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Most β shifts have bad ergodic properties

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1. Introduction

More than 30 years ago Rényi [1] introduced the representations of real numbers with an arbitrary base $\beta > 1$ as a generalization of the p-adic representations. One of the most studied problems in this field is the link between expansions to base β and ergodic properties of the corresponding β-shift.
In this paper we will follow the bibliography of F. Blanchard [2] and give an affirmative answer to a question on the size of the set of real numbers β having the worst ergodic properties of their β-shifts.

2. β-expansions

Throughout this paper we denote by $[x]$ and $\{x\}$ the integer and the fractional part respectively, of the real number x.
Let $\beta > 1$ be a real number.

Definition 2.1. The expansion of a number $x \in [0,1]$ in base β is a sequence of integers out of $\{0, 1, \ldots, [\beta]\}$

$$\{i_n\}_1^\infty = \{i_n(x, \beta)\}_1^\infty,$$

defined by one of the following equivalent properties:

(1) For all $n \geq 0$
$$\sum_{k>n} \frac{i_n}{\beta^k} < \frac{1}{\beta^n}$$
(2) $i_1 = [\beta x]$ $\quad i_2 = [\beta \{\beta x\}]$ $\quad i_3 = [\beta \{\beta \{\beta x\}\}] \ldots$
(3) If $T_\beta : [0,1] \to [0,1)$ is the transformation $T_\beta(x) = \beta x \pmod 1$ then

$$i_n = [\beta T_\beta^{n-1}(x)] \quad n > 0.$$

We endow the set $\{0, \ldots, [\beta]\}^{\mathbb{N}}$ with the lexicographical order ($<_{lex}$ or simply $<$) the product topology and the one–sided shift operator σ:

$$\sigma(i_1 i_2 \ldots i_n \ldots) = i_2 i_2 \ldots i_{n+1} \ldots$$

Moreover we extended the lexicographical ordering to finite blocks:

$$i_1 \ldots i_n <_{lex} j_1 \ldots j_m \quad \text{iff}$$

$$i_1 \ldots i_n 00 \cdots <_{lex} j_1 \ldots j_m [\beta][\beta] \ldots$$

Definition 2.2. The closure of the set of all β-expansions of $x \in [0,1]$ is called the β-shift S_β.

Remark. S_β is σ invariant.
Parry [3] proved that the β–shift S_β is totally determined by its expansions of 1:

Theorem 2.1. *If $\{i_n(1,\beta)\}$ is not finite (i.e. it won't terminate with zeros only) then $\{s_n\} \in \{0,\dots,[\beta]\}^{\mathbb{N}}$ belongs to S_β if and only if*

$$\sigma^k\{s_n\} <_{lex} \{i_n(1,\beta)\} \qquad k > 0$$

If $\{i_n(1,\beta)\} = i_1 \dots i_M 00\dots$ then $\{s_n\}$ belongs to S_β if and only if

$$\sigma^k\{s_n\} <_{lex} i_1 \dots i_{M-1}(i_M - 1)i_1 \dots i_{M-1}(i_M - 1)i_1 \dots \quad k > 0$$

□

According to this theorem we say a word $(j_1 \dots j_m)$ is allowed iff

$$\sigma^k(j_1 \dots j_m) < \{i_n(1,\beta)\} \quad k = 0,1,\dots,m-1$$

Moreover he proved:

Theorem 2.2. *A sequence $\{s_n\} \in \{0,1,\dots,[\beta]\}^{\mathbb{N}}$ is an expansion of 1 for some β iff*
$\sigma^k\{s_n\} <_{lex} \{s_n\}$ $(k > 0)$ and then β is unique.
The map $\pi : \beta \mapsto \{i_n(1,\beta)\}$ is monotone increasing.

□

A more detail survey can be found in [2],[3].

3. Ergodic properties of the β–shifts

In this chapter we give a brief summary of a part of Blanchard's paper. For more details and literature see [2].
The link between topological properties of $\{i_n(1,\beta)\}$ and ergodic properties of S_β is completely known. For this we look at the following classes:

Class C_1: S_β is a subshift of finite type.

