Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e. V.

PREPRINT ISSN 0946 – 8633

Hysteresis in phase–field models with thermal memory

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submitted: 16th June 1999

Preprint No. 497

Berlin 1999

1991 Mathematics Subject Classification. 35K55, 80A22, 47H30

Key words and phrases. Phase transitions, hysteresis operators, phase-field models, heat conduction with memory, integrodifferential equations

Abstract. Phase-field systems as mathematical models for phase transitions have drawn increasing attention in recent years. However, while capable of capturing many of the experimentally observed phenomena, they are only of restricted value in modelling hysteresis effects occurring during phase transition processes. To overcome this shortcoming, a new approach has recently been proposed by the last two authors which is based on the mathematical theory of hysteresis operators developed in the past fifteen years. In this paper this approach is extended to cases where the material exhibits an additional thermal memory, i.e., where the heat flux contains a time convolution of the spatial gradient of temperature. It is shown that the corresponding system of field equations admits a unique strong solution that depends continuously on the data of the system.

1. Introduction

This paper is devoted to the study of initial-boundary value problems for systems of partial differential equations of the form

$$\mu w_t + f_1[w] + f_2[w]\,\vartheta \;=\; 0\,, \tag{1.1}$$
$$(\vartheta + F_1[w])_t - \kappa_1\,\Delta\vartheta - \kappa_2\,\Delta(m * \vartheta) \;=\; \psi(x,t,\vartheta)\,, \tag{1.2}$$

in $\Omega \times (0,T)$, where Ω is a bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary, $T > 0$ is some final time, $\mu > 0$, $\kappa_1 > 0$, $\kappa_2 \geq 0$ are physical constants, and, for functions a, b of one variable $t > 0$, $a * b$ denotes the time convolution which we define by

$$(a * b)(t) = \int_0^t a(t-s)\,b(s)\,ds, \quad t \geq 0, \tag{1.3}$$

whenever this is meaningful.

Systems of the form (1.1–2) arise as *phase-field equations* from the mathematical study of *phase transitions* and have been studied repeatedly in the literature when f_1, f_2, F_1, ψ are (possibly nonlinear) smooth functions of their respective variables; also cases where (1.1) is replaced by an inclusion as in the so-called *relaxed Stefan model* (see, for instance, [9]) have been under continuing study. We only refer to the monographs [3] and [20] for the case $\kappa_2 = 0$ and to [2, 4–8] for the case $\kappa_2 > 0$, respectively.

Models where f_1, f_2, F_1, ψ are *functions or graphs* are of restricted value in cases where, due to cycling loads, the phase transition may run in both directions. In such a situation usually *hysteresis effects* – like *undercooling* or *overheating* in a solid-liquid transition – occur. In a series of previous papers [15–17], the last two authors have proposed a new approach to incorporate the occurrence of hysteresis effects into the model by assuming that f_1, f_2, F_1 are *hysteresis operators* (for the notion of hysteresis operators see the monographs [3, 13, 14, 19]) instead of *functions*. The possible occurrence of hysteresis effects is not the only reason to consider hysteresis operators in (1.1): in fact, it has been pointed out in [15–17] that already classical models like the relaxed Stefan model (cf. [9]) can be brought into the form (1.1) with suitable hysteresis operators f_1, f_2, F_1; in this connection, the quantity w can be interpreted as a sort of *time-integrated memory* that the system keeps with respect to changes of the *thermodynamic force* acting on the system. For details, we refer the reader to [15–17].

In this paper, we extend the results of [15] obtained for the case $\kappa_2 = 0$ to the case $\kappa_2 > 0$. In physical terms, this means that the heat flux $\mathbf{q}$ is no longer assumed in the Fourier form $\mathbf{q} = -\kappa_1 \nabla\vartheta$; instead it is assumed that $\mathbf{q}$ contains an additional term that accounts for the thermal evolution during the past history, that is, we put

$$\mathbf{q}(x,t) = -\kappa_1 \nabla\vartheta(x,t) - \kappa_2 \int_{-\infty}^{t} m(t-s)\, \nabla\vartheta(x,s)\, ds\,, \tag{1.4}$$

with a given smooth function m. Assuming that ϑ is known for $t \le 0$, we may rewrite (1.4) as

$$\mathbf{q}(x,t) = -\nabla\vartheta(x,t) - \kappa_2\,(m * \nabla\vartheta)(x,t) + \mathbf{q}_0(x,t)\,, \tag{1.5}$$

where $\mathbf{q}_0$ is a given function and $m * \nabla\vartheta$ is defined as in (1.3) by

$$(m * \nabla\vartheta)(x,t) \;=\; \int_0^t m(t-s)\, \nabla\vartheta(x,s)\, ds\,. \tag{1.6}$$

Note that (1.4) is just the Gurtin-Pipkin law for the heat flux. Heat flux laws of this and similar types have originally been introduced in order to explain the occurrence of heat waves and to predict the finite speed of thermal disturbances. To give an idea of the interest of this subject and of the number of involved material scientists, we refer to the review papers [11] and [12]; for thermodynamic considerations, we refer to [10].

