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Singularly perturbed reaction - diffusion systems in case of exchange of stabilities

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Abstract

We study singularly perturbed elliptic and parabolic differential equations under the assumption that the associated equation has intersecting families of equilibria (exchange of stabilities). We prove by means of the method of asymptotic lower and upper solutions that the asymptotic behavior with respect to the small parameter changes near the curve of exchange of stabilities. The application of that result to systems modelling fast bimolecular reactions in a heterogeneous environment implies a transition layer (jumping behavior) of the reaction rate. This behavior has to be taken into account for identification problems in reaction systems.

1 Introduction

Mathematical models of reaction–diffusion processes are of increasing interest in different fields of applications, for example in reaction kinetics, biology, astrophysics. There are important classes of processes where boundary layers as well as internal layers of different structures arise. In these cases the corresponding mathematical models represent systems of singularly perturbed differential equations. The motivation for our investigations comes from the papers [4, 5] where spatially homogeneous bimolecular reaction systems of the form

$$\begin{array}{rcll} A & \to & C_1 & (g_1(u)), \\ B & \to & C_2 & (g_2(v)), \\ A+B & \to & C & (\bar{r}(u,v)) \end{array} \tag{1.1}$$

have been studied. Here, g_1, g_2 and $\bar{r}$ are the reaction rates depending on the concentrations u and v of the species A and B respectively. The reaction rate $\bar{r}(u,v)$ is assumed to be fast. To express this fact we represent $\bar{r}(u,v)$ in the form $\bar{r}(u,v) = r(u,v)/\varepsilon$ where ε is a small positive parameter. According to the mass-action kinetics, the time evolution of the concentrations u and v of the reaction system (1.1) is governed by the system of ordinary differential equations

$$\begin{aligned} \frac{du}{dt} &= I_a(t) - g_1(u) - \frac{r(u,v)}{\varepsilon}, \\ \frac{dv}{dt} &= I_b(t) - g_2(v) - \frac{r(u,v)}{\varepsilon}, \end{aligned} \tag{1.2}$$

where $I_a(t)$ and $I_b(t)$ are the input flows of the species A and B respectively, g_1 and g_2 are reaction rates of normally existing slow reactions. In [4, 5] it has been shown that under certain assumptions the reaction rate $\bar{r}(u(t), v(t))$ jumps at some time points. In what follows we exhibit a corresponding phenomenon for systems in a heterogeneous environment.
If we add in (1.2) the Laplacian and replace the time-depending inputs $I_a(t)$ and $I_b(t)$

by space-depending ones, then we get the following system of parabolic differential equations (for convenience we have replaced ε by ε^2)

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial t} &= \Delta \bar{u} + I_a(x) - g_1(\bar{u}) - \frac{r(\bar{u},\bar{v})}{\varepsilon^2}, \\
\frac{\partial \bar{v}}{\partial t} &= \Delta \bar{v} + I_b(x) - g_2(\bar{v}) - \frac{r(\bar{u},\bar{v})}{\varepsilon^2}.
\end{aligned}
\tag{1.3}
$$

A stationary solution of (1.3) satisfies

$$
\begin{aligned}
\Delta \bar{u} &= -I_a(x) + g_1(\bar{u}) + \frac{r(\bar{u},\bar{v})}{\varepsilon^2}, \\
\Delta \bar{v} &= -I_b(x) + g_2(\bar{v}) + \frac{r(\bar{u},\bar{v})}{\varepsilon^2}.
\end{aligned}
\tag{1.4}
$$

After the coordinate transformation $u = \bar{u}, v = \bar{u} - \bar{v}$ and multiplying by ε^2 system (1.3) and (1.4) can be rewritten as

$$
\begin{aligned}
\varepsilon^2\Big(\frac{\partial u}{\partial t} - \Delta u\Big) &= \varepsilon^2(I_a(x) - g_1(u)) - r(u, u-v), \\
\frac{\partial v}{\partial t} - \Delta v &= I_a(x) - I_b(x) - g_1(u) + g_2(u-v),
\end{aligned}
\tag{1.5}
$$

and

$$
\begin{aligned}
\varepsilon^2 \, \Delta u &= -\varepsilon^2(I_a(x) - g_1(u)) + r(u, u-v), \\
\Delta v &= I_b(x) - I_a(x) - g_2(u-v) + g_1(u)
\end{aligned}
\tag{1.6}
$$

respectively. In case when $g_1 \equiv g_2 \equiv 0$ (pure bimolecular reactions) we assume that v can be determined from the second equation in (1.5) and the corresponding initial and boundary conditions or from (1.6) and the corresponding boundary conditions such that we arrive at the equations

$$
\varepsilon^2\Big(\frac{\partial u}{\partial t} - \Delta u\Big) = \varepsilon^2 I_a(x) - r(u, u - v(x,t))
\tag{1.7}
$$

or

$$
\varepsilon^2 \, \Delta u = -\varepsilon^2 I_a(x) + r(u, u - v(x))
\tag{1.8}
$$

respectively.

Motivated by the equations (1.7) and (1.8) we investigate in what follows the equations

$$
\varepsilon^2\Big(\frac{\partial u}{\partial t} - \Delta u\Big) = f(u, x, t, \varepsilon),
\tag{1.9}
$$

and

$$\varepsilon^2 \Delta u = f(u,x,\varepsilon) \tag{1.10}$$

under the assumption that the corresponding degenerate equation $f = 0$ has two solutions with respect to u which intersect. This property implies an exchange of stabilities of these solutions considered as equilibria of the corresponding associated equation. Such situation is typical for reaction kinetics. In section 2 we study a Neumann problem for the singularly perturbed equation (1.10) for dim $x = 2$. The case dim $x = 1$ has been studied in [1] for a scalar equation, and in [2] for singularly perturbed systems. In section 3 we consider the singularly perturbed parabolic equation (1.9) for dim $x = 1$. In the elliptic as well as in the parabolic case we are able to prove a change of the asymptotic behavior of the solution with respect to ε near the curve of exchange of stabilities. The method to establish our results is the method of asymptotic lower and upper solutions. The obtained results are illustrated by means of examples from the reaction kinetics with fast reaction rates.