Proposition 3.1. *$\beta \in C_1$ iff $\{i_n(1,\beta)\}$ is finite.*
This is fullfilled for instance for $\beta = \frac{1+\sqrt{5}}{2}$

Proposition 3.2. *([3]): C_1 is dense in $(1,\infty)$*

Class C_2: S_β is sofic.

Proposition 3.3. *$\beta \in C_2$ iff $\{i_n(1,\beta)\}$ is ultimately periodic.*
For $\beta = \frac{3+\sqrt{5}}{2}$ the expansion of 1 is ultimately periodic but not finite.

Proposition 3.4. *If S_β is sofic, then β is a Perron number.*

Corollary 3.1. *The class C_2 is at most countable.*

Class C_3: Class S_β is specified.

Proposition 3.5. *$\beta \in C_3$ iff there exists an $n \in \mathbb{N}$ such that all strings of 0's in $\{i_n(1,\beta)\}$ have length less than n. This means the origin is not an accumulation point of the orbit of 1 under T_β.*

Class C_4: Class S_β is synchronizing.

Proposition 3.6. *$\beta \in C_4$ iff $\{i_n(1,\beta)\}$ does not contain some allowed word. That is the orbit of 1 under T_β is not dense in $[0,1]$.*

Class C_5: Class S_β has non of the above properties.
That is the orbit of 1 is dense and consequently $\{i_n(1,\beta)\}$ contains all allowed words.

One of the questions raised up by Blanchard is that of the size of the classes C_3, C_4 and C_5.
The main results of this paper is to answer this question.

Definition 3.1. A subset C of $(1,\infty)$ is said to be residual iff it contains a countable intersection of open and dense sets. The complement of a residual set is called meager.

Our aim is to prove the following

Theorem 3.1. (1) *C_5 is residual in $(1,\infty)$.*
(2) *C_5 has full Lebesque measure in $(1,\infty)$.*
(3) *C_3 has Hausdorff dimension 1.*
(4) *$C_4 \backslash C_3$ has Hausdorff dimension 1.*

4. Proof of the results

We start with a couple of definitions which reflect the properties of Theorem A.

Definition 4.1. A block $B = [i_1, \ldots, i_m]$ is called admissible iff
$\sigma^k B < B$
$(k = 1, \ldots, m-1)$.

De ion 4.2. For an arbitrary block $B = [i_1, \ldots, i_m]$ we define the admissability range of β by

$$AR(B) = \{\beta \in (1,\infty) | \sigma^k B < \{i_m(1,\beta)\}, \quad k = 1, \ldots, m-1\}$$

and the admissible block range by

$$\mathcal{AR}(B) = \{C | C \text{ is admissible and } \sigma^k B < C \quad k = 1, \dots, m-1\}.$$

Remark. $\mathcal{AR}(B)$ are the starting blocks of the expansions of 1 for $\beta \in AR(B)$.

Lemma 4.1. *Let $B = [i_1, \dots, j_m]$ be an admissible block. Then the cylinder set*

$$C_B = \{\beta \in (1, \infty) | i_k(1, \beta) = j_k \quad k = 1, \dots, m\}$$

is the half-open interval $[\beta_1, \beta_2)$ with $\beta_1 = \beta_1(B)$ the only solution in $(1, \infty)$ of the equation

$$\beta = j_1 + \frac{j_2}{\beta} + \dots + \frac{j_m}{\beta^{m-1}}$$

and $\beta_2 = \beta_2(B)$ is the limit of the unique solutions in $(1, \infty)$ of the equations

$$\beta_N = j_1 + \frac{j_2}{\beta} + \dots + \frac{j_N}{\beta^{m+N-1}} \quad N \in \mathbb{N}$$

where $[j_1, \dots, j_m \ j_{m+1} \dots j_N]$ is the maximal admissible block of length N starting with entries $[j_1, \dots, j_m]$.
The diameter of C_B is at most $\frac{1}{\beta_2^{n-1}}$.

