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A Super-Brownian Motion with a Single Point Catalyst

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Summary. A one-dimensional continuous measure-valued process $\{\mathcal{X}_t; t \geq 0\}$ is discussed, where branching occurs only at a single point catalyst described by the Dirac δ-function δ_c. A (spatial) density field $\{x_t(z); t>0, z \neq c\}$ exists which is jointly continuous. At a fixed time $t>0$, the density $x_t(z)$ at z degenerates to 0 stochastically as z approaches the catalyst's position c. On the other hand, the occupation time process $\mathcal{Y}_t := \int_0^t dr\ \mathcal{X}_r(\cdot)$ has a (spatial) occupation density field $\{y_t(z); t \geq 0, z \in \mathbb{R}\}$ which is jointly continuous even at c and non-vanishing there. Moreover, the corresponding "occupation density measure" $dy_t(c) =: \lambda^c(dt)$ at c has carrying Hausdorff-Besicovitch dimension one. Roughly speaking, density of mass, arriving at c normally dies immediately, whereas creation of density mass occurs only on a singular time set. Starting initially with a unit mass concentrated at c, the total occupation time measure $\mathcal{Y}_\infty$ equals in law a random multiple of the Lebesgue measure where that factor is just the total occupation density at the catalyst's position and has a stable distribution with index 1/2. The main analytical tool is a reaction diffusion equation (cumulant equation) in which δ-functions enter in three ways, namely as coefficient δ_c of the quadratic reaction term (describing the point-catalytic medium), as Cauchy initial condition (leading to basic solutions), and as external force term (related to the occupation density).

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1. INTRODUCTION AND RESULTS

1.1. Introduction

In Dawson and Fleischmann (1991) a one-dimensional *superprocess* $\mathcal{X} = \{\mathcal{X}_t; t\geq 0\}$ was constructed in which critical branching occurs only in the presence of some *catalysts.* These are an infinite system of weighted points, stochastically fluctuating both in time and space. The catalysts are densely situated in space and have an infinite overall density of weights.[1)] (Heuristically, the model makes sense, since the underlying motion process has a positive occupation density at the point catalyst's locations since the space dimension is one by assumption.[2)])

In spite of that singular and highly fluctuating nature of the random medium, in Dawson, Fleischmann, and Roelly (1991) it was shown that at a *fixed* time $t>0$ the random measure $\mathcal{X}_t$ on $\mathbb{R}$ is *absolutely continuous*, i.e. with probability one there exists a representation $\mathcal{X}_t(\mathrm{d}z) = x_t(z)\mathrm{d}z$ with $x_t \geq 0$ a measurable function. But the paper left the question open as to what properties the random density x_t or even a density process x would have.

To attack this problem, *in the present paper* we focus our attention on the extremely simple case of a single, non-moving and non-random catalyst, described by the Dirac δ-function δ_c, where $c\in\mathbb{R}$ is fixed once and for all. Consequently, branching occurs only at site c, namely according to the simplest continuous state Galton-Watson process, but with an "infinite rate" (in the sense of δ-functions), whereas off c we merely have a deterministic dispersion of population mass by means of the heat flow. More precisely, we consider the (time-homogeneous) *super-Brownian motion* $\mathcal{X}$ related to the non-

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1) For a physical discussion of *fractal catalysts* we refer to Sapoval (1991).

2) For recent results on local times on superprocesses, see Barlow et al. (1991) and Adler and Lewin (1990).

linear (formal) equation

$$(1.1.1)\qquad \begin{cases} \frac{\partial}{\partial t}u(t,z) = \kappa\ \Delta u(t,z) - \delta_c(z)\ u^2(t,z), & t\geq 0,\ z\in\mathbb{R}, \\ u(0,z) = \varphi(z), \quad z\in\mathbb{R},\ \varphi\in\mathbf{G}_+, \end{cases}$$

via its Laplace transition functional

$$(1.1.2)\qquad \mathbb{E}\{\exp(\mathcal{X}_t,-\varphi)\,|\,\mathcal{X}_s=\mu\} = \exp(\mu,-u(t-s)), \qquad 0\leq s\leq t,\ \varphi\in\mathbf{G}_+,\ \mu\in\mathcal{M}_f.$$

Here $\kappa>0$ is the diffusion constant, the (one-dimensional) Laplacian Δ acts on the space variable z, and the set $\mathbf{G}$ just contains all those continuous functions φ defined on $\mathbb{R}$ having "Gaussian decay", that is, $|\varphi(z)|\exp[c_\varphi z^2]$, $z\in\mathbb{R}$, is bounded for some constant $c_\varphi>0$. Moreover, $\mathcal{M}_f$ denotes the set of all finite (non-negative) measures defined on $\mathbb{R}$, equipped with the topology of *"exp-vague"* convergence, i.e. the coarsest topology such that all functions $\mu\mapsto(\mu,\varphi)$, $\varphi\in\mathbf{G}_+$, will be continuous, where (μ,φ) abbreviates the integral $\int\mu(dz)\ \varphi(z)$. By the way, in Dawson and Fleischmann (1992, 1991) *mild* solutions u to (1.1.1), and the superprocess $\mathcal{X}$ were constructed (even in more generality) by means of approximating δ_c by the smooth functions $p(\varepsilon,(\cdot)-c)$, $\varepsilon>0$, where $p(\varepsilon,\cdot)$ is the symmetric *Gaussian* density

$$(1.1.3)\qquad p(\varepsilon,z) := (4\pi\kappa\varepsilon)^{-1/2}\exp[-z^2/4\kappa\varepsilon], \qquad z\in\mathbb{R},$$

and $\varepsilon\to 0$ (see also Section 2 below). Of course, $\mathcal{X}$ can also be understood as a high density limit of a particle model (diffusion approximation; for the constant branching rate case, see e.g. Le Gall (1991) and the references therein).

As already mentioned, by the results in [7], for fixed $t>0$ the random measure $\mathcal{X}_t$ has almost surely a density function x_t on $\mathbb{R}$. In order to get a preliminary feeling for its properties, we suggest the following *heuristic* considerations. Formally, we can interpret the density $x_t(z)$ at site z as $(\mathcal{X}_t,\delta_z)$, for Lebesgue almost all $z\in\mathbb{R}$ (recall the notion of a derivative of a measure). Keeping in mind the well-known formulas for the moments of $(\mathcal{X}_t,\varphi)$, where for the moment $\mathcal{X}$ denotes a superprocess with regular branching rate $\rho(z)$ instead of $\delta_c(z)$, then by the formal substitution $\rho\mapsto\delta_c$ and $\varphi\mapsto\delta_z$ we arrive at the following *expectation and covariance formulas:*

$$\mathbb{E}\{x_t(z)\,|\,\mathcal{X}_0=\mu\} = \int\mu(da)\;p(t,z-a) = [\mu*p(t)](z) \tag{1.1.4}$$

$$\mathbb{C}\mathrm{ov}\{x_s(z),x_t(z')\,|\,\mathcal{X}_0=\mu\} = 2\int\mu(da)\int_0^s dr\;p(r,c-a)p(s-r,z-c)p(t-r,z'-c), \tag{1.1.5}$$

$0<s\le t$, $z,z'\neq c$. Now, for fixed $t>0$, $a,c\in\mathbb{R}$, we get

$$\mathbb{V}\mathrm{ar}\{x_t(z)\,|\,\mathcal{X}_0=\delta_a\} \sim \text{const}\;\big|\log|z-c|\big| \quad \text{as} \quad z\to c, \tag{1.1.6}$$

i.e. the *variance of the random density* x_t *blows up as* z *approaches the catalyst's position* c. Roughly speaking, in the vicinity of the catalyst the density of $\mathcal{X}_t$ is highly fluctuating. This very vague idea, of course, raises the question as to how the density x_t actually behaves as $z\to c$. Also, the absolute continuity for fixed t does not exclude the possibility that $\mathcal{X}_t$ could be pathological on a set of time points t of Lebesgue measure zero.

In the present paper we will give some results related to these problems. In particular, this will also provide some probabilistic insight into the basic nature of measure-valued branching processes (superprocesses) in point catalytic media.

From a technical point of view, we will prove the existence of density fields and study their continuity properties. For the model with *regular* branching rate (instead of a point-catalytic medium), the path continuity of the superprocess with continuous branching component was demonstrated by Watanabe (1968) and in a very general form given by Fitzsimmons (1988). In the one-dimensional case, the existence of (spatial) densities at a fixed time $t>0$ was first proved by Roelly-Coppoletta (1986) and, in more generality, in Fleischmann (1988). Joint continuity of a density field was first obtained in

Konno and Shiga (1988), and for the corresponding occupation time process is due to Sugitani (1989). Our approach is rather close to the latter two references using Kolmogorov's continuity criterion.

The *outline* of the paper is as follows. Next we formulate our main results, followed by a discussion of the longtime behavior of the process. Results on the cumulant equation are mainly collected in Section 2. The following section is devoted to our superprocess and its density field, whereas the final section deals with the occupation time process and occupation densities.

Remark 1.1.7. Let us mention at this point another interesting topic excluded here, namely the question of existence of (non-degenerate) superprocesses in *multi-dimensional singular media* which possess *absolutely continuous states.* This will be the subject of a forthcoming paper of the authors. □

1.2. Main Results

First of all we remark that we will always interpret equation (1.1.1) (or related ones) in its *mild* form, i.e. as an *integral equation,*

(1.2.0) $$u(t,z) = S_t\varphi(z) - \int_0^t \mathrm{d}r\ p(t-r,c-z)\ u^2(r,c), \qquad t\geq 0,\ z\in\mathbb{R},$$

where $\{S_t;t\geq 0\}$ denotes the Brownian semigroup corresponding to (1.1.3), i.e. the Markov semigroup with "generator" $\kappa\Delta$ and transition density p; for more details, see Section 2 below. Although the super-Brownian motion related to this equation does not fit[3] into the very general formulation of Fitzsimmons (1988), as in the latter the continuity in time of the branching component yields the continuity in time of the whole measure-valued process:

Theorem 1.2.1 **(path continuity).** *The time-homogeneous Markov process* $\mathcal{X} = [X,\mathbb{P}_\mu,\mu\in\mathcal{M}_f]$ *determined by equation* (1.1.1) *(i.e.* (1.2.0)*) via the Laplace*

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3) As pointed out in Dawson et al. (1991), our super-Brownian motion can be viewed as a special case of a model in Dynkin (1991).

transition functional (1.1.2) *can be constructed on the space* $\mathbf{C}[\mathbb{R}_+,\mathcal{M}_f]$ *of continuous finite measure-valued trajectories. The following expectation and covariance formulas hold* $(0\le s\le t,\ \mu\in\mathcal{M}_f,\ \varphi,\psi\in\mathbf{G})$:

(1.2.2) $$\mathbb{E}_\mu(\mathcal{X}_t,\varphi) = (\mu S_t,\varphi)$$

(1.2.3) $$\mathbb{C}\mathrm{ov}_\mu[(\mathcal{X}_s,\varphi),(\mathcal{X}_t,\psi)] = 2\int\mu(\mathrm{d}a)\int_0^s \mathrm{d}r\ p(r,c-a)\ S_{s-r}\varphi(c)\ S_{t-r}\psi(c).$$

The next result establishes the existence of a density field which can be chosen to be continuous in time and space excluding the catalyst's position c:

Theorem 1.2.4 **(jointly continuous density).** *There is a version of* $\mathcal{X}$ *such that there exists a sample joint continuous random field* $x = \{x_t(z);t>0,z\ne c\}$ *satisfying*
$$\mathcal{X}_t(\mathrm{d}z) = x_t(z)\mathrm{d}z \quad \text{for all } t>0, \quad \mathbb{P}_\mu\text{-a.s.}, \quad \mu\in\mathcal{M}_f.$$
The state x_t *at time* $t>0$ *of the time-homogeneous Markov process* x *has the Laplace function*

(1.2.5) $$\mathbb{E}_\mu\exp[-\textstyle\sum_{i=1}^k x_t(z_i)\theta_i] = \exp(\mu,-u(t)), \qquad t>0,\ \theta_i\ge 0,\ z_i\ne c,\ 1\le i\le k,$$

where $u\ge 0$ *satisfies*
$$\frac{\partial}{\partial t}u = \kappa\Delta u - \delta_c u^2, \qquad u\Big|_{t=0+} = \textstyle\sum_{i=1}^k \theta_i\delta_{z_i}$$
(see Proposition 2.3.2 below). In particular, the expectation and covariance formulas (1.1.4) *and* (1.1.5) *hold.*

Recall that according to (1.1.6) the variance of the continuous density $x_t(z)$ blows up as $z\to c$. Opposed to this, the following holds:

Theorem 1.2.6 **(vanishing density at the catalyst's position).** *For fixed* $t>0$,
$$x_t(z) \underset{z\to c}{\longrightarrow} 0 \quad \text{in probability.}$$

Consequently, at a fixed time $t>0$, approaching the catalyst's position c *the "increasingly fluctuating" random density* x_t *degenerates stochastically to* 0 (opposed to the non-degeneration of its expectation $\mathbb{E}x_t$ according to (1.1.4)); in particular, the probability for $x_t(z)$ to be large will become very small as $z \to c$. Heuristically this can be explained as follows: Since at c the branching rate is "infinite", population mass which is eventually present at c will be killed with "overwhelming" probability leading to the fact that $\mathcal{X}_t$ almost surely has density $x_t(c) := 0$ at c (with an exceptional set depending on t).