2 The steady state problem

2.1 Formulation of the problem. Assumptions.

Let D be a bounded open simply connected region in R^2 with smooth boundary Γ. By $\partial/\partial n$ we denote the derivative along the inner normal of Γ. Let I_{ε_1} be the interval $(0, \varepsilon_1)$ where $\varepsilon_1 \ll 1$.
We consider the singularly perturbed nonlinear boundary value problem

$$\begin{aligned} &\varepsilon^2 \Delta u = f(u,x,\varepsilon) \qquad \text{for} \quad x \in D, \\ &\frac{\partial u}{\partial n} - \lambda(x) u = 0 \qquad \text{for} \quad x \in \Gamma, \end{aligned} \tag{2.1}$$

where f and λ are assumed to obey

(A_0). $f : R \times \bar{D} \times \bar{I}_{\varepsilon_1} \to R$ and $\lambda : \Gamma \to R$ are sufficiently smooth.

To investigate existence and asymptotic behavior of a solution to (2.1) we use the following equations closely related to (2.1), namely the degenerate equation

$$f(u,x,0) = 0 \tag{2.2}$$

and the associated equation

$$\frac{d^2 u}{d\xi^2} = f(u,x,0) \tag{2.3}$$

in which $x = (x_1, x_2)$ is considered as a parameter.
Concerning the degenerate equation we suppose

(A_1). The degenerate equation (2.2) has two smooth solutions $u = \varphi_1(x)$ and $u = \varphi_2(x)$ defined for $x \in \bar{D}$, and there exists a smooth closed Jordan curve $\mathcal{C}$ located in D such that

$$\begin{array}{lll} \varphi_1(x) = \varphi_2(x) & \textit{for} & x \in \mathcal{C}, \\ \varphi_1(x) > \varphi_2(x) & \textit{for} & x \in D_1 \cup \Gamma, \\ \varphi_1(x) < \varphi_2(x) & \textit{for} & x \in D_2, \end{array}$$

where D_2 is the simply connected region bounded by $\mathcal{C}$, and $D_1 = D \setminus \overline{D}_2$ (see Fig 1).

Assumption (A_1) says that the surfaces $u = \varphi_1(x)$ and $u = \varphi_2(x)$ intersect at a curve whose projection into the region D is the curve $\mathcal{C}$. This property implies that the standard theory of singularly perturbed systems cannot be applied, at least near $\mathcal{C}$. To describe the behavior of a solution of (2.1) near $\mathcal{C}$ it is convenient to introduce local coordinates near $\mathcal{C}$. To this end we fixe some point P on $\mathcal{C}$, and introduce the coordinate s as the arclength on $\mathcal{C}$ measured from P in mathematically positive direction. The coordinate r is introduced in such a way that $|r|$ is the distance on the normal to $\mathcal{C}$ where $r \equiv 0$ characterizes the curve $\mathcal{C}$, $r > 0$ represents points in the interior of the region bounded by $\mathcal{C}$, and $r < 0$ represents points in the exterior of $\mathcal{C}$ (see Fig 1). By a δ-neighborhood of $\mathcal{C}$ we mean the set of all points satisfying $|r| \leq \delta$. It is obvious that there is a $\delta^* > 0$ such that (s, r) represents a local coordinate system in any δ-neighborhood of $\mathcal{C}$ with $\delta \leq \delta^*$.

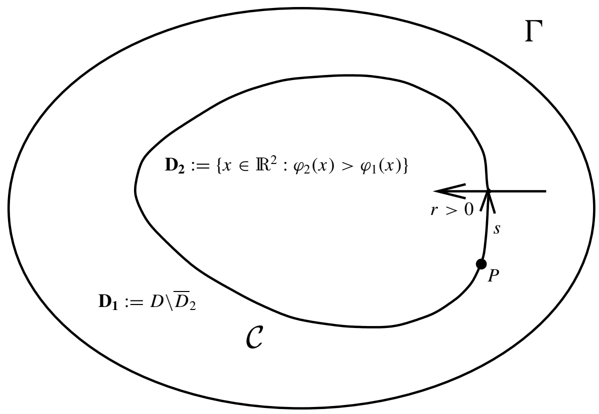

Fig. 1: Intersection of $u = \varphi_1(x)$ and $u = \varphi_2(x)$ at $\mathcal{C}$ in D

From (A_1) we get

$$\frac{\partial \varphi_2(x)}{\partial r} - \frac{\partial \varphi_1(x)}{\partial r} \geq 0 \quad \text{for } x \in \mathcal{C}. \tag{2.4}$$

Note that the surfaces $u = \varphi_1(x)$ and $u = \varphi_2(x)$ are families of equilibria of the associated equation (2.3). An equilibrium point $\bar{u}(x)$ of (2.3) is called conditionally stable if the relation $f_u(\bar{u}(x), x, 0) > 0$ holds. The following assumption describes an exchange of stabilities of the families $\varphi_1(x)$ and $\varphi_2(x)$ of equilibria of (2.3) at the curve $\mathcal{C}$.