Proof. Because $j_1 j_2 \dots j_m 000 \dots$ is the smallest sequence in the set $\pi(C_B)$ it follows from the monotonicity of π that β_1 is the least number in C_B. The second part of the lemma follows from the fact that each orbit of 1 begins with admissible blocks only and from the characterization 1 of β–expansions in definition 2.1.
The last part is a consequence of the fact that β_2 fulfills the inequation

$$\beta_2 \leq j_1 + \frac{j_2}{\beta} + \dots + \frac{j_m + 1}{\beta^{m-1}}$$

this follows again from the characterization 1 in definition 2.1. and the equality:

$$|\beta_2 - \beta_1| \leq \left| j_1 + \frac{j_2}{\beta_2} + \dots + \frac{j_m + 1}{\beta_2^{m-1}} - \left(j_1 + \frac{j_2}{\beta_1} + \dots + \frac{j_m}{\beta_1^{m-1}} \right) \right|$$

□

Definition 4.3. An admissible block $[i_1, \dots, i_n]$ is called N–delaying for some natural number N iff for all blocks $B = [j_1, \dots, j_m]$, such that $[i_1, \dots, i_n] \in \mathcal{AR}(B)$ the block

$$[i_1, \dots, i_n, \underbrace{0, \dots, 0}_{N}, j_1, \dots, j_m]$$

is admissable.

In the next lemma we give a necessary condition for a block not beeing N–delaying.

Lemma 4.2. *If an admissible block C is not N–delaying it has the form*

$$C = [i_n, \ldots, i_m, \underbrace{0, \ldots, 0}_{N+1}, i_{m+N+2}, \ldots, i_M, i_1, \ldots, i_m]$$

for some $m \in \mathbb{N}$, $M > m + N + 2$.

Proof. Let $B = [1]$ then $C \in \mathcal{AR}(B)$ but

$$\sigma^M([i_1, \ldots, i_m, 0, \ldots, 0, i_{m+N+2}, \ldots, i_M, i_1, \ldots, i_m, \underbrace{0, \ldots, 0}_{N}, 1]) > C, \underbrace{0, \ldots, 0}_{N}, B.$$

So C is not N–delaying. On the other hand if C has not that form it follows from the admissability of C that if B is s.t. $C \in \mathcal{AR}(B)$ then

$$\sigma^k(C, \underbrace{0, \ldots, 0}_{N}, B) < C, 0, \ldots, 0, B$$

$k = 1, \ldots,$ length of $[C, 0, \ldots, 0, B]$. □

Lemma 4.3. *Let $B = [i_1, \ldots, i_n]$ be a N–delaying block. Then the cylinder set C_B has diameter at least $\frac{1}{\beta_2(B)^{n+N}}$.*

Proof. Since B is N–delaying the only solutions $\underline{\beta}$ and $\overline{\beta}$ of the equations

$$\beta = \quad i_1 + \tfrac{i_2}{\beta} + \cdots + \tfrac{i_n}{\beta^{n-1}} \qquad \text{and}$$

$$\beta = \quad i_1 + \tfrac{i_2}{\beta} + \cdots + \tfrac{i_n}{\beta^{n-1}} + \tfrac{1}{\beta^{N+n}} \qquad \text{respectively}$$

are contained in C_B. But they fulfill the inequality

$$|\overline{\beta} - \underline{\beta}| > \overline{\beta}^{-(N+n)} > \beta_2(B)^{-(N+n)}$$

□

Lemma 4.4. *There is a constant $A > 0$ such that for each natural M the set $D_{M,L}$ of cylinder sets in $[N-1, N]$ of length M which are not L–delaying has measure at most*
$A(N-1)^L$.

Proof. By lemma 4.2. a non–L–delaying block of length M has the form $[N-1, j_2, \ldots, j_k, \underbrace{0, \ldots, 0}_{L+1}, i_1, \ldots, i_m, N-1, \ldots, j_k]$ with $j_k \neq 0, 2k + L + m = M$.
Each cylinder set $C_L = C_{[N-1, j_2, \ldots, j_k, \underbrace{0, \ldots, 0}_{L+1}]}$ has length at most $\beta_2(C_L)^{-(k+L)}$ (lemma 4.3.).