On the other hand, having in mind the dynamics of the $\mathcal{X}$-process, as long as it is not extinct, there will be a permanent flow of absolutely continuous mass into c, where not only a killing takes place as just described, but also a production of mass according to the critical continuous state branching mechanism. By the "infinite" branching rate, the latter effect happens with a "very small" probability, and the set of time points t where a production of population mass will actually occur (which will be smeared out by the heat flow) should be "very thin".

Our next result will in fact constitute that despite that a.s. degeneration $x_t(c)=0$ at fixed time points t as described in Theorem 1.2.6, our super-Brownian motion $\mathcal{X}$ has a positive occupation density even at the catalyst's position c. In fact, by the sample path continuity of the $\mathcal{X}$-process, we may introduce the *occupation time process* $\mathcal{Y} = \{\mathcal{Y}_t; t \geq 0\}$ *related to* $\mathcal{X}$, defined by $\mathcal{Y}_t = \int_0^t ds\, \mathcal{X}_s$, or more precisely, by

$$(\mathcal{Y}_t, \varphi) = \int_0^t ds\, (\mathcal{X}_s, \varphi), \qquad \varphi \in \mathbf{G}_+.$$

Of course, by the integration, $\mathcal{Y}$ is smoother than $\mathcal{X}$, and

$$y_t(z) := \int_0^t ds\, x_s(z), \qquad t \geq 0,\ z \neq c, \tag{1.2.7}$$

yields a density field of $\mathcal{Y}$, which is $\mathbb{P}_\mu$-a.s., $\mu \in \mathcal{M}_f$, jointly continuous on $\mathbb{R}_+ \times \{z \neq c\}$. It remains to determine its behavior at the catalyst's position c.

Theorem 1.2.8 (everywhere continuous occupation density). *There is a version of* $\mathcal{X}$ *such that the density field* y *of* $\mathcal{Y}$ *defined by* (1.2.7) *extends continuously to all of* $\mathbb{R}_+ \times \mathbb{R}$. *Moreover,* (1.2.9) $\quad \mathbb{E}_\mu \exp[-\sum_{i=1}^{k} \mathcal{Y}_t(z_i)\theta_i] = \exp(\mu,-u(t)), \qquad t\ge 0,\ \theta_i\ge 0,\ z_i\in\mathbb{R},\ 1\le i\le k,$

where $u\ge 0$ *solves*

(1.2.10) $$\frac{\partial}{\partial t}u = \kappa\Delta u - \delta_c u^2 + \sum_{i=1}^{k} \theta_i\delta_{z_i}, \qquad u\Big|_{t=0+} = 0,$$

(see Lemma 2.2.2 below). The following expectation, variance, and covariance formulas hold $(0\le s\le t<s'\le t',\ z,z'\in\mathbb{R},\ \mu\in\mathcal{M}_f)$:

(1.2.11) $$\mathbb{E}_\mu \mathcal{Y}_t(z) = \int\mu(da) \int_0^t ds\ p(s,z-a),$$

(1.2.12) $$\mathbb{V}\mathrm{ar}_\mu[\mathcal{Y}_t(z)-\mathcal{Y}_s(z)] = 2\int\mu(da) \int_0^t d\tau\ p(\tau,c-a) \left[\int_{\tau\vee s}^t dr\ p(r-\tau,z-c)\right]^2,$$

(1.2.13) $$\mathbb{C}\mathrm{ov}_\mu[\mathcal{Y}_t(z)-\mathcal{Y}_s(z),\ \mathcal{Y}_{t'}(z')-\mathcal{Y}_{s'}(z')]$$
$$= 2\int\mu(da) \int_s^t dr \int_{s'}^{t'} dr' \int_0^r d\tau\ p(\tau,c-a)\ p(r-\tau,z-c)\ p(r'-\tau,z'-c),$$

We call $\mathcal{Y}_t(z)$ the *occupation density* of $\mathcal{X}$ at z during the time period $[0,t]$. Note that the expectation formula implies that even at the catalyst's position the occupation density $\mathcal{Y}_t(c)$ cannot be identically 0, which is in contrast to the a.s. vanishing random density $x_t(c)$ at c for *fixed* t, in the sense of Theorem 1.2.6. Note also that the variance of $\mathcal{Y}$ is finite even at the catalyst's position, in contrast to the blowing up effect (1.1.6).

For each $z\in\mathbb{R}$, the sample *monotone* stochastic process $\{\mathcal{Y}_t(z);t\ge 0\}$ determines some locally finite continuous random measure $d\mathcal{Y}_t(z) =: \lambda^z(dt)$ defined on $\mathbb{R}_+$, which we call the *occupation density measure at* z. By the definition (1.2.7) it is a.s. an absolutely continuous measure on the time parameter set $\mathbb{R}_+$, as long as $z\ne c$. What can be said on the occupation density measure λ^c at the catalyst's position? Heuristically it measures just the "thin" time set where there is a non-vanishing population density at c. Theorem 1.2.6 suggests that this measure has to be *singular* a.s. Nevertheless, in the next result we will show that λ^c has a support of "full dimension".

Let us first recall the definition of the *Hausdorff-Besicovitch dimens-*

ion $d = dim(A) \in [0,1]$ of a subset A of $\mathbb{R}$: It is defined by the requirement that

$$\liminf_{\delta\to 0+} \left\{\sum_k (diam(B_k))^\rho;\ \bigcup_k B_k \supset A,\ diam(B_k)<\delta\right\}$$

equals $+\infty$ for $\rho\in(0,d)$ whereas it vanishes for $\rho\in(d,1]$. Here (B_k) is a countable covering of A by closed intervals B_k with diameter smaller than δ. (For more details, we refer to Billingsley (1965), §14.) Furthermore, a measure μ defined on $\mathbb{R}$ (more precisely defined on the Borel σ-algebra $\mathcal{R}$ in $\mathbb{R}$) is said to have *carrying (Hausdorff-Besicovitch) dimension* $cardim(\mu)$ $= d$ if d is the smallest number such that $dim(A)=d$ for some $A\in\mathcal{R}$ with $\mu(\mathbb{R}\setminus A)=0$. Now we are ready to formulate our next result:

Theorem 1.2.14 **(carrying dimension).** *Given* $\mathcal{X}_0(\mathbb{R})>0$, *the occupation density measure* λ^c *at the catalyst's position has a.s. carrying (Hausdorff-Besicovitch) dimension one.*

It is interesting to compare this with the usual Brownian local time (at a fixed point) which determines a singular random measure with carrying dimension $1/2$.

Remark 1.2.15. The results of this paper suggest that in the "original" catalytic superprocess of [5] with a *dense* set of point catalysts the corresponding occupation time process has a sample jointly continuous (spatial) density, too. In fact, in the singular situation (if the branching rate is not regular), the occupation time process $\mathcal{Y}$, or more precisely, its density process y seems to be an essential even for the formulation of the model. For instance, the *martingale problem* for our super-Brownian motion with a single point catalyst at c, should be posed as follows: For $\varphi \in G\cap\mathcal{D}(\Delta)$,

$$M_t(\varphi) := (\mathcal{X}_t,\varphi) - (\mathcal{X}_0,t) - \int_0^t \mathrm{d}r\ (\mathcal{X}_r,\ \kappa\Delta\varphi), \qquad t\geq 0,$$

is a martingale with quadratic variation process

$$\langle\!\langle M(\varphi)\rangle\!\rangle_t := 2\ \varphi^2(\mathrm{c})\ y_t(\mathrm{c}), \qquad t\geq 0,$$

where

$$\int_0^t dr\ (\mathcal{X}_r,\varphi) - \int dz\ \mathcal{Y}_t(z)\ \varphi(z) = 0, \qquad t\geq 0.$$

That is, this quadratic variation disappears if $\varphi(c)=0$, and it cannot be defined simply in terms of $(\mathcal{X}_t,\varphi)$ as in the case of a regular branching rate (Fitzsimmons (1988)). □

1.3. Further Properties: Longterm Behavior

In this subsection we establish some further properties of the processes $\mathcal{X}$ and $\mathcal{Y}$, specifically concerning their asymptotic properties as time tends to infinity.

Up to this point the superprocess $\mathcal{X}$ is defined in the space of *finite* measures. But assume for the moment, that the super-Brownian motion $\mathcal{X}$ *starts off at time* 0 *with the Lebesgue measure* denoted by ℓ, that is $\mathcal{X}_0(dz) = \ell(dz)$. Then

$$(\mathcal{X}_t,\varphi) \underset{t\to\infty}{\longrightarrow} 0 \quad \text{stochastically,} \quad \text{for each } \varphi\in\mathbf{G}_+,$$

i.e. $\mathcal{X}_t$ suffers *local extinction.* In fact (recall that the Laplace transform of $\mathcal{X}_t$ is given by (1.1.2) in conjunction with equation (1.1.1)):

Proposition 1.3.1 **(local extinction).** *For all* $\varphi\in\mathbf{G}_+$, *we have* $\int dz\ u(t,z) \underset{t\to\infty}{\longrightarrow} 0$, *where* $u\geq 0$ *is the solution to* (1.1.1).

This property is interesting in that the *single* catalyst finally kills off all the mass in any bounded region. Consequently, from this point of view, the branching component dominates the spatial diffusion of mass (since in the pure diffusion case $\mathcal{X}_t\equiv\ell$ holds). Of course, from the intuitive viewpoint, the recurrence of the one-dimensional Brownian motion is necessary here. On the other hand, note that "stability of second order" of a branching random walk on the square lattice with (critical binary) branching only at the origin is established in Matthes, Siegmund-Schultze, and Wakolbinger (1992), Example 3.7.

Our next result concerns the limiting behavior as $t\to\infty$ of the occupation time $\mathcal{Y}_t$ and the occupation densities $y_t(z) = \lambda^z([0,t])$ at z.

Theorem 1.3.2. *Assume that* $\mathcal{X}_0=\delta_c$ *and fix* $z\in\mathbb{R}$ *(for instance* $z=c$*).*

(i) **(total occupation density):** $y_t(z)$ *converges in distribution as* $t\to\infty$ *to some stable random variable* $y_\infty(z)$ *with index* $1/2$ *determined by its Laplace function* $\mathbb{E}\exp[-\theta y_\infty(z)] = \exp[-\sqrt{\theta}]$, $\theta\geq 0$.

(ii) **(total occupation time):** $\mathcal{Y}_t$ *converges in distribution as* $t\to\infty$ *to* $\mathcal{Y}_\infty := y_\infty(c)\ell$, *a random multiple of the Lebesgue measure* ℓ.

We call $\mathcal{Y}_\infty$ the *total occupation time* and $y_\infty(z) = \lambda^z(\mathbb{R})$ the *total occupation density* at $z\in\mathbb{R}$. It is interesting to compare these results with the corresponding properties resulting from the "individual mechanisms" in the model. In fact, if we drop the branching mechanism, then $\mathcal{Y}_t(dz)$ equals the "potential measure" $\left(\int_0^t ds\ p(s,z-c)\right)dz$ which approximates $\sqrt{t}\ \ell(dz)$ as $t\to\infty$ (except a constant factor). On the other hand, if we omit the diffusion mechanism (or replace the point catalytic branching rate by a constant rate) then $\mathcal{Y}_t(\mathbb{R})$ forms the occupation time process of the simplest continuous state Galton-Watson process, which has in law a stable random limit variable ζ with index $1/2$ as $t\to\infty$; see Dawson and Fleischmann (1988), p. 198. Hence, our point catalytic model combines and reflects features of both mechanisms resulting in $\mathcal{Y}_\infty \overset{\mathcal{D}}{=} \zeta\ell$. In other words, adding a point catalyst to the pure diffusion situation leads to a reduction and randomization of the "uniform" limiting mass.

2. The Cumulant Equation

In this section we will collect some basic facts on the (integral) equation (1.1.1), but in a certain more general form.