(A$_2$).

$$\begin{array}{lll} f_u(\varphi_1(x), x, 0) > 0, \ f_u(\varphi_2(x), x, 0) < 0 & \textit{for} & x \in D_1 \cup \Gamma, \\ f_u(\varphi_1(x), x, 0) < 0, \ f_u(\varphi_2(x), x, 0) > 0 & \textit{for} & x \in D_2. \end{array}$$

Now we define the function $\hat{u}(x)$ by

$$\hat{u}(x) \ := \ \begin{cases} \varphi_1(x) & \text{for } x \in \overline{D}_1, \\ \varphi_2(x) & \text{for } x \in D_2. \end{cases} \tag{2.5}$$

It follows from assumption (A$_1$) that

$$\hat{f}(x) \ \equiv \ f(\hat{u}(x), x, 0) \equiv 0 \quad \text{for } x \in \overline{D}, \tag{2.6}$$

according to assumption (A$_2$) we have

$$\begin{aligned} \hat{f}_u(x) &\equiv f_u(\hat{u}(x), x, 0) > 0 \quad \text{for } x \in \overline{D} \backslash \mathcal{C}, \\ \hat{f}_u(x) &\equiv 0 \quad \text{for } x \in \mathcal{C}. \end{aligned} \tag{2.7}$$

Definition 1. *Under the assumptions* (A$_1$), (A$_2$), *the function $\hat{u}$ defined by* (2.5) *is referred to as the composed stable solution to the degenerate equation* (2.2).

Note that $\hat{u}(x)$ is continuous in $\bar{D}$ and smooth in D_1 and D_2 but not necessarily smooth on $\mathcal{C}$.

To be able to investigate the behavior of $u(x, \varepsilon)$ as ε tends to zero we need the assumption

(A$_3$).

$$\hat{f}_{uu}(x) \equiv f_{uu}(\hat{u}(x), x, 0) > 0 \quad \textit{for } x \in \mathcal{C}.$$

The concept of lower and upper solutions to the boundary value problem (2.1) plays a central role in our approach.

Definition 2. *The functions $\alpha(x, \varepsilon), \beta(x, \varepsilon)$ which are continuous in $\bar{D} \times I_{\varepsilon_2}$, $(\varepsilon_2 \leq \varepsilon_1)$, are called lower and upper solutions respectively to the boundary value problem* (2.1) *if for all $\varepsilon \in I_{\varepsilon_2}$ they satisfy the following conditions*

(i) *α and β are continuously differentiable with respect to $x \in \overline{D}_1$ and twice continuously differentiable with respect to $x \in D_1 \cup \mathcal{C}$ and $x \in \overline{D}_2$,* (*ii*) $\frac{\partial\alpha}{\partial r}(x,\varepsilon)\Big|_{+0} - \frac{\partial\alpha}{\partial r}(x,\varepsilon)\Big|_{-0} \geq 0,\ \frac{\partial\beta}{\partial r}(x,\varepsilon)\Big|_{+0} - \frac{\partial\beta}{\partial r}(x,\varepsilon)\Big|_{-0} \leq 0$ *for* $x \in \mathcal{C}$, *where* $\partial/\partial r$ *denotes the differentiation along the inner normal of* $\mathcal{C}$.

(*iii*) $L_\varepsilon\alpha(x,y) := \varepsilon^2\Delta\alpha(x,\varepsilon) - f(\alpha,x,\varepsilon) \geq 0,\ L_\varepsilon\beta(x,\varepsilon) \leq 0$ *for* $x \in D_1 \cup \mathcal{C}$ *and for* $x \in \overline{D}_2$.

(*iv*) $\frac{\partial\alpha}{\partial n}(x,\varepsilon) - \lambda(x)\alpha(x,\varepsilon) \geq 0,\ \frac{\partial\beta}{\partial n}(x,\varepsilon) - \lambda(x)\beta(x,\varepsilon) \leq 0$ *for* $x \in \Gamma$, *where* $\partial/\partial n$ *denotes the differentiation along the inner normal of* Γ.

Note that $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ may be non-smooth in x on the curve $\mathcal{C}$.

It is known (see, for example, [6]) that if there exist ordered lower and upper solutions to (2.1) i.e., they satisfy the inequality

$$\alpha(x,\varepsilon) \leq \beta(x,\varepsilon) \quad \text{for } (x,\varepsilon) \in \overline{D} \times I_{\varepsilon_2}, \tag{2.8}$$

then problem (2.1) has a solution $u(x,\varepsilon)$ satisfying

$$\alpha(x,\varepsilon) \leq u(x,\varepsilon) \leq \beta(x,\varepsilon) \quad \text{for } (x,\varepsilon) \in \overline{D} \times I_{\varepsilon_2}.$$

2.2 Existence and asymptotic behavior of a solution to (2.1).

We distinguish the cases that f depends on ε and that f does not depend on ε. In what follows we study the case that f depends on ε.

Additionally to the assumptions (A_0)–(A_3) we suppose

(A_4)

$$\hat{f}_\varepsilon(x) \equiv f_\varepsilon(\hat{u}(x), x, 0) < 0 \quad \textit{for } x \in \mathcal{C}.$$

Theorem 1. *Assume hypotheses* (A_0)–(A_4) *to be valid. Then, for sufficiently small* ε, *the boundary value problem* (2.1) *has a solution* $u(x,\varepsilon)$ *satisfying*

$$\lim_{\varepsilon\to 0} u(x,\varepsilon) = \hat{u}(x) \quad \textit{for } x \in \overline{D}. \tag{2.9}$$

Moreover, it holds

$$u(x,\varepsilon) - \hat{u}(x) \quad = \quad \begin{cases} O(\sqrt{\varepsilon}) & \textit{for } x \in D_\delta, \\ O(\varepsilon) & \textit{for } x \in \overline{D}\backslash D_\delta, \end{cases} \tag{2.10}$$

where D_δ *is a* δ*-neighborhoud of the curve* $\mathcal{C}$, *and* δ *is any fixed positive number sufficiently small.*

Proof. To prove our theorem we apply the technique of lower and upper solutions. For the construction of lower and upper solutions we use the composed stable solution $\hat{u}(x)$ defined in (2.5).

