Biodiversity of Catalytic Super-Brownian Motion

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Abstract

In this paper we investigate the structure of the equilibrium state of three–dimensional catalytic super-Brownian motion where the catalyst is itself a classical super-Brownian motion. We show that the reactant has an infinite local biodiversity or genetic abundance. This contrasts the finite local biodiversity of the equilibrium of classical super-Brownian motion.

Another question we address is that of extinction of the reactant in finite time or in the long–time limit in dimensions $d = 2, 3$. Here we assume that the catalyst starts in the Lebesgue measure and the reactant starts in a finite measure. We show that there is extinction in the long–time limit if $d = 2$ or 3. There is, however, no finite time extinction if $d = 3$ (for $d = 2$ this problem is left open). This complements a result of Dawson and Fleischmann (1997a) for $d = 1$ and again contrasts the behaviour of classical super-Brownian motion.

As a key tool for both problems we show that in $d = 3$ the reactant matter propagates everywhere in space immediately.

1 Introduction and results

Catalytic super-Brownian motion (CSBM) X^ϱ is the measure–valued (finite variance) branching diffusion on $\mathbb{R}^d$ where the local branching rate is given by a space–time varying medium ϱ, the so–called catalyst. For a survey on CSBM and a variety of different spatial branching models see Klenke (1999). The case on which we focus here is where ϱ is a random sample of classical super-Brownian motion (SBM). In order that the reactant is non–degenerate we have to restrict to $d \le 3$.

This model has been constructed in Dawson and Fleischmann (1997a) and has been considered under various aspects, for instance, also in Dawson and Fleischmann (1997b) and Fleischmann and Klenke (1999). This paper is meant to be concise – not self–contained. So we skip the usual heuristics and repetitive constructions and only refer to the above mentioned papers.

1.1 Biodiversity

The main subject of this paper is the *local biodiversity* or *genetic abundance* of the equilibrium states in $d = 3$. The investigation of biodiversity is a booming field in biology. Roughly speaking, biodiversity is a measure for the number of species per square meter in an ecosystem. Our ecosystem is the reactant of three–dimensional catalytic super–Brownian motion in a steady state. Before we make mathematical statements about its biodiversity we have to fix this notion.

It is well known that (if $d \ge 3$) SBM has a unique ergodic equilibrium with intensity i_c ($i_c > 0$). We denote by $\mathbb{P}_{-\infty,i_c\ell}$ the law of the corresponding equilibrium process $(\varrho_t)_{t\in\mathbb{R}}$. For fixed ϱ consider X^ϱ started at time t in $i_r\ell$, $i_r > 0$, (ℓ is the Lebesgue measure) and denote its law by $P^\varrho_{t,i_r\ell}$. Letting $t \to -\infty$ one obtains $P^\varrho_{-\infty,i_r\ell}$ and $(\varrho_t, X^\varrho_t)_{t\in\mathbb{R}}$ is the (bivariate) equilibrium process (see [DF97b]):

$$\mathbb{P}_{-\infty,i_c\ell}[P^\varrho_{-\infty,i_r\ell}[(\varrho_{T+t}, X^\varrho_{T+t})_{t\in\mathbb{R}} \in \bullet] \in \bullet] \quad \text{is independent of } T \in \mathbb{R}. \tag{1.1}$$

Furthermore $\mathbb{P}_{-\infty,i_c\ell}$–almost surely $E^\varrho_{-\infty,i_r\ell}[X^\varrho_0] = i_r\ell$, and $P^\varrho_{-\infty,i_r\ell}[X^\varrho_0 \in \bullet]$ is infinitely divisible.

Note that the outlined $\mathbb{P}$ refers to the medium and the italic P to the reactant. All other quantities' laws will be denoted by a bold $\mathbf{P}$. Expectations will be denoted by the symbol E in the respective font.

We are interested in the number of families that contribute to $X^\varrho_0(B)$ for, say, the unit ball B. To make this notion precise recall that an infinitely divisible random measure Y (with values in $\mathcal{M}(\mathbb{R}^d)$, the space of Radon measures on $\mathbb{R}^d$) has a cluster representation

$$Y = \alpha + \sum_i \chi_i, \tag{1.2}$$

where $\alpha \in \mathcal{M}(\mathbb{R}^d)$ is the deterministic component of Y (or the essential infimum of Y). The $\chi_i \in \mathcal{M}(\mathbb{R}^d)$ are the "points" of a Poissonian point process on $\mathcal{M}(\mathbb{R}^d) \setminus \{0\}$ with intensity measure $\mathbf{Q}$ which is called the *canonical measure* of Y. We can reformulate (1.2) as the classical Lévy-Hincin formula for the Laplace transforms

$$-\log \mathbf{E}\big[e^{-\langle Y,\varphi\rangle}\big] = \langle \alpha, \varphi\rangle + \int_{\mathcal{M}(\mathbb{R}^d)} \mathbf{Q}(d\chi)(1 - e^{-\langle \chi,\varphi\rangle}). \tag{1.3}$$

Here $\varphi \in C_c^+(\mathbb{R}^d)$ (the space of nonnegative continuous functions on $\mathbb{R}^d$ with compact support) and $\langle \mu, \varphi\rangle$ denotes the integral $\int \varphi\, d\mu$. We also write $||\mu|| = \langle \mu, \mathbf{1}\rangle$ for the total mass of μ.

If $\alpha = 0$ then the number of families in B (that is $\#\{i : \chi_i(B) > 0\}$) has a Poisson distribution with expectation $\mathbf{Q}(\chi :\ \chi(B) > 0)$. If $\alpha(B) > 0$ then a "continuum of families" contributes to $Y(B)$. This motivates the following definition.

Definition 1.1 We say that the *local biodiversity* of the infinitely divisible random measure Y is

- *finite*, if $\alpha = 0$ and $\mathbf{Q}(\chi :\ \chi(B) > 0) < \infty$ for every compact set B,
- *countably infinite*, if $\alpha = 0$ and $\mathbf{Q}(\chi :\ \chi(B) > 0) = \infty$ for every open set $B \neq \emptyset$,
- *uncountably infinite*, if $\alpha(B) > 0$ for every open set $B \neq \emptyset$.