Since $N-1>0$ by the same argument as in lemma 4.3. the cylinder sets $C_{[N-1,\ldots,j_k,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m]}$ have length at least $\beta_2(C_L)^{-(k+L+m)}$. Consequently , there are at most $\beta_2(C_2)^m$ of them. Now

$$D_{M,L} \subset \bigcup_k \bigcup_{[N-1,\ldots,j_k]} \bigcup_{[i_1,\ldots,i_m]} C_{[N-1,\ldots,j_k,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m,N-1,\ldots,j_k]}$$

Where the union is taken over all blocks such that

$$[N-1,\ldots,j_n,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m,N-1,\ldots,j_k]$$

is admissable. Then the Lebesque measure of $D_{M,L}$ can be estimated as follows:

$$\begin{aligned} L(D_{M,L}) &\leq \sum_k \sum_{[N-1,\ldots,j_k]} \sum_{[i_1,\ldots,i_m]} L(C_{[N-1,\ldots,j_k,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m,j_1,\ldots,j_k]} \\ &\leq \sum_k \sum_{[N-1,\ldots,j_k]} \beta_2(C_L)^m L(C_{[N-1,\ldots,j_k,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m,j_1,\ldots,j_k]}) \end{aligned}$$

Using again the same arguments as above and $j_k \neq 0$ we obtain

$$\begin{aligned} &\frac{L(C_{[N-1,\ldots,j_k]})}{L(C_{[N-1,\ldots,j_k,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m,N-1,\ldots,j_k]})} \geq \\ \geq\ &\frac{\beta_2(C_{[N-1,\ldots,j_k]})^{-(k-2)}}{\beta_2(C_{[N-1,\ldots,j_k,\underbrace{0,\ldots,0}_{L+1},i_1,\ldots,i_m,N-1,\ldots,j_k]})^{-M-1}} \geq \\ \geq\ &\beta_2(C_{[N-1,\ldots,j_k]})^{k+L+m+1}. \end{aligned}$$

So we can continue

$$\begin{aligned} L(D_{M,L}) &\leq \sum_k \sum_{[N-1,\ldots,j_k]} L(C_{[N-1,\ldots,j_k]}) \cdot \beta_2(C_L)^m \beta_2(C_{[N-1,\ldots,j_n]})^{-(k+L+m+1)} \\ &\leq \sum_k \sum_{[N-1,\ldots,j_k]} L(C_{[N-1,\ldots,j_k]}) \cdot \beta_2(C_{[N-1,\ldots,j_k]})^{-(k+L+1)} \\ &\leq \sum_k \sum_{[N-1,\ldots,j_k]} L(C_{[N-1,\ldots,j_k]}) \cdot (N-1)^{-(k+L+1)} \\ &\leq \sum_k (N-1)^{-(k+L+1)} \\ &\leq A(N-1)^L \end{aligned}$$

□

Remark. For $N = 2$ the above estimates are useless. But if we subdivide $(1, 2]$ into a countable number of subintervals we can achieve on each such subinterval corresponding estimates.

Lemma 4.5. *Let $B = [\ell_1, \dots, \ell_m]$ be an admissable block, then*

$$L\left(W_B \cap \mathcal{AR}(B)\right) = 0$$

where $W_B = \{\beta \in (1, \infty) | \{i_n(1, \beta)\}$ does not contain B as a subword. $\}$

Proof. Let $k, N, L \in \mathbb{N}, N > 2$ be given. We denote by $CD_{n,L} (n \in \mathbb{N})$ the complement of the set $D_{n,L}$ in lemma 4.4. We will construct inductively sets W_n all containing $W_{B_R} \cap AR(B_k)$, where $B_k = [\ell_1, \dots, \ell_m, \underbrace{0, \dots, 0}_{k}, 1]$. Obviously, $AR(B_{k+1}) \supset AR(B_R)$ and

$$\bigcup_{k=0}^{\infty} AR(B_k) = AR(B).$$

Moreover

$$W_{B_k} \cap AR(B_k) \supset W_B \cap AR(B_k).$$

We set

$$W_0 = [N - 1, N] \cap AR(B_k)$$

Assume that we have constructed all set W_i up to step $n - 1$. We consider the set

$$\widetilde{W}_{n-1} = W_{n-1} \cap D_{n-1,L}.$$

For each $C_{[N-1,\dots,i_{n-1}]} \subset D_{n-1,L} \cap AR(B_k)$ we consider the cylinder $C_{[N-1,\dots,i_{n-1},\underbrace{0,\dots,0}_{L},\ell_1,\dots,\ell_m]}$.