It follows from (2.4) that $\hat{u}(x)$ fulfills on $\mathcal{C}$ the condition (ii) of Definition 2 for the lower solution $\alpha(x,\varepsilon)$. But in case

$$\frac{\partial\varphi_2}{\partial r}(x) - \frac{\partial\varphi_1}{\partial r}(x) > 0 \quad \text{for } x \in \mathcal{C}$$

$\hat{u}(x)$ cannot be used as an upper solution since it does not fulfill condition (ii) for $\beta(x,\varepsilon)$. Therefore, we construct an upper solution by smoothing $\hat{u}(x)$ as follows.

Let $\omega \in C^2(R,[0,1])$ be such that

$$\omega(\varrho) \;=\; \begin{cases} 0 & \text{for} \quad \varrho \le -1, \\ \in (0,1) & \text{for} \quad -1 < \varrho < 1, \\ 1 & \text{for} \quad \varrho \ge 1. \end{cases} \tag{2.11}$$

By means of $\omega(\varrho)$ we define the function $\tilde{u}(x,\varepsilon)$ for $(x,\varepsilon) \in D \times I_{\varepsilon_1}$ as follows:

$$\tilde{u}(x,\varepsilon) \;:=\; \begin{cases} \varphi_1(x) + \omega(\frac{r}{\varepsilon})(\varphi_2(x) - \varphi_1(x)) & \text{for} \quad x \in D_\delta, \\ \varphi_1(x) & \text{for} \quad x \in \overline{D}_1 \backslash D_\delta, \\ \varphi_2(x) & \text{for} \quad x \in D_2 \backslash D_\delta, \end{cases} \tag{2.12}$$

where (s,r) are local coordinates in D_δ. It is obvious that $\tilde{u}(x,\varepsilon)$ is twice continuously differentiable in x. If we represent $\tilde{u}(x,\varepsilon)$ in the form

$$\tilde{u}(x,\varepsilon) \;=\; \hat{u}(x) + v(x,\varepsilon) \tag{2.13}$$

then, taking into account $\varphi_2(x) - \varphi_1(x) = O(|r|)$ in D_δ, it is easy to show that $v(x,\varepsilon)$ satisfies

$$v(x,\varepsilon) \;=\; \begin{cases} O(\varepsilon) & \text{for} \quad x \in D_\delta, \\ 0 & \text{for} \quad x \in \overline{D} \backslash D_\delta, \end{cases} \tag{2.14}$$

moreover we have

$$\varepsilon^2 \Delta \tilde{u}(x,\varepsilon) \;=\; \begin{cases} O(\varepsilon) & \text{for} \quad x \in D_\delta, \\ O(\varepsilon^2) & \text{for} \quad x \in \overline{D} \backslash D_\delta. \end{cases} \tag{2.15}$$

By (2.14) and (2.15) there exists positive constants c_1 and c_2 such that for sufficiently small ε the inequalities

$$|v(x,\varepsilon)| \le c_1\varepsilon, \quad \varepsilon^2 |\Delta\tilde{u}(x,\varepsilon)| \le c_2\varepsilon \quad \text{for } x \in D_\delta \tag{2.16}$$

hold.

Now we construct an upper solution $\beta(x,\varepsilon)$ to (2.1) by using the smooth function $\tilde{u}(x,\varepsilon)$. To this end we introduce a local coordinate system (σ, n) in a sufficiently small δ-neighborhood Γ_δ of Γ, $\Gamma_\delta \subset D, \Gamma_\delta \cap D_\delta = \emptyset$, in the same way as we have introduced local coordinates (s, r) near $\mathcal{C}$. We use the twice continuously differentiable cut-off function $\kappa_a : R \to [0, 1], a > 0$, satisfying

$$\kappa_a(\varrho) \quad := \quad \begin{cases} 1 & \text{for} \quad |\varrho| \le a/2, \\ \in (0,1) & \text{for} \quad a/2 < |\varrho| < a, \\ 0 & \text{for} \quad |\varrho| \ge a \end{cases} \tag{2.17}$$

to define the following functions we need to construct upper and lower solutions to (2.1):

$$h(x, \varepsilon) \quad := \quad \begin{cases} (\sqrt{\varepsilon} - \varepsilon)\kappa_\delta(r) + \varepsilon & \text{for} \quad x = (s, r) \in D_\delta, \\ \varepsilon & \text{for} \quad x \in \overline{D} \backslash D_\delta, \end{cases} \tag{2.18}$$

$$z(x, \varepsilon, k) := \begin{cases} \varepsilon \exp\left(-\frac{kn}{\varepsilon}\right) \kappa_\delta(n) & \text{for} \quad x = (\sigma, n) \in \Gamma_\delta, \\ 0 & \text{for} \quad x \in \overline{D} \backslash \Gamma_\delta. \end{cases} \tag{2.19}$$

Now we define an upper solution $\beta(x, \varepsilon)$ to (2.1) as

$$\beta(x, \varepsilon) \quad := \quad \tilde{u}(x, \varepsilon) + b_\beta h(x, \varepsilon) + z(x, \varepsilon, k_\beta) \tag{2.20}$$

where b_β and k_β are some positive numbers to be chosen in an appropriate way later.