In what follows, we aim to show that the phase-field system (1.1–2) with hysteresis operators f_1, f_2, F_1 instead of functions or graphs, complemented by suitable initial and boundary conditions, admits a unique strong solution that depends continuously on the data of the system. To simplify the notation, we shall always assume that $\kappa_1 = \kappa_2 = 1$; this has no bearing on the mathematical analysis.

2. Statement of the problem

We consider in $\Omega \times (0,T)$ the system of equations

$$\mu w_t + f_1[w] + f_2[w]\,\vartheta \;=\; 0\,, \tag{2.1}$$

$$(\vartheta + F_1[w])_t - \Delta(\vartheta + m * \vartheta) \;=\; \psi(x,t,\vartheta)\,, \tag{2.2}$$

coupled with the initial conditions

$$w(x,0) = w^0(x), \quad \vartheta(x,0) = \vartheta^0(x), \quad \text{for } \ (x,t) \in \Omega\,, \tag{2.3}$$

and with the no-flux boundary condition

$$(\nabla\vartheta + m * \nabla\vartheta)\cdot \mathbf{n} = 0, \quad \text{on } \ \partial\Omega \times (0,T)\,, \tag{2.4}$$

where $\mathbf{n}$ denotes the unit outward normal vector field to $\partial\Omega$, $\mu > 0$ and $T > 0$ are given numbers. Moreover, the memory kernel m, the function ψ, and the hysteresis operators f_1, f_2, F_1 satisfy the precise assumptions specified below. Note that any hysteresis operator is *causal*, i.e. for every $w \in C[0,T]$ and $t \in (0,T)$, the value of the output function at time t depends only on the restriction of the input to the interval $[0,t]$. Finally, w^0 and ϑ^0 are prescribed initial data.

Remark 2.1. Different boundary conditions could be considered, e.g., Dirichlet or mixed boundary conditions, with some minor changes in the sequel. Moreover, we could also deal with the 1D and 2D cases. ■

We now state our precise assumptions on the hysteresis operators f_1, f_2, F_1, the right-hand side ψ and the memory kernel m. For the sake of convenience, we use the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } \ 0 < t \leq T. \tag{2.5}$$

The following hypotheses are natural in the context of hysteresis operators, see the abovementioned monographs and the papers [15–17].

Hypothesis 2.2. *We assume that f_1, f_2 map $C[0,T]$ into itself and that there exists a constant K_1 such that the inequalities*

$$|f_i[w_1](t) - f_i[w_2](t)| \leq K_1 \max_{0 \leq s \leq t} |w_1(s) - w_2(s)|, \quad i = 1, 2, \tag{2.6}$$

$$|f_2[w](t)| \leq K_1 \tag{2.7}$$

hold for every $w_1, w_2, w \in C[0,T]$ and $t \in [0,T]$. ■

Hypothesis 2.3. *We assume that F_1 maps $H^1(0,T)$ into itself and that there exist a constant K_2 and a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\left| \frac{d}{dt} F_1[w](t) \right| \leq K_2 \left| \frac{dw}{dt}(t) \right| \quad \text{a.e. in } (0,T), \quad \forall\, w \in H^1(0,T), \tag{2.8}$$

$$|F_1[w_1](t) - F_1[w_2](t)| \leq \varphi(R) \, \|w_1 - w_2\|_{H^1(0,T)} \tag{2.9}$$

$$\text{for any } R > 0 \text{ and } w_1, w_2 \in H^1(0,T) \text{ with } \|w_i\|_{H^1(0,T)} \leq R.$$

Moreover, we assume that $\psi : Q_T \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\psi_0 := \psi(\cdot, \cdot, 0) \in L^2(Q_T) \quad \text{and} \quad |\psi_\vartheta(x,t,\vartheta)| \leq K_2 \text{ a.e. in } Q_T \times \mathbb{R}. \ ■ \tag{2.10}$$

Hypothesis 2.4. *We assume that*

$$m \in W^{1,1}(0,T), \tag{2.11}$$

and that there exists a constant $\alpha_0 > 0$ such that

$$\int_0^t (v + m * v)(s)\, v(s)\, ds \geq \alpha_0 \int_0^t v^2(s)\, ds \quad \forall\, v \in L^2(0,t) \quad \forall\, t \in [0,T]. \ ■ \tag{2.12}$$

Condition (2.12) can be illustrated by the example $m(t) = a \exp(-\varepsilon t)$ for some $a, \varepsilon \geq 0$. Then the heat conduction law (1.4) for $\kappa_1 = \kappa_2 = 1$ can be formally written in the form

$$\mathbf{q}_t + \varepsilon \mathbf{q} = -\nabla \vartheta_t - (a + \varepsilon) \nabla \vartheta .$$

It is easy to check that, in this case, condition (2.12) is fulfilled with $\alpha_0 = 1$.

Remark 2.5. As done in [15], we extend the meaning of $f_i[w]$ and $F_1[w]$ allowing w to depend also on the space variable and setting for instance

$$F_1[w](x,t) := F_1[w(x,\cdot)](t).$$

We now state our existence and continuous dependence results. Uniqueness follows obviously.

