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Homogenization of scalar wave equations with hysteresis

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Abstract

The paper deals with a scalar wave equation of the form $\rho u_{tt} = (\mathcal{F}[u_x])_x + f$, where $\mathcal{F}$ is a Prandtl-Ishlinskii operator and ρ, f are given functions. This equation describes longitudinal vibrations of an elastoplastic rod. The mass density ρ and the Prandtl-Ishlinskii distribution function η are allowed to depend on the space variable x. We prove existence, uniqueness and regularity of solution to a corresponding initial-boundary value problem. The system is then homogenized by considering a sequence of equations of the above type with spatially periodic data ρ^ε and η^ε, where the spatial period ε tends to 0. We identify the homogenized limits ρ^* and η^* and prove the convergence of solutions u^ε to the solution u^* of the homogenized equation.

Introduction

Homogenization is a mathematical method used in modelling composite materials with periodic structure. If the spatial period is very small, that is, if the spatial microstructure is too fine, one might wish to reduce the computational complexity by replacing the rapidly varying coefficients by, say, constant ones, corresponding to an idealized homogeneous material, which at the macroscopic level exhibits a qualitatively and quantitatively similar behaviour. The approach proposed by I. Babuška [2] consists in considering a sequence of heterogeneous constitutive laws with diminishing periods. One looks for a limit homogeneous constitutive law which, when coupled with the balance equations, gives a solution which is a limit of solutions to the original heterogeneous problems. An interested reader can find more information e. g. in [2, 4, 14, 3] and many others.

In this paper, we deal with a homogenization problem for uniaxial longitudinal vibrations of an elastoplastic rod governed by the one-dimensional quasilinear wave equation, where the constitutive law is considered in the form of a (spatially inhomogeneous) Prandtl-Ishlinskii hysteresis operator. The hysteresis approach to elastoplasticity is an alternative to the method of monotone operators (explained in detail e. g. in [1]) and it seems to be useful especially in connection with problems of stability and asymptotic behaviour of solutions. A systematic mathematical investigation of Prandtl-Ishlinskii operators started relatively recently (see for instance [5, 9, 11, 15]), although the model itself was introduced much earlier ([13, 8]). These operators are rate-independent, and the system is hyperbolic in the sense of bounded speed of propagation, see [11].

The paper is organized as follows. In Section 1, we introduce briefly the problem. Section 2 is devoted to a detailed survey of the theory of hysteresis operators. Special attention is paid to spatially inhomogeneous Prandtl-Ishlinskii operators which, to the authors' knowledge, have not been studied yet in a sufficient generality. In particular, the convergence result in Proposition 2.12, which is substantial for the homogenization argument, seems to be new. In Section 3, we prove by space semidiscretization that the inhomogeneous problem admits a unique strong solution. The fact that shocks do not occur in quasilinear hyperbolic equations involving hysteresis operators with convex hysteresis loops has already been discussed in [11]; here the same result is obtained under weaker hypotheses on the space dependence.

In Section 4, we derive an explicit form of the homogenized Prandtl-Ishlinskii operator (it was already derived in [7] without proof of convergence of the solutions) such that the solution to the limit initial-boundary value problem is a limit of the solutions to the 'periodic' problems. In particular, this result enables us to interpret the underlying rheological structure of the homogeneous Prandtl-Ishlinskii model as a homogenized limit of simple one-yield elastoplastic elements periodically distributed along the rod.

1 Formulation of the problem

Let us consider a thin rod occupying a space interval $J = (0, \ell)$ during a time interval $I = (0, T)$. The longitudinal displacement at the space point x and time t will be denoted by $u = u(x,t)$. The vibration of the rod is described in Lagrange coordinates by the equation of motion

$$\rho\, u_{tt} = \sigma_x + f\,, \tag{1.1}$$

where $\rho = \rho(x)$ is the mass density, $\sigma = \sigma(x,t)$ the stress and $f = f(x,t)$ the volume force density. The material behaviour is characterized by a constitutive relation between the stress σ and the strain $e = u_x$, which we consider here in the form

$$\sigma = \mathcal{F}(e)\,, \tag{1.2}$$

where $\mathcal{F}$ is the spatially inhomogeneous Prandtl-Ishlinskii operator with a space dependent density function $\eta = \eta(x,r)$, $x \in J$, $r \geq 0$, which will be described in detail in Section 2. The problem is hyperbolic in the sense of bounded speed of propagation, see [11]. In particular, if $\mathcal{F}$ is a positive multiple of the identity operator (i.e. in the purely elastic case $\eta(x,r) \equiv k$), then Eq. (1.1) becomes the well-known linear wave equation.

The equation is completed with static boundary conditions. For the sake of simplicity, we consider a fixed end of the rod at $x = 0$ and a free end at $x = \ell$, that is,

$$u(0,t) = \sigma(\ell,t) = 0 \qquad \text{for } \ t \in I\,. \tag{1.3}$$

Initial conditions will be chosen in the form

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = u_1(x) \quad \text{for } \ x \in J\,. \tag{1.4}$$

We first consider a heterogeneous rod, where the material parameters ρ and η depend on x in a discontinuous way. In Section 3, Theorem 3.2, we prove that the spatially inhomogeneous problem admits a unique strong solution.

In Section 4, we consider a material with periodic structure. The mass density ρ and the Prandtl-Ishlinskii density η are assumed to be periodic in x with a period ε. As a simplest example we may consider a rod composed of two materials A and B distributed in layers of thickness $d_A\varepsilon$ and $d_B\varepsilon$ with $d_A + d_B = 1$, see Fig. 1. More specifically, we consider a family $\{\rho^\varepsilon, \eta^\varepsilon\}_{\varepsilon>0}$ of material parameters of the form

$$\rho^\varepsilon(x) = \rho(x/\varepsilon), \quad \eta^\varepsilon(x,r) = \eta(x/\varepsilon, r), \quad x \in J, \ r > 0, \tag{1.5}$$

where ρ, $\eta(\cdot, r) : \mathbb{R} \to (0,\infty)$ are periodic functions for all $r \geq 0$ with period 1, that is,

$$\rho(y+1) = \rho(y), \quad \eta(y+1, r) = \eta(y,r), \quad \forall y \in \mathbb{R}, \ r > 0. \tag{1.6}$$

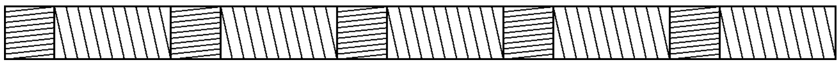

Figure 1: Layered rod – material with periodic structure.

Letting ε tend to 0, we obtain a family of initial-boundary value problems of the form (1.1) – (1.4) and a corresponding family of solutions u^ε. Our main result (Theorem 4.5) consists in proving that the solutions u^ε converge under natural hypotheses to a solution u^* of an initial-boundary value problem of the same type with homogeneous parameters $\rho^* =$ const. and $\eta^* = \eta^*(r)$. While it is obvious that ρ^* is nothing but the average of ρ, the relation between η^* and η involves a more detailed analysis of Prandtl-Ishlinskii operators. In particular, we derive an explicit formula for the homogenized operator $\mathcal{F}^*$ based on the inverse operator $\mathcal{F}^{-1}$ to $\mathcal{F}$. This formula was already intuitively derived in [7].

2 Hysteresis operators

2.1 Stop and play operators

One of the basic elements of the theory of hysteresis operators is borrowed from continuum mechanics, more precisely, from Prandtl's model for elastic-perfectly plastic constitutive laws as on Fig. 2, where u and s represent the (scalar) strain and stress, respectively. In mathematical terms, it can be formally described as the input-output relation between $u \in W^{1,1}(I)$ and $s \in W^{1,1}(I)$ defined by the variational inequality

$$\begin{cases} s(t) \in [-r, r] & \text{for every } \ t \in \bar{I}, \\ (\dot{s}(t) - \dot{u}(t))(\phi - s(t)) \geq 0 & \text{for a.e. } \ t \in I \text{ and every } \ \phi \in [-r,r], \\ s(0) = s^0, \end{cases} \tag{2.1}$$

where $r > 0$ and $s^0 \in [-r, r]$ are given and the dot denotes derivative with respect to t. The model corresponds to a rheological combination (cf. Fig. 3) of one linearly elastic element (represented by a spring) in series with a dry friction term.

We list below some basic well-known analytical properties of the Prandtl model and its extensions. A detailed discussion on this subject can be found in the monographs [5, 9, 11, 15]. We do not treat here its vectorial or tensorial counterparts, where the interval $[-r, r]$ is replaced by an arbitrary convex closed subset Z. The analytical properties of

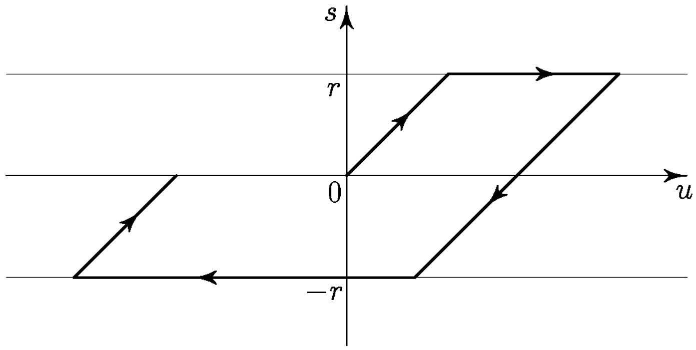

Figure 2: Strain-stress diagram for Prandtl's model with yield point r and unit elasticity modulus.

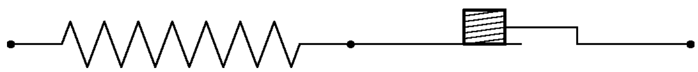

Figure 3: Rheological scheme for Prandtl's model.

the model then depend substantially on the geometry of the set Z and a survey can be found in [6].

In order to make the presentation consistent and more accessible, we sketch at least some main ideas of the proofs that are elementary enough.