Since $\tilde{u}(x, \varepsilon)$ is smooth it follows from (2.18) - (2.20) that $\beta(x, \varepsilon)$ is also smooth and satisfies condition (ii) in Definition 2 for an upper solution.
From (2.18) and (2.19) we get the existence of positive numbers $\bar{c}_1, \bar{c}_2$ such that for sufficiently small ε

$$\begin{aligned} |\Delta h(x, \varepsilon)| \le \bar{c}_1 \sqrt{\varepsilon}, \quad \varepsilon^2 |\Delta z(x, \varepsilon)| \le \bar{c}_2 \varepsilon \quad \text{for} \quad x \in \overline{D}, \\ 0 \le z(x, \varepsilon) \le \varepsilon \quad \text{for} \quad x \in \overline{D}. \end{aligned} \tag{2.21}$$

Now we check that $\beta(x, \varepsilon)$ satisfies the inequality (iii) in Definition 2. From (2.20), (2.13), and (2.6) it follows

$$\begin{aligned} L_\varepsilon \beta(x, \varepsilon) \quad \equiv \quad & \varepsilon^2 \Delta \beta(x, \varepsilon) - f(\beta, x, \varepsilon) = \varepsilon^2 \Delta \Big(\tilde{u}(x, \varepsilon) + b_\beta h(x, \varepsilon) + z(x, \varepsilon, k_\beta) \Big) \\ & - \hat{f}_u(x) \Big(b_\beta h(x, \varepsilon) + z(x, \varepsilon, k_\beta) + v(x, \varepsilon) \Big) \\ & - \frac{1}{2} \hat{f}_{uu}(x) \Big(b_\beta h(x, \varepsilon) + z(x, \varepsilon, k_\beta) + v(x, \varepsilon) \Big)^2 \\ & - \hat{f}_\varepsilon(x) \varepsilon + o(\varepsilon). \end{aligned} \tag{2.22}$$

Our aim is to prove $L_\varepsilon \beta(x, \varepsilon) \le 0$ for $x \in D$ and for sufficiently small ε.

First we estimate $L_\varepsilon \beta(x, \varepsilon)$ in the region $D_{\delta/2}$. According to (2.18), (2.19) and (2.14) we have for $x \in D_{\delta/2}$

$$L_\varepsilon\beta(x,\varepsilon) \equiv \varepsilon^2\Delta\tilde{u}(x,\varepsilon) - \hat{f}_u(x)\Big(b_\beta\sqrt{\varepsilon}+v(x,\varepsilon)\Big) - \frac{1}{2}\hat{f}_{uu}(x)\Big(b_\beta\sqrt{\varepsilon}+v(x,\varepsilon)\Big)^2 - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \tag{2.23}$$

By (2.16) we have for $x \in D_{\delta/2}$ and for sufficiently small ε

$$b_\beta\sqrt{\varepsilon}+v(x,\varepsilon) \geq 0.$$

Thus, according to (2.7), we may omit the second term in (2.23) in order to estimate $\beta(x,\varepsilon)$ from above.

From hypothesis (A_3) and from our smoothness properties it follows the existence of positive constants c_3 and c_4 such that

$$\begin{aligned} &\hat{f}_{uu}(x) \geq c_3 \qquad \text{for } x \in D_{\delta/2},\ \delta \text{ sufficiently small},\\ &|\hat{f}_\varepsilon(x)| \leq c_4 \qquad \text{for } x \in \overline{D}. \end{aligned} \tag{2.24}$$

From (2.23), (2.16), (2.14), (2.24) we obtain

$$L_\varepsilon\beta(x,\varepsilon) \leq (c_2 - \frac{c_3}{2}b_\beta^2 + c_4)\varepsilon + o(\varepsilon).$$

Therefore, for sufficiently large b_β we have $L_\varepsilon\beta(x,\varepsilon) \leq 0$ for $x \in D_{\delta/2}$.

Next, we estimate $L_\varepsilon\beta(x,\varepsilon)$ in $D_\delta^{(2)} := D_\delta \backslash D_{\delta/2}$. From (2.7) it follows that there exists a positive constant c_5 such that

$$\hat{f}_u(x) \geq c_5 > 0 \qquad \text{for } x \in D_\delta^{(2)}. \tag{2.25}$$

By (2.19), (2.22) we have for $x \in D_\delta^{(2)}$

$$\begin{aligned} L_\varepsilon\beta(x,\varepsilon) \equiv\ & \varepsilon^2\Big(\Delta\tilde{u}(x,\varepsilon)+b_\beta\Delta h(x,\varepsilon)\Big) - \hat{f}_u(x)\Big(b_\beta h(x,\varepsilon)+v(x,\varepsilon)\Big)\\ -\ & \frac{1}{2}\hat{f}_{uu}(x)\Big(b_\beta h(x,\varepsilon)+v(x,\varepsilon)\Big)^2 - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \end{aligned} \tag{2.26}$$

Let δ be so small that we have $\hat{f}_{uu}(x) \geq 0$ for $x \in D_\delta$. From (2.26), (2.16), (2.21), (2.25), and (2.24) it follows

$$L_\varepsilon\beta(x,\varepsilon) \leq (c_2 - c_5 b_\beta + c_5 c_1 + c_4)\varepsilon + o(\varepsilon).$$

Therefore, for sufficiently large b_β we have $L_\varepsilon\beta(x,\varepsilon) \leq 0$ for $x \in D_\delta^{(2)}$.