2.1. Prerequisites

Fix a time interval $I = [L,T]$, $L<T$. Let $\mathbf{G}^I$ denote the set of all

continuous mappings $u\colon I \mapsto \mathbf{G}$ which are *dominated* in the sense that $|u(t,x)| \le f_u(x)$, $t\in I$, $x\in\mathbb{R}$, for some $f_u\in\mathbf{G}_+$. We equip $\mathbf{G}$ and $\mathbf{G}^I$ with the supremum norm of uniform convergence, denoted by $\|\bullet\|_\infty$ (in both cases), resulting into normed subspaces of the Banach spaces $\mathbf{C}_0(\mathbb{R})$ and $\mathbf{C}_0(I\times\mathbb{R})$.

Particular subsets of $\mathbf{G}^I$ are given by the following families (recall that S denotes the Brownian semigroup):

$$(2.1.1)\qquad [s,x]\in I\times\mathbb{R} \mapsto S_{T-s}\varphi(x) =: (S^I\varphi)(s,x), \qquad \varphi\in\mathbf{G},$$

$$[s,x]\in I\times\mathbb{R} \mapsto \int_s^T \mathrm{d}r\ p(r-s,y-x), \qquad y\in\mathbb{R}.$$

Note that S^I is a linear contraction operator of $\mathbf{G}$ into $\mathbf{G}^I$. Observe also that in the second case, for fixed y, a dominating function is given by $c\ (T-L)\ p(T,y-(\bullet))$, for a sufficiently large constant c.

Finally, let Ω^I denote the set of all (non-negative) *kernels* $\omega(t,\mathrm{d}x)$ from I into $\mathbb{R}$ (that is ω is a non-negative function defined on $I\times\mathcal{R}$ which is measurable in the first variable and a finite measure in the second) with the following property: the mapping

$$(2.1.2)\qquad [s,x]\in I\times\mathbb{R} \mapsto \int_s^T \mathrm{d}r \int\omega(r,\mathrm{d}y)\ p(r-s,y-x) =: (W^I\omega)(s,x)$$

belongs to $\mathbf{G}^I_+$. Natural examples are *absolutely continuous* kernels $\omega(t,\mathrm{d}x) = \psi(t,x)\mathrm{d}x$ with the property that the (measurable) *density kernel* $\psi\ge 0$ is dominated by some $f_\psi\in\mathbf{G}_+$; or the kernels $\omega(t,\mathrm{d}x) = f(t)\delta_z(\mathrm{d}x)$ with $z\in\mathbb{R}$ and a bounded (measurable) function $f\ge 0$. In Ω^I we introduce a notion of *convergence* $\omega_n \xrightarrow[n\to\infty]{} \omega$ by the requirement that $W^I\omega_n \xrightarrow[n\to\infty]{} W^I\omega$ in $\mathbf{G}^I_+$. Note that in the case of absolutely continuous kernels $\omega_n(t,\mathrm{d}x) = \psi_n(t,x)\mathrm{d}x$ the convergence $\omega_n \xrightarrow[n\to\infty]{} \omega$ automatically holds if $\psi_n \xrightarrow[n\to\infty]{} \psi$ in $\mathbf{G}^I$.

2.2. Basic Setting

Given $\varphi\in\mathbf{G}_+$, $\omega_1,\omega_2\in\Omega^I$, and $\xi\in\mathcal{M}_f$, instead of (1.1.1) we now consider the more general *integral equation*

$$(2.2.1)\qquad u(s,x) = S_{T-s}\varphi(x) + \int_s^T \mathrm{d}r \int\omega_1(r,\mathrm{d}y)\ p(r-s,y-x)$$

$$- \int_s^T dr \int \omega_2(r,dy)\ p(r-s,y-x)\ u(r,y)\ -\ \int_s^T dr \int \xi(dy)\ p(r-s,y-x)\ u^2(r,y),$$

$[s,x] \in I\times\mathbb{R}$, or more formally,

$$-\frac{\partial}{\partial s}u = \kappa\Delta u + \omega_1 - \omega_2 u - \xi u^2, \qquad u\big|_{s=T} = \varphi.$$

If $\omega_2=0$, and ω_1 is absolutely continuous with a dominated density kernel, then (2.2.1) is a special case of an equation studied in [6]. Analyzing the proofs of the Theorems 2.6, 2.10, 2.11, and 2.13 there, one can check that the proofs remain valid in the more general setting concerning ω_1 and ω_2 if one incorporates the obvious changes (in particular, also the transition from the forward formulation there to the backward one here). In other words, the following two lemmas can be derived.

Lemma 2.2.2. *To the given* $[\varphi,\omega_1,\omega_2,\xi] \in \mathbf{G}_+\times\Omega^I\times\Omega^I\times\mathcal{M}_f$, *there exists a unique element* $u =: U^I[\varphi,\omega_1,\omega_2,\xi]$ *in* $\mathbf{G}_+^I$ *satisfying the non-linear equation* (2.2.1). *Moreover,* u *monotonously depends on its "parameters"* $\varphi,\omega_1,\omega_2,\xi$.

If $\varphi_n\to\varphi$ as $n\to\infty$ is valid in $\mathbf{G}_+$ and all functions are dominated in the sense that $\varphi_n\le f$, $n\ge 1$, for some $f\in\mathbf{G}_+$, then we will refer to this as *dominated convergence*. Each ξ in $\mathcal{M}_f$ can be approximated by the absolutely continuous measures

(2.2.3) $$(\xi S_\varepsilon)(dy) := \left(\int \xi(dx)\ p(\varepsilon,y-x)\right) dy \in \mathcal{M}_f, \qquad \varepsilon>0.$$

For this particular *weak* convergence $\xi S_\varepsilon \underset{\varepsilon\to 0}{\Longrightarrow} \xi$ in $\mathcal{M}_f$ we will use the term *approximating convergence.*

Lemma 2.2.4. $U^I[\varphi,\omega_1,\omega_2,\xi]$ *continuously depends on its parameters* $[\varphi,\omega_1,\omega_2,\xi] \in \mathbf{G}_+\times\Omega^I\times\Omega^I\times\mathcal{M}_f$ *in the sense of dominated convergence in* $\mathbf{G}_+$, *the previously defined convergence in* Ω^I, *and approximating convergence in* $\mathcal{M}_f$.

Remark 2.2.5. The *backward formulation* in equation (2.2.1) is adequate to include inhomogeneous "data" as ω_1,ω_2 (which, for instance, will be needed to write down the Laplace transform of the occupation density measures). On the other hand, if ω_1,ω_2 do not depend on the space variable, i.e. all "parameters" entering into the equation are homogeneous, then it is often profitable to switch to the forward setting, in particular when dealing with scaling properties, and we will frequently exploit this. □

2.3. Fundamental Solutions

At this place we recall the existence of basic solutions of the non-linear equation (1.1.1) established in [7], Theorem 3.5. Since in our case the catalytic medium is constant in time, for convenience we may switch from the time-inhomogeneous and backward setting there to a homogeneous forward one by replacing the considered time interval $[L,T)$ by $[-T,0)$ and reversing the time (cf. Remark 2.2.5). Consequently, we fix a finite half open time interval $J = (0,T]$, $T>0$. We also introduce the normed space $\mathbf{G}^J$ of all continuous mappings $u\colon J \mapsto \mathbf{G}$ with $\|u\|_J := \int_J ds\ \|u(s)\|_\infty < \infty$.

Let Θ denote the set of all those measures ϑ in $\mathcal{M}_f$ which are either atomic with a finite set of atoms (i.e. the support of ϑ consists of a finite set) or which are absolutely continuous with a density function dominated by some $f_\vartheta \in \mathbf{G}_+$. Note that $\vartheta \mapsto \{\vartheta * p(t); t\in J\}$ continuously maps Θ into $\mathbf{G}_+^J$ (indeed, proceed as in the proof of Lemma 3.1 in [7] by using the fact that, for t fixed, $\delta_y * p(t)$ is bounded above by a function in $\mathbf{G}_+$, uniformly for each bounded set of y). The *uniform distribution* on the interval $[-\varepsilon,+\varepsilon]$ serves as an example of a measure in Θ, let q_ε denote its density function, $\varepsilon>0$.

Given $\vartheta\in\Theta$, instead of (2.2.1) we now consider the equation

(2.3.1) $$u(t,x) = \vartheta * p(t)(x) - \int_0^t dr\ p(t-r,c-x)\ u^2(r,c), \qquad t\in J,\ x\in\mathbb{R},$$

or more formally,
$$\frac{\partial}{\partial t}u = \kappa\Delta u - \delta_c u^2, \qquad u\Big|_{t=0+} = \vartheta.$$

Proposition 2.3.2 **(basic solutions).** *To each* $\vartheta\in\Theta$ *which does not have an* *atom at* c, *there exists a unique element* $u =: U^J[\vartheta,0,0,\delta_c]$ *in* $\mathbf{G}_+^J$ *satisfying equation* (2.3.1). *Moreover,*

$$U^J[\vartheta_1+\vartheta_2*q_\varepsilon(x)\mathrm{d}x,0,0,\delta_c] \underset{\varepsilon\to 0}{\longrightarrow} U^J[\vartheta_1+\vartheta_2,0,0,\delta_c] \quad \textit{for all such } \vartheta_1,\vartheta_2.$$

Proof. See [7], Theorem 3.5, with the obvious changes required by the slightly more general set Θ (allowing a finite set of atoms instead of a single one) and the "splitting" of ϑ within the continuity assertion. ■

For instance, if $\vartheta=\delta_z$, $z\neq c$, then the previous lemma establishes the existence of *basic solutions.* This, of course, is also a place where our restriction to a model in space dimension one is essential. By the way, the restriction $z\neq c$ cannot be dropped, see Remark 3.5.4 below.

2.4. An Asymptotic Property

Here we will present a lemma which will later be used to show the "degeneration" of density at the catalyst's position and also to show local extinction. We are dealing with asymptotic properties of solutions $u = U^J[\vartheta,0,0,\delta_0]$ to (2.3.1) according to Proposition 2.3.2 (with δ_c replaced by δ_0). Since the end point T of the interval $J = (0,T]$ is arbitrary, formally we may and shall switch to $J = (0,+\infty)$. Write $\|\mu\|$ for the total mass $\mu(\mathbb{R})$ of a measure $\mu\in\mathcal{M}_f$.

Lemma 2.4.1. *Let* $\vartheta\in\Theta$ *with* $\vartheta(\{0\})=0$. *Then the solution* $u=U^J[\vartheta,0,0,\delta_0]$ *to* (2.3.1) *according to Proposition* 2.3.2 *with* $J = (0,+\infty)$ *satisfies*

(i) $\sqrt{t}\, u(t,\sqrt{t}x_t) \underset{t\to\infty}{\longrightarrow} 0$ *whenever* $x_t \underset{t\to\infty}{\longrightarrow} x$ *in* $\mathbb{R}$,

(ii) $\int_0^\infty \mathrm{d}r\; u^2(r,0) = \|\vartheta\|$.

Proof. Using the self-similarity

$$Kp(K^2r,Ky) = p(r,y), \qquad r,K>0,\ y\in\mathbb{R}, \tag{2.4.2}$$

of Gaussian densities, we get from (2.3.1)

$$\sqrt{t}\, u(t,\sqrt{t}x) = \int\vartheta(\mathrm{d}y)\ (4\pi\kappa)^{-1/2} \exp[-(t^{-1/2}y-x)^2/4\kappa] \tag{2.4.3}$$

$$- \int_0^t dr\ p(1-r/t,x)\ u^2(r,0), \qquad t>0,\ x\in\mathbb{R}.$$

Integrating over x yields

$$\|u(t)\|_1 := \int dx\ u(t,x) = \|\vartheta\| - \int_0^t dr\ u^2(r,0), \qquad t\geq 0,$$

which implies that

$$\int_0^\infty dr\ u^2(r,0) \leq \|\vartheta\| < \infty. \tag{2.4.4}$$

By the way, this already gives

$$\liminf_{t\to\infty} \sqrt{t}\ u(t,0) = 0. \tag{2.4.5}$$

Replacing x by x_t in (2.4.3), by dominated convergence the first term on the r.h.s. of (2.4.3) converges as $t\to\infty$, namely to $p(1,x)\ \|\vartheta\|$. Next we want to establish that the second term in (2.4.3) will also converge.