For every input $u \in W^{1,1}(I)$ and initial condition $s^0 \in [-r, r]$, problem (2.1) has a unique solution $s \in W^{1,1}(I)$. We can therefore define the solution operator $\mathcal{S}_r : [-r, r] \times W^{1,1}(I) \to W^{1,1}(I)$ by the formula

$$\mathcal{S}_r[s^0, u] := s\,. \tag{2.2}$$

It is convenient to introduce also its complement

$$\mathcal{P}_r[s^0, u] := u - \mathcal{S}_r[s^0, u]\,. \tag{2.3}$$

The argument of the operators is written in square brackets to indicate the functional dependence, since they map a function to a function. The operators $\mathcal{S}_r$ and $\mathcal{P}_r$ are called the *stop* and *play*, respectively, with threshold r. In each interval of monotonicity $[t_0, t_1]$ of the input function u, the outputs are explicitly given by the formulas

$$\mathcal{S}_r[s^0, u](t) \;=\; \min\{r,\, \max\{-r,\, \mathcal{S}_r[s^0, u](t_0) + u(t) - u(t_0)\}\}\,, \tag{2.4}$$

$$\mathcal{P}_r[s^0, u](t) \;=\; \max\{u(t) - r,\, \min\{u(t) + r,\, \mathcal{P}_r[s^0, u](t_0)\}\}\,, \tag{2.5}$$

which have traditionally been used as alternative definitions of the stop and play on piecewise monotone inputs, see [5, 9]. The following inequalities hold:

Proposition 2.1 *For $s_1^0, s_2^0 \in [-r, r]$ and $u_1, u_2 \in W^{1,1}(I)$ put $p_i := \mathcal{P}_r[s_i^0, u_i]$ and $s_i := \mathcal{S}_r[s_i^0, u_i]$, $i = 1, 2$. Then we have*

(i) $(\dot p_1(t) - \dot p_2(t))(s_1(t) - s_2(t)) \;\geq\; 0$ *for a.e.* $t \in I$,

(ii) $|p_1(t) - p_2(t)| \ \le\ \max\left\{|p_1(0) - p_2(0)|\,, \max_{0\le s\le t} |u_1(s) - u_2(s)|\right\} \quad \forall t \in \bar{I}\,.$

Sketch of the proof. (i) For $i = 1,2$ we have by definition $\dot{p}_i(t)(s_i(t) - \phi) \ge 0$ a.e. for all $\phi \in [-r, r]$. Putting $\phi = s_{3-i}(t)$ and summing up the resulting inequalities for $i = 1,2$, we obtain the assertion.

(ii) For $0 \le \tau \le t \le T$ put

$$V_t(\tau) := \max\left\{|p_1(\tau) - p_2(\tau)|^2\,, \max_{0\le s\le t} |u_1(s) - u_2(s)|^2\right\}\,,$$

and assume that for some $0 < \tau < t$ we have

$$\frac{\partial}{\partial\tau} V_t(\tau) \ >\ 0\,.$$

This assumption implies

$$|p_1(\tau) - p_2(\tau)| \ >\ \max_{0\le s\le t} |u_1(s) - u_2(s)|\,, \tag{2.6}$$

$$(\dot{p}_1(\tau) - \dot{p}_2(\tau))(p_1(\tau) - p_2(\tau)) \ >\ 0\,. \tag{2.7}$$

From (i) and Ineq. (2.7) it follows that $(s_1(\tau) - s_2(\tau))(p_1(\tau) - p_2(\tau)) \ge 0$, hence

$$(p_1(\tau) - p_2(\tau))^2 \ \le\ (p_1(\tau) - p_2(\tau))(u_1(\tau) - u_2(\tau))\,,$$

which contradicts inequality (2.6). We therefore have $V_t(t) \le V_t(0)$ and Proposition 2.1 is proved. ❐

Remark 2.2 Part (ii) of Proposition 2.1 states that the play (and therefore also the stop) can be extended to Lipschitz continuous mappings from $[-r, r] \times C(\bar{I})$ to $C(\bar{I})$, where $C(\bar{I})$ denotes the space of continuous functions from $\bar{I}$ to $\mathbb{R}$. In the sequel, we will mainly work with these extensions, still using the same notation.

To simplify the presentation, we consider special initial configurations of the stop and play operators. They consist in choosing

$$s^0 := \operatorname{sign}(u(0)) \min\{|u(0)|, r\} \tag{2.8}$$

in the variational problem (2.1). In materials sciences, this corresponds to the initially unperturbed (or *virgin*) reference state. In some applications, for instance to problems of fatigue accumulation, it is substantial to consider more general initial states, and an interested reader can find a detailed analysis in [5] or [11]. Here, the results do not depend on the choice of s^0.

This enables us to consider the stop and play as operators from $C(\bar{I})$ to $C(\bar{I})$ and to write simply $\mathcal{S}_r[u]$, $\mathcal{P}_r[u]$ instead of $\mathcal{S}_r[s^0, u]$, $\mathcal{P}_r[s^0, u]$.

The stop and play are obviously *locally monotone hysteresis operators* in the following sense:

Definition 2.3 *An operator* Φ *acting in some space* $R(\bar{I})$ *of functions from* $\bar{I}$ *to* $\mathbb{R}$ *is called a* hysteresis operator *if it is*

— rate-independent, *that is, if for every* $u \in R(\bar{I})$ *and every nondecreasing mapping* α *of* $\bar{I}$ *onto* $\bar{I}$ *such that* $u_\alpha(t) := u(\alpha(t))$ *belongs to* $R(\bar{I})$, *we have*

$$\Phi[u_\alpha](t) = \Phi[u](\alpha(t)) \qquad \text{for all } t \in \bar{I}, \tag{2.9}$$

and

— causal, *that is, if the implication*

$$u(t) = v(t) \quad \forall\, t \in [0, t_0] \;\Rightarrow\; \Phi[u](t_0) = \Phi[v](t_0). \tag{2.10}$$

holds for every $u, v \in R(\bar{I})$ *and* $t_0 \in \bar{I}$.

The operator Φ *is said to be* locally monotone, *if*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi[u](t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}u(t) \;\geq\; 0 \tag{2.11}$$

whenever the derivatives exist.

The local monotonicity of the stop and play has a particular form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_r[u] \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_r[u] \;=\; 0 \qquad \text{a.e. } t \in I, \tag{2.12}$$

hence $\mathrm{d}\mathcal{P}_r[u](t)/\mathrm{d}t = \dot{u}(t)$, $\mathrm{d}\mathcal{S}_r[u](t)/\mathrm{d}t = 0$ or vice versa.
We endow the space $C(\bar{I})$ with a system of seminorms

$$\|u\|_{[0,t]} \;:=\; \max_{0 \leq s \leq t} |u(s)| \qquad \text{for } u \in C(\bar{I}) \quad \text{and} \quad t \in \bar{I}. \tag{2.13}$$

From Proposition 2.1 we immediately obtain the following estimate.

Corollary 2.4 *For* $u_1, u_2 \in C(\bar{I})$ *put* $p_i := \mathcal{P}_r[u_i]$, $s_i := \mathcal{S}_r[u_i]$, $i = 1, 2$. *Then for all* $t \in \bar{I}$ *we have*

$$\begin{aligned} |p_1(t) - p_2(t)| \;&\leq\; \|u_1 - u_2\|_{[0,t]}\,, \\ |s_1(t) - s_2(t)| \;&\leq\; 2\,\|u_1 - u_2\|_{[0,t]}\,. \end{aligned} \tag{2.14}$$

2.2 Prandtl-Ishlinskii operators

One practical drawback of Prandtl's model in Fig. 2 consists in an instantaneous transition from the purely elastic to the purely plastic regime. In 'real' elastoplastic materials, this transition zone is smooth, see [12]. Prandtl [13] and Ishlinskii [8] therefore proposed to combine rheological elements from Fig. 3 corresponding to different values $r_1 < r_2 < \ldots < r_n < \infty$ of the yield point in parallel, as on Figure 4. The purely elastic element corresponding to $r = \infty$ accounts for the *kinematic hardening*.

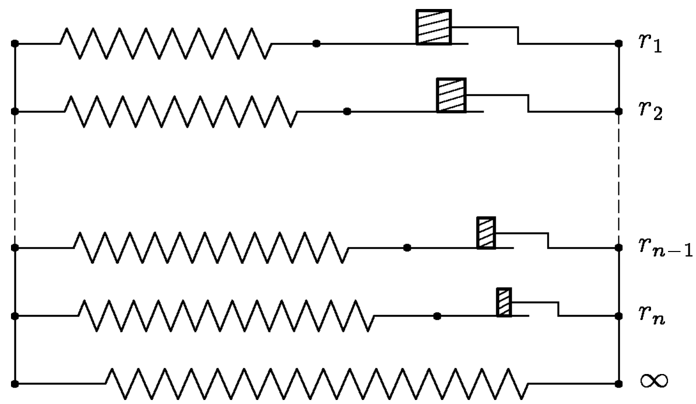

Figure 4: Rheological structure of the Prandtl-Ishlinskii model.

According to Eq. (2.2), the strain-stress law for the Prandtl-Ishlinskii model can be written in operator form as

$$\sigma \;=\; \mathcal{F}[e] \;:=\; \alpha_\infty\, e + \sum_{i=1}^{n} \alpha_i\, \mathcal{S}_{r_i}[e]\,, \tag{2.15}$$

where α_i are given nonnegative empirical constants.

In fact, there is no reason to restrict the model to finitely many yield points. For a mathematical treatment, it is more convenient to work with more general constitutive operators. This leads us to the following definition:

Definition 2.5 *Let us introduce the following sets of functions*

$$\begin{aligned} PI^- \;&:=\; \{\eta : [0,\infty] \to (0,\infty)\,; \\ &\qquad \eta \;\; \textit{bounded, nonincreasing, right-continuous, } \eta(\infty) > 0\}\,, \\ PI^+ \;&:=\; \{\zeta : [0,\infty) \to (0,\infty)\,; \\ &\qquad \zeta \;\; \textit{bounded, nondecreasing, right-continuous, } \zeta(0) > 0\} \end{aligned}$$

called admissible Prandtl-Ishlinskii distribution functions.

For a given function $\eta \in PI^-$, the operator $\mathcal{F}$ defined by the Stieltjes integral

$$\mathcal{F}[e] \;:=\; \eta(\infty)\, e - \int_0^\infty \mathcal{S}_r[e]\, \mathrm{d}\eta(r)\,, \tag{2.16}$$

is called a Prandtl-Ishlinskii operator of stop type.

Indeed, the case (2.15) is included in the above definition; it suffices to put $r_0 := 0$, $r_{n+1} := \infty$, $\alpha_{n+1} := \alpha_\infty$, and

$$\eta(r) := \sum_{i=k}^{n+1} \alpha_i \quad \text{for } r \in [r_{k-1}, r_k)\,, \quad k = 1, \ldots, n+1\,. \tag{2.17}$$

In particular, constant functions η correspond to purely elastic constitutive law.