Theorem 2.6. *Assume Hypotheses 2.2, 2.3, and 2.4. Then, for every $w^0 \in L^\infty(\Omega)$ and $\vartheta^0 \in H^1(\Omega) \cap L^\infty(\Omega)$, Problem (2.1–4) has a solution (w, ϑ) such that*

$$\begin{aligned} &w, \vartheta \in L^\infty(Q_T), \\ &\vartheta_t, \Delta\vartheta \in L^2(Q_T), \quad w_t \in L^\infty(Q_T), \end{aligned}$$

and (2.1–4) are satisfied almost everywhere. ■

Theorem 2.7. *Assume Hypotheses 2.2, 2.3, and 2.4. Let $w_i^0 \in L^\infty(\Omega)$, $\vartheta_i^0 \in H^1(\Omega) \cap L^\infty(\Omega)$, and $\psi_i : Q_T \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$, be given functions. Let each of the functions $\psi = \psi_1$ and $\psi = \psi_2$ satisfy (2.10), and let there exist a function $d_\psi \in L^2(Q_T)$ such that*

$$\left|\psi_1(x,t,\vartheta^1) - \psi_2(x,t,\vartheta^2)\right| \le d_\psi(x,t) + K_2\left|\vartheta^1 - \vartheta^2\right| \tag{2.13}$$

for a.e. $(x,t,\vartheta^i) \in Q_T \times \mathbb{R}$, $i = 1,2$. Let (w_i, ϑ_i) be solutions to (2.1–4) corresponding to the data w_i^0, ϑ_i^0, ψ_i, $i = 1,2$. Then there exists a constant $C > 0$ such that, for all $t \in [0,T]$,

$$\iint_{Q_t} |\vartheta_1 - \vartheta_2|^2(x,s)\,dx\,ds$$
$$\le C\left\{t\left(\|w_1^0 - w_2^0\|^2_{L^2(\Omega)} + \|\vartheta_1^0 - \vartheta_2^0\|^2_{L^2(\Omega)}\right) + \iint_{Q_t} d_\psi^2(x,s)\,dx\,ds\right\}, \tag{2.14}$$

$$\int_\Omega \|w_1 - w_2\|^2_{H^1(0,T)}(x)\,dx$$
$$\le C\left\{\|w_1^0 - w_2^0\|^2_{L^2(\Omega)} + \|\vartheta_1^0 - \vartheta_2^0\|^2_{L^2(\Omega)} + \iint_{Q_T} d_\psi^2(x,t)\,dx\,dt\right\}. \ ■ \tag{2.15}$$

Theorems 2.6 and 2.7 will be proved in Section 4. The next section is devoted to some preliminary remarks on an abstract integro–differential equation. In the sequel, we widely use the elementary inequality

$$ab \le \sigma a^2 + \frac{1}{4\sigma}b^2 \quad \forall\, a, b \in \mathbb{R} \quad \forall\, \sigma > 0$$

and the well–known Young inequality

$$\|a * v\|_{L^2(0,T;X)} \le \|a\|_{L^1(0,T)}\,\|v\|_{L^2(0,T;X)} \quad \forall\, a \in L^1(0,T) \quad \forall\, v \in L^2(0,T;X) \tag{2.16}$$

where X is a Banach space.

3. An integro–differential equation

Let V and H be two Hilbert spaces and assume that V is a dense linear subspace of H. Then we identify H with a dense subspace of V' in a usual way and write

$$V \subset H \subset V'.$$

In this framework, we consider the integro–differential problem

$$u' + A(u + m * u) = f\,, \tag{3.1}$$
$$u(0) = u^0\,, \tag{3.2}$$

where $A : V \to V'$ is a continuous linear operator.

Sufficient conditions for the well–posedness of (3.1–2) are well–known. Indeed, one can find a very general theory, e.g., in [1]. Nevertheless, for the reader's convenience, we sketch the proof of the simple result stated below. We introduce the notation

$$\|\cdot\| := \|\cdot\|_V\,, \quad |\cdot| := \|\cdot\|_H\,, \quad (\,\cdot\,,\,\cdot\,) := (\,\cdot\,,\,\cdot\,)_H, \quad \text{and} \quad \langle\,\cdot\,,\,\cdot\,\rangle := {}_{V'}\langle\,\cdot\,,\,\cdot\,\rangle_V \tag{3.3}$$

and observe that $\langle u, v\rangle = (u, v)$ for every $u \in H$ and $v \in V$.