Take any constant $\eta\in(0,1)$. Then from equation (2.3.1),

$$u(r,0) \leq \vartheta * p(r)(0) \leq \|\vartheta\|\ p(r,0) \leq \|\vartheta\|\ p(\eta t,0), \qquad r\geq\eta t>0.$$

Thus

$$\int_{\eta t}^t dr\ p(1-r/t,x_t)\ u^2(r,0) \leq \|\vartheta\|^2\ p^2(\eta t,0) \int_{\eta t}^t dr\ p(1-r/t,0)$$

resulting in a negligible error term since the latter expression equals $const\ \sqrt{1-\eta}/\eta$ and converges to 0 as $\eta\to 1$. On the other hand, for fixed $\eta\in(0,1)$, using (2.4.4), we get

$$\int_0^\infty dr\ 1\{r\leq\eta t\}\ p(1-r/t,x_t)\ u^2(r,0) \xrightarrow[t\to\infty]{} p(1,x) \int_0^\infty dr\ u^2(r,0)$$

by dominated convergence, since $1\{r\leq\eta t\}\ p(1-r/t,x_t)$ tends to $p(1,x)$ as $t\to\infty$ and is uniformly bounded by $p(1-\eta,0)<\infty$. Summarizing, we showed that

$$\lim_{t\to\infty} \sqrt{t}\ u_1(t,\sqrt{t}x_t) = p(1,x) \left(\|\vartheta\| - \int_0^\infty dr\ u_1^2(r,0) \right).$$

From (2.4.5) we conclude that in the case $x_t\equiv x=0$ the r.h.s. will disappear. But then it is identically zero, and the proof is finished. ■

2.5. Signed Solutions around the Origin

Later on we also need to have solutions of the cumulant equation in the case of some *"signed"* initial data. Then, generally speaking, the solutions will *explode* in a finite time. Therefore we have to restrict our consideration to those initial functions which are sufficiently "small". The route we will follow here is related to Fleischmann and Kaj (1992) where in the case of the "classical" equation in constant medium an implicit function theorem approach is used.

In order to switch to a Banach space setting, we introduce the following spaces. Let Φ denote the set of all real-valued continuous functions φ defined on $\mathbb{R}$ such that $e^{|z|}\varphi(z)$ has a *finite limit* as $|z|\to\infty$. We endow Φ with the norm $\|\varphi\| := \sup\{e^{|z|}\varphi(z);z\in\mathbb{R}\}$, $\varphi\in\Phi$, resulting into a Banach space. Moreover, for fixed $I:=[0,T]$, $T>0$, we introduce the Banach space Φ^{I} of all continuous maps u of I into Φ equipped with the norm $\|u\|_{\mathrm{I}} := \sup\{\|u(t)\|;\ t\in I\}$. The spaces Φ and Φ^{I} become *Banach algebras* with respect to the pointwise product of functions. Note also that $\mathbf{G}\subset\Phi$ and $\mathbf{G}^{\mathrm{I}}\subset\Phi^{\mathrm{I}}$, and that the topologies in $\mathbf{G},\mathbf{G}^{\mathrm{I}}$ induced by Φ,Φ^{I}, respectively, are stronger since $\|\varphi\|_\infty\le\|\varphi\|$, $\varphi\in\Phi$.

Analogously to (2.1.1) and (2.1.2), set

$$(S^{\mathrm{I}}\varphi)(t,x) := S_t\varphi(x),\quad (W^{\mathrm{I}}\psi)(t,x) := \int_0^t \mathrm{d}s\ S_s\psi(x),\qquad 0\le t\le T,\ x\in\mathbb{R},\ \varphi,\psi\in\Phi,$$

and define

$$(2.5.1)\qquad (H^{\mathrm{I}}(u))(t,x) := \int_0^t \mathrm{d}r\ p(t-r,c-x)\ u(r,c),\qquad 0\le t\le T,\ x\in\mathbb{R},\ u\in\Phi^{\mathrm{I}}.$$

By standard arguments one gets the following properties:

Lemma 2.5.2. *S^{I}, W^{I} and H^{I} are bounded linear operators of Φ and Φ^{I}, respectively, into Φ^{I}.*

For the fixed $T>0$ and given "signed" $\varphi,\psi\in\Phi$ we now consider

$$(2.5.3)\qquad u(t,x) = S_t\varphi(x) + \int_0^t \mathrm{d}s\ S_s\psi(x) - \int_0^t \mathrm{d}r\ p(t-r,c-x)\ u^2(r,c),\qquad 0\le t\le T,\ x\in\mathbb{R},$$

or in a symbolic form,

$$\frac{\partial}{\partial t}u = \kappa\Delta u + \psi - \delta_c u^2,\qquad u\big|_{t=0} = \varphi.$$

Lemma 2.5.4. *There are positive numbers ε_1 and ε_2 such that for each pair $\varphi,\psi\in\Phi$ with $\|\varphi\|+\|\psi\| < \varepsilon_1$ there exists exactly one element $u =: U^{\mathrm{I}}[\varphi,\psi,0,\delta_c]$ in Φ^{I} with $\|u\|_{\mathrm{I}}<\varepsilon_2$ satisfying the equation* (2.5.3). *More-*

over, u *is analytic as a functional of* $[\varphi,\psi]$ *(in that range considered).*

Of course, in this real Banach space setting, *analyticity* at a point means that the power series expansion at that point has a positive radius of convergence; see e.g. Zeidler (1986), Section 8.2. Note also that in the case $\varphi,\psi\in\mathbf{G}_+$ the solution according to Lemma 2.5.4 coincides with the solution in $\mathbf{G}_+^{\mathrm{I}}$ according to Lemma 2.2.2.

Proof. Set

$$F(\varphi,\psi,u) := u - S^{\mathrm{I}}\varphi - W^{\mathrm{I}}\psi + H^{\mathrm{I}}(u^2), \qquad [\varphi,\psi,u] \in \Phi\times\Phi\times\Phi^{\mathrm{I}},$$

with $S^{\mathrm{I}},W^{\mathrm{I}}$, and H^{I} defined immediately before Lemma 2.5.2. Then we may rewrite equation (2.5.3) as $F(\varphi,\psi,u)=0$. For given $[\varphi,\psi]$, we will solve this equation with the help of the implicit function theorem. Since Φ^{I} is a Banach algebra, by Lemma 2.5.2 F maps $\Phi\times\Phi\times\Phi^{\mathrm{I}}$ continuously into Φ^{I}. Furthermore, at each point $[\varphi,\psi,u] \in \Phi\times\Phi\times\Phi^{\mathrm{I}}$ we get the following first partial (Fréchet) derivative of F with respect to u:

(2.5.5) $$D_u^1F(\varphi,\psi,u)v = v + 2H^{\mathrm{I}}(uv), \qquad v\in\Phi^{\mathrm{I}}.$$

Consequently, this partial derivative is linear in u and continuous in $[\varphi,\psi,u]$ (recall Lemma 2.5.2 and that Φ^{I} is a Banach algebra). But trivially, $F(0,0,0) = 0$, and $D_u^1F(0,0,0)$ is the identity operator, hence is bijective. Therefore, the existence and uniqueness claim follows from the *implicit function theorem*, see, for instance, Zeidler (1986), Theorem 4.B, (a) and (b). Now, the first partial derivative of F with respect to $[\varphi,\psi]$ is given by

$$D^1_{[\varphi,\psi]}F(\varphi,\psi,u)[\xi,\zeta] = - S^{\mathrm{I}}\xi - W^{\mathrm{I}}\zeta, \qquad \xi,\zeta\in\Phi,$$

hence is independent of $[\varphi,\psi]$. Combined with (2.5.5), the first partial derivative $D^1_{[\varphi,\psi,u]}F(\varphi,\psi,u)$ exists and is even continuous in $[\varphi,\psi,u]$. Next,

$$D_u^2F(\varphi,\psi,u)vw = 2H^{\mathrm{I}}(vw), \qquad v,w\in\Phi^{\mathrm{I}},$$

that is $D_u^2F(\varphi,\psi,u)$ is independent of $[\varphi,\psi,u]$. Consequently, all higher partial derivatives of F with respect to $[\varphi,\psi,u]$ will disappear (roughly speaking, F is a polynomial in $[\varphi,\psi,u]$). Therefore $F(\varphi,\psi,u)$ is analytic in $[\varphi,\psi,u]$ and the claimed analyticity property follows too; see Zeidler (1986), Corollary 4.23. ■

2.6. Estimates for Derivatives to a Parameter

In order to estimate later higher moments of the random processes, at this point we want to provide some estimates for higher derivatives with respect to a parameter θ at $\theta=0$ of solutions of the cumulant equation in the "signed setting" of (2.5.3). But let us first introduce some terminology which will be useful here and later.

Convention 2.6.1. Let be given a set Ξ, an open neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}$, and a function $f\colon \Xi\times\mathcal{U} \mapsto \mathbb{R}$. For $n\geq 0$, we will write $\mathrm{D}^n f$ for the n-th partial derivative of f with respect to the second variable $\theta\in\mathcal{U}$ provided that it exists. By an abuse of notation we set $f^{(n)} := \mathrm{D}^n f\big|_{\theta=0}$ for the n-th partial derivative *taken at* $\theta=0$. We will interpret θ as a parameter, and will often suppress it in notation. □

Fix $T>0$ and for the moment $\varphi\in\mathbf{G}\subset\Phi$. According to Lemma 2.5.4 there are positive numbers ε_1 and ε_2 such that for $|\theta|<\varepsilon_1$, there is exactly one solution $u=u_\theta$ in Φ^{I} of

$$u(t,x) = \theta S_t\varphi(x) - \int_0^t \mathrm{d}r\ p(t-r,\mathrm{c}-x)\ u^2(r,\mathrm{c}), \qquad 0\leq t\leq T,\ x\in\mathbb{R}, \tag{2.6.2}$$

or, in a symbolic form, of

$$\frac{\partial}{\partial t}u = \kappa\Delta u - \delta_{\mathrm{c}}u^2, \qquad u\big|_{t=0} = \theta\varphi,$$

satisfying $\|u\|_{\mathrm{I}}<\varepsilon_2$. Set

$$v(t,x) := \theta S_t\varphi(x) - u(t,x), \qquad 0\leq t\leq T,\ x\in\mathbb{R},\ |\theta|<\varepsilon_1. \tag{2.6.3}$$

Recall that according to the Convention 2.6.1 we denote by $v^{(k)}$ the k-th derivative of v with respect to θ, taken at $\theta=0$, (which exists by the analyticity property in Lemma 2.5.4). Put

(2.6.4) $$\|S\varphi\|_T := \sup\{|S_r\varphi(c)|;0<r\leq T\}, \qquad \varphi\in\mathbf{G},\ T>0.$$

Lemma 2.6.5. *There are constants* $c_k>0,\ k\geq 2,$ *such that the power series* $\sum_{k\geq 2} c_k\theta^k,\ \theta>0,$ *has a positive radius of convergence and that*

$$\|v^{(k)}(t)\|_\infty \leq k!\, c_k\, \|S\varphi\|_T^k\, t^{(k-1)/2}, \qquad 0\leq t\leq T,\ \varphi\in\mathbf{G},\ k\geq 2,$$

(with v *defined in* (2.6.3) *and* (2.6.2)*).*

Proof. By definition,

$$v(t,x) = \int_0^t dr\ p(t-r,c-x)\ u^2(r,c), \qquad 0\leq t\leq T,\ x\in\mathbb{R},\ \varphi\in\mathbf{G},\ |\theta|<\varepsilon_1.$$

Hence,

(2.6.6) $$v^{(n)}(t,x) = \int_0^t dr\ p(t-r,c-x)\left(\sum_{i=0}^n \binom{n}{i} u^{(n-i)}u^{(i)}\right)(r,c), \qquad n\geq 0.$$

The analyticity of u implies, in particular, continuity at $\theta=0$, hence $u^{(0)}=0$, and thus $v^{(0)}=0=v^{(1)}$. On the other hand, differentiating (2.6.3) we get $u^{(1)}=S\varphi$ and $u^{(k)}=-v^{(k)}$, $k\geq 2$. Inserting this into (2.6.6) yields

(2.6.7) $$v^{(2)}(t,x) = 2\int_0^t dr\ p(t-r,c-x)[S_r\varphi(c)]^2 \leq 2\|S\varphi\|_T^2 \int_0^t dr\ p(t-r,c-x),$$

$$v^{(k)}(t,x) = \int_0^t dr\ p(t-r,c-x)\left(-2kS\varphi\ v^{(k-1)} + \sum_{2\leq i\leq k-2}\binom{k}{i} v^{(k-i)}v^{(i)}\right)(r,c), \qquad k\geq 3.$$