An important practical question consists in identifying the function η from physical measurements. The usual approach is to increase monotonically the load from zero to some final value and to plot the corresponding strain-stress graph called the *initial loading curve*. So, assume that e increases in $\bar{I}$ from the starting value $e(0) = 0$. Then, at time t, we have by Eq. (2.4) for every $r > 0$

$$\mathcal{S}_r[e](t) = \min\{e(t), r\},$$

and formula (2.16) yields

$$\mathcal{F}[e](t) \;=\; \eta_\infty\, e(t) - \int_0^{e(t)} r\, \mathrm{d}\eta(r) - e(t) \int_{e(t)}^{\infty} \mathrm{d}\eta(r) = \int_0^{e(t)} \eta(r)\, \mathrm{d}r\,. \tag{2.18}$$

Given an *increasing concave* experimental initial loading curve $\sigma = \varphi(e)$, Eq. (2.18) says that it determines uniquely the Prandtl-Ishlinskii operator (2.16) through the relation

$$\eta(r) \;:=\; \varphi'(r) \equiv \frac{\mathrm{d}}{\mathrm{d}r}\varphi(r)\,. \tag{2.19}$$

In the Prandtl-Ishlinskii model, all secondary branches of hysteresis loops have the same shape, namely $\sigma = \sigma^* + 2\varphi((e - e^*)/2)$ for an increasing branch, $\sigma = \sigma^* - 2\varphi((e^* - e)/2)$ for a decreasing branch, where (e^*, σ^*) is a turning point, cf. Fig. 5.

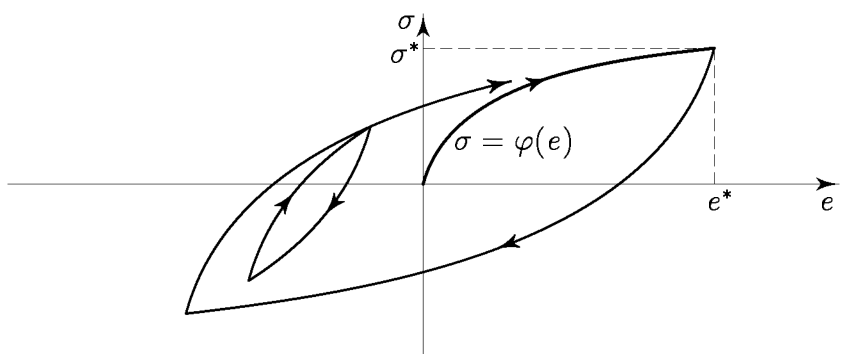

Figure 5: A diagram of the Prandtl-Ishlinskii operator.

Prandtl-Ishlinskii operators have a very specific property, namely that they are invertible and the inverse has the same structure. This result goes back to [10] in the time-periodic case. We need the following version which can be found in [11], Corollary II.3.4.

Theorem 2.6 *Let $\varphi : [0,\infty) \to [0,\infty)$ be a concave increasing function, $\varphi(0) = 0$, $\varphi(\infty) = \infty$, $\varphi'(0) < \infty$, $\varphi'(\infty) > 0$ and let $\psi = \varphi^{-1} : [0,\infty) \to [0,\infty)$ be its inverse. Let $\eta := \varphi'$, $\zeta := \psi'$ be the right-continuous representatives of their respective derivatives. Then $\eta \in PI^-$, $\zeta \in PI^+$ and the operator $\mathcal{G} : C(\bar{I}) \to C(\bar{I})$ defined by the formula*

$$\mathcal{G}[\sigma] \;:=\; \zeta(0)\,\sigma + \int_0^\infty \mathcal{P}_r[\sigma]\, \mathrm{d}\zeta(r) \quad \textit{for} \quad \sigma \in C(\bar{I}) \tag{2.20}$$

is the inverse operator to $\mathcal{F}$ given by formula (2.16).

In the situation of Theorem 2.6, we say that $(\eta, \zeta) \in PI^- \times PI^+$ form a *pair of adjoint Prandtl-Ishlinskii distribution functions*. The operator $\mathcal{G}$ is called *Prandtl-Ishlinskii operator of play type*.

In terms of the underlying mechanical construction, Theorem 2.6 can be interpreted in such a way that the rheological models on Fig. 4 and Fig. 6 are equivalent.

Prandtl-Ishlinskii operators are locally monotone hysteresis operators according to Definition 2.3. More precisely, Theorem 2.6 and Eq. (2.12) imply the following inequalities:

Proposition 2.7 *Let $\mathcal{F}$ be as in Definition 2.5, and let α, β be positive constants such that*

$$\beta \le 1/\eta(0), \qquad \alpha \le \eta(\infty). \tag{2.21}$$

For $e \in W^{1,1}(I)$ put $\sigma := \mathcal{F}[e]$. Then for a.e. t we have

$$\begin{array}{rcccl} \alpha\, \dot{e}^2(t) & \le & \dot{e}(t)\,\dot{\sigma}(t) & \le & \frac{1}{\beta}\,\dot{e}^2(t)\,, \\ \beta\,\dot{\sigma}^2(t) & \le & \dot{e}(t)\,\dot{\sigma}(t) & \le & \frac{1}{\alpha}\,\dot{\sigma}^2(t)\,. \end{array} \tag{2.22}$$

Indeed, if $(\eta, \zeta) \in PI^- \times PI^+$ is a pair of adjoint Prandtl-Ishlinskii distribution functions, then, due to the identities $\zeta(0) = 1/\eta(0)$ and $\zeta(\infty) = 1/\eta(\infty)$, Ineqs. (2.21) are equivalent to

$$0 < \beta \le \zeta(0)\,, \qquad 0 < \alpha \le 1/\zeta(\infty)\,. \tag{2.23}$$

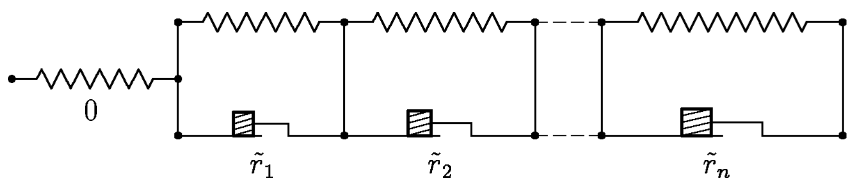

Figure 6: Rheological structure of the Prandtl-Ishlinskii operator of play type.

As an immediate consequence of Theorem 2.6 and Corollary 2.4, we have the following Lipschitz estimates:

Proposition 2.8 *Let $\mathcal{F}, \alpha, \beta$ be as in Proposition 2.7. For given functions $e_1, e_2 \in C(\bar{I})$ put $\sigma_i := \mathcal{F}[e_i]$ for $i = 1, 2$. Then for every $t \in \bar{I}$ we have*

$$|\sigma_1(t) - \sigma_2(t)| \;\le\; \left(\frac{2}{\beta} - \alpha\right) \|e_1 - e_2\|_{[0,t]}\,, \tag{2.24}$$

$$|e_1(t) - e_2(t)| \;\le\; \frac{1}{\alpha}\,\|\sigma_1 - \sigma_2\|_{[0,t]}\,. \tag{2.25}$$

For every input function $\sigma \in C(\bar{I})$ and every fixed time $t \in \bar{I}$, the function $\pi_t : [0, \infty) \to \mathbb{R}$ defined by the formula

$$\pi_t(r) := \begin{cases} \mathcal{P}_r[\sigma](t) & \text{for } r > 0\,, \\ \sigma(t) & \text{for } r = 0\,, \end{cases} \tag{2.26}$$

represents the *memory state of the system* at time t. It has the following properties (see Proposition II.2.5 of [11]).

Proposition 2.9 *Let $\sigma \in C(\bar{I})$ and $t \in \bar{I}$ be given. Then the function π_t defined by Eq. (2.26) is Lipschitz continuous with coefficient 1 and we have*

$$\left\{ \begin{array}{ll} \pi_t(r) = 0 & \textit{for } r \geq \|\sigma\|_{[0,t]} \\ \left|\dfrac{\partial}{\partial r}\pi_t(r)\right| = 1 & \textit{for a.e. } r \in (0, \|\sigma\|_{[0,t]})\,. \end{array} \right. \tag{2.27}$$

This result enables us to estimate the difference of two Prandtl-Ishlinskii operators of play type in the following way:

Proposition 2.10 *Let $\zeta_1, \zeta_2 \in PI^+$ be given, and let $\mathcal{G}_1$, $\mathcal{G}_2$ be the corresponding operators of the form (2.20). Let $\sigma_1, \sigma_2 \in C(\bar{I})$ be arbitrary functions. Then for every $t \in \bar{I}$ we have*

$$\|\mathcal{G}_1[\sigma_1] - \mathcal{G}_2[\sigma_2]\|_{[0,t]} \leq \zeta_1(\infty)\ \|\sigma_1 - \sigma_2\|_{[0,t]} + \int_0^{\|\sigma_2\|_{[0,t]}} |\zeta_1(r) - \zeta_2(r)|\,\mathrm{d}r\,. \tag{2.28}$$

Proof. Using the integration-by-parts formula for the Stieltjes integral, we have for all $t \in \bar{I}$

$$\begin{aligned} (\mathcal{G}_1[\sigma_1] - \mathcal{G}_2[\sigma_2])(t) \ &= \ \zeta_1(0)(\sigma_1 - \sigma_2)(t) \\ &+ \ \int_0^\infty (\mathcal{P}_r[\sigma_1] - \mathcal{P}_r[\sigma_2])(t)\,\mathrm{d}\zeta_1(r) \ - \int_0^\infty \frac{\partial}{\partial r}\mathcal{P}_r[\sigma_2](t)\,(\zeta_1 - \zeta_2)(r)\,\mathrm{d}r\,, \end{aligned} \tag{2.29}$$

and Ineq. (2.28) follows from Corollary 2.4 and Proposition 2.9. ❐

2.3 Energy inequalities

The energy dissipation is a typical feature of hysteresis phenomena. To introduce it as a mathematical concept, we have to define an internal energy functional $U \geq 0$ corresponding to the constitutive law $\sigma = \mathcal{F}[e]$ or equivalently $e = \mathcal{G}[\sigma]$. The second principle of thermodynamics then requires that the dissipation rate q has to satisfy

$$q := \sigma\,\dot{e} - \dot{U} \ \geq \ 0\,. \tag{2.30}$$

If we choose e as state variable (input) and $\sigma = \mathcal{F}[e]$ as state function (output), we define a continuous family of internal parameters $\sigma_r := \mathcal{S}_r[e]$ which correspond to individual stress components in the rheological construction of Fig. 4. It is assumed that no internal energy can be stored in the dry friction elements; the internal energy U of the system is then defined as the total internal energy of the individual elastic elements, that is, in operator form,

$$U = \mathcal{U}[e] := \frac{1}{2}\left(\eta(\infty)e^2 - \int_0^\infty \sigma_r^2\,\mathrm{d}\eta(r)\right) = \frac{1}{2}\left(\eta(\infty)e^2 - \int_0^\infty (\mathcal{S}_r[e])^2\,\mathrm{d}\eta(r)\right). \tag{2.31}$$

Conversely, if σ is the input and $e = \mathcal{G}[\sigma]$ is the output, then we choose the strain components $e_r := \mathcal{P}_r[\sigma]$ to be the internal parameters and the total internal energy has the form

$$U = \mathcal{V}[\sigma] := \frac{1}{2}\left(\zeta(0)\sigma^2 + \int_0^\infty e_r^2\,\mathrm{d}\zeta(r)\right) = \frac{1}{2}\left(\zeta(0)\sigma^2 + \int_0^\infty (\mathcal{P}_r[\sigma])^2\,\mathrm{d}\zeta(r)\right). \tag{2.32}$$

It can be shown that formulas (2.31) and (2.32) are equivalent. It is easy to check that for every $e, \sigma \in W^{1,1}(I)$ we have

$$\begin{aligned} \mathcal{F}[e]\,\dot{e} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}[e] &= -\int_0^\infty \mathcal{S}_r[e]\,\frac{\partial}{\partial t}\mathcal{P}_r[e]\,\mathrm{d}\eta(r) \geq 0 \quad \text{a.e.}, & (2.33)\\ \sigma\,\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}[\sigma] - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}[\sigma] &= \int_0^\infty \mathcal{S}_r[\sigma]\,\frac{\partial}{\partial t}\mathcal{P}_r[\sigma]\,\mathrm{d}\zeta(r) \geq 0 \quad \text{a.e.}, & (2.34) \end{aligned}$$

hence the model is consistent with inequality (2.30).