Finally, we estimate $L_\varepsilon\beta(x,\varepsilon)$ in $D\backslash D_\delta$. Taking into account (2.14), (2.18) we have by (2.22)

$$L_\varepsilon\beta(x,\varepsilon) \equiv \varepsilon^2\Big(\Delta\tilde{u}(x,\varepsilon)+\Delta z(x,\varepsilon,k_\beta)\Big) - \hat{f}_u(x)\Big(b_\beta\varepsilon + z(x,\varepsilon,k_\beta)\Big) - \frac{1}{2}\hat{f}_{uu}(x)\Big(b_\beta\varepsilon+z(x,\varepsilon,k_\beta)\Big)^2 - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \tag{2.27}$$

From (2.27), (2.15), (2.21), (2.25), and (2.24) we get

$$L_\varepsilon\beta(x,\varepsilon) \le (\overline{c}_2 - c_5 b_\beta + c_4)\varepsilon + o(\varepsilon).$$

Thus, for sufficiently large b_β we have $L_\varepsilon\beta(x,\varepsilon)\le 0$ for $x\in D\backslash D_\delta$.

Taking into account that $\lambda(x),\varphi_1(x)$ and $\frac{\partial\varphi_1}{\partial x}(x)$ are bounded on $\mathcal{C}$ we get from (2.20),(2.5) and (2.19) for $x\in\Gamma$ and for sufficiently large k_β

$$\frac{\partial\beta}{\partial n}(x,\varepsilon) - \lambda(x)\beta(x,\varepsilon) = \frac{\partial\varphi_1}{\partial n}(x) - k_\beta - \lambda(x)\Big(\varphi_1(x)+b_\beta\varepsilon+\varepsilon\Big) < 0,$$

i.e. the inequality (iv) for the upper solution $\beta(x,\varepsilon)$ in Definition 2 is fulfilled.

Consequently, the function $\beta(x,\varepsilon)$ defined in (2.20) satisfies the conditions (iii) and (iv) in Definition 2 for an upper solution.

Now we construct a lower solution $\alpha(x,\varepsilon)$ in the form

$$\alpha(x,\varepsilon) := \hat{u}(x) - b_\alpha\varepsilon - z(x,\varepsilon,k_\alpha) \tag{2.28}$$

where the positive constants b_α and k_α have to be chosen in an appropriate way. Note that $\alpha(x,\varepsilon)$ may be non-smooth on the curve $\mathcal{C}$, but according to (2.4) it satisfies the condition (ii) in Definition 2.

For $L_\varepsilon\alpha$ we get analogously to (2.22)

$$\begin{aligned} L_\varepsilon\alpha &\equiv \varepsilon^2\Delta\alpha(x,\varepsilon) - f(\alpha(x,\varepsilon),x,\varepsilon) = \\ &= \varepsilon^2\Delta\Big(\hat{u}(x) - z(x,\varepsilon,k_\alpha)\Big) + \hat{f}_u(x)\Big(b_\alpha\varepsilon + z(x,\varepsilon,k_\alpha)\Big) - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \end{aligned} \tag{2.29}$$

First, we consider $L_\varepsilon\alpha$ in the region D_δ for sufficiently small δ. From (2.29), (2.19), (2.7) and under our smoothness assumptions we get

$$L_\varepsilon\alpha(x,\varepsilon) = \varepsilon^2\Delta\hat{u}(x) + \hat{f}_u(x)b_\alpha\varepsilon - \hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon) \ge -\hat{f}_\varepsilon(x)\varepsilon + o(\varepsilon). \tag{2.30}$$

By assumption (A_4) there is a positive constant c_6 such that

$$-\hat{f}_\varepsilon(x) \ge c_6 \quad \text{for } x\in D_\delta. \tag{2.31}$$

Thus, from (2.30) and (2.31) we get

$$L_\varepsilon\alpha(x,\varepsilon) \ge 0 \quad \text{for} \quad x\in D_\delta.$$

Finally, we study $L_\varepsilon \alpha(x,\varepsilon)$ in $D \setminus D_\delta$. By (2.28), (2.21), (2.24), and (2.25) we have

$$L_\varepsilon \alpha \geq (-\bar{c}_2 + c_5 b_\alpha - c_4)\varepsilon + o(\varepsilon).$$

Therefore, for sufficiently large b_α we obtain

$$L_\varepsilon \alpha(x,\varepsilon) \geq 0 \quad \text{for} \quad x \in D \setminus D_\delta.$$

From (2.28), (2.5), and (2.19) we get for $x \in \Gamma$

$$\frac{\partial \alpha}{\partial n}(x,\varepsilon) - \lambda(x)\alpha(x,\varepsilon) = \frac{\partial \varphi_1}{\partial n}(x) + k_\alpha - \lambda(x)\big(\varphi_1(x) - b_\alpha \varepsilon - \varepsilon\big).$$

Thus, for sufficiently large k_α, $\alpha(x,\varepsilon)$ satisfies the inequality (iv) in Definition 2 for a lower solution.

Consequently, the function $\alpha(x,\varepsilon)$ defined in (2.28) satisfies all conditions in Definition 2 for a lower solution.

From (2.20) and (2.28) it follows for sufficiently small ε that $\beta(x,\varepsilon) > \hat{u}(x)$ and $\alpha(x,\varepsilon) < \hat{u}(x)$ in $\overline{D}$. Thus, $\alpha(x,\varepsilon)$ and $\beta(x,\varepsilon)$ are ordered lower and upper solutions to (2.1). Therefore, we can conclude that for sufficiently small ε there exists a solution $u(x,\varepsilon)$ of (2.1) satisfying

$$\alpha(x,\varepsilon) \leq u(x,\varepsilon) \leq \beta(x,\varepsilon) \quad \text{for } x \in \overline{D}.$$

The relations (2.18) – (2.20) and (2.28) show that the relations (2.10) and consequently (2.9) for $u(x,\varepsilon)$ are fulfilled. This completes the proof of Theorem 1.