Now,

(2.6.8) $$p(s,y) \leq p(s,0) = p(1,0)\ s^{-1/2}, \qquad s>0,\ y\in\mathbb{R},$$

and for all constants $\rho\geq 0$,

(2.6.9) $$\int_0^t dr\ (t-r)^{-1/2}\ r^\rho = t^{\rho+1/2}\int_0^1 dr\ (1-r)^{-1/2}\ r^\rho \leq 2\ t^{\rho+1/2}, \qquad t\geq 0.$$

Let $\{c_k;k\geq 1\}$ be the unique solution of the following recursive system:

(2.6.10) $$c_1 := 1, \qquad c_k := 4\ p(1,0)\sum_{1\leq i\leq k-1} c_{k-i}c_i, \qquad k\geq 2.$$

Note that the corresponding power series $g(\theta) := \sum_{k\geq 1} c_k\theta^k$, $\theta\in\mathbb{R}$, satisfies the quadratic equation $g(\theta) - \theta = 4\ p(1,0)\ g^2(\theta)$, which can be solved for $|\theta|$ sufficiently small. Using (2.6.8), (2.6.9), and (2.6.10), the claim easily follows from (2.6.7) by induction on k. ■

To fixed $T>0$ and $\psi\in\mathbf{G}\subset\Phi$, according to Lemma 2.5.4 there are positive numbers ε_1 and ε_2 such that for $|\theta|<\varepsilon_1$, there is exactly one solution $u=u_\theta$ in Φ^I of

(2.6.11) $$u(t,x) = \theta\int_0^t dr\, S_r\psi(x) - \int_0^t dr\, p(t-r,c-x)\, u^2(r,c), \qquad 0\le t\le T,\ x\in\mathbb{R},$$

or, in a symbolic form, of

$$\frac{\partial}{\partial t}u = \kappa\Delta u + \theta\psi - \delta_c u^2, \qquad u\Big|_{t=0} = 0,$$

satisfying $\|u\|_I < \varepsilon_2$. Set

(2.6.12) $$v(t,x) := \theta\int_0^t dr\, S_r\psi(x) - u(t,x), \qquad 0\le t\le T,\ x\in\mathbb{R},\ |\theta|<\varepsilon_1,$$

(2.6.13) $$\|W\psi\|_T := \left|\int_0^T dr\, S_r\psi(c)\right|, \qquad \psi\in\mathbf{G},\ T>0.$$

Denote again by $v^{(k)}$ the k-th derivative of v with respect to θ, taken at $\theta=0$. Then analogously to Lemma 2.6.5 we get the following result.

Lemma 2.6.14. *There are constants* $c_k>0$, $k\ge 2$, *such that the power series* $\sum_{k\ge 2} c_k\theta^k$, $\theta>0$, *has a positive radius of convergence and that*

$$\|v^{(k)}(t)\|_\infty \le k!\, c_k\, \|W\psi\|_T^k\, t^{(k-1)/2}, \qquad 0\le t\le T,\ \psi\in\mathbf{G},\ k\ge 2,$$

(with v *defined in* (2.6.12) *and* (2.6.11)*).*

3. Continuity of Our Superprocess and Its Density Field

3.1. Construction of the Density Field

The purpose of this subsection is to give a rigorous justification of the formal transition from $(\mathcal{X}_t,\varphi)$ to $(\mathcal{X}_t,\delta_z)$ which (heuristically) describes the density $x_t(z)$ of the random measure $\mathcal{X}_t$ at z (for almost all z).

From the construction in [5] we know that there *exists* a time-homogeneous Markov process $\mathcal{X} = [\mathcal{X},\ \mathbb{P}_\mu,\ \mu\in\mathcal{M}_f]$ defined on the σ-field generated by the cylinder subsets of $(\mathcal{M}_f)^{[0,\infty)}$ and determined by the Laplace transition functional

(3.1.1) $$\mathbb{E}_\mu\exp(\mathcal{X}_t,-\varphi) = \exp(\mu,-u(t)), \qquad t\ge 0,\ \mu\in\mathcal{M}_f,\ \varphi\in\mathbf{G}_+,$$

with u the solution to

(3.1.2) $$\frac{\partial}{\partial t}u = \kappa\Delta u - \delta_c u^2, \qquad u\Big|_{t=0} = \varphi$$

(in $\mathbf{G}_+^I$). Write $J:=(0,+\infty)$. From the Markov property we immediately get the following formula.

Lemma 3.1.3. *For each finite collection* $0<t_1<\ldots<t_n$ *and* $\varphi_1,\ldots,\varphi_n\in\mathbf{G}_+$,

$$\mathbb{E}_\mu \exp\left[-\sum_{i=1}^n (\mathcal{X}_{t_i},\varphi_i)\right] = \exp(\mu,\ -A_n[t_1,\ldots,t_n;\varphi_1,\ldots,\varphi_n]),\qquad \mu\in\mathcal{M}_f, \tag{3.1.4}$$

where A_n *is recursively defined by* $A_1[t,\varphi] := U^J[\varphi,0,0,\delta_c](t)$ *with* $U^J[\varphi,0,0,\delta_c] = u$ *the solution to* (3.1.2), *and, for* $n\geq 2$,

$$A_n[t_1,\ldots,t_n;\varphi_1,\ldots,\varphi_n] := A_1\left[t_1,\ \varphi_1+A_{n-1}[t_2-t_1,\ldots,t_n-t_1;\varphi_2,\ldots,\varphi_n]\right].$$

The point is now that by Proposition 2.3.2 the right hand side of the formula (3.1.4) makes sense if we replace the φ_i by $\vartheta_i\in\Theta$ with $\vartheta_i(\{c\})=0$. In particular, we can do this for ϑ_i of the form

$$\vartheta_i = \sum_{j=1}^n [\theta_{i,j}\varphi_{i,j} + \theta'_{i,j}\delta_{z(i,j)}*\varphi_\varepsilon],\quad \varphi_{i,j}\in\mathbf{G}_+,\quad \theta_{i,j},\theta'_{i,j}\geq 0,\quad z(i,j)\neq c,\quad \varepsilon>0,$$

$1\leq i,j\leq n$. Moreover, by the continuity assertion in Proposition 2.3.2, the limit transition $\varepsilon\to 0$ makes sense leading to well-defined expressions in terms of solutions to (3.1.2), which additionally converge to 0 as $\theta_{i,j},\theta'_{i,j} \longrightarrow 0$ (use domination by the heat solution). Therefore, we arrive at Laplace transforms of certain random vectors. By consistency and Kolmogorov's extension theorem we can finally *construct a random family*

$$[\mathcal{X},x] = \{[(\mathcal{X}_t,\varphi),x_t(z)];\ t>0,\ \varphi\in\mathbf{G}_+,\ z\neq c\}$$

defined on some complete probability space $[\Omega,\mathcal{F},\mathcal{P}_\mu]$ which *satisfies the assertions in Lemma* 3.1.3 provided that we adopt the following convention:

Convention 3.1.5. In Lemma 3.1.3, the φ_i may be replaced by

$$\vartheta_i = \sum_{j=1}^n [\theta_{i,j}\varphi_{i,j} + \theta'_{i,j}\delta_{z(i,j)}],\quad \varphi_{i,j}\in\mathbf{G}_+,\quad \theta_{i,j},\theta'_{i,j}\geq 0,\quad z(i,j)\neq c,\quad 1\leq i,j\leq n,$$

by reading $(\mathcal{X}_{t_i},\vartheta_i)$ as $\sum_{j=1}^n [\theta_{i,j}(\mathcal{X}_{t_i},\varphi_{i,j}) + \theta'_{i,j}x_{t_i}(z(i,j))]$; that is, we formally identify $x_t(z)$ with $(\mathcal{X}_t,\delta_z)$, $t>0,\ z\neq c$. □

Note that in particular this covers the representation (1.2.5) in our Theorem 1.2.4. Consequently, we constructed a "density field" x, and it remains to construct continuous versions of $\mathcal{X}$ and x and to rigorously identify x with the density field of $\mathcal{X}$ as claimed in Theorem 1.2.4.

3.2. Some Moment Estimates

The proof of the existence of continuous versions of our processes will be based on Kolmogorov's continuity criterion involving some higher moment estimates. In fact, despite the singular branching rate δ_c, the random field $[\mathcal{X},x]$ has moments of all orders (recall that $z\neq c$ in the definition of the density field x), since our model is based on the simplest possible continuous state branching mechanism.

As a preparation for such proofs, in this subsection we obtain some moment estimates. We fix our attention to a finite time interval $[0,T]$, $T>0$. Set $\mathcal{Z}_t := \mathcal{X}_0 S_t - \mathcal{X}_t$, $0\le t\le T$. Note that $\mathbb{E}_\mu \mathcal{Z}_t \equiv 0$ since $\mathbb{E}_\mu \mathcal{X}_t = \mu S_t$ (which, for instance, will follow from formula (3.2.4) below). Also recall the notation (2.6.4).

Lemma 3.2.1. *Fix $T>0$. To each $k\ge2$ there exists a constant C_k such that*

$$\left|\mathbb{E}_\mu(\mathcal{Z}_t,\varphi)^k\right| \le C_k\, t^{k/4}\, \|S\varphi\|_T^k \sum_{i=1}^{k-1} \|\mu\|^i, \qquad 0\le t\le T,\ \mu\in\mathcal{M}_f,\ \varphi\in\mathbf{G}. \tag{3.2.2}$$

Proof. Start by considering a non-negative $\varphi\in\mathbf{G}$. By formula (3.1.1) combined with the notation (2.6.3),

$$\mathbb{E}_\mu \exp(\mathcal{Z}_t,\theta\varphi) = \exp(\mu,v(t)), \qquad 0\le t\le T,\ \mu\in\mathcal{M}_f,\ \varphi\in\mathbf{G}_+,\ \theta\ge0. \tag{3.2.3}$$

(Note that these exponential moments exist finitely, since $\mathcal{Z}_t\le\mu S_t$ $\mathbb{P}_\mu$-a.s.) Differentiate this identity once with respect to θ to get

$$\mathbb{E}_\mu(\mathcal{Z}_t,\varphi)\exp(\mathcal{Z}_t,\theta\varphi) = (\mu,\mathrm{D}^1 v(t))\ \mathbb{E}_\mu \exp(\mathcal{Z}_t,\theta\varphi)$$

For $k\ge2$, differentiate this now $(k-1)$ times at $\theta=0+$ to arrive at

$$\mathbb{E}_\mu(\mathcal{Z}_t,\varphi)^k = \sum_{j=0}^{k-1}\binom{k-1}{j}(\mu,v^{(k-j)}(t))\ \mathbb{E}_\mu(\mathcal{Z}_t,\varphi)^j \tag{3.2.4}$$

(recall the Convention 2.6.1). Since $\mathbb{E}_\mu(\mathcal{Z}_t,\varphi) = v^{(1)}(t) \equiv 0$, the summands for $j=1$ and $j=k-1$ disappear. Hence, for $\varphi\in\mathbf{G}_+$,

$$\mathbb{E}_\mu(\mathcal{Z}_t,\varphi)^k = (\mu,v^{(k)}(t)) + \sum_{2\le j\le k-2}\binom{k-1}{j}(\mu,v^{(k-j)}(t))\ \mathbb{E}_\mu(\mathcal{Z}_t,\varphi)^j, \qquad k\ge2. \tag{3.2.5}$$

Now we want to verify that the latter formula is valid also for *"signed"* functions $\varphi\in\mathbf{G}$. In fact, first let $\varphi\in\mathbf{G}_+$ have the form $\varphi = a\varphi_1+b\varphi_2$ with

$\varphi_1,\varphi_2\in\mathbf{G}_+$ and $a,b\in[0,1]$. Then, for fixed $\varphi_1,\varphi_2\in\mathbf{G}_+$, the expectation expressions in (3.2.5) are polynomials in $a,b\in[0,1]$. Simultaneously, each $v^{(k)}(t)$, $k\geq 2$, $0\leq t\leq T$, is a polynomial in $a,b\in[0,1]$. In fact, this follows from the recursion formulas (2.6.7). However, all expressions remain meaningful if we switch to $a,b\in[-1,1]$ and they continue to be polynomials. Hence, the validity of (3.2.5) for those non-negative $\varphi = a\varphi_1+b\varphi_2$, $\varphi_1,\varphi_2\in\mathbf{G}_+$, $a,b\in[0,1]$, continues to hold for "signed" $a,b\in[-1,1]$. Specializing to $a=1$, $b=-1$, $\varphi_1=\varphi_+$, and $\varphi_2=\varphi_-$, yields the desired claim, i.e. (3.2.5) holds for all $\varphi\in\mathbf{G}$.