Hysteresis operators admit a *second order energy inequality* which is related to the convexity of hysteresis loops. A detailed discussion on this subject can be found in Section II.4 of [11]. We need here the following simplified version which follows directly from the definition without referring to the geometry of hysteresis loops.

Theorem 2.11 *Let $\mathcal{F}, \alpha, \beta$ be as in Proposition 2.7 and let $e \in W^{2,1}(0,T)$ be given. Then $\sigma = \mathcal{F}[e]$ belongs to $W^{1,\infty}(0,T)$ and for every $t \in \bar{I}$ we have*

$$\int_0^t \ddot{e}(\tau)\,\dot{\sigma}(\tau)\,\mathrm{d}\tau \;\geq\; \frac{\alpha}{2}\,\dot{e}^2(t) - \frac{1}{2\beta}\,\dot{e}^2(0)\,. \tag{2.35}$$

Proof. The fact that σ belongs to $W^{1,\infty}(0,T)$ follows immediately from Proposition 2.7. For an arbitrary $h > 0$ sufficiently small, $r > 0$ and $t \in [0, T-h]$ put $e_1(t) := e(t+h)$, $e_2(t) := e(t)$, $\sigma^0_{1r} := \mathcal{S}_r[e](h)$, $\sigma^0_{2r} := \mathcal{S}_r[e](0)$, $\sigma_r(t) := \mathcal{S}_r[e](t)$. Then for each $t \in [0, T-h]$ we have $\mathcal{S}_r[\sigma^0_{1r}, e_1](t) = \sigma_r(t+h)$, $\mathcal{S}_r[\sigma^0_{2r}, e_2](t) = \sigma_r(t)$, and Proposition 2.1 (i) yields

$$(\dot{e}(t+h) - \dot{e}(t))\,(\sigma_r(t+h) - \sigma_r(t)) \;\geq\; \frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,|\sigma_r(t+h) - \sigma_r(t)|^2 \qquad \text{a.e.}$$

Integrating the above inequality we obtain for every $0 \leq s < t \leq T-h$

$$\begin{aligned} &\int_s^t (\dot{e}(\tau+h) - \dot{e}(\tau))\,(\sigma_r(\tau+h) - \sigma_r(\tau))\,\mathrm{d}\tau \\ &\qquad \geq\; \frac{1}{2}\,\left(|\sigma_r(t+h) - \sigma_r(t)|^2 - |\sigma_r(s+h) - \sigma_r(s)|^2\right)\,. \end{aligned} \tag{2.36}$$

Proposition II.2.8 and Corollary II.2.9 of [11] enable us to justify the following formal computation which consists in dividing inequality (2.36) by h^2, letting h tend to 0 and using the fact that $\dot{\sigma}_r^2 = \dot{\sigma}_r \dot{e}$ a.e. according to identity (2.12). We conclude that for a.e. $s < t$ and every $r > 0$ except for two values at most, we have

$$\int_s^t \ddot{e}(\tau)\,\dot{\sigma}_r(\tau)\,\mathrm{d}\tau \;\geq\; \frac{1}{2}\,(\dot{e}(t)\,\dot{\sigma}_r(t) - \dot{e}(s)\,\dot{\sigma}_r(s))\,. \tag{2.37}$$

Integrating with respect to $-\mathrm{d}\eta(r)$ we obtain for a.e. $s < t$

$$\int_s^t \ddot{e}(\tau)\,\dot{\sigma}(\tau)\,\mathrm{d}\tau \;\geq\; \frac{1}{2}\,(\dot{e}(t)\,\dot{\sigma}(t) - \dot{e}(s)\,\dot{\sigma}(s))\,, \tag{2.38}$$

and the assertion follows from Proposition 2.7. ❐

2.4 Spatially dependent hysteresis operators

It is natural to consider elastoplastic materials whose constitutive properties are spatially inhomogeneous, that is, the Prandtl-Ishlinskii constitutive law $\sigma = \mathcal{F}[e]$ may be different at different points $x \in J = (0, \ell)$. In other words, we allow the Prandtl-Ishlinskii distribution function η to depend on x. Given an input function $e : J \times I \to \mathbb{R}$, we define the output σ by the formula

$$\sigma(x,t) := \mathcal{F}[e](x,t) = \eta(x,\infty)\, e(x,t) \;-\; \int_0^\infty \mathcal{S}_r[e(x,\cdot)](t)\,\mathrm{d}_r\eta(x,r) \tag{2.39}$$

for $(x,t) \in J \times I$. It makes sense for every $x \in J$, for which the function $t \mapsto e(x,t)$ is continuous and the function $r \mapsto \eta(x,r)$ belongs to PI^-. For an input function σ, we similarly define the output of the inverse operator

$$e(x,t) := \mathcal{G}[\sigma](x,t) = \zeta(x,0)\,\sigma(x,t) \;+\; \int_0^\infty \mathcal{P}_r[\sigma(x,\cdot)](t)\,\mathrm{d}_r\zeta(x,r)\,, \tag{2.40}$$

where $(\eta(x,\cdot), \zeta(x,\cdot)) \in PI^- \times PI^+$ is a pair of adjoint Prandtl-Ishlinskii distribution functions.

Analogously as in Proposition 2.10, the space-dependent operators are continuous with respect to input functions from the space $C(\bar{J} \times \bar{I})$ of continuous functions on $\bar{J} \times \bar{I}$, endowed with the usual sup-norm $\|\cdot\|_\infty$. Below we give a more substantial result on the continuity with respect to weak-star convergence of the Prandtl-Ishlinskii distribution functions.

Proposition 2.12 *Let ζ_n, $n \in \mathbb{N}$, be a sequence of functions in $L^\infty(J \times (0,\infty)$ such that $\zeta_n(x,\cdot) \in PI^+$ for every $n \in \mathbb{N}$ and a.e. $x \in J$, and let there exist positive constants α, β such that for every $n \in \mathbb{N}$, $r > 0$ and a.e. $x \in J$ we have*

$$\beta \le \zeta_n(x,r) \le 1/\alpha\,. \tag{2.41}$$

Assume that ζ_n converge to ζ in $L^\infty(J \times (0,\infty))$ weakly-star as $n \to \infty$.

Let σ_n, $n \in \mathbb{N}$, be a sequence in $L^\infty(I \times J))$ such that $\sigma_n(x,\cdot) \in C(\bar{I})$ for a.e. $x \in J$ and $\|\sigma_n - \sigma\|_\infty \to 0$ as $n \to \infty$. Let $\mathcal{G}_n$, $\mathcal{G}$ be the operators corresponding to ζ_n, ζ, respectively, according to Eq. (2.40).

Then $\mathcal{G}_n[\sigma_n](\cdot,t)$ converge to $\mathcal{G}[\sigma](\cdot,t)$ for every $t \in I$ in $L^\infty(J)$ weakly-star as $n \to \infty$.

Proof. The function $\zeta(x,\cdot)$ is obviously nondecreasing for a.e. $x \in J$ and $\beta \le \zeta(x,r) \le 1/\alpha$ a.e., hence the operator $\mathcal{G}$ is well defined. By Eq. (2.29), we have for a.e. $x \in J$ and every $t \in I$

$$\begin{aligned}(\mathcal{G}_n[\sigma_n] - \mathcal{G}[\sigma])(x,t) \;&=\; \zeta_n(x,0)(\sigma_n - \sigma)(x,t) + \\ &+\; \int_0^\infty (\mathcal{P}_r[\sigma_n(x,\cdot)] - \mathcal{P}_r[\sigma(x,\cdot)])(t)\,\mathrm{d}_r\zeta_n(x,r) \\ &-\; \int_0^\infty \frac{\partial}{\partial r}\mathcal{P}_r[\sigma(x,\cdot)](t)\,(\zeta_n - \zeta)(x,r)\,\mathrm{d}r\,.\end{aligned} \tag{2.42}$$

The first two terms of the right-hand side of Eq. (2.42) converge uniformly to 0 by Corollary 2.4 and Ineq. (2.41), the third term converges weakly-star to 0 by Proposition 2.9, and Proposition 2.12 follows. ❒

3 The spatially inhomogeneous wave equation

3.1 Statement of the problem

We rewrite Eqs. (1.1), (1.2) in the form of a first order system

$$\left\{\begin{array}{rcl} \rho\, v_t &=& \sigma_x + f\,, \\ e_t &=& v_x\,, \\ e &=& \mathcal{G}[\sigma] \end{array}\right. \tag{3.1}$$

in $J \times I$, where $\rho = \rho(x)$ is a given mass density, $v = u_t$ is the velocity, $f = f(x,t)$ is a given volume force density and $\mathcal{G}$ is a Prandtl-Ishlinskii operator of play type of the form (2.40) with a given density $\zeta = \zeta(x,r)$.
The boundary and initial conditions (1.3), (1.4) are considered here in the form

$$v(0,t) = \sigma(\ell,t) = 0 \qquad \text{for} \quad t \in I\,, \tag{3.2}$$

$$v(x,0) = v^0(x)\,, \quad \sigma(x,0) = \sigma^0(x) \qquad \text{for} \quad x \in J\,. \tag{3.3}$$

This is indeed formally equivalent to (1.3), (1.4). It suffices to put $v^0(x) := u_1(x)$ and, according to the choice (2.8) of the initial conditions for the stop operators,

$$\sigma^0(x) := \int_0^{e^0(x)} \eta(x,r)\,\mathrm{d}r\,, \qquad e^0(x) := u_0'(x)\,,$$

where $(\eta(x,\cdot), \zeta(x,\cdot))$ is a pair of adjoint Prandtl-Ishlinskii distribution functions for a.e. $x \in J$.
The data are assumed to satisfy the following requirements:

Hypothesis 3.1

(i) $\rho \in L^\infty(J)$ *and there exist constants* $\rho_m, \rho_M > 0$ *such that* $\rho_m \le \rho(x) \le \rho_M$ *for a.e.* $x \in J$,

(ii) $\zeta \in L^\infty(J \times (0,\infty))$, *the function* $\zeta(x,\cdot)$ *belongs to* PI^+ *for a.e.* $x \in J$ *and there exist* $\alpha, \beta > 0$ *such that* $\beta \le \zeta(x,r) \le 1/\alpha$ *for a.e.* $x \in J$,

(iii) $f \in W^{1,1}(I; L^2(J))$,

(iv) $\sigma^0, v^0 \in W^{1,2}(J)$, $\sigma^0(\ell) = v^0(0) = 0$.