Note that this distinction is exhaustive if the distribution of Y is translation invariant.

As a trivial example we would like to mention $Y = \mathcal{S}_t\mu$, where $\mathcal{S}_t$ is the heat flow at time $t > 0$ and $\mu \neq 0$ is a finite measure. In this case obviously Y has uncountably infinite local biodiversity.

As a second example we consider in $d \geq 3$ the equilibrium super–Brownian motion $(\varrho_t)_{t\in\mathbb{R}}$. It is easily verified that ϱ_0 has finite local biodiversity. In fact, for general Y to have finite local biodiversity it is sufficient and necessary that

$$\mathbf{P}[Y(B) = 0] > 0 \quad \text{for any compact set } B. \tag{1.4}$$

This follows from the simple observation that if $\alpha = 0$

$$\mathbf{Q}(\chi :\ \chi(B) > 0) = -\log \mathbf{P}[Y(B) = 0]. \tag{1.5}$$

Coming back to the equilibrium of super–Brownian motion, it is easily verified that for every compact set B, $\mathbb{P}_{-\infty,i_c\ell}[\varrho_0(B) = 0] > 0$.

The situation is quite different for CSBM and this is the content of our main result.

Theorem 1 (Biodiversity) *Let $d = 3$. For $\mathbb{P}_{-\infty,i_c\ell}$–almost all ϱ the random measure X_0^ϱ (under the distribution $P^\varrho_{-\infty,i_r\ell}$) has countably infinite local biodiversity.*

The intuitive reasons for this behaviour are that

(i) In three dimensions the catalyst ϱ lives on such a thin time–space set that small amounts of reactant mass can percolate to B along catalyst free regions. In contrast, this is not possible for classical SBM: Here too small portions of mass (immigrating from outer space) get killed before they reach B.

(ii) The catalyst is not that thin that the reactant could sustain a deterministic component. Thus its genetic abundance is not as "rich" as that of the heat flow.

1.2 Instantaneous propagation of matter

They key ingredient for the proof of Theorem 1 is an instantaneous propagation of the reactant matter. Like the heat flow, the three–dimensional reactant spills out mass everywhere in space immediately. This property contrasts the compact support property of classical SBM (see Iscoe (1988)) and, for example, one–dimensional CSBM where the (time–homogeneous) catalyst is a certain stable random measure (see Dawson, Li and Mueller (1995)). An instantaneous propagation of matter for a super-Lévy process was first established by Perkins (1990); see also Evans and Perkins (1991) for a generalization and a slimmer proof.

Before we formulate our proposition we introduce some notation which helps defining CSBM in a somewhat more general setup. Let $\mathcal{M}_f(\mathbb{R}^d) = \{\mu \in \mathcal{M}(\mathbb{R}^d) : \|\mu\| < \infty\}$, and define the space of tempered measures $\mathcal{M}'(\mathbb{R}^d) = \bigcup_{p>d} \mathcal{M}_p(\mathbb{R}^d)$, where

$$\mathcal{M}_p(\mathbb{R}^d) = \{\mu \in \mathcal{M}(\mathbb{R}^d) : \langle \mu, (1 + \|\bullet\|^2)^{-p/2} \rangle < \infty\}.$$

$\mathcal{M}'(\mathbb{R}^d)$ is the state space for both ϱ and X^ϱ. Denote by $\mathbb{P}_{t,\mu}$ the law of ϱ started at time t in the state $\mu \in \mathcal{M}'(\mathbb{R}^d)$ and for given ϱ by $P^\varrho_{t,m}$ the law of X^ϱ started at time t in the state $m \in \mathcal{M}'(\mathbb{R}^d)$. Let $\ll$ denote absolute continuity and $\approx$ equivalence of measures.

In order that X^ϱ can be defined properly an additional restriction applies to μ. The crucial property is that we can define the collision local time (see Evans and Perkins (1994), Theorem 4.1) of a Brownian particle with ϱ started in μ as a so–called *nice branching functional*. We call such a μ *admissible*. The class of admissible μ has not been characterized yet. However μ is known to be admissible if, for example, it is "η–diffusive" in the sense of [FK99]. Here we only mention that the Lebesgue measure ℓ, any $\mu \ll \ell$ with bounded density and $\mathbb{P}_{t,i_c\ell}$–almost all ϱ_0 are admissible, where $t \in [-\infty, 0)$ if $d = 3$ and $t \in (-\infty, 0)$ if $d = 2$. (This has been shown for in [FK99] only for $t \in (-\infty, 0)$ but follows easily for $d = 3$ and $t = -\infty$. In fact, η–diffusivity is essentially a local property. However, for any compact $B \subset \mathbb{R}^3$, the total variation $\|\mathbb{P}_{t,i_c\ell}[\varrho_0|_B \in \bullet] - \mathbb{P}_{-\infty,i_c\ell}[\varrho_0|_B \in \bullet]\|_{TV}$ tends to 0 as $t \to -\infty$, as can be seen by a simple cluster decomposition, e.g.)

Now we can formulate our proposition on the instantaneous propagation of the reactant matter.

Proposition 1.2 (Instantaneous propagation of matter) *Assume $d = 3$, that $\mu \in \mathcal{M}'(\mathbb{R}^d)$ is admissible and that $m \in \mathcal{M}'(\mathbb{R}^d)$, $m \neq 0$. Then for all $t > 0$,*

$$\mathbb{E}_{0,\mu}[P^\varrho_{0,m}[\ell \ll X^\varrho_t \,|\, X^\varrho_t \neq 0]] = 1. \tag{1.6}$$

Together with the result of [FK99] saying that $\mathbb{P}_{0,\mu}$–a.s. the reactant's states X^ϱ_t are absolutely continuous w.r.t. ℓ we get

Corollary 1.3 *Assume $d = 3$, that $\mu \in \mathcal{M}'(\mathbb{R}^d)$ is admissible and that $m \in \mathcal{M}'(\mathbb{R}^d)$, $m \neq 0$. Then for all $t > 0$,*

$$\mathbb{E}_{0,\mu}[P^\varrho_{0,m}[\ell \approx X^\varrho_t \,|\, X^\varrho_t \neq 0]] = 1. \tag{1.7}$$

The reason why Proposition 1.2 is true is that in $d = 3$ the catalyst is so thin that it will not hit thin (time–space) cylinders connecting two points. Through those tubes reactant mass propagates from one point to all other points in space immediately. It might seem reasonable to expect such a behaviour also for $d = 2$. However here the catalyst does hit the tubes (more formally: in $d = 2$ lines are not polar for super–Brownian motion). In order to mimic an argument as for $d = 3$ one would have to establish a percolation argument for the complement of the time–space support of two–dimensional super–Brownian motion.

