$	$0.8664 \cdot 10^{-1}$ $0.3542 \cdot 10^{-1}$	$0.8786 \cdot 10^{-2}$ $0.7693 \cdot 10^{-2}$	$0.9705 \cdot 10^{-3}$ $1.685 \cdot 10^{-3}$	$1.018 \cdot 10^{-4}$ $3.622 \cdot 10^{-4}$
$t = 0.9$	> 5 $5.910 \cdot 10^{-1}$	$8.094 \cdot 10^{-1}$ $8.109 \cdot 10^{-2}$	$8.265 \cdot 10^{-2}$ $8.656 \cdot 10^{-3}$	$8.817 \cdot 10^{-3}$ $8.918 \cdot 10^{-4}$

We simulate the solution to (7.10)-(7.12) by two algorithms: the algorithm (5.1)-(5.2) and the algorithm based on the layer method (4.1) and linear interpolation. The last one in the case of the problem (7.10)-(7.12) has the form

$$\bar v(0,x) = \begin{cases} (1 - x/L)^{2(q+1)/q}, \ x \in [0,L], \\ 0, \ x \in (L,\infty), \end{cases} \tag{7.13}$$

$$\bar v(t_{k+1}, x_j) = \frac{1}{2}\bar v(t_k, x_j - (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}) + \frac{1}{2}\bar v(t_k, x_j + (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}),$$

$$\text{if } x_j - (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h} \geq 0;$$

$$\bar v(t_{k+1}, x_j) = \frac{1}{1+\sqrt{\bar\lambda_k}}(1 - t_{k+1-\bar\lambda_k})^{-(q+1)/q} + \frac{\sqrt{\bar\lambda_k}}{1+\sqrt{\bar\lambda_k}}\bar v(t_k, x_j + (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}),$$

$$\bar\lambda_k = \left(\frac{x_j}{(\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h}}\right)^2, \ \text{if } x_j - (\bar v(t_k,x_j))^{q/2(q+1)} \cdot \sqrt{h} < 0;$$

$$\bar v(t_{k+1}, x) = \frac{x_{j+1} - x}{h_x}\,\bar v(t_{k+1}, x_j) + \frac{x - x_j}{h_x}\,\bar v(t_{k+1}, x_{j+1}), \ x_j \leq x \leq x_{j+1}, \tag{7.14}$$

$$j = 0,1,2,\ldots, \ k = 1,\ldots,N,$$

where $x_j = j \cdot h_x$, $t_k = k \cdot h$.

In our tests we take $h_x = h$. Tables 2 and 3 give numerical results obtained by using the algorithm (7.13)-(7.14). The algorithm (5.1)-(5.2) gives similar results and they are omitted here. Let us mark that in the experiments the algorithm (7.13)-(7.14) based on the layer method (4.1) behaves as an algorithm of the order $O(h)$ while due to Theorem 4.1 the layer method (4.1) has the accuracy order $O(h^{1/2})$ only (see also Remark 4.3). Table 2 presents the errors

$$err_{\bar v}(t,h) := \max_j |\bar v(t,x_j) - v(t,x_j)|,$$

$$err_{\bar u}(t,h) := \max_j |\bar u(t,x_j) - u(t,x_j)|, \ \bar u(t,x_j) = (\bar v(t,x_j))^{1/(q+1)}.$$

For times t which are close to the explosion time $t = 1$ the functions $u(t,x)$ and $v(t,x)$ take big values and the absolute errors become fairly large. In Table 3 we present the relative error

$$\delta(t,h) := \frac{err_{\bar u}(t,h)}{u(t,0)}$$

TABLE 3. Quasilinear equation with power law nonlinearities. Dependence of the relative error $\delta(t,h)$ in h and t for the algorithm (7.13)-(7.14) under $q=1.5$.

	$h=10^{-1}$	$h=10^{-2}$	$h=10^{-3}$	$h=10^{-4}$
$t=0.9$	$1.273\cdot10^{-1}$	$1.747\cdot10^{-2}$	$1.865\cdot10^{-3}$	$1.921\cdot10^{-4}$
$t=0.99$	—	$1.392\cdot10^{-1}$	$1.789\cdot10^{-2}$	$1.913\cdot10^{-3}$
$t=0.999$	—	—	$1.398\cdot10^{-1}$	$1.801\cdot10^{-2}$
$t=0.9999$	—	—	—	$1.400\cdot10^{-1}$

at times close to the explosion.

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