We now state the main result of this section.

Theorem 3.2 *Let Hypothesis 3.1 hold. Then there exist uniquely determined functions* $v, \sigma \in C(\bar{J} \times \bar{I})$ *and* $e \in L^2(J; C(\bar{I}))$ *such that* $e_t, v_t, \sigma_t, v_x, \sigma_x \in L^\infty(I; L^2(J))$, *conditions (3.2), (3.3) hold pointwise and Eqs. (3.1) are satisfied almost everywhere. Moreover, there exists a constant* $B > 0$ *depending only on* $\alpha, \beta, \rho_m, \rho_M$, *the norm of* f *in* $W^{1,1}(I; L^2(J))$ *and the norms of* σ^0, v^0 *in* $W^{1,2}(J)$ *such that the norms of* $e_t, v_t, \sigma_t, v_x, \sigma_x$ *in* $L^\infty(I; L^2(J))$ *are estimated from above by the constant* B.

The rest of this section is devoted to the proof of Thm. 3.2 which extends the results of [11], where the function ζ in Hypothesis 3.1 is required to be continuous with respect to x.

The argument consists of several steps. We first discretize Eqs. (3.1) in space and solve for each partition of the interval J the corresponding system of ODEs. Then we derive upper bounds independent of the discretization parameter for the discrete solutions and use compact embeddings for passing to the limit. Finally, we check that the limit functions are the unique solutions of the problem.

3.2 Space discretization

For a fixed integer $n \in \mathbb{N}$, we divide the interval $J = (0, \ell)$ into an equidistant partition of size $h = \ell/n$, and consider the system

$$\left\{ \begin{array}{rcl} \rho_k \dot{v}_k &=& \frac{1}{h}(\sigma_{k+1} - \sigma_k) + f_k\,, \\ \dot{e}_k &=& \frac{1}{h}(v_k - v_{k-1})\,, \\ e_k &=& \mathcal{G}_k[\sigma_k] \end{array} \right. \tag{3.4}$$

for $k = 1, \ldots, n-1$ with unknown functions $v_1, \ldots, v_{n-1}$, $\sigma_1, \ldots, \sigma_{n-1}$, where the dot denotes derivative with respect to t. We prescribe 'boundary conditions'

$$v_0 = 0\,, \qquad \sigma_n = 0 \tag{3.5}$$

and initial conditions

$$v_k(0) = v^0(kh)\,,\ \sigma_k(0) = \sigma^0(kh)\,,\ e_k(0) = \mathcal{G}_k[\sigma_k](0)\,, \quad k = 1, \ldots, n-1\,, \tag{3.6}$$

where for $k = 1, \ldots, n$ we define

$$\rho_{k-1} := \frac{1}{h}\int_{(k-1)h}^{kh} \rho(x)\,\mathrm{d}x\,, \tag{3.7}$$

$$f_{k-1}(t) := \frac{1}{h}\int_{(k-1)h}^{kh} f(x,t)\,\mathrm{d}x\,, \tag{3.8}$$

$$\mathcal{G}_k[\sigma] := \zeta_k(0)\,\sigma + \int_0^\infty \mathcal{P}_r[\sigma]\,\mathrm{d}\zeta_k(r) \tag{3.9}$$

for an arbitrary function $\sigma \in C(\bar{I})$, where

$$\zeta_k(r) := \frac{1}{h}\int_{(k-1)h}^{kh} \zeta(x,r)\,\mathrm{d}x \quad \text{for} \ \ r \geq 0\,. \tag{3.10}$$

The functions ζ_k in Eqs. (3.9), (3.10) are obviously nondecreasing and fulfil the inequalities

$$\beta \leq \zeta_k(r) \leq 1/\alpha \qquad \forall\, r \geq 0\,,\ k = 1, \ldots, n\,, \tag{3.11}$$

For the sake of completeness, we recall the following existence result for the system (3.4) – (3.10) which is analogous as in Section III.2 of [11].

Proposition 3.3 *Let Hypothesis 3.1 holds. Then the system (3.4) – (3.6) admits a unique global solution in I such that $v_k \in W^{2,1}(I)$, $e_k \in W^{3,1}(I)$ and $\sigma_k \in W^{1,\infty}(I)$ for $k = 1, \ldots, n-1$.*

Proof. Let $\mathcal{F}_k = (\mathcal{G}_k)^{-1}$ be the inverse operator to $\mathcal{G}_k$ for $k = 1, \ldots, n$ according to Thm. 2.6. System (3.4) – (3.10) is then of the form

$$\dot{Y} = \mathcal{T}[Y], \quad Y(0) = Y^0, \tag{3.12}$$

where $Y = (Y_1, \ldots, Y_{2n-2})$, with

$$Y_j = v_j, \quad (\mathcal{T}[Y])_j = ((\mathcal{F}_{j+1}[e_{j+1}] - \mathcal{F}_j[e_j])/h + f_j)/\rho_j \quad \text{for } j = 1, \ldots, n-1,$$

$$Y_j = e_{j+1-n}, \quad (\mathcal{T}[Y])_j = (v_{j+1-n} - v_{j-n})/h \quad \text{for } j = n, \ldots, 2n-2.$$

By Proposition 2.8, there exists a constant L_n such that for every $Y, Z \in C(\bar{I}; \mathbb{R}^{2n-2})$ and every $t \in \bar{I}$ we have

$$|\mathcal{T}[Y](t) - \mathcal{T}[Z](t)| \le L_n \|Y - Z\|_{[0,t]}, \tag{3.13}$$

with natural extension of the notation from $C(\bar{I})$ to $C(\bar{I}; \mathbb{R}^{2n-2})$. Let us define an auxiliary mapping K from $C(\bar{I}; \mathbb{R}^{2n-2})$ to $C(\bar{I}; \mathbb{R}^{2n-2})$ by the formula

$$K[Y](t) := Y^0 + \int_0^t \mathcal{T}[Y](\tau)\, d\tau. \tag{3.14}$$

Solutions of Eq. (3.12) can be identified with fixed points of the mapping K. By induction, the $p-th$ iterate K^p of K fulfils the inequality

$$|K^p[Y](t) - K^p[Z](t)| \le \frac{(L_n t)^p}{p!} \|Y - Z\|_\infty,$$

for every $Y, Z \in C(\bar{I}; \mathbb{R}^{2n-2})$ and every $t \in \bar{I}$. For p sufficiently large, K^p is a contraction, hence K admits a unique fixed point by the Banach Contraction Principle. The regularity follows from a usual bootstrapping argument: since f_k and σ_k are continuous, the first equation of (3.4) yields $v_k \in C^1(\bar{I})$, from the second one it follows that $e_k \in C^2(\bar{I})$. The third equation together with Proposition 2.9 imply that $\sigma_k \in W^{1,\infty}(I)$. Taking into account the fact that $f_k \in W^{1,1}(I)$, we can repeat the procedure and obtain the assertion. ❒

3.3 Estimates

With the intention to pass to the limit as $n \to \infty$ in the spatially discrete system (3.4), we derive for its solutions estimates independent of n. Throughout this subsection, we denote by $C_1, C_2, \ldots$ any constants depending exclusively on $\alpha, \beta, \rho_m, \rho_M$, the norm of f in $W^{1,1}(I; L^2(J))$ and the norms of σ^0, v^0 in $W^{1,2}(J)$.

Lemma 3.4 *The solution $\{v_k, e_k, \sigma_k\}_{k=1}^{n-1}$ to Problem (3.4) – (3.6) satisfies for every $t \in I$ the estimate*

$$h \sum_{k=1}^{n-1} \left(\dot{v}_k^2 + \dot{e}_k^2 + \dot{\sigma}_k^2 + \left(\frac{\sigma_{k+1} - \sigma_k}{h}\right)^2 + \left(\frac{v_k - v_{k-1}}{h}\right)^2 \right)(t) \le C, \tag{3.15}$$

where $C > 0$ is a constant which depends only on $\alpha, \beta, \rho_m, \rho_M$, the norm of f in $W^{1,1}(I; L^2(J))$ and the norms of σ^0, v^0 in $W^{1,2}(J)$.

Proof. We differentiate Eqs. (3.4) with respect to t, multiply the first equation by $\dot v_k(t)$, the second equation by $\dot\sigma_k(t)$, and sum up over $k = 1, \dots, n-1$. Using the conditions (3.5) we obtain

$$\sum_{k=1}^{n-1} \left(\rho_k\, \ddot v_k(t)\, \dot v_k(t) + \ddot e_k(t)\, \dot\sigma_k(t)\right) \;=\; \sum_{k=1}^{n-1} \dot f_k(t)\, \dot v_k(t) \tag{3.16}$$

for a.e. $t \in I$. We now fix any $t \in I$ and integrate Eq. (3.16) from 0 to t. This yields

$$\sum_{k=1}^{n-1} \Big(\frac{\rho_k}{2}\, \dot v_k^2(t) \;+\; \int_0^t \ddot e_k(\tau)\, \dot\sigma_k(\tau)\, \mathrm{d}\tau\Big) \;\le \tag{3.17}$$

$$\le\; \sum_{k=1}^{n-1} \frac{\rho_k}{2}\, \dot v_k^2(0) + \left(\left\|\sum_{k=1}^{n-1} \dot v_k^2\right\|_{[0,t]}\right)^{1/2} \int_0^t \left(\sum_{k=1}^{n-1} \dot f_k^2(\tau)\right)^{1/2} \mathrm{d}\tau\,.$$

To estimate the right-hand side of Ineq. (3.17), we first notice that for a.e. $\tau \in I$ we have

$$\sum_{k=1}^{n-1} \dot f_k^2(\tau) = \frac{1}{h^2} \sum_{k=1}^{n-1} \left(\int_{kh}^{(k+1)h} f_t(x,\tau)\, \mathrm{d}x\right)^2 \le \frac{1}{h} \int_J f_t^2(x,\tau)\, \mathrm{d}x\,, \tag{3.18}$$

hence

$$\int_I \left(h \sum_{k=1}^{n-1} \dot f_k^2(\tau)\right)^{1/2} \mathrm{d}\tau \le \|f_t\|_{L^1(I, L^2(J))}\,. \tag{3.19}$$