1.3 Finite mass extinction

Another question we address in this paper is that of long–term extinction or finite time extinction of finite reactant masses in a catalyst started in Lebesgue measure. More precisely, assume that $\varrho_0 = i_c\ell$, $i_c > 0$, and $X_0 = m \in \mathcal{M}_f(\mathbb{R}^d)$. Is it true that $X_t^\varrho \to 0$, $\mathbb{E}_{0,i_c\ell}[P^\varrho_{0,m}]$–almost surely or even $\mathbb{E}_{0,i_c\ell}[P^\varrho_{0,m}[X_t^\varrho = 0]] \to 1$ as $t \to \infty$?

The corresponding question for classical SBM is very simple to answer. Assume for the moment that the SBM ϱ is started with a finite initial measure $\mu \in \mathcal{M}_f(\mathbb{R}^d)$. Then the total mass process $(\|\varrho_t\|)_{t\geq 0}$ is simply Feller's branching diffusion with initial mass $\|\mu\|$ (this is the diffusion on $[0,\infty)$ with infinitesimal generator $\sqrt{2x}\frac{\partial^2}{\partial x^2}$). Hence $\mathbb{P}_{0,\mu}[\varrho_t = 0] = \exp(-\|\mu\|/t)$ and we have extinction in finite time:

$$\lim_{t\to\infty} \mathbb{P}_{0,\mu}[\varrho_t = 0] = 1, \qquad \mu \in \mathcal{M}_f(\mathbb{R}^d).$$

This is contrasted by the behaviour of the reactant X^ϱ. In [DF97a], Theorem 5, it is shown that if $d = 1$ then for $\mathbb{P}_{0,i_c\ell}$–a.a. ϱ under $P^\varrho_{0,m}$ the total mass process $(\|X_t^\varrho\|)_{t\geq 0}$ is an L^2–bounded martingale and hence converges almost surely to a random variable with full expectation $\|m\|$ and finite variance (persistence of second order).

However for $d = 2, 3$ the reactant's behaviour is quite different. In the long run the catalyst is not so scarce as in $d = 1$ and so we do not have persistence of finite reactant mass, not even long–term survival. However, we neither have extinction in finite time (at least for $d = 3$). Here is our result.

Theorem 2 (Finite mass extinction) *Let $d = 3$ and $m \in \mathcal{M}_f(\mathbb{R}^d)$, $m \neq 0$. Then there is no finite time extinction for the reactant:*

$$\mathbb{E}_{0,i_c\ell}[P^\varrho_{0,m}[X_t^\varrho \neq 0]] = 1, \qquad t \geq 0. \tag{1.8}$$

However, for $d = 2$ or $d = 3$ there is extinction in the long–term limit:

$$\mathbb{E}_{0,i_c\ell}\left[P^\varrho_{0,m}\left[\lim_{t\to\infty} \|X_t^\varrho\| = 0\right]\right] = 1. \tag{1.9}$$

The reason why we do not have finite time extinction is simple to explain. The key is the instantaneous propagation of matter (Proposition 1.2). At time $t = 0$ reactant mass is instantaneously spilled everywhere in space. For every $t > 0$ and $\varepsilon \in (0, t)$ there are tubes $(x + B) \times [\varepsilon, t]$ in the complement of the time–space support of ϱ. In these tubes the reactant can survive until time t as it dominates heat flow with absorption at the boundary of $x + B$. We will convert this idea into a rigorous proof in Section 3.

Remark Statement (1.8) depends on the assumption $d = 3$ only by the instantaneous propagation of matter property (1.6). It would be true also for $d = 2$ if one could show (1.6) also for $d = 2$ which seems to be a reasonable statement.

1.4 Outline

The rest of the paper is organized as follows. In Section 2 we give the short proof of Proposition 1.2. In Section 3 we prove Theorem 1. It takes some technical effort involving exit measures to turn the reasoning below Theorem 2 into a rigorous proof. This is the content of Section 4.

2 Instantaneous propagation of matter

Here we prove Proposition 1.2.

The rough idea is that any two time–space points (t, x) and (t, y) can be connected by a straight line that is not hit by $\mathrm{supp}(\varrho)$. (We denote by $\mathrm{supp}(\varrho) \subset [0, \infty) \times \mathbb{R}^d$ the closed support of the measure $dt\varrho_t(dx)$.) Hence also a time–space neighbourhood of this line is catalyst free. If there is reactant matter around y at time t then a small amount has percolated according to the heat flow through the "tube" to x. If we condition on $X_t^\varrho \neq 0$ then there is some y such that there is reactant matter around (t, y). Thus there is some matter around (t, x) also and we are done. The following lines make this idea rigorous.

For $t > 0$ fixed, $x, y \in \mathbb{R}^d$ and $\varepsilon \in (0, t)$ define the tube

$$T(t, x, y, \varepsilon) = \big\{(s, z) : \ s \in (t - \varepsilon, t + \varepsilon), \ z \in \mathbb{R}^d, \exists \alpha \in [0, 1] : \ |z - (\alpha x + (1 - \alpha)y)| < \varepsilon\big\}. \tag{2.1}$$

Since lines are polar for the time–space support $\operatorname{supp}(\varrho)$ of three–dimensional super–Brownian motion (see Dynkin (1992), Theorem 3.5.B) we have

$$\lim_{\varepsilon \to 0} \mathbb{P}_{0, i_c \ell}\left[\int_{T(t,x,y,\varepsilon)} ds\varrho_s(dz) \ = 0\right] = 1. \tag{2.2}$$

Thus with probability 1, for any $x, y \in \mathbb{Q}^3$ (the three–dimensional rational numbers) there exists a random number $\varepsilon(x, y) > 0$ such that $\int_{T(t,x,y,\varepsilon(x,y))} ds\varrho_s(dz) \ = 0$.