Clearly, $[N - 1, \dots, i_{n-1}, \underbrace{0, \dots, 0}_{L}, \ell_1, \dots, \ell_m, \underbrace{0, \dots, 0}_{k}, 1]$ is in $\mathcal{AR}(B_k)$ and admissable and hence, by the standard arguments used before

$$\frac{L(C_{[N-1,\dots,i_{n-1}]})}{L(C_{[N-1,\dots,i_{n-1},\underbrace{0,\dots,0}_{L},\ell_1,\dots,\ell_m,\underbrace{0,\dots,0}_{k},1]})} \leq N^{L+m+k}$$

The set W_n we define as

$$W_n = (D_{n-1,L} \cap W_{n-1}) \cup \left(\widetilde{W}_{n-1} \setminus \bigcup_{CD_{n-1,L} \cap AR(B_k)} C_{[N-1,\dots,i_{n-1},\underbrace{0,\dots,0}_{L},\ell_1,\dots,\ell_m,\underbrace{0,\dots,0}_{k},1]} \right)$$

According to the above estimates and lemma 4.4. the measures of W_n fulfill the inequalities:

$$
\begin{aligned}
L(W_n) &\leq L(D_{n-1,L} \cap W_{n-1}) + \left(1 - \left(\tfrac{1}{N}\right)^{L+m+k}\right) L(W_{n-1}) \\
&\leq A(N-1)^L + \left(1 - \left(\tfrac{1}{N}\right)^{L+m+k}\right)^n
\end{aligned}
$$

Hence

$$
L\left(\bigcap_{n=0}^{\infty} W_n\right) \leq A(N-1)^L .
$$

Using that $W_{B_k} \cap AR(B_k) \subset \bigcap\limits_{n=0}^{\infty} W_n$ for all L and letting L tend to infinity we can derive

$$
L(W_{B_k} \cap AR(B_k)) = 0.
$$

Finally the observations at the beginning of the proof give the desired result. □

From the arbitrary choice of B and the remark after lemma 4.4. we get as a corollary:

Proposition 4.1. *C_5 has full Lebesque measure.*

Remark. Carrying out a slightly more detailed analysis in the proof of the previous lemma, we are able to show that the Hausdorff dimension of the set $\tilde{W}_B = \{\beta \in W_B | \{i_n(1,\beta\}$ contains all allowed words except $B\} \cap AR(B)$ equals the Hausdorff dimension of

$$
W_B \cap AR(B).
$$

Our next step is to prove the claims 3) and 4) of the main theorem.
The crucial point in the proof is the estimation of the Hausdorff dimension of a class of special Cantor sets.

5. Special Cantor sets

We want to construct a class of Cantor sets which enable us to prove the statements 2 and 3 of the main theorem.
The construction will be made by inductively deleting intervals.
Let the natural numbers m and N be fixed.
In the first step we cancel one of the open intervals

$$
\text{interior}\left(C_{[N-1, i_1, \ldots, i_m]}\right) = W_{[N-1]}
$$

with $[N-1, i_1, \dots, i_m]$ an admissable block. After leaving out all intervals up to step n we proceed as follows:
In each remaining interval of the form

$$C_{[N-1,j_1,\dots,j_{n-1}]} \quad ([N-1, j_1, \dots, j_{n-1}] \text{ is admissable})$$

we delete exactly one interval

$$W_{[N-1,j_1,\dots,j_{n-1}]} = \text{ interior } \left(C_{[N-1,j_1,\dots,j_{n-1},\ell_1,\dots,\ell_m]}\right)$$

with $[N-1, j_1, \dots, j_{n-1}, \ell_1, \dots, \ell_m]$ an admissable block. This procedure gives rise to a Cantor set U_m in $[N-1, N]$.
In the next lemma we give a lower bound of its Hausdorff dimension.