Remark 1. *In equation* (1.7) *which models a process in reaction kinetics we have* $\hat{f}_\varepsilon(x) \equiv 0$. *That means assumption* (A_4) *is not valid. In such cases we may replace hypotheses* (A_4) *by the following condition:*

(A_5). The composed stable solution $\hat{u}(x)$ of the degenerate equation (2.2) *is a lower solution for* (2.1), *i.e.*

$$\begin{aligned} &(i) \qquad L_\varepsilon \hat{u}(x,\varepsilon) \geq 0 \quad \textit{for } x \in D \setminus \mathcal{C} \quad \textit{and} \quad \varepsilon \in I_{\varepsilon_1}, \\ &(ii) \qquad \frac{\partial \hat{u}}{\partial n}(x) - \lambda(x)\hat{u}(x) \geq 0 \quad \textit{for } x \in \Gamma. \end{aligned}$$

It is easy to verify that under the assumptions (A_0) - (A_3) *and* (A_5) *Theorem 1 remains true.*

Remark 2. *In the subsets $\overline{D}_1 \setminus D_\delta$ and $D_2 \setminus D_\delta$ we can derive an asymptotic expansion of any order in ε for the solution $u(x,\varepsilon)$ by means of standard theory for singularly perturbated problems provided the function f is sufficiently smooth.*
In $\overline{D}_1 \setminus D_\delta$ the asymptotic expansion reads

$$u(x,\varepsilon) = \varphi_1(x) + \varepsilon\overline{u}_1(x) + \ldots + \varepsilon^m\overline{u}_m(x) + \varepsilon\Pi_1\left(\sigma, \frac{n}{\varepsilon}\right) + \ldots + \varepsilon^m\Pi_m\left(\sigma, \frac{n}{\varepsilon}\right) + O(\varepsilon^{m+1}) \tag{2.32}$$

where

$$\begin{aligned}
&\overline{u}_1(x) = -\hat{f}_u^{-1}(x)\hat{f}_\varepsilon(x), \\
&\overline{u}_2(x) = \hat{f}_u^{-1}(x)\left[\Delta\varphi_1(x) - \frac{1}{2}\hat{f}_{\varepsilon\varepsilon}(x) - \hat{f}_{u\varepsilon}(x)\overline{u}_1(x) - \frac{1}{2}\hat{f}_{uu}(x)\overline{u}_1^2(x)\right], \\
&\ldots\ .
\end{aligned} \tag{2.33}$$

$\Pi_i(\sigma, \frac{n}{\varepsilon}), i = 1, 2, \ldots,$ *are boundary layer functions which can be constructed by means of the standard theory and which satisfy*

$$\left|\Pi_i\left(\sigma, \frac{n}{\varepsilon}\right)\right| \le c\exp\left(-\frac{\kappa n}{\varepsilon}\right), \ \ i = 0, 1, \ldots, m, \tag{2.34}$$

where c and κ are some positive constants, σ and n are local coordinates near Γ. In $D_2 \setminus D_\delta$ the asymptotic expansion of $u(x,\varepsilon)$ has the form

$$u(x,\varepsilon) = \varphi_2(x) + \varepsilon\overline{u}_1(x) + \ldots + \varepsilon^m\overline{u}_m(x) + O(\varepsilon^{m+1}). \tag{2.35}$$

Here, the functions $\overline{u}_i(x)$ $(i = 1, \ldots, m)$ are defined as in (2.33) *if we replace there $\varphi_1(x)$ by $\varphi_2(x)$.*

From (2.32) and (2.35) we obtain the following corollary which we need to estimate the jumping behavior of the reaction rates (see subsection 2.3).

Corollary 1. *Under the assumptions of Theorem 1 we have*

$$\Delta u(x,\varepsilon) = \Delta\hat{u}(x) + O(\varepsilon) \quad \text{for } x \in D \setminus (\Gamma_\delta \cup D_\delta). \tag{2.36}$$

Proof. We prove (2.36) for $x \in D_1 \setminus (\Gamma_\delta \cup D_\delta)$. From (2.32) and (2.34) we get for $m = 2$

$$u(x,\varepsilon) = \varphi_1(x) + \varepsilon\overline{u}_1(x) + \varepsilon^2\overline{u}_2(x) + O(\varepsilon^3) \equiv U_2(x,\varepsilon) + O(\varepsilon^3).$$

Consequently,

$$\begin{aligned}
&\Delta(u(x,\varepsilon) - U_2(x,\varepsilon)) = \frac{1}{\varepsilon^2}f(U_2(x,\varepsilon) + O(\varepsilon^3), x, \varepsilon) - \Delta U_2(x,\varepsilon) \\
&= \{f(U_2(x,\varepsilon) + O(\varepsilon^3), x, \varepsilon) - f(U_2(x,\varepsilon), x, \varepsilon) + f(U_2(x,\varepsilon), x, \varepsilon) - \varepsilon^2\Delta U_2(x,\varepsilon)\}/\varepsilon^2.
\end{aligned} \tag{2.37}$$

Obviously we have

$$f(U_2(x,\varepsilon)+O(\varepsilon^3),x,\varepsilon)-f(U_2(x,\varepsilon),x,\varepsilon)=O(\varepsilon^3).$$

By means of (2.33) we get

$$f(U_2(x,\varepsilon),x,\varepsilon)-\varepsilon^2\Delta U_2(x,\varepsilon)=O(\varepsilon^3).$$

Therefore, we obtain from (2.37)

$$\Delta(u(x,\varepsilon)-U_2(x,\varepsilon))=O(\varepsilon).$$

By using the obvious relation

$$\Delta U_2(x,\varepsilon)=\Delta\varphi_1(x)+O(\varepsilon)$$

we get $\Delta u(x,\varepsilon)=\Delta\varphi_1(x)+O(\varepsilon)$, i.e. the relation (2.36) holds for $x\in D_1\setminus(\Gamma_\delta\cup D_\delta)$. For $x\in D_2\setminus D_\delta$, relation (2.36) can be proved in a similar way.