If $k=2$, then (3.2.2) directly follows from Lemma 2.6.5. Assume that the estimate (3.2.2) holds for $n=2,\dots,k-1$ with $k\geq 3$. Then from (3.2.5) and Lemma 2.6.5 we get

$$\left|\mathbb{E}_\mu(\mathcal{Z}_t,\varphi)^k\right| \leq \text{const}\ \|S\varphi\|_T^k\left(\|\mu\|t^{(k-1)/2} + \sum_{2\leq j\leq k-2} t^{(k-j-1)/2}t^{j/4}\sum_{i=1}^{j-1}\|\mu\|^{i+1}\right).$$

But $(k-j-1)/2 + j/4$ is certainly $\geq k/4$, for all j in that range of summation, and (3.2.2) follows by induction on k. ∎

Lemma 3.2.1 will now be used to estimate the *even* moments of the *increments* of the process $\mathcal{Z}$:

Lemma 3.2.6. *Fix* $k\geq 1$, $T>0$, *and* $\mu\in\mathcal{M}_f$. *Then there exists a constant* c *such that*

$$\mathbb{E}_\mu(\mathcal{Z}_{t+h}-\mathcal{Z}_t,\ \varphi)^{2k} \leq c\ [\|S(S_h\varphi-\varphi)\|_T^{2k} + h^{k/2}\|S\varphi\|_T^{2k}],\qquad 0\leq t\leq t+h\leq T,\quad \varphi\in\mathbf{G}.$$

Proof. For $0\leq t\leq t+h\leq T$, from $(\mu S_{t+h},\varphi) = (\mu S_t, S_h\varphi)$ we conclude that

$$(\mathcal{Z}_{t+h}-\mathcal{Z}_t,\ \varphi) = (\mathcal{Z}_t,\ S_h\varphi-\varphi) + (\mathcal{X}_tS_h-\mathcal{X}_{t+h},\ \varphi). \tag{3.2.7}$$

Apply the elementary inequality

$$|x+y|^n \leq 2^{n-1}(|x|^n+|y|^n),\qquad x,y\in\mathbb{R},\ n\geq 0, \tag{3.2.8}$$

the Markov property, and time-homogeneity to get

$$\mathbb{E}_\mu(\mathcal{Z}_{t+h}-\mathcal{Z}_t,\ \varphi)^{2k} \leq \text{const}\left(\mathbb{E}_\mu(\mathcal{Z}_t,\ S_h\varphi-\varphi)^{2k} + \mathbb{E}_\mu\mathbb{E}_{\mathcal{X}_t}(\mathcal{Z}_h,\varphi)^{2k}\right).$$

By Lemma 3.2.1, we may continue with

(3.2.9) $$\leq \text{const}\left(\|S(S_h\varphi-\varphi)\|_T^{2k} + h^{k/2}\ \|S\varphi\|_T^{2k}\ \Sigma_{i=1}^{2k}\ \mathbb{E}_\mu\|\mathcal{X}_t\|^i\right).$$

Again by (3.2.8) and Lemma 3.2.1,

(3.2.10) $$(\mathbb{E}_\mu\|\mathcal{X}_t\|^i)^2 \leq \mathbb{E}_\mu\|\mathcal{X}_t\|^{2i} \leq \text{const}\left(\mathbb{E}_\mu(\mathcal{Z}_t(\mathbb{R}))^{2i} + \|\mu\|^{2i}\right) \leq \text{const}.$$

Thus, the sum in the second term of (3.2.9) can be absorbed into the constant, and the proof is complete. ■

3.3. Path Continuity of Our Super-Brownian Motion $\mathcal{X}$

This subsection is devoted to the **Proof of Theorem** 1.2.1, i.e. we want to show that the superprocess $\mathcal{X}$ can be realized on the space of continuous $\mathcal{M}_f$-valued trajectories.

Let $\mathcal{D}_0$ denote a subset of the domain of definition of the "generator" $\kappa\Delta$ of the strongly continuous semi-group S, which is a dense subset of $\mathbf{G}_+$ (in the supremum norm $\|\bullet\|_\infty$). Fix $T>0$, $\mu\in\mathcal{M}_f$, and for the moment $\varphi\in\mathcal{D}_0$. Then

(3.3.1) $$\|S(S_h\varphi-\varphi)\|_T \leq \|S_h\varphi-\varphi\|_\infty \leq \text{const}\ h\ \|\Delta\varphi\|_\infty = \text{const}\ h.$$

Therefore Lemma 3.2.6 yields

(3.3.2) $$\mathbb{E}_\mu(\mathcal{Z}_{t+h}-\mathcal{Z}_t,\ \varphi)^{2k} \leq \text{const}\ [h^{2k} + h^{k/2}] \leq \text{const}\ h^{k/2}, \qquad 0\leq t\leq t+h\leq T.$$

Applying this to $k=3$, with the help of *Kolmogorov's criterion* we conclude that the real-valued process $\{(\mathcal{Z}_t,\varphi);0\leq t\leq T\}$ has a version which has $\mathbb{P}_\mu$-a.s. continuous sample paths, for each fixed $\mu\in\mathcal{M}_f$ and $\varphi\in\mathcal{D}_0$; see e.g. Ikeda and Watanabe (1981), Corollary 1.4.3. But $\mathbf{G}$ is separable, therefore $\mathcal{Z} = \mu S-\mathcal{X}$ is $\mathbb{P}_\mu$-a.s. continuous. However, $\mu S = \{\mu S_t;0\leq t\leq T\}$ is also continuous (even in the weak topology in $\mathcal{M}_f$), and we get the desired continuity assertion for $\mathcal{X}$.

If $s<t$, then the covariance formula follows from the Markov property and twice applying (3.2.5) and (2.6.7) for $k=2$. In the case $s=t$, consider the two-parameter $[\theta_1,\theta_2]$-Laplace function with $\varphi,\psi\in\mathbf{G}_+$, partially differentiate to both parameters and evaluate at $\theta_1=\theta_2=0$ etc., we omit the stan-

dard details. The generalization to signed test functions can be provided by immediate calculations (or working from the beginning with exponential moments and with the "signed" equation according to Lemma 2.5.4). This finishes the proof of Theorem 1.2.1. ∎

Remark 3.3.3. Note that the *rough* moment estimates (3.3.2) imply *sample Hölder continuity* of the orders $1/4 - \varepsilon > 0$ $(\varepsilon>0)$ for the real-valued processes $\{(\mathcal{X}_t,\varphi);t\geq 0\}$, for fixed $\varphi\in\mathbf{G}$; cf. Gihman and Skorohod (1980), Corollary 3.5.1). This can be contrasted with the known $1/2 - \varepsilon$ sample Hölder continuity of the usual super-Brownian motion, see Perkins (1992). Actually, we obtained the moment estimates in a unified form in order to cover all three cases we need to establish sample continuity, and no effort was made to get estimates which would produce the optimal Hölder index. □

3.4. Continuity of the Density Field

The purpose of this subsection is to provide the **Proof of Theorem** 1.2.4. Our starting point is the random family $[\mathcal{X},x]$ constructed in Subsection 3.1. To apply a two-parameter version of Kolmogorov's theorem, see e.g. Walsh (1986), Corollary 1.2, we will work with the moment estimates of the time and space increments separately, and we may restrict our attention to the space component $\mathbb{R}_\varepsilon := \{y\in\mathbb{R}, |y-c|\geq\varepsilon\}$, for a fixed $\varepsilon>0$.

Recalling the Convention 3.1.5 and the continuity properties, the arguments of Subsection 3.2 remain completely valid, if we replace $\varphi\in\mathbf{G}$ by δ_y with $y\in\mathbb{R}_\varepsilon$ fixed. In particular, Lemma 3.1.3 immediately yields the representation (1.2.5) of the Laplace functional. In Lemma 3.2.1, for $t>0$ we must interpret $(\mathcal{Z}_t,\varphi)$ as $z_t(y) := \mathcal{X}_0{*}p(t)(y) - x_t(y)$, and $\|S\varphi\|_T$, as

$$\sup_{0<t\leq T}\{p(t,y-c)\} \leq \text{const } \varepsilon^{-1} = \text{const}, \quad y\in\mathbb{R}_\varepsilon,$$

where we used the elementary fact that the function $s\mapsto s^\rho e^{-s}$ is bounded on $\{s;s\geq\varepsilon'\}$, for each fixed constant $\rho\geq 0$ and $\varepsilon'>0$. Approximating δ-functions by functions from $\mathbf{G}_+$, by continuity in the case of a single δ-function, the estimate (3.2.2) can be extended to hold for $\varphi = \delta_y - \delta_{y'}$, $y,y' \in \mathbb{R}_\varepsilon$, and for $\delta_y * p(h) - \delta_y$, $y \in \mathbb{R}_\varepsilon$, $0 \le h \le T$. Then $\|S(\delta_y - \delta_{y'})\|_T$ has to be interpreted as

$$\sup_{0<t\le T} |p(t,y-c) - p(t,y'-c)| = |y-y'| \sup_{0<t\le T} |\frac{\partial}{\partial y} p(t,y+\theta(y'-y)-c)|$$

where $\theta \in [0,1]$ depends on t,y,y',c. Again using the elementary boundedness effect, we conclude that the latter supremum expression is finite, uniformly in θ and $y,y' \in \mathbb{R}_\varepsilon$. Hence,

$$\sup_{0<t\le T} |p(t,y-c) - p(t,y'-c)| \le \text{const } |y-y'|, \quad y,y' \in \mathbb{R}_\varepsilon. \tag{3.4.1}$$

Similarly, we interpret $\|S(S_h\delta_y - \delta_y)\|_T$ as

$$\sup_{0<t\le T} |p(t+h,y-c) - p(t,y-c)| = h \sup_{0<t\le T} |\frac{\partial}{\partial h} p(t+\theta h,y-c)| \le \text{const } h \tag{3.4.2}$$

$y \in \mathbb{R}_\varepsilon$, $0 \le h \le T$.

Setting $\mathcal{Y}_t := \mathcal{X}_0 * p(t) - x_t$, $0<t\le T$, by Lemma 3.2.1 and (3.4.1) for fixed $k\ge 1$ (and μ,ε,T) we get

$$\mathbb{E}_\mu [\mathcal{Y}_t(y) - \mathcal{Y}_t(y')]^{2k} \le \text{const } |y-y'|^{2k} \le \text{const } |y-y'|^{k/2}, \tag{3.4.3}$$

$0<t\le T$, $y,y' \in \mathbb{R}_\varepsilon$. Similarly, by Lemma 3.2.6 and (3.4.2) we obtain

$$\mathbb{E}_\mu [\mathcal{Y}_{t+h}(y) - \mathcal{Y}_t(y)]^{2k} \le \text{const } (h^{2k} + h^{k/2}) \le \text{const } h^{k/2}, \tag{3.4.4}$$

$0<t\le t+h\le T$, $y \in \mathbb{R}_\varepsilon$ (also for fixed T,k,ε,μ).

Take $k=5$, then (3.4.3) and (3.4.4) and Kolmogorov's Theorem yield the desired existence of a continuous version of $\mathcal{Y} = \mu * p - x$ on $[\varepsilon,T] \times \mathbb{R}_\varepsilon$ where $0<\varepsilon<T$, hence of x on $\{t>0, z\ne c\}$.

The expectation and covariance formulas follow in a similar way as for the $\mathcal{X}$ process (or formally by replacing φ,ψ in (1.2.2) and (1.2.3) by δ-functions).

The constructed continuous field x is really the desired density field. In fact, as in the proof of the basic Lemma 1.15 in [7], Section 2, we get that $\mathbb{P}_\mu \times \ell$-almost everywhere $\langle \mathcal{X}_t, \delta_z * q_\varepsilon \rangle$ converges to the existing density $x'_t(z)$, say, of $\mathcal{X}_t$ at $z\ne c$ as $\varepsilon \to 0$, for fixed $t>0$. Hence, for

Lebesgue almost all z we get convergence in distribution towards $x'_t(z)$. By uniqueness of the limit, $x'_t(z)$ must coincide with $x_t(z)$, for a.a. z, according to the Convention 3.1.5. Thus, x yields the claimed continuous density field of $\mathcal{X}$, and the proof of Theorem 1.2.4 is complete. ■

3.5. Mass Density Zero at the Catalyst's Position

Here we are going to present the **Proof of Theorem** 1.2.6. Fix $t>0$. From Theorem 1.2.4 we know that

(3.5.1) $$\mathbb{E}\exp[-\theta x_t(z)] = \mathbb{E}\exp\left[-\int \mathcal{X}_0(da)\, u_z(t,a)\right], \qquad t>0,\ z\neq c,\ \theta\geq 0,$$

where u_z (for fixed $z\neq c$ and $\theta\geq 0$) satisfies equation (2.3.1) with $\vartheta = \theta\delta_z$ and $J=(0,+\infty)$. To prove that (3.5.1) tends to 1 as $z\to c$, it is enough to show that, for fixed $\theta\geq 0$, we have $\langle \mathcal{X}_0, u_z(t,\cdot)\rangle \xrightarrow[z\to c]{} 0$ a.s. Since by assumption $\mathcal{X}_0$ is a finite random measure, and

$$u_z(t,y) \leq \theta p(t,z-y) \leq \theta p(t,0) < \infty,$$

by bounded convergence it suffices to prove that approaching the catalyst's position the solutions to (2.3.1) will degenerate:

(3.5.2) $$u_z(t,a) \xrightarrow[z\to c]{} 0, \qquad a\in\mathbb{R}.$$

By the spatial homogeneity of the motion component in the model, without loss of generality we may assume that $c=0$. Also, by the symmetry of the Gaussian density (1.1.3), we may restrict our attention to $z>0$.