We know that X_t^ϱ is absolutely continuous with respect to Lebesgue measure and that its density function ξ_t^ϱ is continuous off $\operatorname{supp}(\varrho)$ and solves the heat equation (see [FK99]). Let $S_t = \{z \in \mathbb{R}^3 : \ (t, z) \in \operatorname{supp}(\varrho)\}$. Assume now that $X_t^\varrho \neq 0$. Since $\ell(S_t) = 0$ a.s. it suffices to show for $x \in S_t^c := \mathbb{R}^3 \setminus S_t$ that $\xi_t^\varrho(x) > 0$. By the continuity of the density we need to show this only for all $x \in \mathbb{Q}^3 \cap S_t^c$. Note that $X_t^\varrho \neq 0$ implies that there exists some $y \in \mathbb{Q}^3 \cap S_t^c$ with $\xi_t^\varrho(y) > 0$. Since $T(t, x, y, \varepsilon(x, y)) \to [0, \infty)$, $(s, z) \mapsto \xi_s^\varrho(z)$ solves the heat equation we have in fact $\xi_s^\varrho(z) > 0$ for all $(s, z) \in T(t, x, y, \varepsilon(x, y))$. □

3 Infinite Biodiversity

In this section we prove Theorem 1. The statement we have to show consists of two parts: (i) the deterministic component of X_0^ϱ vanishes and (ii) the canonical measure of $\{\chi : \ \chi(B) > 0\}$ is infinite for every open set $B \neq \emptyset$.

3.1 Vanishing deterministic component

We start with the proof of the first statement. Recall that we consider the bivariate process $(\varrho_t, X_t^\varrho)_{t\in\mathbb{R}}$ in the equilibrium. That is, ϱ is sampled according to $\mathbb{P}_{-\infty, i_c\ell}$ and for given ϱ the law of X^ϱ is $P^\varrho_{-\infty, i_r\ell}$. For convenience we agree that for fixed ϱ we let be defined all random variables related to superprocesses in the catalytic medium ϱ on the same sufficiently large underlying probability space whose law we denote by P^ϱ. For the deterministic and random component of a random measure Y over this probability space we simply write $\det^\varrho Y := \operatorname{essinf} Y$ and $\operatorname{ran}^\varrho Y := Y - \det^\varrho Y$.

Lemma 3.1 *Let $d = 3$. For $\mathbb{P}_{-\infty, i_c\ell}$–almost all ϱ*

$$\det{}^\varrho X_0^\varrho = 0.$$

Proof It suffices to show that the expected deterministic component $\mathbb{E}_{-\infty, i_c\ell}[\det^\varrho X_0^\varrho]$ disappears. Assume the contrary. Then, by the spatial shift–invariance of $\mathbb{P}_{-\infty, i_c\ell}$ we have

$$\mathbb{E}_{-\infty, i_c\ell}[\det{}^\varrho X_0^\varrho] = b\ell \qquad \text{for some } b \in (0, i_r]. \tag{3.1}$$

Fix a sample ϱ such that $\det^\varrho X_0^\varrho \neq 0$. Given X_0^ϱ introduce independent CSBM $X^{\varrho,d} = (X_t^{\varrho,d})_{t\geq 0}$ and $X^{\varrho,r} = (X_t^{\varrho,r})_{t\geq 0}$ (in the medium ϱ) with initial states $X_0^{\varrho,d} = \det^\varrho X_0^\varrho$ and $X_0^{\varrho,r} = \operatorname{ran}^\varrho X_0^\varrho$. By the branching property we may assume that

$$X_1^\varrho = X_1^{\varrho,d} + X_1^{\varrho,r}. \tag{3.2}$$

We claim that

$$\det{}^\varrho X_1^{\varrho,r} = 0. \tag{3.3}$$

In fact, fix a compact $A \subset \mathbb{R}^3$ and an $\varepsilon > 0$. For $R > 0$, the fixed ϱ, and given $X_0^{\varrho,r}$, as before we decompose $X^{\varrho,r}$ into independent catalytic super-Brownian motions $X^{\varrho,r,i}$, $i = 1, 2$, with catalyst ϱ and initial states

$$X_0^{\varrho,r,1} = X_0^{\varrho,r}\mathbf{1}_{B(0,R)}, \quad X_0^{\varrho,r,2} = X_0^{\varrho,r}\mathbf{1}_{B(0,R)^c},$$

where $B(x,R)$ is the open ball with radius R centered at $x \in \mathbb{R}^d$. Assume that

$$X_1^{\varrho,r} = X_1^{\varrho,r,1} + X_1^{\varrho,r,2}. \tag{3.4}$$

Note that also $\det{}^\varrho X_1^{\varrho,r} = \det{}^\varrho X_1^{\varrho,r,1} + \det{}^\varrho X_1^{\varrho,r,2}$. By the Markov property and the expectation formula, (recall that $\mathcal{S}_t$ is the heat flow)

$$E^\varrho[X_1^{\varrho,r,2}(A)] = E^\varrho[\langle \mathcal{S}_1(X_0^{\varrho,r}\mathbf{1}_{B(0,R)^c}), \mathbf{1}_A\rangle] = E^\varrho[\langle X_0^{\varrho,r}\mathbf{1}_{B(0,R)^c}, \mathcal{S}_1\mathbf{1}_A\rangle].$$

Hence we can choose R large enough such that

$$E^\varrho[X_1^{\varrho,r,2}(A)] \le \varepsilon \tag{3.5}$$

(recall that $X_1^{\varrho,r,2} \le X_1^{\varrho,r} \le X_1^\varrho$, where the latter term has P^ϱ–expectation $i_r\ell$).