Applying the same argument to $h \sum f_k^2$, we thus obtain

$$\int_I \left(h \sum_{k=1}^{n-1} \left(f_k^2(\tau) + \dot f_k^2(\tau)\right)\right)^{1/2} \mathrm{d}\tau \le C_1\,. \tag{3.20}$$

On the other hand, for every function $F \in W^{1,1}(I)$ and every $0 \le s < t \le T$ we have $F(t) \le F(s) + \int_s^t |F'(\tau)| \mathrm{d}\tau$. This yields in particular

$$\left(\sum_{k=1}^{n-1} f_k^2(t)\right)^{1/2} \;\le\; \left(\sum_{k=1}^{n-1} f_k^2(s)\right)^{1/2} + \int_s^t \left(\sum_{k=1}^{n-1} \dot f_k^2(\tau)\right)^{1/2} \mathrm{d}\tau\,,$$

and integrating over s we obtain from Ineq. (3.20) that

$$h \left\|\sum_{k=1}^{n-1} f_k^2\right\|_{[0,T]} \;\le\; C_2\,. \tag{3.21}$$

From Theorem 2.11 we infer that

$$\int_0^t \ddot e_k(\tau)\, \dot\sigma_k(\tau)\, \mathrm{d}\tau \;\ge\; \frac{\alpha}{2} \dot e_k^2(t) - \frac{1}{2\beta} \dot e_k^2(0)\,, \qquad \forall\, t \in I,\; k = 1, \dots, n\,. \tag{3.22}$$

Combining Ineqs. (3.17), (3.20) and (3.22), we have for every $t \in I$

$$h \sum_{k=1}^{n-1} (\rho_k \, \dot{v}_k^2(t) \; + \; \alpha \, \dot{e}_k^2(t)) \; \le \tag{3.23}$$

$$\le \; h \sum_{k=1}^{n-1} \left(\rho_k \, \dot{v}_k^2(0) + \frac{1}{\beta} \dot{e}_k^2(0) \right) + C_3 \left(\left\| h \sum_{k=1}^{n-1} \dot{v}_k^2 \right\|_{[0,t]} \right)^{1/2} .$$

To estimate the right-hand side of Ineq. (3.23), we use Eqs. (3.4) which yield for $k = 1, \ldots, n-1$ that

$$\begin{aligned} \rho_k \, |\dot{v}_k(0)| \; &\le \; \left| \frac{\sigma_{k+1}(0) - \sigma_k(0)}{h} \right| + \; |f_k(0)| \\ &\le \; \frac{1}{h} \int_{kh}^{(k+1)h} |\sigma_x^0(x)| \, \mathrm{d}x + \; |f_k(0)| \, , \end{aligned} \tag{3.24}$$

$$|\dot{e}_k(0)| \; \le \; \left| \frac{v_k(0) - v_{k-1}(0)}{h} \right| \; \le \; \frac{1}{h} \int_{(k-1)h}^{kh} |v_x^0(x)| \, \mathrm{d}x \, . \tag{3.25}$$

The obvious inequality

$$\rho_m \le \rho_k \le \rho_M \qquad \forall k = 0, \ldots, n-1 \tag{3.26}$$

together with (3.25), (3.26) and (3.21) implies that

$$h \sum_{k=1}^{n-1} \left(\rho_k \, \dot{v}_k^2(0) + \frac{1}{\beta} \dot{e}_k^2(0) \right) \; \le \; C_4 \, . \tag{3.27}$$

From Ineq. (3.23) we now easily conclude that

$$h \sum_{k=1}^{n-1} \left(\dot{v}_k^2(t) + \dot{e}_k^2(t) \right) \; \le \; C_5 \tag{3.28}$$

independently of t. To complete the proof, it suffices to use again Eqs. (3.4) and Ineqs. (3.21), (2.22). ❐

3.4 Passage to the limit

For each fixed $n \in \mathbb{N}$, we construct approximate solutions to Eqs. (3.1) as piecewise constant or piecewise linear interpolates of solutions to the semidiscrete system (3.4). For $t \in I$, $r \ge 0$, $x \in [(k-1)h, kh)$, $h = \ell/n$, $k = 1, \ldots, n$ we define the functions (continuously extended to $x = \ell$)

$$\begin{aligned} \overline{\rho}^{(n)}(x) \; &:= \; \rho_{k-1} \, , \\ \overline{\zeta}^{(n)}(x, r) \; &:= \; \zeta_k(r) \, , \\ \overline{f}^{(n)}(x, t) \; &:= \; f_{k-1}(t) \, , \end{aligned}$$

$$
\begin{aligned}
\overline{v}^{(n)}(x,t) &:= v_{k-1}(t)\,,\\
\overline{e}^{(n)}(x,t) &:= e_k(t)\,,\\
\overline{\sigma}^{(n)}(x,t) &:= \sigma_k(t)\,,\\
\widehat{v}^{(n)}(x,t) &:= v_{k-1}(t) + \left(\frac{x}{h} - (k-1)\right)(v_k(t) - v_{k-1}(t))\,,\\
\widehat{\sigma}^{(n)}(x,t) &:= \sigma_{k-1}(t) + \left(\frac{x}{h} - (k-1)\right)(\sigma_k(t) - \sigma_{k-1}(t))\,,\\
\widetilde{\sigma}^{(n)}(x,t) &:= \begin{cases} \widehat{\sigma}^{(n)}(x,t) & \text{for } x \in [h,\ell]\,,\\ \sigma_1(t) & \text{for } x \in [0,h)\,, \end{cases}
\end{aligned}
$$

where we put $e_n(t) := 0$, $v_n(t) := v_{n-1}(t)$, $\sigma_0(t) := \sigma_1(t) + h\, f_0(t)$. We also introduce the interpolated Prandtl-Ishlinskii operator

$$
\overline{\mathcal{G}}^{(n)}[\sigma](x,t) := \overline{\zeta}^{(n)}(x,0)\,\sigma(x,t) + \int_0^\infty \mathcal{P}_r[\sigma(x,\cdot)](t)\,\mathrm{d}_r\overline{\zeta}^{(n)}(x,r)
$$

for each input σ such that $\sigma(x,\cdot) \in C(\bar{I})$ for a.e. $x \in J$.

The above functions have been chosen in order to satisfy the system

$$
\left\{
\begin{aligned}
\overline{\rho}^{(n)}\,\overline{v}_t^{(n)} &= \widehat{\sigma}_x^{(n)} + \overline{f}^{(n)}\,,\\
\overline{e}_t^{(n)} &= \widehat{v}_x^{(n)}\,,\\
\overline{e}^{(n)} &= \overline{\mathcal{G}}^{(n)}[\overline{\sigma}^{(n)}]
\end{aligned}
\right.
\tag{3.29}
$$

almost everywhere in $J \times I$, together with boundary conditions

$$
\widehat{v}(0,t) = \widehat{\sigma}(\ell,t) = 0 \quad \text{for} \quad t \in I\,. \tag{3.30}
$$

The following estimate is crucial for passing to the limit as $n \to \infty$.

Proposition 3.5 *There exists a constant $C_6 > 0$ such that for every $n \in \mathbb{N}$ we have*

$$
\begin{aligned}
\max_{t\in I} \int_J (|\overline{e}_t^{(n)}|^2 + |\overline{v}_t^{(n)}|^2 + |\widehat{v}_t^{(n)}|^2 + |\widetilde{\sigma}_t^{(n)}|^2 \qquad\qquad & \qquad (3.31)\\
+|\widehat{v}_x^{(n)}|^2 + |\widehat{\sigma}_x^{(n)}|^2 + |\widetilde{\sigma}_x^{(n)}|^2)(x,t)\,\mathrm{d}x \;\le\; C_6\,. &
\end{aligned}
$$

Proof. For $x \in [(k-1)h, kh]$ we have $|\widehat{v}_t^{(n)}(x,t)|^2 \le \dot{v}_{k-1}^2(t) + \dot{v}_k^2(t)$ for $k \le n-1$, $|\widehat{v}_t^{(n)}(x,t)|^2 = \dot{v}_{n-1}^2(t)$ for $k = n$ and analogously $|\widetilde{\sigma}_t^{(n)}(x,t)|^2 \le \dot{\sigma}_{k-1}^2(t) + \dot{\sigma}_k^2(t)$ for $k \ge 2$. For $x \in [0,h]$ we have $|\widetilde{\sigma}_t^{(n)}(x,t)|^2 = \dot{\sigma}_1^2(t)$ and $|\widehat{\sigma}_x^{(n)}(x,t)|^2 = f_0^2(t)$. This yields for any $t \in I$

$$
\begin{aligned}
&\int_J (|\overline{e}_t^{(n)}|^2 + |\overline{v}_t^{(n)}|^2 + |\widehat{v}_t^{(n)}|^2 + |\widetilde{\sigma}_t^{(n)}|^2 + |\widehat{v}_x^{(n)}|^2 \qquad (3.32)\\
&\qquad +|\widehat{\sigma}_x^{(n)}|^2 + |\widetilde{\sigma}_x^{(n)}|^2)(x,t)\,\mathrm{d}x\\
&\le\; h \sum_{k=1}^{n-1} \left(\dot{e}_k^2 + 3\,\dot{v}_k^2 + 2\,\dot{\sigma}_k^2 + 2\,\left(\frac{\sigma_{k+1} - \sigma_k}{h}\right)^2 + \left(\frac{v_k - v_{k-1}}{h}\right)^2 \right)(t)\\
&\qquad +h\, f_0^2(t)\,.
\end{aligned}
$$

The assertion now follows from Lemma 3.4 and Ineq. (3.21). ❐

We are now ready to prove Thm. 3.2.