Using the self-similarity (2.4.2) of Gaussian densities, by uniqueness of the solution to (2.3.1) one easily verifies the *self-similarity property*

(3.5.3) $$K u_{Kz}(K^2 r, Ky) = u_z(r,y), \qquad K,r,z>0,\ y\in\mathbb{R}.$$

Applying this first to $K=t^{-1/2}$ we note that, in showing (3.5.2), without loss of generality we may assume that $t=1$. Next we apply (3.5.3) to $K=z^{-1}$ and make a change of variables to $s:=z^{-2}$. Then instead of (3.5.2) we have to prove that $\sqrt{s}\, u_1(s,\sqrt{s}a) \xrightarrow[s\to\infty]{} 0$, $a\in\mathbb{R}$. But this immediately follows from Lemma 2.4.1 (i). ■

Remark 3.5.4. The integral equation (2.3.1), applied to the "limiting" situation $\vartheta=\theta\delta_c$ (which is excluded in Proposition 2.3.2) does *not* describe the random density of $\mathcal{X}_t$ at c (which we proved to be zero in the sense of Theorem 1.2.6) since it does *not* have a *non-negative* solution at all. In fact, for $x=c$ we get (dropping the constants) the "ordinary" equation

$$v(t) = t^{-1/2} - \int_0^t \mathrm{d}r\ (t-r)^{-1/2}\ v^2(r), \qquad t>0,$$

which fails to have a non-negative solution. □

3.6. Local Extinction

This section is devoted to the **Proof of Proposition** 1.3.1. Without loss of generality we may again assume that $c=0$. We have to show that $\|u(t)\|_1 \underset{t\to\infty}{\longrightarrow} 0$, for fixed $\varphi\in\mathbf{G}_+$, where

$$(3.6.1) \qquad u(t,x) = S_t\varphi(x) - \int_0^t \mathrm{d}s\ p(t-s,c-x)\ u^2(s,c), \qquad t\ge 0,\ x\in\mathbb{R}.$$

Integrating the equation with the Lebesgue measure $\ell(\mathrm{d}x)$ yields

$$\|u(t)\|_1 = \|\varphi\|_1 - \int_0^t \mathrm{d}s\ u^2(s,0), \qquad t\ge 0.$$

Then the claim follows from Lemma 2.4.1 (ii) with $\vartheta(\mathrm{d}x) = \varphi(x)\mathrm{d}x$. ∎

Remark 3.6.2. Another consequence of Lemma 2.4.1 is the following *extinction property:*

$$\{\langle\mathcal{X}_t,\varphi\rangle \,|\, \mathcal{X}_0=\sqrt{t}\,\delta_a\} \xrightarrow[t\to\infty]{\mathcal{D}} 0, \qquad \varphi\in\mathbf{G}_+,\ a\in\mathbb{R}.$$

Consequently, although here the process starts with an increasing initial mass, nevertheless it will become locally extinct. In fact, only use the relations (3.1.1) and (3.6.1) in conjunction with Lemma 2.4.1 (i) in the case $x_t := t^{-1/2}a$ and $\vartheta(\mathrm{d}x) = \varphi(x)\mathrm{d}x$ (and $c=0$ without loss of generality). □

4. Occupation Time Processes and Densities

4.1. Occupation Times

In order to formulate a more general time-space random measure process $\mathfrak{Y}$ which has the occupation time process $\mathcal{Y}$ as its "marginal", we need the following definitions.

For $t\geq 0$, let $\mathcal{M}_f^t$ denote the set of all finite measures on $[0,t]\times\mathbb{R}$, and write

$$(\nu,\psi)_{[s,s']} := \int_{[s,s']\times\mathbb{R}} \nu(\mathrm{d}[r,x])\ \psi(r,x), \qquad 0\leq s\leq s'\leq t,\ \nu\in\mathcal{M}_f^t,\ \psi\in\mathbf{G}_+^{[0,t]}.$$

Note that such ψ can be considered as the density kernel of a kernel ω in $\Omega^{[0,t]}$, and then identify ψ and ω. Set

$$(\mathfrak{Y}_t,\psi) = \int_0^t \mathrm{d}s\ (\mathcal{X}_s,\ \psi(s)), \qquad t\geq 0,\ \psi\in\mathbf{G}_+^{[0,t]}.$$

Based on the Lemmas 2.2.2 and 2.2.4, we get the following representation for the joint distribution of the *super-Brownian motion* $\mathcal{X}$ *in the point-catalytic medium* δ_c and its related *occupation time measure process* $\mathfrak{Y}$:

Proposition 4.1.1 **(occupation time).** *The (time-inhomogeneous) Markov process* $[\mathcal{X},\mathfrak{Y}] =: \left[[\mathcal{X},\mathfrak{Y}],\ \mathbb{P}_{s,[\mu,\nu]},\ s\in\mathbb{R}_+,\ \mu\in\mathcal{M}_f,\ \nu\in\mathcal{M}_f^s\right]$ *has Laplace transition functional*

$$(4.1.2)\qquad \mathbb{E}_{s,[\mu,\nu]} \exp\left[(\mathcal{X}_t,-\varphi) + (\mathfrak{Y}_t,-\psi)_{[0,t]}\right] = \exp\left[(\nu,-\psi)_{[0,s]} + (\mu,-u(s))\right],$$

$0\leq s\leq t$, $\mu\in\mathcal{M}_f$, $\nu\in\mathcal{M}_f^s$, $\varphi\in\mathbf{G}_+$, $\psi\in\mathbf{G}_+^{[0,t]}$, *where* u *is the solution to*

$$u(s,x) = S_{t-s}\varphi(x) + \int_s^t \mathrm{d}r \int\mathrm{d}y\ \psi(r,y)\ p(r-s,y-x) - \int_s^t \mathrm{d}r\ p(r-s,c-x)\ u^2(r,c),$$

$0\leq s\leq t$, $x\in\mathbb{R}$, *or formally, to*

$$-\frac{\partial}{\partial s}u = \kappa\Delta u + \psi - \delta_c u^2, \qquad u\big|_{s=t} = \varphi.$$

(That is, $u = U^{[0,t]}[\varphi,\psi,0,\delta_c]$ is the solution of (2.2.1) in the case $I=[0,t]$, $\omega_1(r,\mathrm{d}x) = \psi(r,x)\mathrm{d}x$, $\omega_2=0$, and $\vartheta=\delta_c$.)

In the constant branching rate case, i.e. if for the coefficient ϑ of the non-linear term we formally have $\vartheta(\mathrm{d}x) = const\ \ell(\mathrm{d}x)$, this representation was given in Iscoe (1986), Theorem 3.2. In our framework, the generalization is straightforward, and we leave the details to the reader.

Note that by definition $\mathfrak{Y}_t$ is a random measure on the product space $[0,t]\times\mathbb{R}$, for each $t\geq 0$. Its marginal measure $\mathcal{Y}_t(B) := \mathfrak{Y}_t([0,t]\times B)$, $B\in\mathcal{R}$, is the usual *occupation time at time* t of the Subsections 1.2 and 1.3 above.

4.2. Occupation Densities

The purpose of this subsection is to deal with the **Proof of the occupation density Theorem** 1.2.8. The method will again be to apply Kolmogorov's criterion, and we proceed in the same way as above.

Fix $T>0$, $\mu\in\mathcal{M}_f$. From Proposition 4.1.1 and the Lemmas 2.2.2 and 2.2.4 we obtain the following formulas:

(4.2.1) $$\mathbb{E}_\mu \exp[-(y_{t+h}(z)-y_t(z))\theta] = e^{-(\mu,u(0))}, \qquad 0\le t\le t+h\le T,\ z\ne c,\ \theta\ge 0,$$

where $\mathbb{E}_\mu = \mathbb{E}_{0,[\mu,0]}$ and $u=u_{t,h,z}$ solves

(4.2.2) $$u(s,x) = \theta \int_{s\vee t}^{t+h} dr\ p(r-s,z-x) - \int_s^{t+h} dr\ p(r-s,c-x)\ u^2(r,c),$$

$0\le s\le t+h$, $x\in\mathbb{R}$, or, in a more symbolic form,

$$-\frac{\partial}{\partial s}u = \kappa\Delta u + \theta 1_{[t,t+h]}\delta_z - \delta_c u^2, \qquad 0\le s\le t+h, \qquad u\Big|_{s=t+h} = 0.$$

That is, $u = u_{t,h,z} = U^{[0,t+h]}[0,\theta 1_{[t,t+h]}\delta_z,0,\delta_c]$ is the solution of (2.2.1) in the case $I=[0,t+h]$, $\omega_1(r,dx) = \theta\ 1_{[t,t+h]}(r)\ \delta_z(dx)$, $\omega_2=0$, and $\vartheta=\delta_c$.

Set

(4.2.3) $$v(s,x) := \theta\int_{s\vee t}^{t+h} dr\ p(r-s,z-x) - u(s,x), \qquad 0\le s\le t+h,\ x\in\mathbb{R},\ \theta\ge 0,\ z\ne c.$$

Then for $v^{(k)}=v^{(k)}_{t,h,z}$, the k-th derivative of $v=v_{t,h,z}$ at $\theta=0$, we get:

Lemma 4.2.4. *Fix $T>0$. To each $k\ge 2$ there exists a constant c_k such that*

(4.2.5) $$\|v^{(k)}_{t,h,z}(s)\|_\infty \le c_k\ h^{k/2}, \qquad 0\le s,t\le t+h\le T,\ z\ne c.$$

Proof. By definition,

(4.2.6) $$v(s,x) = \int_s^{t+h} dr\ p(r-s,c-x)\ u^2(r,c), \qquad 0\le s,t\le t+h,\ x\in\mathbb{R},\ \theta\ge 0.$$

Hence,

(4.2.7) $$v^{(n)}(s,x) = \int_s^{t+h} dr\ p(r-s,c-x)\left(\sum_{i=0}^n \binom{n}{i} u^{(n-i)}u^{(i)}\right)(r,c), \qquad n\ge 0.$$

Since u is non-negative, (4.2.2) implies $u^{(0)}\equiv 0$, thus $v^{(0)}\equiv 0\equiv v^{(1)}$. On the other hand, differentiating (4.2.3) we obtain

(4.2.8) $$u^{(1)}(s,x) = \int_{s\vee t}^{t+h} dr\ p(r-s,z-x) \le \text{const}\ [\sqrt{t+h-s} - \sqrt{(t-s)_+}] \le \text{const}\ \sqrt{h},$$

uniformly in all those s,t,h,x,z, and $u^{(k)}=-v^{(k)}$, $k\ge 2$. Inserting this into (4.2.7) yields

$$(4.2.9)\qquad v^{(2)}(s,x) \le \text{const} \int_s^{t+h} dr\ (r-s)^{-1/2}\ h \le \text{const}\ h,$$

$$|v^{(k)}(t,x)| \le \text{const} \int_s^{t+h} dr\ (r-s)^{-1/2}\Big(\sqrt{h}\,|v^{(k-1)}| + \textstyle\sum_{2\le i\le k-2}|v^{(k-i)}v^{(i)}|\Big)(r,c), \qquad k\ge 3.$$

The claim now easily follows by induction on k. ∎

Set

$$\mathcal{Z}_t(z) := \int \mathcal{X}_0(\mathrm{da}) \int_0^t dr\ p(r,z-a) - \mathcal{Y}_t(z), \qquad 0\le t\le T,\ z\ne c.$$

Note that $\mathbb{E}_\mu \mathcal{Z}_t(z)\equiv 0$. Similar to the derivation of (3.2.5) we get

$$\mathbb{E}_\mu[\mathcal{Z}_{t+h}(z)-\mathcal{Z}_t(z)]^k = (\mu, v^{(k)}(0))$$
$$+ \sum_{2\le j\le k-2}\binom{k-1}{j}(\mu, v^{(k-j)}(0))\ \mathbb{E}_\mu[\mathcal{Z}_{t+h}(z)-\mathcal{Z}_t(z)]^j, \qquad 0\le t\le t+h\le T,\ z\ne c,\ k\ge 2,$$

with $v=v_{t,h,z}$ from (4.2.3). By induction on k, Lemma 4.2.4 then implies

$$(4.2.10)\qquad \mathbb{E}_\mu[\mathcal{Z}_{t+h}(z)-\mathcal{Z}_t(z)]^k \le \text{const}\ h^{k/2}, \qquad 0\le t\le t+h\le T,\ z\ne c,\ k\ge 2,$$

(recall that $T>0$ and $\mu\in\mathcal{M}_f$ are fixed).