Since $X_0^{\varrho,r}$ has a vanishing deterministic component its infinite divisibility implies that also $\|X_0^{\varrho,r,1}\| = X_0^{\varrho,r}(B(0,R))$ does not have a deterministic component. Thus for all $\delta > 0$ we have $P^\varrho[\|X_0^{\varrho,r,1}\| < \delta/2] > 0$. Noting that $(\|X_t^{\varrho,r,1}\|)_{t\ge 0}$ is a martingale we get

$$P^\varrho\Big[\|X_1^{\varrho,r,1}\| \le 2\|X_0^{\varrho,r,1}\| \;\Big|\; \|X_0^{\varrho,r,1}\|\Big] \ge \frac{1}{2} \qquad \text{almost surely,}$$

and thus

$$P^\varrho[\|X_1^{\varrho,r,1}\| < \delta] \ge \frac{1}{2}P^\varrho[\|X_0^{\varrho,r,1}\| < \delta/2] > 0.$$

Hence $\det{}^\varrho X_1^{\varrho,r,1} = 0$. Combining this with (3.5) the claim (3.3) follows.

Recall that we fixed ϱ such that $X_0^{\varrho,d} = \det{}^\varrho X_0^\varrho \neq 0$. This implies $\mathrm{ran}^\varrho X_1^{\varrho,d} \neq 0$ with positive P^ϱ–probability, thus $E^\varrho[\mathrm{ran}^\varrho X_1^{\varrho,d}] \neq 0$. By translation invariance of $\mathbb{P}_{-\infty,i_c\ell}$ there exists an $a > 0$ such that

$$\mathbb{E}_{-\infty,i_c\ell}[E^\varrho[\mathrm{ran}^\varrho X_1^{\varrho,d}]] = a\ell. \tag{3.6}$$

Finally, by the decomposition (3.2) and by (3.3), $\det{}^\varrho X_1^\varrho = \det{}^\varrho X_1^{\varrho,d} = X_1^{\varrho,d} - \mathrm{ran}^\varrho X_1^{\varrho,d}$, and therefore we can build the annealed expectation to obtain

$$b\ell = \mathbb{E}_{-\infty,i_c\ell}[E^\varrho[\det{}^\varrho X_1^\varrho]] = \mathbb{E}_{-\infty,i_c\ell}[E^\varrho[X_1^{\varrho,d}]] - \mathbb{E}_{-\infty,i_c\ell}[E^\varrho[\mathrm{ran}^\varrho X_1^{\varrho,d}]] = (b-a)\ell. \tag{3.7}$$

This is clearly a contradiction and finishes the proof. □

3.2 The equilibrium reactant charges every set

We complete the proof of Theorem 1 by showing that the reactant's canonical measure is infinite on $\{\chi :\ \chi(B) > 0\}$ for any open set $B \neq \emptyset$. Recall from (1.5) that is is enough to show that the reactant charges any open set:

$$\mathbb{E}_{-\infty,i_c\ell}[P^\varrho_{-\infty,i_r\ell}[X_0^\varrho(B) = 0]] = 0 \qquad \text{for any open } B \neq \emptyset. \tag{3.8}$$

However this follows from the instantaneous propagation of matter (Proposition 1.2), and Theorem 1 is now completely proved. □

4 Finite mass extinction

In this section we prove Theorem 2. The proofs of the two statements (no finite time extinction but long–term extinction) are methodologically different and we present them in separate subsections.

4.1 No finite time extinction

Recall that here ϱ is distributed according to $\mathbb{P}_{0,i_c\ell}$ and X^ϱ according to $P^\varrho_{0,m}$ for some non–vanishing $m \in \mathcal{M}_f(\mathbb{R}^d)$.

In order to show that there is no finite time extinction we rely again on the instantaneous propagation of reactant matter (Proposition 1.2). Additionally we need the following property of the support $\operatorname{supp}(\varrho) \subset [0,\infty) \times \mathbb{R}^d$ of the the measure $dt\varrho_t(dx)$. Recall that $B(x,r) \subset \mathbb{R}^d$ is the open ball of radius r centered at x.

Lemma 4.1 (Empty tubes in the catalyst) *Assume $d \geq 2$. For every $t > 0$, $\varepsilon \in (0,t)$ and $\mathbb{P}_{0,i_c\ell}$–a.a. ϱ there exists a $z \in \mathbb{Z}^d$ such that* $\operatorname{supp}(\varrho) \cap ([\varepsilon,t] \times B(z,1)) = \emptyset$.

Proof Fix $t > \varepsilon > 0$. For $z \in \mathbb{R}^d$ define the event

$$A(z) = \big\{\operatorname{supp}(\varrho) \cap ([\varepsilon,t] \times B(z,1)) = \emptyset\big\}.$$

Fix $R > 0$ and note that by the branching property of SBM we have

$$\mathbb{P}_{0,i_c\ell}[A(z)] = \mathbb{P}_{0,i_c\ell\mathbf{1}_{B(z,R)}}[A(z)] \cdot \mathbb{P}_{0,i_c\ell\mathbf{1}_{B(z,R)^c}}[A(z)]. \tag{4.1}$$

Obviously,

$$\mathbb{P}_{0,i_c\ell\mathbf{1}_{B(z,R)}}[A(z)] \geq \mathbb{P}_{0,i_c\ell\mathbf{1}_{B(z,R)}}[\varrho_\varepsilon = 0] > 0, \quad R > 0. \tag{4.2}$$

For the other factor in (4.1) we need an estimate on the range of SBM (see Dawson, Iscoe and Perkins (1989), Theorem 3.3a): Fix $R > 2t^{1/2}$. Then there exists a $c > 0$ such that for $x \in \mathbb{R}^d$

$$f_z(x) := \mathbb{P}_{0,\delta_x}[A(z)^c] \leq c\exp(-\|x-z\|^2/2t). \tag{4.3}$$