Proof of Theorem 3.2. From Proposition 3.5 it follows that there exist functions $v, \sigma \in C(\bar{J} \times \bar{I})$ such that $v_t, \sigma_t, v_x, \sigma_x \in L^\infty(I; L^2(J))$ and a subsequence $\left\{\widehat{v}^{(n_j)}, \widetilde{\sigma}^{(n_j)}\right\}$ of $\left\{\widehat{v}^{(n)}, \widetilde{\sigma}^{(n)}\right\}$ such that

$$\begin{aligned}
\widehat{v}_t^{(n_j)} &\to v_t \quad \text{weakly-star in } L^\infty(I; L^2(J)),\\
\widehat{v}_x^{(n_j)} &\to v_x \quad \text{weakly-star in } L^\infty(I; L^2(J)),\\
\widetilde{\sigma}_t^{(n_j)} &\to \sigma_t \quad \text{weakly-star in } L^\infty(I; L^2(J)),\\
\widetilde{\sigma}_x^{(n_j)} &\to \sigma_x \quad \text{weakly-star in } L^\infty(I; L^2(J)),
\end{aligned}$$

and, by compact embedding,

$$\begin{aligned}
\widehat{v}^{(n_j)} &\to v \quad \text{uniformly in } C(\bar{J} \times \bar{I}),\\
\widetilde{\sigma}^{(n_j)} &\to \sigma \quad \text{uniformly in } C(\bar{J} \times \bar{I})
\end{aligned}$$

as $j \to \infty$. We moreover have for every $n \in \mathbb{N}$ and $(x,t) \in J \times I$

$$\begin{aligned}
|\widetilde{\sigma}^{(n)}(x,t) - \widehat{\sigma}^{(n)}(x,t)| &\le h\, f_0(t) \le C_7 \sqrt{h},\\
|\widetilde{\sigma}^{(n)}(x,t) - \overline{\sigma}^{(n)}(x,t)| &\le \max_{1\le k\le n-1} |\sigma_{k+1} - \sigma_k|\\
&\le \left(\sum_{k=1}^{n-1} (\sigma_{k+1} - \sigma_k)^2\right)^{1/2} \le C_8 \sqrt{h},\\
|\widehat{v}^{(n)}(x,t) - \overline{v}^{(n)}(x,t)| &\le \max_{1\le k\le n-1} |v_k - v_{k-1}|\\
&\le \left(\sum_{k=1}^{n-1} (v_k - v_{k-1})^2\right)^{1/2} \le C_9 \sqrt{h}.
\end{aligned}$$

We conclude that the subsequences can be chosen in such a way that

$$\begin{aligned}
\overline{v}_t^{(n_j)} &\to v_t \quad \text{weakly-star in } L^\infty(I; L^2(J)),\\
\widehat{\sigma}_x^{(n_j)} &\to \sigma_x \quad \text{weakly-star in } L^\infty(I; L^2(J)),\\
\widehat{\sigma}^{(n_j)} &\to \sigma \quad \text{uniformly in } C(\bar{J} \times \bar{I}),\\
\overline{\sigma}^{(n_j)} &\to \sigma \quad \text{uniformly in } L^\infty(I \times J),\\
\overline{v}^{(n_j)} &\to v \quad \text{uniformly in } L^\infty(I \times J).
\end{aligned}$$

We now check that v, σ is a solution of the system (3.1) – (3.3). By construction, we have

$$\begin{aligned}
\overline{\rho}^{(n)} &\to \rho \quad \text{strongly in } L^p(J) \text{ for every } p \ge 1 \text{ and weakly-star in } L^\infty(J),\\
\overline{f}^{(n)} &\to f \quad \text{strongly in } L^1(I; L^2(J)),\\
\overline{\zeta}^{(n)} &\to \zeta \quad \text{strongly in } L^p(I \times J) \text{ for every } p \ge 1\\
&\qquad \text{and weakly-star in } L^\infty(I \times J).
\end{aligned}$$

We are in the situation of Proposition 2.12. In fact, we can prove more. Eq. (2.42) (with σ_n replaced by $\overline{\sigma}^{(n_j)}$ etc.) implies that

$$\overline{\mathcal{G}}^{(n_j)}[\overline{\sigma}^{(n_j)}] \to \mathcal{G}[\sigma] \quad \text{strongly in } L^p(J; C(\bar{I})) \text{ for every } p \geq 1\,.$$

This enables us to conclude that

$$\begin{aligned} \overline{e}^{(n_j)} &\to e = \mathcal{G}[\sigma] \quad \text{strongly in } L^p(J; C(\bar{I})) \text{ for every } p \geq 1\,, \\ \overline{e}_t^{(n_j)} &\to e_t \qquad \text{weakly-star in } L^\infty(I; L^2(J))\,, \end{aligned}$$

and passing to the limit in Eqs. (3.4) we see that Eqs. (3.1) hold. The boundary conditions are obviously fulfilled as a consequence of Eqs. (3.30) and of the uniform convergence of the approximate solutions. For each $n \in \mathbb{N}$ and $x \in J$ we moreover have

$$|\overline{v}^{(n)}(x,0) - v^0(x)| \leq C_{10}\sqrt{h}\,, \quad |\overline{\sigma}^{(n)}(x,0) - \sigma^0(x)| \leq C_{11}\sqrt{h}\,,$$

and the initial conditions (3.3) follow again from the uniform convergence.

To complete the proof of Thm. 3.2, it remains to prove that the solution is unique. Assume that v_1, σ_1, v_2, σ_2 are two solutions of the system (3.1) – (3.3). Then we have

$$\left\{ \begin{aligned} \rho\,(v_1 - v_2)_t &= (\sigma_1 - \sigma_2)_x\,, \\ (e_1 - e_2)_t &= (v_1 - v_2)_x\,, \\ e_i &= \mathcal{G}[\sigma_i] \quad i = 1,2 \end{aligned} \right. \tag{3.33}$$

almost everywhere in $J \times I$. By Proposition 2.1 (i) we have

$$(\mathcal{P}_r[\sigma_1] - \mathcal{P}_r[\sigma_2])_t(\sigma_1 - \sigma_2) \geq \frac{1}{2}\frac{\partial}{\partial t}(\mathcal{P}_r[\sigma_1] - \mathcal{P}_r[\sigma_2])^2$$

for every $r > 0$ and a.e. $(x,t) \in J \times I$, hence

$$\begin{aligned} &(e_1 - e_2)_t(\sigma_1 - \sigma_2) \\ &\geq \frac{1}{2}\frac{\partial}{\partial t}\left(\zeta(x,0)\,(\sigma_1 - \sigma_2)^2 + \int_0^\infty (\mathcal{P}_r[\sigma_1] - \mathcal{P}_r[\sigma_2])^2\,\mathrm{d}_r\zeta(x,r)\right) \quad \text{a.e.} \end{aligned} \tag{3.34}$$

Multiplying the first equation of (3.33) by $v_1 - v_2$, the second one by $\sigma_1 - \sigma_2$, integrating over $J \times (0,t)$ for any $t \in I$ and using Ineq. (3.34), we obtain

$$\int_J \left(\rho_m\,(v_1 - v_2)^2 + \beta\,(\sigma_1 - \sigma_2)^2\right)(x,t)\,\mathrm{d}x \leq 0\,,$$

hence $\sigma_1 = \sigma_2$, $v_1 = v_2$. Theorem 3.2 is proved. ❐

4 Homogenization

4.1 Wave equation with weakly convergent parameters

The homogenization result is based on the weak convergence statement in Thm. 4.2 below. Let us consider a sequence of initial-boundary value problems on $J \times I$

$$\left\{ \begin{array}{rcl} \rho^n v_t^n & = & \sigma_x^n + f^n , \\ e_t^n & = & v_x^n , \\ e^n & = & \mathcal{G}^n[\sigma^n] , \end{array} \right. \tag{4.1}$$

$$v^n(0,t) = \sigma^n(\ell,t) = 0 \qquad \text{for } \ t \in I , \tag{4.2}$$

$$v^n(x,0) = v_n^0(x) , \quad \sigma^n(x,0) = \sigma_n^0(x) \qquad \text{for } \ x \in J \tag{4.3}$$

in $J \times I$ for $n \in \mathbb{N}$, where $\mathcal{G}^n$ are Prandtl-Ishlinskii operators of the form

$$\mathcal{G}^n[\sigma](x,t) = \zeta^n(x,0)\,\sigma(x,t) \, + \, \int_0^\infty \mathcal{P}_r[\sigma(x,\cdot)](t)\,\mathrm{d}_r\zeta^n(x,r) \tag{4.4}$$

analogous to Eq. (2.40).

We shall prove that the corresponding sequence of solutions v^n, e^n, σ^n converges to the solution of a problem of the same type, namely

$$\left\{ \begin{array}{rcl} \rho^* v_t^* & = & \sigma_x^* + f^* , \\ e_t^* & = & v_x^* , \\ e^* & = & \mathcal{G}^*[\sigma^*] , \end{array} \right. \tag{4.5}$$

$$v^*(0,t) = \sigma^*(\ell,t) = 0 \qquad \text{for } \ t \in I , \tag{4.6}$$

$$v^*(x,0) = v_*^0(x) , \quad \sigma^*(x,0) = \sigma_*^0(x) \qquad \text{for } \ x \in J , \tag{4.7}$$

where $\mathcal{G}^*$ is Prandtl-Ishlinskii operator defined by

$$\mathcal{G}^*[\sigma](x,t) = \zeta^*(x,0)\,\sigma(x,t) \, + \, \int_0^\infty \mathcal{P}_r[\sigma(x,\cdot)](t)\,\mathrm{d}_r\zeta^*(x,r) . \tag{4.8}$$

We make the following hypotheses:

Hypothesis 4.1

(i) $\rho^n, \rho^* \in L^\infty(J)$, *and there exist* $\rho_m, \rho_M > 0$, $\rho \in L^\infty(J)$ *such that* $\rho_m \le \rho^n(x) \le \rho_M$ *a.e. for each* $n \in \mathbb{N}$ *and*

$\rho^n \to \rho^*$ *weakly-star in* $L^\infty(J)$,

(ii) $\zeta^n, \zeta^* \in L^\infty(J \times (0,\infty))$, *the functions* $\zeta^n(x,\cdot)$ *belong to* PI^+ *for each* $n \in \mathbb{N}$ *and a.e.* $x \in J$, *and there exist* $\alpha, \beta > 0$ *such that* $\beta \le \zeta^n(x,r) \le 1/\alpha$ *a.e. for* $n \in \mathbb{N}$ *and*

$\zeta^n \to \zeta^*$ *weakly-star in* $L^\infty(J \times (0,\infty))$,

(iii) $f^n, f^* \in W^{1,1}(I; L^2(J))$ *for each* $n \in \mathbb{N}$ *and*

$f^n \to f^*$ *in* $W^{1,1}(I; L^2(J))$ *weakly,*

(iv) $\sigma_n^0, v_n^0 \in W^{1,2}(J)$, $\sigma_n^0(\ell) = v_n^0(0) = 0$ *for* $n \in \mathbb{N}$, *and there exist* $\sigma_*^0, v_*^0 \in W^{1,2}(J)$ *such that*

$\sigma_n^0 \to \sigma_*^0$, $v_n^0 \to v_*^0$ *weakly in* $W^{1,2}(J)$.