Proposition 3.1. *With the above notation, assume that $A \in \mathcal{L}(V; V')$ is symmetric and denote its norm by M. Assume moreover that*

$$m \in L^1(0, T)\,, \tag{3.4}$$

and that there exist constants $\alpha, \lambda > 0$ such that

$$\langle Av, v\rangle \geq \alpha\,\|v\|^2 - \lambda\,|v|^2 \qquad \forall\, v \in V, \tag{3.5}$$

$$\int_0^t \langle A(v + m * v)(s), v(s)\rangle\, ds \geq \alpha \int_0^t \|v(s)\|^2\, ds - \lambda \int_0^t |v(s)|^2\, ds \tag{3.6}$$
$$\forall\, v \in L^2(0, T; V),$$

$$\int_0^t \big((v + m * v)(s), v(s)\big)\, ds \geq \alpha \int_0^t |v(s)|^2\, ds \qquad \forall\, v \in L^2(0, T; H), \tag{3.7}$$

for every $t \in [0, T]$. Then, for any $f \in L^2(0, T; H)$ and $u^0 \in V$, there exists a unique solution u to (3.1–2) satisfying

$$u \in C^0([0, T]; V) \cap H^1(0, T; H) \quad \textit{and} \quad Au \in L^2(0, T; H). \tag{3.8}$$

Moreover, the following estimate holds:

$$\|u\|_{L^\infty(0,T;V)} + \|u'\|_{L^2(0,T;H)} + \|Au\|_{L^2(0,T;H)} \leq C_0 \left(\|f\|_{L^2(0,T;H)} + \|u^0\|\right)\,, \tag{3.9}$$

where C_0 depends only on T, $\|m\|_{L^1(0,T)}$, α, λ, and M. ■

Proof. Uniqueness is easy to prove. As far as existence is concerned, we just show the a priori estimate (3.9) formally. A complete proof could rely, e.g., on a Galerkin method.

Moreover, note that the condition $u \in C^0([0,T];V)$ is a well–known consequence of (3.5) and the other regularity requirements of (3.8). In the sequel, we use the same symbol C_0 for different constants which have the properties specified in the statement.

Testing (3.1) with u, integrating over $(0,t)$, and using (3.6), we obtain

$$\begin{aligned} &\frac{1}{2}|u(t)|^2 + \alpha \int_0^t \|u(s)\|^2\, ds - \lambda \int_0^t |u(s)|^2\, ds \\ &\le \frac{1}{2}|u^0|^2 + \int_0^t \big(f(s), u(s)\big)\, ds \\ &\le \frac{1}{2}|u^0|^2 + \frac{1}{2}\int_0^t |u(s)|^2\, ds + \frac{1}{2}\int_0^T |f(s)|^2\, ds \end{aligned}$$

which implies

$$\|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)} \le C_0 \left(\|f\|_{L^2(0,T;H)} + |u^0|\right) \tag{3.10}$$

via Gronwall's lemma. Now, we test (3.1) with Au and use (3.5) and (3.7). We get

$$\begin{aligned} &\frac{\alpha}{2}\|u(t)\|^2 - \frac{\lambda}{2}|u(t)|^2 + \alpha \int_0^t |Au(s)|^2\, ds \\ &\le \frac{1}{2}\langle Au^0, u^0\rangle + \int_0^t \big(f(s), Au(s)\big)\, ds \\ &\le \frac{M}{2}\|u^0\|^2 + \frac{\alpha}{2}\int_0^t |Au(s)|^2\, ds + \frac{2}{\alpha}\int_0^T |f(s)|^2\, ds. \end{aligned}$$

Using (3.10), we derive the second estimate

$$\|u\|_{L^\infty(0,T;V)} + \|Au\|_{L^2(0,T;H)} \le C_0 \left(\|f\|_{L^2(0,T;H)} + \|u^0\|\right). \tag{3.11}$$

Finally, we test (3.1) with u'. We have

$$\begin{aligned} &\int_0^t |u'(s)|^2\, ds \\ &= -\frac{1}{2}\langle Au(t), u(t)\rangle + \frac{1}{2}\langle Au^0, u^0\rangle - \int_0^t \big((m * Au)(s), u'(s)\big)\, ds + \int_0^t \big(f(s), u'(s)\big)\, ds \\ &\le \frac{1}{2}\left(\|u(t)\|^2 + \|u^0\|^2\right) + \int_0^T |(m * Au)(s)|^2\, ds + \frac{1}{2}\int_0^t |u'(s)|^2\, ds + \int_0^T |f(s)|^2\, ds. \end{aligned}$$

On the other hand, the Young inequality (2.16) yields

$$\int_0^T |(m * Au)(s)|^2\, ds \le \|m\|^2_{L^1(0,T)} \|Au\|^2_{L^2(0,T;H)}.$$

Hence, using (3.11), we conclude

$$\|u'\|_{L^2(0,T;H)} \le C_0 \left(\|f\|_{L^2(0,T;H)} + \|u^0\|\right). \blacksquare \tag{3.12}$$

We now deal with the particular situation we are interested in, i.e., we choose V, H, and A according to the problem we want to solve. We set

$$V = H^1(\Omega) \quad \text{and} \quad H = L^2(\Omega)\,, \tag{3.13}$$

$$\langle Au, v\rangle = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx \quad \forall\, u, v \in V. \tag{3.14}$$

Note that (3.5) holds and that (3.6–7) are fulfilled if m satisfies (2.12). Hence, Proposition 3.1 can be applied. In this case, α and λ depend on the constant α_0 which appears in (2.12), whence also C_0 does. We now prove the following boundedness property for the solution