Noting that $\log(1-s) > -2s$ for $s \in [0,\frac{1}{2}]$ we see that for R sufficiently large

$$\begin{aligned} \mathbb{P}_{0,i_c\ell\mathbf{1}_{B(z,R)^c}}[A(z)] &= \exp\left(i_c\int_{B(z,R)^c}\log(1-f_z(x))dx\right) \\ &\geq \exp\left(-2ci_c\int_{B(z,R)^c}\exp(-\|x-z\|^2/2t)\,dx\right) \;\; > 0. \end{aligned} \tag{4.4}$$

Thus $\mathbb{P}_{0,i_c\ell}[A(z)] > 0$ and we can infer from the ergodic theorem (note that $\mathbb{P}_{0,i_c\ell}$ is spatially ergodic) $\mathbb{P}_{0,i_c\ell}[\cup_{z\in\mathbb{Z}^d}A(z)] = 1$. □

With this lemma we are almost done. Recall that we specify on $d = 3$. Fix $\delta \in (0,1)$ and $\varepsilon \in (0,t)$ such that $\mathbb{P}_{0,i_c\ell}[P^\varrho_{0,m}[X^\varrho_\varepsilon \neq 0] > 1-\delta] > 1-\delta$. Now choose such a ϱ and an X^ϱ_ε that are in the described event. We may assume that $\varrho \in \cup_{z\in\mathbb{Z}^3}A(z)$. Let $z \in \mathbb{Z}^3$ such that $\varrho \in A(z)$. By Proposition 1.2 we have $X^\varrho_\varepsilon(B(z,\frac{1}{2})) > 0$. Denote by $(\mathcal{S}^z_s)_{s\geq 0}$ the semigroup of heat flow with absorption at $\mathbb{R}^3 \setminus B(z,1)$. Then we can estimate

$$X^\varrho_t(B(z,1)) \geq \|\mathcal{S}^z_{t-\varepsilon}X^\varrho_\varepsilon\| \geq \|\mathcal{S}^z_{t-\varepsilon}(\mathbf{1}_{B(z,\frac{1}{2})}X^\varrho_\varepsilon)\| > 0. \tag{4.5}$$

Hence

$$\mathbb{P}_{0,i_c\ell}[P^\varrho_{0,m}[X^\varrho_t \neq 0] > 1-\delta] > 1-\delta.$$

Now let $\delta \to 0$. This shows the first assertion of Theorem 2.

4.2 Long–term extinction

In this subsection we show the second statement of Theorem 2. We first outline the idea of the proof.

An "infinitesimal particle" of X^ϱ performs Brownian motion W on $\mathbb{R}^d$ whose law and expectation is denoted by $\mathbf{P}_x$ and $\mathbf{E}_x$ respectively, $x \in \mathbb{R}^d$. The branching along such a reactant's path W is governed by the collision local time $L_{[W,\varrho]}$ of W with the catalyst ϱ. This can be defined for $d \leq 3$ as the L^2–limit (see [EP94] and [DF97a])

$$L_{[W,\varrho]}(0,t) = \lim_{\varepsilon \to 0} \int_0^t ds \int_{\mathbb{R}^d} \varrho_s(dz) p_\varepsilon(z, W_s), \qquad t \geq 0, \tag{4.6}$$

where $(p_t)_{t>0}$ is the family of standard Brownian transition densities. (For $d \geq 4$, $\mathrm{supp}(\varrho)$ is polar for W; that is, W does not collide with ϱ, and X^ϱ degenerates to the heat flow.)

Loosely speaking the idea is that by a fixed large time T most infinitesimal reactant particles have collected a large amount of collision local time, say at least K. With a high probability (namely the extinction probability of Feller's branching diffusion at time K) all these particles have died. The expected number of particles that have collected less collision local time is $\mathbf{E}_0[\mathbb{P}_{0,i_c\ell}[L_{[W,\varrho]}(0,T) < K]]$ which tends to 0 as $T \to \infty$.

Infinite total collision local time

However intuitively appealing and verbally simple to describe the idea is, we need some technical effort to make it rigorous. We start by showing that the collision local time $L_{[W,\varrho]}$ increases in fact to ∞ almost surely if $d = 2, 3$. Note that this contrasts with dimension $d = 1$ where $L_{[W,\varrho]}(0,\infty) < \infty$ almost surely (see [DF97a, Proposition 8]). The difference between dimension one and two is that ϱ dies out locally almost surely if $d = 1$ while it does so only in probability if $d = 2$. In the latter case the clusters recur to visit the window of observation at arbitrarily late times. Of course, for $d = 3$ there is no extinction and a law of large numbers applies.

Proposition 4.2 *Let $d = 2, 3$. Then*

$$\mathbf{E}_x[\mathbb{P}_{0,i_c\ell}[L_{[W,\varrho]}(0,\infty) = \infty]] = 1, \qquad x \in \mathbb{R}^d. \tag{4.7}$$

Proof By spatial homogeneity of Brownian motion and the law of ϱ it suffices to consider $x = 0$. For $d = 3$ the claim follows from a law of large numbers (see [DF97b]):

$$\mathbf{E}_0[\mathbb{P}_{0,i_c\ell}[\lim_{t\to\infty} t^{-1} L_{[W,\varrho]}(0,t) = i_c]] = 1. \tag{4.8}$$

For $d = 2$ there is no law of large numbers. Rather $L_{[W,\varrho]}$ is self similar in the sense that (see [DF97b])

$$\mathbf{E}_0[\mathbb{P}_{0,i_c\ell}[L_{[W,\varrho]}(0,t) \in \bullet]] = \mathbf{E}_0[\mathbb{P}_{0,i_c\ell}[T^{-1} L_{[W,\varrho]}(0,Tt) \in \bullet]], \qquad t, T > 0. \tag{4.9}$$

Since $\mathbf{E}_0[\mathbb{E}_{0,i_c\ell}[L_{[W,\varrho]}(0,1)]] = i_c > 0$ we have that

$$\mathbf{E}_0[\mathbb{P}_{0,i_c\ell}[L_{[W,\varrho]}(0,\infty) < \infty]] = \mathbf{E}_0[\mathbb{P}_{0,i_c\ell}[L_{[W,\varrho]}(0,1) = 0]] < 1. \tag{4.10}$$