Lemma 5.1. *Let U_m be a Cantor set constructed as descripted above. Then*

$$\dim_H U_m \geq \frac{\log((N-1)^m - 1)}{\log(N-1)^m}.$$

Proof. We want to associate to U_m a measure μ in order to use Frostmans lemma (see appendix).
Let the construction – and therefore U_m – be fixed.
We define μ by assigning to any cylinder set its value:

$$\mu([N-1, N]) = 1$$

If $C_{[N-1,i_1,\dots,i_n]}$ is a remaining interval then, obviously, is $C_{[N-1,i_1,\dots,i_{n-m+1}]}$ and they are nonempty. So we assign to the interval $C_{[N-1,i_1,\dots,i_n]}$ the measure

$$\mu\left(C_{[N-1,i_1,\dots,i_{n-m}]}\right) \frac{\operatorname{diam}\left(C_{[N-1,\dots i_n]}\right)}{\operatorname{diam}\left(C_{[N-1,i_1,\dots,i_{n-m}]}\right)}.$$

Thus μ can be extended to a measure concentrated on U_m. Let $\beta \in U_m$ have as the expansion of 1 the sequence $\{i_n(1,\beta)\} = \{i_n\}$.
Then the construction implies the property

$$\mu\left(C_{[N-1,\dots,i_n]}\right) \leq a_1 \prod_{k=1}^{[\frac{n}{m}]} \frac{\operatorname{diam}\left(C_{[N-1,\dots,i_n]}\right)}{\left(1 - \frac{\operatorname{diam}\left(W_{[N-1,\dots i_{n-(k-1)m}]}\right)}{\operatorname{diam}\left(C_{[N-1,\dots,i_{n-km}]}\right)}\right)}$$

With the help of similar arguments to that of lemma 4.3. we conclude

$$\operatorname{diam}\left(C_{[N-1,\dots,i_n]}\right) \leq (N-1)^{-n}$$

and

$$\frac{\operatorname{diam}\left(W_{[N-1,\dots,i_{n-(k-1)m}]}\right)}{\operatorname{diam}\left(C_{[N-1,\dots,i_{n-km}]}\right)} \leq (N-1)^{-m}$$

There exists a constant a_2 such that for every interval $I_\varepsilon(\beta)$ with centre β and diameter ε one can find a number $p = p(\varepsilon)$ fitting the inequalities

$$a_2^{-1}\mu(I_\varepsilon(\beta)) \leq \mu(C_{[N-1,\dots,i_p]}) \leq a_2\mu(I_\varepsilon(\beta))$$

and

$$a_2^{-1}\cdot\varepsilon \leq \operatorname{diam}(C_{[N-1,\dots,i_p]}) \leq a_2\cdot\varepsilon.$$

So we get

$$\frac{\log\mu(I_\varepsilon(\beta))}{\log\varepsilon} \geq \frac{\log a_2\mu(C_{[N-1,\dots,i_p]})}{\log a_2^{-1}\operatorname{diam}(C_{[N-1,\dots,i_p]})} \geq$$

$$\geq\ 1+\frac{\log(a_3\prod_{k=1}^{[\frac{n}{m}]}\frac{1}{1-(N-1)^{-m}})}{\log(a_4(N-1)^{-n})}.$$

Consequently

$$\liminf_{\varepsilon\to 0}\frac{\log\mu(I_\varepsilon(\beta))}{\log\varepsilon} \geq \frac{\log((N-1)^m-1)}{\log(N-1)^m}$$

for all $\beta\in U_m$ and Frostmans lemma implies this lemma. $\square$

Corollary 5.1. $\dim_H C_3 = 1$.

Proof. We specify the construction of U_m by cancelling the intervals ending with a string of m zeros. Then the lemma gives the statement for the limit $m\mapsto\infty$. $\square$

Corollary 5.2. $\dim_H(C_4\backslash C_3) = 1$.