Proposition 3.2. *Let the hypotheses of Proposition 3.1 hold and assume the notation* (3.13–14)*, where Ω is a bounded open subset of $\mathbb{R}^3$ with Lipschitz boundary. Assume moreover*

$$m \in L^2(0,T), \quad f \in L^\infty(0,T;H), \quad \text{and} \quad u^0 \in V \cap L^\infty(\Omega)\,, \tag{3.15}$$

as well as condition (2.12) *for m. Then the solution u to* (3.1–2) *belongs to $L^\infty(Q_T)$ and satisfies the estimate*

$$\|u\|_{L^\infty(Q_T)} \le C_1 \left(\|f\|_{L^\infty(0,T;H)} + \|u^0\| + \|u^0\|_{L^\infty(\Omega)} \right), \tag{3.16}$$

where C_1 depends only on Ω, T, α_0, and $\|m\|_{L^2(0,T)}$. ∎

Proof. Assume first $m = 0$. Then it suffices to apply [18, Th. 7.1, p. 181] with $n = 3$, $r = \infty$, and $q = 2$, with just one modification. Indeed, the argument of [18] deals with Dirichlet boundary conditions and uses inequality [18, (3.4) p. 75] to derive the estimate. On the other hand, this inequality still holds even though the functions involved do not vanish at the boundary, provided that Ω is bounded and Lipschitz and we allow the constant to depend also on Ω.

In the general case, we write (3.1) in the form

$$u' + Au = f - m * Au\,,$$

and we can apply the first part of the proof provided that we estimate the norm of the convolution. To do that, we use the Cauchy inequality

$$\|m * Au\|_{L^\infty(0,T;H)} \le \|m\|_{L^2(0,T)} \, \|Au\|_{L^2(0,T;H)}\,,$$

and owing to (3.9) we derive

$$\|m * Au\|_{L^\infty(0,T;H)} \le c \left(\|f\|_{L^2(0,T;H)} + \|u^0\| \right)$$

where c depends only on the quantities specified in the statement. ∎

Remark 3.3. The choice (3.13–14) corresponds to the Neumann condition, of course. However, we could deal with different boundary conditions, as we already observed in Remark 2.1.

Remark 3.4. The same argument works if equation (3.1) is replaced with

$$u' + A(u + m * u) + \nu u = f \tag{3.17}$$

where ν is a given positive number.

4. Existence and continuous dependence

This section is devoted to the proof of Theorems 2.6 and 2.7. We follow [15] and just modify the argument when necessary. Moreover, we keep the notation used in Section 3, in particular (3.3) and (3.13–14).

First of all, we solve (2.1) with respect to w for a given ϑ. Under the Hypothesis 2.2, by Lemmas 3.1 and 3.2 of [15], to every $\vartheta \in L^1(0,T)$ and $w^0 \in \mathbb{R}$ we can associate a unique solution $w \in W^{1,1}(0,T)$ to the problem

$$\mu \frac{dw}{dt} + f_1[w] + f_2[w]\,\vartheta = 0 \quad \text{a.e. in } (0,T) \qquad \text{and} \qquad w(0) = w^0. \tag{4.1}$$

Moreover, the estimate

$$\left| \frac{dw}{dt}(t) \right| \le c \left(1 + |w^0| + \|\vartheta\|_{L^1(0,t)} + |\vartheta(t)| \right) \tag{4.2}$$

holds for a.a. $t \in (0,T)$ and a constant c independent of t, w^0, and ϑ. In particular, $w \in H^1(0,T)$ whenever $\vartheta \in L^2(0,T)$ and, following [15] with $p = 2$, we define the causal operator

$$\mathcal{P} : \mathbb{R} \times L^2(0,T) \to H^1(0,T)$$

by means of

$$\mathcal{P}[w^0, \vartheta] = w\,, \tag{4.3}$$

where w is the corresponding solution to (4.1). We now allow both w^0 and ϑ to depend on x. Hence, we can define the operator

$$\mathcal{V} : D(\mathcal{V}) \to L^2(\Omega; H^1(0,T))$$

through the formulas

$$D(\mathcal{V}) := L^2(\Omega) \times L^2(Q_T) \tag{4.4}$$

$$\mathcal{V}[w^0, \vartheta](x,t) := F_1\left[\mathcal{P}[w^0(x), \vartheta(x,\cdot)]\right](t) \tag{4.5}$$

for a.a. $x \in \Omega$ and $t \in [0,T]$. Clearly, $\mathcal{V}$ is causal and can be seen as a family of operators mapping $L^2(\Omega) \times L^2(Q_t)$ into $L^2(\Omega; H^1(0,t))$ for every $t \in [0,T]$. Arguing as in [15], we have the following result