We are done if we can show a suitable 0-1 law for the l.h.s. of (4.10). This is a spin–off of the subsequent lemma which then finishes the proof of the proposition. □

Lemma 4.3 (0-1 law) *Assume $d \leq 3$. Denote by $A(W)$ the set*

$$A(W) = \{\varrho : \ L_{[W,\varrho]}(t,\infty) = 0 \ \textit{for some} \ t > 0 \ \}. \tag{4.11}$$

Then one of the following two alternatives holds

$$\mathbf{P}_x[W : \mathbb{P}_{0,i_c\ell}[\varrho \in A(W)] = 0] = 1, \qquad x \in \mathbb{R}^d, \tag{4.12}$$

or

$$\mathbf{P}_x[W : \mathbb{P}_{0,i_c\ell}[\varrho \in A(W)] = 1] = 1, \qquad x \in \mathbb{R}^d. \tag{4.13}$$

Proof Again by spatial homogeneity, either alternative holds if it does for $x = 0$.

We first show that

$$\mathbf{P}_0[\mathbb{P}_{0,i_c\ell}[\varrho \in A(W)] \in \{0,1\}] = 1. \tag{4.14}$$

In fact, let $\varrho = \sum_{z \in \mathbb{Z}^d} \varrho^z$ be a decomposition of ϱ into independent SBM starting in $\varrho_0^z = \ell \mathbf{1}_{z+[0,1)^d}$. Since $\mathbb{P}_{0,i_c\ell}[\sum_{z \in Z} \varrho_t^z = 0] \to 1$ as $t \to \infty$, for any finite $Z \subset \mathbb{Z}^d$, the event $A(W)$ is in the completion of the tail field:

$$A(W) \in \bigcap_{n \in \mathbb{N}} \sigma(\varrho^z, \ \|z\| \geq n) \qquad (\mathrm{mod}\ \mathbb{P}_{0,i_c\ell}). \tag{4.15}$$

The tail field is $\mathbb{P}_{0,i_c\ell}$–trivial and (4.14) follows.

Assume now that (4.12) does not hold. Hence $\mathbf{P}_0[W : \mathbb{P}_{0,i_c\ell}[\varrho \in A(W)] > 0] > 0$ and by (4.14)

$$\mathbf{P}_0[W : \mathbb{P}_{0,i_c\ell}[\varrho \in A(W)] = 1] > 0. \tag{4.16}$$

Fix $t > 0$. Note that $A(W) = \{\varrho : \ \varrho_{t+\bullet} \in A(W_{t+\bullet})\}$, hence we have

$$\mathbb{P}_{0,i_c\ell}[\varrho \in A(W)] = \mathbb{P}_{0,i_c\ell}[\varrho_{t+\bullet} \in A(W_{t+\bullet})] = \mathbb{P}_{0,i_c\ell}[\varrho_{t+\bullet} \in A(W_{t+\bullet} - W_t)], \tag{4.17}$$

where in the last step we used the spatial translation invariance of $\mathbb{P}_{0,i_c\ell}$. Hence $W \mapsto \mathbb{P}_{0,i_c\ell}[\varrho \in A(W)]$ is measurable w.r.t. the tail field of the increments of W and thus constant. By (4.16) we get (4.13). $\square$

Remark 4.4 Note that the proof of Proposition 4.2 shows for $d = 2$ even the stronger statement that for all $x \in \mathbb{R}^2$ and $\mathbf{E}_x\mathbb{P}_{0,i_c\ell}$–almost surely

$$L_{[W,\varrho]}(0,t) > 0 \quad \text{and} \quad L_{[W,\varrho]}(t,\infty) = \infty, \qquad t > 0. \tag{4.18}$$

Exit measures

Now we make precise the idea of the collision local time collected by individual "infinitesimal particles" from the introduction of this subsection. Note that the idea of using exit measures for this purpose has been employed successfully also by Dawson, Fleischmann and Mueller (1998) (see also Fleischmann and Mueller (1997)).

Choose a sample ϱ according to $\mathbb{P}_{0,i_c\ell}$. Recall that $d = 2, 3$. From Proposition 4.2 it follows that the following stopping times of W are $\mathbf{P}_x$–almost surely finite:

$$\tau_K = \inf\{t > 0 : L_{[W,\varrho]}(0,t) \geq K\}, \qquad K \geq 0. \tag{4.19}$$

For each of these stopping times we could define Dynkin's stopped measure, which is approximately what we want. However this is a (random) measure on the particles' *path space* and needs a construction of historical CSBM. This is not too hard to do but we prefer to follow a slightly more elementary route.

We would like to consider the joint process of W and its collision local time $L_{[W,\varrho]}$. It will, however, be convenient to introduce the *trivariate* (time–homogeneous) Markov process $\widetilde{W} = (W, L, I)$ on $\mathbb{R}^d \times [0,\infty) \times [0,\infty)$, where for $t \geq 0$

$$\begin{aligned} I_t &= t + I_0, \\ L_t &= L_0 + L_{[W,\varrho]}(I_0, I_t), \end{aligned} \tag{4.20}$$

with $I_0, L_0 \geq 0$. For this enriched process $\widetilde{W}$, started in $(W_0, 0, 0)$, each τ_K is an *exit time*:

$$\tau_K = \inf\{t > 0 : \widetilde{W}_t \notin A\}, \tag{4.21}$$

where

$$A = \mathbb{R}^d \times [0, K) \times [0, \infty).$$

We can define the catalytic branching process $\widetilde{X}^\varrho$ which is the catalytic superprocess (on $\mathbb{R}^d \times [0,\infty) \times [0,\infty)$) with underlying motion $\widetilde{W}$, with "critical binary" branching and with branching functional L. For an initial state $\widetilde{m} \in \mathcal{M}_f(\mathbb{R}^d \times [0,\infty) \times [0,\infty))$ we denote its law by $P^\varrho_{\widetilde{m}}$. Note that for

$$\begin{aligned} &\widetilde{m} \text{ concentrated on } \mathbb{R}^d \times \{0\} \times \{0\} \text{ with} \\ &\widetilde{m}(\bullet \times [0,\infty) \times [0,\infty)) = m(\bullet) \end{aligned} \tag{4.22}$$

$X^\varrho_T(\bullet)$ and $\widetilde{X}^\varrho_T(\bullet \times [0,\infty) \times \{T\})$ coincide in distribution. Hence we are done if we can show that

$$\limsup_{T\to\infty} P^\varrho_{\widetilde{m}}[\|\widetilde{X}^\varrho_T\| > \varepsilon] = 0, \qquad \varepsilon > 0. \tag{4.23}$$

(Note that the $P^\varrho_{0,m}$–almost sure convergence statement (1.9) of Theorem 2 follows from this as $(\|X^\varrho_t\|)_{t\geq 0}$ is a nonnegative martingale and hence almost surely convergent.)