Sketch of the Proof. By the remark after proposition 4.1. we see that
$\dim_H\{\beta|\{i_n(1,\beta)\}$ contains all allowed words but the allowed word
$B\} = \dim_H\{\beta|\{i_n(1,\beta\}$ does not contain $B\}$ holds.
Let now $\{B_n\}$ be a sequence of words whose length tends to infinity and containing only strings of zeros of a commonly bounded length. If U_n is the special Cantor set constructed by leaving out the word B_n whenever it is possible then U_n is contained in the set $\{\beta|\{i_n(1,\beta)\}$ contains all allowed words but $B_n\}$. Since the Hausdorff dimension of U_n tends to 1 this implies the corollary.
$\square$

6. Proof of statement 1 of the main theorem

Proposition 6.1. *C_5 is residual.*

Proof. We build a set $G \subseteq C_5$ which is the countable intersection of open and dense sets.
For a natural n and an admissable block $[i_1, \ldots, i_n]$ we consider the lexicographicly ordered sequence of all allowed blocks $\{B_k\}_{k=1}^m$. It is easy to check that the block

$$[i_1, \ldots, i_n, 0, \ldots, 0, B_{k-1}, \underbrace{0, \ldots, 0}_{k! \cdot n}, B_k, 0, \ldots, 0, B_m] = B([i_1, \ldots, i_n])$$

is admissable. Therefore the set

$$\bigcup_{\substack{[i_1, \ldots, i_n] \\ \text{is admissable}}} C_{B([i_1, \ldots, i_n])}$$

is open. Moreover, for each natural N the set

$$G_N = \bigcup_{n \geq N} \bigcup_{\substack{[i_1, \ldots, i_n] \\ \text{is admissable}}} C_{B([i_1, \ldots; i_n])}$$

is open and dense in $(1, \infty)$. Obviously the set

$$\bigcap_{N=1}^{\infty} G_N = G$$

is residual and contained in C_5 because for each $\beta \in G$ and each natural number m all allowed words up to length m occure in the expansion of 1 $\{i_n(1, \beta)\}$. $\square$

The theorem is now the summary of the above propositions.

7. Concluding remarks

We found a quite complete hierarchy in the classes introduced by F. Blanchard. This leads to the following picture:

$$\emptyset \neq C_1 \subseteq C_2 \subseteq C_3 \subseteq C_4 \subseteq (1, \infty) \quad C_5 = (1, \infty) \backslash C_4.$$

Their sizes are indicated in the diagram below

	C_1	$C_2\backslash C_1$	$C_3\backslash C_2$	$C_4\backslash C_3$	C_5
Hausdorff dimension	0	0	1	1	1
Lebesque measure	0	0	0	0	full measure
Baire category	countable and dense	countable and dense	meager and dense	meager and dense	residual

The transition from C_4 to C_5 is quite strong in size. It goes both from zero to full Lebesque measure and from a meager to a residual set. This indicates to look for a subset of C_5 exhibiting only one "jump" in size. In a forthcoming paper we introduce one of such subsets, namely a set having full Lebesque measure but being meager. The set we are looking at is that of all β where the expansion of one is normal ([4]).

8. Appendix: Hausdorff dimension

For general definitions and results in the theory of Hausdorff dimension see [5]. Let $X \subset \mathbb{R}$. An at most countable collection of sets is called an ε–cover of Y if it covers Y and the diameters of all its members are less then ε. For $s \in \mathbb{R}^+$ the s–dimensional outer Hausdorff measure of X is defined by

$$m_s(X) = \liminf_{\varepsilon \to 0} \left\{ \sum \text{ diameter } (\cup_i)^s | \{U_i\} \text{ is an } \varepsilon - \text{cover of } X \right\}.$$

There is a unique critical value s_0 for which the s–dimensional Hausdorff measure jumps from infinity to zero:

$$s_0 = \inf \{ s | m_s(X) = 0 \}.$$

This value is called the Hausdorff dimension $\dim_H(X)$ of X. A very usefull tool to get estimations of the Hausdorff dimension from below is Frostman's lemma:

Lemma 8.1. *If μ is a probability measure concentrated on X and*

$$\liminf_{\varepsilon \to 0} \frac{\log \mu(I_\varepsilon(x))}{\log \varepsilon} \geq s$$

for all $x \in X$ and $I_\varepsilon(x)$–the interval centered at x with diameter ε, then

$$\dim_H(X) \geq s. \qquad \square$$

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