Proposition 4.1. *Let Hypotheses 2.2 and 2.3 hold. Then there exist a constant $C_3 > 0$ and a function $\widetilde{\psi} : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for every $R > 0$, every $t \in [0,T]$, and every $(w^0, \vartheta), (w_i^0, \vartheta_i) \in D(\mathcal{V})$ satisfying $\|w_i\|_{L^\infty(\Omega)} \le R$ and $\|\vartheta_i\|_{L^\infty(Q_t)} \le R$, $i = 1,2$, it holds*

$$\|\mathcal{V}[w^0, \vartheta]_t\|_{L^2(Q_t)} \le C_3 \left(1 + |w^0| + \|\vartheta\|_{L^2(Q_t)}\ , \right) \tag{4.6}$$

$$\|\mathcal{V}[\vartheta]_t\|_{L^\infty(0,T;H)} \le C_3 \left(1 + |w^0| + \|\vartheta\|_{L^\infty(0,T;H)}\ , \right) \tag{4.7}$$

$$\|\mathcal{V}[w_1^0, \vartheta_1] - \mathcal{V}[w_2^0, \vartheta_2]\|_{L^2(\Omega; L^\infty(0,t))} \le \widetilde{\psi}(R) \left(|w_1^0 - w_2^0| + \|\vartheta_1 - \vartheta_2\|_{L^2(Q_t)} \right). \ \blacksquare \tag{4.8}$$

At this point, we can replace the system (2.1–2) and the first Cauchy condition in (2.3) with a single equation by means of the operator $\mathcal{V}$. Therefore, (2.1–4) is formally equivalent to the problem of finding ϑ such that

$$(\vartheta + \mathcal{V}[w^0, \vartheta])_t + A(\vartheta + m * \vartheta) = \psi(\vartheta), \tag{4.9}$$

$$\vartheta(0) = \vartheta^0, \tag{4.10}$$

where $\psi(\vartheta)$ stands for the function $(x,t) \mapsto \psi(x,t,\vartheta(x,t))$. Note that (4.9) contains both a partial differential equation and a Neumann boundary condition. More precisely (see [15, Th. 4.2]), Theorem 2.6 is equivalent to the following statement:

Theorem 4.2. *Assume Hypotheses 2.2, 2.3, 2.4, and take $w^0 \in L^\infty(\Omega)$ and $\vartheta^0 \in H^1(\Omega) \cap L^\infty(\Omega)$. Then there esists a unique ϑ such that*

$$\vartheta \in L^\infty(Q_T), \quad \vartheta_t, \Delta\vartheta \in L^2(Q_T), \tag{4.11}$$

which satisfies (4.9–10). ■

Proof. We write $\mathcal{V}[\vartheta]$ in place of $\mathcal{V}[w^0, \vartheta]$, for the sake of simplicity. We follow the argument used in the proof of [15, Th. 4.2] and modify it when necessary. We start from ϑ^0 and define ϑ^k by induction through

$$\vartheta^k_t + A(\vartheta^k + m * \vartheta^k) + \vartheta^k = \Psi_k, \tag{4.12}$$

$$\vartheta^k(0) = \vartheta^0, \tag{4.13}$$

for $k \geq 1$, where

$$\Psi_k := \vartheta^{k-1} + \psi(\vartheta^{k-1}) - \mathcal{V}[\vartheta^{k-1}]_t. \tag{4.14}$$

We have to show that these equations actually define the sequence $\{\vartheta^k\}$. Indeed, by (2.10) and (4.6–7), we have

$$\|\psi(\vartheta^{k-1})\|_{L^2(Q_T)} \leq |\psi_0| + K_2 \|\vartheta^{k-1}\|_{L^2(Q_T)}, \tag{4.15}$$

$$\|\mathcal{V}[\vartheta^{k-1}]_t\|_{L^2(Q_T)} \leq C_3 \left(1 + |w^0| + \|\vartheta^{k-1}\|_{L^2(Q_T)},\right) \tag{4.16}$$

$$\|\psi(\vartheta^{k-1})\|_{L^\infty(0,T;H)} \leq |\psi_0| + K_2 \|\vartheta^{k-1}\|_{L^\infty(0,T;H)}, \tag{4.17}$$

$$\|\mathcal{V}[\vartheta^{k-1}]_t\|_{L^\infty(0,T;H)} \leq C_3 \left(1 + |w^0| + \|\vartheta^{k-1}\|_{L^\infty(0,T;H)}\right). \tag{4.18}$$

Hence, Propositions 3.1 and Remark 3.4 ensure that the sequence $\{\vartheta^k\}$ is well defined.

We now derive some a priori estimates. From (4.15–16) we deduce

$$\|\Psi_k\|_{L^2(0,T;H)} \leq c\left(1 + \|\vartheta^{k-1}\|_{L^2(0,T;H)}\right)$$

for any k and some c independent of k. Hence, Proposition 3.1 and Remark 3.4 imply

$$\|\vartheta^k\|^2_{L^\infty(0,T;H)} \leq c\left(1 + \|\vartheta^{k-1}\|^2_{L^2(0,T;H)}\right)$$

with a new c. Replacing T by t, we obtain

$$|\vartheta^k(t)|^2 \le c\left(1+\int_0^t |\vartheta^{k-1}(s)|^2\,ds\right) \qquad \forall\, t\in[0,T] \quad \forall\, k\,,$$