For an exit time τ of a domain $A \subset \mathbb{R}^d \times [0,\infty) \times [0,\infty)$ we can define the *exit measure* $\widetilde{X}^\varrho_\tau$ by the following procedure. Let $\widetilde{X}^{\varrho,\tau}$ be defined as $\widetilde{X}^\varrho$ but with $\widetilde{W}_{\tau\wedge\bullet}$ instead of $\widetilde{W}$ as motion (and with $L_{\tau\wedge\bullet}$ as branching functional). That is, the infinitesimal particles get frozen when they reach the boundary ∂A of A. Finally, let $\widetilde{X}^\varrho_\tau$ be the measure that is supported by ∂A and that is obtained as the monotone limit

$$\widetilde{X}^\varrho_\tau = \lim_{t\to\infty} \widetilde{X}^{\varrho,\tau}_t(\partial A \cap \bullet). \tag{4.24}$$

Note that for $\widetilde{m}$ as in (4.22) and τ a finite exit time we have

$$E^\varrho_{\widetilde{m}}[\widetilde{X}^\varrho_\tau] = \int m(dx)\mathbf{P}_x[\widetilde{W}_\tau \in \bullet]. \tag{4.25}$$

For the exit measures $\widetilde{X}^\varrho_\tau$ we have the so–called *special Markov property* which amounts to saying that if $\tau \leq \sigma$ are exit times of $\widetilde{W}$ then we obtain $\widetilde{X}^\varrho_\sigma$ from $\widetilde{X}^\varrho_\tau$ by starting the process afresh (cf. Dynkin (1991b), Theorem 1.6, and Dynkin (1991a), Theorem 1.5). More precisely, if for $\varphi \in C^+_c(\mathbb{R}^d \times [0,\infty) \times [0,\infty))$ and $x \in \mathbb{R}^d \times [0,\infty) \times [0,\infty)$ we define

$$(V^\varrho_\sigma \varphi)(x) = -\log E^\varrho_{\delta_x}[\exp(-\langle \widetilde{X}^\varrho_\sigma, \varphi\rangle)], \tag{4.26}$$

then

$$E^{\varrho}_{\widetilde{m}}[\exp(-\langle \widetilde{X}^{\varrho}_{\sigma}, \varphi\rangle)] = E^{\varrho}_{\widetilde{m}}[\exp(-\langle \widetilde{X}^{\varrho}_{\tau}, V^{\varrho}_{\sigma}\varphi\rangle)]. \tag{4.27}$$

In particular, if $\partial A = A^1 \cup A^2$, $A^1 \cap A^2 = \emptyset$, and $\widetilde{X}^{\varrho,1}$ and $\widetilde{X}^{\varrho,2}$ are independent processes (given their initial states) of the type introduced above with $\widetilde{X}^{\varrho,i}_0 = \widetilde{X}^{\varrho}_{\tau}(A^i \cap \bullet)$, $i = 1, 2$, then $\widetilde{X}^{\varrho}_{\sigma}$ is equal in distribution to their sum,

$$\widetilde{X}^{\varrho}_{\sigma} \stackrel{\mathrm{d}}{=} \widetilde{X}^{\varrho,1}_{\sigma} + \widetilde{X}^{\varrho,2}_{\sigma}. \tag{4.28}$$

Now we come back to our concrete situation. Here $\sigma \equiv T$ and $\tau = \tau_K \wedge T$, $A = \mathbb{R}^d \times [0, K) \times [0, T)$, $\partial A = A^1 \cup A^2$, where $A^1 = \mathbb{R}^d \times [0, K) \times \{T\}$ and $A^2 = \mathbb{R}^d \times \{K\} \times [0, T]$. Assume that $\widetilde{m}$ is as in (4.22). Hence, using (4.25) we get

$$E^{\varrho}[\|\widetilde{X}^{\varrho,1}_T\|] = E^{\varrho}_{\widetilde{m}}[\widetilde{X}^{\varrho}_{\tau_K \wedge T}(A^1)] = \int m(dx)\mathbf{P}_x[\tau_K > T] \to 0, \quad T \to \infty. \tag{4.29}$$

On the other hand $\widetilde{X}^{\varrho}_{\tau_K \wedge T}(A^2) = \widetilde{X}^{\varrho}_{\tau_K}(A^2)$, thus

$$P^{\varrho}[\widetilde{X}^{\varrho,2}_T \neq 0] \leq P^{\varrho}_{\widetilde{m}}[\widetilde{X}^{\varrho}_{\tau_K}(A^2) > 0]. \tag{4.30}$$

Now we employ a result of [DFM98] that the process $(\|\widetilde{X}^{\varrho}_{\tau_t}\|)_{t\geq 0}$ is Feller's branching diffusion. Hence we have $P^{\varrho}_{\widetilde{m}}[\|\widetilde{X}^{\varrho}_{\tau_K}\| > 0] = 1 - e^{-\|m\|/K}$ and thus

$$P^{\varrho}[\|\widetilde{X}^{\varrho,2}_T\| > 0] \leq 1 - e^{-\|m\|/K}. \tag{4.31}$$

Combining (4.28), (4.29) and (4.31), where we first let $T \to \infty$ and then $K \to \infty$, we get that (4.23) holds. □

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