The above hypotheses imply in particular that the data $\rho^n, f^n, \zeta^n, \sigma_n^0, v_n^0$ for $n \in \mathbb{N}$ as well as the limit data $\rho^*, f^*, \zeta^*, \sigma_*^0, v_*^0$ satisfy the assumptions of Thm. 3.2. Therefore each of the problems (4.1) – (4.4) for $n \in \mathbb{N}$, (4.5) – (4.8) admits a unique solution. The proof of Thm. 3.2 and Proposition 2.12 can be modified for proving the following result:

Theorem 4.2 *Let Hypothesis 4.1 hold. Let v^n, e^n, σ^n be solutions to problem (4.1)–(4.4) for $n \in \mathbb{N}$, and let v^*, e^*, σ^* be the solution to the limit problem (4.5)–(4.8). Then $e_t^n \to e_t^*$, $v_t^n \to v_t^*$, $\sigma_t^n \to \sigma_t^*$, $v_x^n \to v_x^*$, $\sigma_x^n \to \sigma_x^*$ weakly-star in $L^\infty(I; L^2(J))$, $v^n \to v^*$, $\sigma^n \to \sigma^*$ uniformly in $C(\bar{J} \times \bar{I})$ and $e^n(\cdot, t) \to e^*(\cdot, t)$ weakly-star in $L^\infty(J)$ for every $t \in I$ as $n \to \infty$.*

Proof. From Thm. 3.2 it follows that the norms of $e_t^n, v_t^n, \sigma_t^n, v_x^n, \sigma_x^n$ in the space $L^\infty(I; L^2(J))$ are bounded above independently of n. There exists therefore functions $e^{**}, v^{**}, \sigma^{**}$ and a subsequence indexed by n_j such that the functions $e_t^{n_j}, v_t^{n_j}, \sigma_t^{n_j}, v_x^{n_j}, \sigma_x^{n_j}$ converge weakly-star to $e_t^{**}, v_t^{**}, \sigma_t^{**}, v_x^{**}, \sigma_x^{**}$, respectively and, by compact embedding, σ^{n_j} and v^{n_j} converge uniformly in $C(\bar{J} \times \bar{I})$ to σ^{**} and v^{**}, respectively.

Using Proposition 2.12 we can pass to the limit in Eqs. (4.1). Indeed, we can pass to the limit in each term of (4.1) consisting of one weakly converging sequence. We can pass to the limit even in the product of $\rho^{n_j} \cdot v_t^{n_j}$ due to the fact that ρ^n are independent of t and v^{n_j} converges uniformly.

We conclude that the limit functions $\sigma^{**}, v^{**}, e^{**}$ satisfy the system (4.5) – (4.8) and the assertion follows from the fact that the solution is unique. ❐

Remark 4.3 In the considered one-dimensional case we obtain uniform convergence of σ^n and v^n and only weak-star convergence of e^n. This is why we use the constitutive relation in the form $e^n = \mathcal{G}^n[\sigma^n]$. We cannot pass to the limit in the constitutive relation $\sigma^n = \mathcal{F}^n[e^n]$, since on the right hand side we have a combination of two weakly converging sequences.

4.2 Homogenization

For a given $\varepsilon > 0$, we consider a system of the form (3.1) – (3.3) with ε-periodic data in the constitutive relations, namely

$$\left\{ \begin{array}{rcl} \rho^\varepsilon v_t^\varepsilon &=& \sigma_x^\varepsilon + f\,, \\ e_t^\varepsilon &=& v_x^\varepsilon\,, \\ e^\varepsilon &=& \mathcal{G}^\varepsilon[\sigma^\varepsilon]\,, \end{array} \right. \tag{4.9}$$

$$v^\varepsilon(0,t) = \sigma^\varepsilon(\ell,t) = 0 \qquad \text{for } \ t \in I\,, \tag{4.10}$$

$$v^\varepsilon(x,0) = v^0(x)\,, \quad \sigma^\varepsilon(x,0) = \sigma^0(x) \qquad \text{for } \ x \in J\,, \tag{4.11}$$

where f, v^0, σ^0 satisfy Hypotheses 3.1 (iii), (iv), and $\rho^\varepsilon, \mathcal{G}^\varepsilon$ have a special form

$$\rho^\varepsilon(x) \;=\; \rho(x/\varepsilon) \qquad \text{for } \ x \in J\,, \tag{4.12}$$

$$\mathcal{G}^\varepsilon[\sigma^\varepsilon](x,t) \;=\; \zeta^\varepsilon(x,0)\,\sigma^\varepsilon(x,t) + \int_0^\infty \mathcal{P}_r[\sigma^\varepsilon(x,\cdot)](t)\,\mathrm{d}_r\zeta^\varepsilon(x,r)\,, \tag{4.13}$$

where

$$\zeta^\varepsilon(x,r) \;=\; \zeta(x/\varepsilon, r) \qquad \text{for } \; x \in J\,,\; r > 0\,, \tag{4.14}$$

with the intention to pass to the limit as $\varepsilon \to 0$. The functions ρ and ζ in Eqs. (4.12), (4.14) are assumed to have the following properties:

Hypothesis 4.4

(i) $\rho \in L^\infty(\mathbb{R})$, *there exist* $\rho_m, \rho_M > 0$ *such that* $\rho_m \le \rho(y) \le \rho_M$ *and* $\rho(y+1) = \rho(y)$ *a.e.,*

(ii) $\zeta \in L^\infty(\mathbb{R} \times (0,\infty))$, *the functions* $\zeta(y,\cdot)$ *belong to* PI^+ *for a.e.* $y \in \mathbb{R}$, *there exist* $\alpha, \beta > 0$ *such that* $\beta \le \zeta(y,r) \le 1/\alpha$, *and* $\zeta(y+1,r) = \zeta(y,r)$ *a.e.*

From Thm. 3.2 we immediately see that under the above hypotheses, the system (4.9) – (4.11) has a unique solution $(v^\varepsilon, e^\varepsilon, \sigma^\varepsilon)$ for each $\varepsilon > 0$. The homogenization result consists in proving that $(v^\varepsilon, e^\varepsilon, \sigma^\varepsilon)$ converge to the solution (v^*, e^*, σ^*) of the homogenized problem (4.5) – (4.7) with

$$\rho^* \;=\; \int_0^1 \rho(y)\,\mathrm{d}y\,, \tag{4.15}$$

$$\mathcal{G}^*[\sigma^*](x,t) \;=\; \zeta^*(0)\,\sigma^*(x,t) + \int_0^\infty \mathcal{P}_r[\sigma^*(x,\cdot)](t)\,\mathrm{d}\zeta^*(r)\,, \tag{4.16}$$

$$\zeta^*(r) \;=\; \int_0^1 \zeta(y,r)\,\mathrm{d}y \qquad \text{for } \; r > 0\,. \tag{4.17}$$

The following statement is a consequence of Thm. 4.2:

Theorem 4.5 *Let Hypotheses 4.4 and 3.1 (iii), (iv) hold, and let* $(v^\varepsilon, e^\varepsilon, \sigma^\varepsilon)$ *for* $\varepsilon > 0$, (v^*, e^*, σ^*) *be the solutions of the problems (4.9) – (4.11), (4.5) – (4.7), respectively. Then* $e^\varepsilon_t \to e^*_t$, $v^\varepsilon_t \to v^*_t$, $\sigma^\varepsilon_t \to \sigma^*_t$, $v^\varepsilon_x \to v^*_x$, $\sigma^\varepsilon_x \to \sigma^*_x$ *weakly-star in* $L^\infty(I; L^2(J))$, $v^\varepsilon \to v^*$, $\sigma^\varepsilon \to \sigma^*$ *uniformly in* $C(\bar{J} \times \bar{I})$ *and* $e^\varepsilon(\cdot,t) \to e^*(\cdot,t)$ *weakly-star in* $L^\infty(J)$ *for every* $t \in I$ *as* $\varepsilon \to 0$.

Proof. According to Theorem 4.2, we only have to prove that $\rho^\varepsilon \to \rho^*$ weakly-star in $L^\infty(J)$ and $\zeta^\varepsilon \to \zeta^*$ weakly-star in $L^\infty(J \times (0,\infty))$ as $\varepsilon \to 0$. For the reader's convenience, we briefly recall the argument which is in fact classical, see e.g. [4].
Since the sequence ρ^ε is bounded in $L^\infty(J)$ and $C_0^\infty(J)$ is dense in $L^1(J)$ it is sufficient to check the convergence $\int_J (\rho^\varepsilon(x) - \rho^*)\varphi(x)\mathrm{d}x \to 0$ only for smooth test function φ with compact support in J.
Let $\varphi \in C_0^\infty(J)$ and let us define primitive function

$$R^\varepsilon(x) = \int_0^x (\rho^\varepsilon(\xi) - \rho^*)\,\mathrm{d}\xi = \int_0^x (\rho(\xi/\varepsilon) - \rho^*)\,\mathrm{d}\xi\,.$$

Since integral $\int (\rho^\varepsilon - \rho^*)\mathrm{d}x$ over any subinterval of length ε equals to zero, the functions $R^\varepsilon(x)$ are ε-periodic and satisfy the estimate $|R^\varepsilon(x)| \le c\cdot\varepsilon$ with a constant independent of ε. Thus integration by parts yields

$$\int_0^\ell \rho^\varepsilon\,\varphi\mathrm{d}x = -\int_0^\ell R^\varepsilon\,\varphi'\,\mathrm{d}x\,,$$

which can be estimated by $c\,\varepsilon\,\int_0^\ell |\varphi'|\,\mathrm{d}x$ and the convergence follows.
In the same way we can check that $\zeta^\varepsilon \to \zeta^*$ weakly-star in $L^\infty(J \times (0,\infty))$ and the assertion follows from Theorem 4.2. ❐

Remark 4.6 The homogenization result can be extended to the case of a sequence of the right hand sides f^ε of the form

$$f^\varepsilon(x,t) = f(x, x/\varepsilon, t)\,,$$

where the function $f(x,y,t)$ is periodic in y and continuous in x.
Theorem 4.5 shows that the original statement of the problem in the form (1.1) – (1.4) was in fact misleading. The natural homogenization takes place in the inverse Prandtl-Ishlinskii constitutive law $e = \mathcal{G}[\sigma]$ rather than in Eq. (1.2). On the other hand, the above analysis gives a physical justification to the inverse Prandtl-Ishlinskii rheological model in Fig. 5. We can consider a 'real' homogeneous elastoplastic material as a limit periodic superposition of elastic-perfectly plastic layers, each of them having a single yield point $r > 0$ and a (possibly infinitesimal) relative thickness $\mathrm{d}\zeta(r)$.

Conclusion

We have proved the existence and uniqueness of global solutions to a spatially inhomogeneous hyperbolic system with hysteresis describing longitudinal oscillations of a heterogeneous elastoplastic rod. Assuming that the spatial structure is periodic with a period ε tending to 0, we derived the form of the homogenized constitutive operator and proved that the solutions to the 'periodic' system converge to the solution of the homogenized one as ε tends to 0. In this one-dimensional case, the homogenized operator $\mathcal{F}^*$ is obtained by 'averaging' the corresponding inverse operators $\mathcal{G}^\varepsilon$ to $\mathcal{G}^*$.

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