and assuming also $c \ge |\vartheta^0|$, we easily derive that

$$|\vartheta^k(t)|^2 \le c\,e^{cT} \qquad \forall\, t\in[0,T] \quad \forall\, k\,,$$

i.e., the sequence $\{\vartheta^k\}$ is bounded in $L^\infty(0,T;H)$. Therefore, taking (4.17–18) into account, we see that Propositions 3.1, 3.2 and Remark 3.4 ensure that the estimate

$$\|\vartheta^k\|_{L^\infty(0,T;V)} + \|\vartheta^k_t\|_{L^2(0,T;H)} + \|\Delta\vartheta^k\|_{L^2(0,T;H)} + \|\vartheta^k\|_{L^\infty(Q_T)} \le C \tag{4.19}$$

holds for any k and for some C independent of k. This estimate yields a convergent subsequence, by well–known weak and weak* compactness results. However, we have to show that the whole sequence $\{\vartheta^k\}$ converges in some topology. Hence, we prove that $\{\vartheta^k\}$ is a Cauchy sequence in $L^2(Q_T)$. In view of using (4.8), we choose a constant $R \ge \|w^0\|_{L^\infty(\Omega)}$ such that $\|\vartheta^k\|_{L^\infty(Q_T)} \le R$ for every k.

We integrate (4.12) with respect to t and subtract the resulting identities corresponding to $k+1$ and k. This yields

$$\vartheta(t) + A\Theta(t) + \Theta(t) = (1*\Psi)(t) - A(m*\Theta)(t) \tag{4.20}$$

where we have set

$$\vartheta := \vartheta^{k+1} - \vartheta^k, \qquad \Theta := 1*\vartheta, \qquad \text{and} \qquad \Psi := \Psi_{k+1} - \Psi_k. \tag{4.21}$$

Testing (4.20) with $\vartheta = \Theta_t$ and integrating over $(0,t)$, we obtain

$$\begin{aligned} &\int_0^t |\vartheta(s)|^2\,ds + \frac{1}{2}\langle A\Theta(t),\Theta(t)\rangle + \frac{1}{2}|\Theta(t)|^2 \\ &= \int_0^t \big((1*\Psi)(s),\vartheta(s)\big)\,ds - \int_0^t \langle A(m*\Theta)(s),\Theta_t(s)\rangle\,ds. \end{aligned} \tag{4.22}$$

By (3.14), the left hand side is given by

$$\int_0^t |\vartheta(s)|^2\,ds + \frac{1}{2}\langle A\Theta(t),\Theta(t)\rangle + \frac{1}{2}|\Theta(t)|^2 = \|\vartheta\|^2_{L^2(Q_t)} + \frac{1}{2}\|\Theta(t)\|^2\,.$$

The first term on the right hand side can be handled as follows:

$$\begin{aligned} &\int_0^t \big((1*\Psi)(s),\vartheta(s)\big)\,ds \le \frac{1}{4}\|\vartheta\|^2_{L^2(Q_t)} + \|1*\Psi\|^2_{L^2(Q_t)} \\ &\le \frac{1}{4}\|\vartheta\|^2_{L^2(Q_t)} + 3\,\|1*(\vartheta^k-\vartheta^{k-1})\|^2_{L^2(Q_t)} \\ &\quad + 3\,\|1*(\psi(\vartheta^k)-\psi(\vartheta^{k-1}))\|^2_{L^2(Q_t)} + 3\,\|\mathcal{V}[\vartheta^k]-\mathcal{V}[\vartheta^{k-1}]\|^2_{L^2(Q_t)}\,. \end{aligned} \tag{4.23}$$

The second term on the right hand side can be treated using the Cauchy inequality, while we can estimate the last two terms owing to (2.10) and (4.8). We have

$$\|1 * (\vartheta^k - \vartheta^{k-1})\|^2_{L^2(Q_t)} \le \int_0^t \Big(\int_0^s |(\vartheta^k - \vartheta^{k-1})(\tau)| \, d\tau \Big)^2 ds \,,$$

$$\le T \int_0^t \|\vartheta^k - \vartheta^{k-1}\|^2_{L^2(Q_s)} \, ds$$

$$\|(1 * (\psi(\vartheta^k) - \psi(\vartheta^{k-1}))\|^2_{L^2(Q_t)} \le K_2^2 T \int_0^t \|\vartheta^k - \vartheta^{k-1}\|^2_{L^2(Q_s)} \, ds \,,$$

$$\|\mathcal{V}[\vartheta^k] - \mathcal{V}[\vartheta^{k-1}]\|^2_{L^2(Q_t)} \le \int_0^t \|\mathcal{V}[\vartheta^k] - \mathcal{V}[\vartheta^{k-1}]\|^2_{L^2(\Omega; L^\infty(0,s))} \, ds$$

$$\le (\widetilde{\psi}(R))^2 \int_0^t \|\vartheta^k - \vartheta^{k-1}\|^2_{L^2(Q_s)} \, ds.$$