Put $(\mathcal{Z}_t,\psi) := \int_0^t ds\ (\mathcal{X}_0 S_s - \mathcal{Y}_t,\ \psi)$, $0\le t\le T$. Based on Lemma 2.6.14, in analogy with Lemma 3.2.1, one obtains the following result.

Lemma 4.2.11. *Fix $T>0$. To each $k\ge 2$ there exists a constant C_k such that*

$$(4.2.12)\qquad \left|\mathbb{E}_\mu(\mathcal{Z}_t,\psi)^k\right| \le C_k\ t^{k/4}\ \|W\psi\|_T^k \sum_{i=1}^{k-1}\|\mu\|^i, \qquad 0\le t\le T,\ \mu\in\mathcal{M}_f,\ \psi\in\mathbf{G}.$$

Again by a continuous transition to $\psi = \delta_z-\delta_{z'}$, $z,z'\ne c$, as in Subsection 3.4 the latter inequality yields

$$(4.2.13)\qquad \mathbb{E}_\mu[\mathcal{Z}_t(z) - \mathcal{Z}_t(z')]^{2k} \le \text{const}\ |z-z'|^{2k}, \qquad 0<t\le T,\ z,z'\ne c,$$

since the potential function $x \mapsto \int_0^T dr\ p(r,x)$ is Lipschitz continuous.

Combining (4.2.10) and (4.2.13), Kolmogorov's criterion implies that the occupation density field $\mathcal{Y}$ has an everywhere sample continuous version. The derivation of expectation and covariance formulas is again standard and will be omitted. This terminates the proof of Theorem 1.2.8. ∎

4.3. A Counterpart to Lemma 2.4.1

To formulate the following counterpart to Lemma 2.4.1 we first note that to each $\vartheta\in\Theta$ by $\omega(t,dx) \equiv \vartheta(dx)$ we get an element ω in Ω^I. We deal with the solution $u = U^I[\varphi,\omega_1,\omega_2,\xi]$ to (2.2.1) as in Lemma 2.2.2, but with

$\varphi=0$, $\omega_1=\vartheta\in\Theta$ (in the sense just described), $\omega_2=0$, and $\xi=\delta_0$. Moreover, by the time-homogeneity in this case, we may again switch to the forward setting:

$$(4.3.1)\qquad u(t,x) = \int_0^t \mathrm{d}r\ \vartheta * p(r)(x) - \int_0^t \mathrm{d}r\ p(t-r,x)\ u^2(r,0),\qquad t>0,\ x\in\mathbb{R},$$

or more formally,

$$\frac{\partial}{\partial t}u = \kappa\Delta u + \vartheta - \delta_0 u^2,\qquad u\big|_{t=0+} = 0.$$

Lemma 4.3.2. *Let* $\vartheta\in\Theta$ *be absolutely continuous with a continuous density function, or atomic with a single atom. Then the solution* u *to* (4.3.1) *satisfies*

(i) $\quad t^{-1/2}u(t,t^{1/2}x_t) \underset{t\to\infty}{\longrightarrow} 0$ *whenever* $x_t \underset{t\to\infty}{\longrightarrow} x$ *in* $\mathbb{R}$,

(ii) $\quad u(t,x) \underset{t\to\infty}{\nearrow}$ *some* $u(\infty,x) < \infty$, $x\in\mathbb{R}$, *and* $u(\infty,0) = \sqrt{\|\vartheta\|}$.

Proof. 1°. First of all, u is monotone non-decreasing in the time variable. In fact, this follows from its probabilistic meaning as "cumulant function"

$$u(t,x) = \log \mathbb{E}_{0,[\delta_x,0]} \exp(\mathcal{Y}_t,-\vartheta),\qquad 0\le s\le t,\ x\in\mathbb{R},$$

in the same sense as the Convention 3.1.5, that is we interpret $(\mathcal{Y}_t,\vartheta)$ as $(\mathcal{Y}_t,\varphi)$ if $\vartheta(\mathrm{d}x) = \varphi(x)\mathrm{d}x$, $\varphi\in\mathbf{G}_+$, or as $\theta y_t(z)$ if $\vartheta = \theta\delta_z$, $\theta\ge 0$, $z\in\mathbb{R}$.

2°. Next, we observe that

$$\|\vartheta\| \int_0^t \mathrm{d}r\ p(r,0) \ge \int_0^t \mathrm{d}r\ \vartheta * p(r)(0).$$

From (4.3.1) with $x=0$, combined with the non-negativity of u we may continue with

$$\ge \int_0^t \mathrm{d}r\ p(t-r,0)\ u^2(r,0) \ge p(t,0) \int_0^t \mathrm{d}r\ u^2(r,0).$$

Hence,

$$2\ \|\vartheta\| \ge t^{-1} \int_0^t \mathrm{d}r\ u^2(r,0),\qquad t>0.$$

By 1°, in the latter expression we may let t tend to infinity to conclude that $u(t,0)$ increases as $t\to\infty$ to some *finite* value denoted by $u(\infty,0)$.

3°. Assume $x_t \underset{t\to\infty}{\longrightarrow} x$ *in* $\mathbb{R}$. By (4.3.1) and the self-similarity (2.4.2),

$$t^{-1/2}u(t,t^{1/2}x_t) = \int_0^1 \mathrm{d}r \int \vartheta(\mathrm{d}y)\ p(r,t^{-1/2}y-x_t) - \int_0^1 \mathrm{d}r\ p(1-r,x_t)\ u^2(tr,0).$$

By dominated convergence, the r.h.s. tends to

$$(4.3.3)\qquad \int_0^1 \mathrm{d}r\ p(r,x)\ (\|\vartheta\| - u^2(\infty,0)).$$

But in the case $x_t \equiv 0$, by 2° the l.h.s. of the former equation converges to 0. Therefore (4.3.3) must be identical to 0.

4°. It remains to show that $u(\infty,x)$ is finite for each fixed $x \in \mathbb{R}$. First observe that to each constant $k>0$ there exist a constant $K>0$ such that

(4.3.4) $$\sup\{|p(t,x) - p(t,y)|;\ |x|,|y| \le k\} \le K\, t^{-3/2}, \qquad t>0.$$

Consider $u(t,0) - u(t,x)$, and write it with the help of (4.3.1). For $0<r\le 1$ use $p(r,y) \le p(r,0)$ whereas for $r>1$ apply (4.3.4) to see that $u(t,0) - u(t,x)$ is bounded in t, for fixed x. This finishes the proof. ∎

4.4. Total Occupation Time and Total Occupation Density

In this subsection we complete the **Proof of Theorem** 1.3.2. Without loss of generality, again we may assume that $c=0$. Fix $\vartheta \in \Theta$ with properties as in Lemma 4.3.2. As in the previous proof, we interpret $(\mathcal{Y}_t,\vartheta)$ as $(\mathcal{Y}_t,\varphi)$ if $\vartheta(dx) = \varphi(x)dx$, $\varphi \in \mathbf{G}_+$, or as $\theta y_t(z)$ if $\vartheta = \theta\delta_z$, $\theta \ge 0$, $z \in \mathbb{R}$. Then from Lemma 4.3.2 (ii) we conclude that

$$\mathbb{E}_{0,[\delta_0,0]} \exp[-(\mathcal{Y}_t,\vartheta)] = e^{-u(t,0)} \underset{t\to\infty}{\longrightarrow} e^{-u(\infty,0)} = \exp[-\sqrt{\|\vartheta\|}].$$

But $\|\vartheta\| = \|\varphi\|_1$ if $\vartheta(dx) = \varphi(x)dx$, $\varphi \in \mathbf{G}_+$, whereas $\|\vartheta\| = \theta$ if $\vartheta = \theta\delta_z$, $\theta \ge 0$, $z \in \mathbb{R}$. Then the claims follow directly. ∎

4.5. Hausdorff Dimension One

This subsection is devoted to the **Proof of Theorem** 1.2.14. Let $\mathbf{C}_+^{\text{comp}}$ denote the set of all continuous non-negative functions f defined on $\mathbb{R}$ having compact support. Write $\mathbf{s}(f)$ and $\mathbf{t}(f)$ for the infimum and supremum of the support of a non-vanishing $f \in \mathbf{C}_+^{\text{comp}}$. Given $\mathcal{X}_0 = \mu$, $\mu \in \mathcal{M}_f$, from Proposition 4.1.1 and the Lemmas 2.2.2 and 2.2.4 we conclude the following formulas for the *occupation density measures* λ^z, $z \in \mathbb{R}$:

(4.5.1) $$\mathbb{E}_{0,[\mu,0]} \exp(\lambda^z,-f) = e^{-(\mu,u(0))}, \qquad f \in \mathbf{C}_+^{\text{comp}},\ f \ne 0,$$

where u solves

(4.5.2) $$u(s,x) = \int_s^{\mathbf{t}(f)} dr\, f(r)\, p(r-s,z-x) - \int_s^{\mathbf{t}(f)} dr\, p(r-s,c-x)\, u^2(r,c),$$

$0 \le s \le \mathbf{t}(f)$, $x \in \mathbb{R}$, or in a more symbolic form,

$$-\frac{\partial}{\partial s} u = \kappa \Delta u + f \delta_z - \delta_c u^2, \quad 0 \le s \le \mathbf{t}(f), \qquad u\big|_{s=\mathbf{t}(f)} = 0.$$

(That is, $u = U^{[0,\mathbf{t}(f)]}[0, f\delta_z, 0, \delta_c]$ is the solution of (2.2.1) in the case $I = [0, \mathbf{t}(f)]$, $\omega_1(r, \mathrm{d}x) = f(r)\delta_z(\mathrm{d}x)$, $\omega_2 = 0$, and $\vartheta = \delta_c$.) In generalization of the formula (1.2.13) one easily justifies the following identity. For $y, z \in \mathbb{R}$ and $f, g \in \mathbf{C}_+^{\mathrm{comp}}$ with $\mathbf{t}(f) < \mathbf{s}(g)$,

$$\mathbb{C}\mathrm{ov}\{(\lambda^y, f), (\lambda^z, g)\} = 2\int \mu(\mathrm{d}a) \int \mathrm{d}s\ f(s) \int \mathrm{d}t\ g(t) \int_0^s \mathrm{d}r\ p(r, c-a) p(s-r, y-c) p(t-r, z-c).$$

Hence, the occupation density measure λ^c at the catalyst's position has the *correlation density*

(4.5.3) $$\mathscr{k}^c(s,t) := 2\int_0^s \mathrm{d}r\ p(s-r,0)\ p(t-r,0) \int \mu(\mathrm{d}a)\ p(r, c-a), \qquad 0<s<t.$$

Assume $\mu \ne 0$. Distinguishing between $r < s/2$ and $r > s/2$, we easily see

$$\mathscr{k}^c(s, s+\varepsilon) \sim \text{const} \int_0^{s/2\varepsilon} \mathrm{d}r\ r^{-1/2}\ (r+\varepsilon)^{-1/2}$$
$$\sim \text{const} \int_0^{s/2\varepsilon} \mathrm{d}r\ r^{-1/2}\ (r+1)^{-1/2} \sim \text{const}\ |\log\varepsilon| \quad \text{as } \varepsilon \to 0,$$

for each fixed $s > 0$. Consequently, the correlation density $\mathscr{k}^c$ of the occupation time measure λ^c has a *logarithmic pole* along the whole diagonal. Then,

$$\int_\varepsilon^T \mathrm{d}s \int_s^{s+1} \mathrm{d}t\ \mathscr{k}^c(s,t)\ (t-s)^{-\gamma} < \infty, \qquad 0<\varepsilon<T,\ 0<\gamma<1,$$

and from *Frostman's lower bound technique* (see, for instance, Zähle (1988), assertion 6.3) follows that λ^c has a.s. carrying dimension one. ∎

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