Hence (4.23) yields that

$$\int_0^t \big((1 * \Psi)(s), \vartheta(s) \big) \, ds \le \frac{1}{4} \|\vartheta\|^2_{L^2(Q_t)} + c \int_0^t \|\vartheta^k - \vartheta^{k-1}\|^2_{L^2(Q_s)} \, ds$$

for any $t \in [0, T]$ and $k \ge 1$ and for some constant c. Finally, we estimate the last integral in (4.22) integrating by parts as follows:

$$- \int_0^t \big\langle A(m * \Theta)(s), \Theta_t(s) \big\rangle \, ds$$

$$= - \big\langle A(m * \Theta)(t), \Theta(t) \big\rangle + \int_0^t \big\langle A(m * \Theta)'(s), \Theta(s) \big\rangle \, ds$$

$$\le \|(m * \Theta)(t)\| \, \|\Theta(t)\| + \int_0^t \|(m * \vartheta)(s)\| \, \|\Theta(s)\| \, ds$$

$$\le \frac{1}{4} \|\Theta(t)\|^2 + \|(m * \Theta)(t)\|^2 + \frac{1}{2} \int_0^t \|\Theta(s)\|^2 \, ds + \frac{1}{2} \int_0^t \|(m * \vartheta)(s)\|^2 \, ds.$$

On the other hand, we observe that

$$\|(m * \Theta)(t)\|^2 \le \|m\|^2_{L^2(0,T)} \int_0^t \|\Theta(s)\|^2 \, ds \,,$$

and treat the last integral in the previous chain using (2.11) and (2.16) this way:

$$\int_0^t \|(m * \vartheta)(s)\|^2 \, ds = \int_0^t \big\| \big((m(0) + 1 * m') * \vartheta \big)(s) \big\|^2 \, ds$$

$$= \int_0^t \|m(0)\Theta(s) + (m' * \Theta)(s)\|^2 \, ds$$

$$\le 2m^2(0) \int_0^t \|\Theta(s)\|^2 \, ds + 2 \int_0^t \|(m' * \Theta)(s)\|^2 \, ds$$

$$\le 2m^2(0) \int_0^t \|\Theta(s)\|^2 \, ds + 2 \, \|m'\|^2_{L^1(0,T)} \int_0^t \|\Theta(s)\|^2 \, ds.$$

At this point, we collect (4.21–22) and all these inequalities and obtain

$$
\begin{aligned}
&\|\vartheta^{k+1} - \vartheta^k\|^2_{L^2(Q_t)} + \|\Theta(t)\|^2 \\
&\leq c \int_0^t \|\vartheta^k - \vartheta^{k-1}\|^2_{L^2(Q_s)}\, ds + c \int_0^t \|\Theta(s)\|^2\, ds
\end{aligned}
$$

for any $k \geq 1$ and $t \in [0, T]$ and for some c independent of k and t. Applying the Gronwall lemma, we get rid of the last integral and deduce the estimate

$$
\|\vartheta^{k+1} - \vartheta^k\|^2_{L^2(Q_t)} \leq c \int_0^t \|\vartheta^k - \vartheta^{k-1}\|^2_{L^2(Q_s)}\, ds
$$

with a new c, whence we easily conclude that $\{\vartheta^k\}$ is a Cauchy sequence in $L^2(Q_T)$. Recalling (4.19), we see that $\{\vartheta^k\}$ converges to a solution to problem (4.9–10) as in [15]. This concludes the proof of Theorem 4.2, thus the proof of Theorem 2.6. ∎

Proof of Theorem 2.7. We write (4.9–10) for both solutions and initial data, then we take the difference and integrate the resulting identity on $(0, t)$. We obtain an equation similar to (4.20), namely

$$
\vartheta(t) + A\Theta(t) = (1 * \Psi)(t) - A(m * \Theta)(t)
$$

with the notation

$$
\begin{aligned}
&\vartheta := \vartheta^1 - \vartheta^2, \quad \Theta := 1 * \vartheta, \quad \Psi := \Psi_1 - \Psi_2 \\
&\Psi_i = \psi(\vartheta^i) - \mathcal{V}[w_i^0, \vartheta^i]_t, \quad i = 1, 2.
\end{aligned}
$$

Then, we can argue as in the second part of the previous proof and get

$$
\begin{aligned}
&\|\vartheta\|^2_{L^2(Q_t)} + \|\Theta(t)\|^2 \\
&\leq c \int_0^t \|\vartheta\|^2_{L^2(Q_s)}\, ds + c \int_0^t \|\Theta(s)\|^2\, ds \\
&\quad + c \left\{ t \left(|w_1^0 - w_2^0|^2 + |\vartheta_1^0 - \vartheta_2^0|^2 \right) + \iint_{Q_t} d_\psi^2(x, s)\, dx\, ds \right\}
\end{aligned}
$$

for any $t \in [0, T]$ and some constant c. Hence, noting that the term in braces is an increasing function of t, we apply the Gronwall lemma and obtain (2.14), while (2.15) follows as in [15] from the properties of the map $\mathcal{P}$ given by (4.3). ∎

Acknowledgements

This work has been supported by M.U.R.S.T. (Italy) and by the Vigoni Program 1998.

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