2.3 Application: The purely bimolecular reaction.

We consider system (1.4) in case $g_1\equiv g_2\equiv 0$ (pure bimolecular reactions) in a bounded open simply connected region D of R^2 with a smooth boundary Γ and assume

$$r(\bar u,\bar v)\equiv k\bar u\bar v$$

where k is a positive constant. Hence, (1.8) reads

$$\varepsilon^2\Delta u = -\varepsilon^2 I_a(x)+ku(u-v(x)),\qquad x\in D \tag{2.38}$$

where $v(x)$ is the solution of the equation

$$\Delta v = I_b(x)-I_a(x) \tag{2.39}$$

(see (1.6)) with corresponding boundary conditions. Concerning $v(x)$ we suppose

$$\begin{aligned} v(x)=0 &\quad \text{for}\quad x\in\mathcal{C},\\ v(x)<0 &\quad \text{for}\quad x\in D_1,\\ v(x)>0 &\quad \text{for}\quad x\in D_2. \end{aligned}$$

As an example we consider the equation

$$\Delta v \;=\; I_b(x) - I_a(x)$$

in the region $D := \{x \in R^2 : x_1^2 + x_2^2 < 4\}$ under the condition $I_b(x) - I_a(x) = 4$ together with the boundary condition

$$\frac{\partial v}{\partial n}(x) - v(x) = 7 \qquad \text{for} \quad x_1^2 + x_2^2 = 4$$

where $\frac{\partial}{\partial n}$ denotes the differentiation in the direction of the inner normal. It easy to verify that $v(x) \equiv 1 - x_1^2 - x_2^2$ solves this boundary value problem. We note that $v(x)$ changes its sign on the circle $\mathcal{C} := x \in R^2 : x_1^2 + x_2^2 = 1$.

Additionally we assume the boundary condition

$$\frac{\partial u}{\partial n} - \lambda(x)u \;=\; 0 \quad \text{for } x \in \Gamma. \tag{2.40}$$

One can easily check that the assumptions (A_1) and (A_2) are fufilled for equation (2.38).

The corresponding composed stable solution reads

$$\hat{u}(x) = \begin{cases} 0 & \text{for} \quad x \in \overline{D}_1, \\ v(x) & \text{for} \quad x \in D_2 \end{cases} \tag{2.41}$$

(see Definition 1). Recalling that $I_a(x)$ and $I_b(x)$ are non-negative functions it is easy to check that $\hat{u}(x)$ is a lower solution to (2.38), (2.40). Indeed, using (2.39) we have

$$\varepsilon^2 \Delta \hat{u} + \varepsilon^2 I_a(x) - k\hat{u}(\hat{u} - v(x)) = \begin{cases} \varepsilon^2 I_a(x) \geq 0 & \text{in } D_1, \\ \varepsilon^2 I_b(x) \geq 0 & \text{in } D_2, \end{cases}$$

and

$$\frac{\partial \hat{u}}{\partial n} - \lambda(x)\hat{u} = 0 \quad \text{for} \quad x \in \Gamma,$$

i.e. assumption (A_5) is satisfied.
Assumption (A_3) also holds as $\hat{f}_{uu}(x) = 2k > 0$. Therefore, by means of Theorem 1 (see Remark 1) we obtain that the boundary value problem (2.38), (2.40) has a solution $u(x, \varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0} u(x, \varepsilon) = \hat{u}(x) \quad \text{for } x \in \overline{D}.$$

For the reaction rate $\tilde{r}(x, \varepsilon) := r(u(x, \varepsilon), u(x, \varepsilon) - v(x))/\varepsilon^2$ we get from (1.8)

$$\tilde{r}(x, \varepsilon) = \Delta u(x, \varepsilon) + I_a(x). \tag{2.42}$$

From (2.42),(2.36),(2.41), and (2.39) we obtain

$$\tilde{r}(x,\varepsilon) = \begin{cases} I_a(x) + O(\varepsilon) & \text{for} \quad x \in D_1 \backslash (\Gamma_\delta \cup D_\delta), \\ I_b(x) + O(\varepsilon) & \text{for} \quad x \in D_2 \backslash D_\delta. \end{cases}$$

Thus, taking into account that δ is any small number we conclude that the reaction rate $\tilde{r}(x,\varepsilon)$ has a jump (transition layer) near the curve $\mathcal{C}$ characterizing the exchange of stabilities.

3 The nonstationary problem

3.1 Existence and asymptotic behavior of the solution.

Let $D := \{(x,t) \in R^2 : 0 < x < 1, 0 < t \le T\}, I_{\varepsilon_1} := \{\varepsilon \in R : 0 < \varepsilon \le \varepsilon_1\}$ where $0 < \varepsilon_1 \ll 1$. We consider the the singularly perturbed initial-boundary value problem

$$L_\varepsilon u \equiv \varepsilon^2(u_t - u_{xx}) - f(u,x,t,\varepsilon) = 0, \quad (x,t) \in D, \tag{3.1}$$

$$u(x,0) = u^0(x)\,, \tag{3.2}$$

$$u_x(0,t) = u_x(1,t) = 0\,, \tag{3.3}$$

under the following assumptions:

(V_0) $f : R \times \bar{D} \times \bar{I}_{\varepsilon_1} \to R$ and $u^0 : [0,1] \to R$ are sufficiently smooth.

(V_1) The degenerate equation $f(u,x,t,0) = 0$ has two smooth roots with respect to u in $\bar{D}$

$$u = \varphi_1(x,t) \text{ and } u = \varphi_2(x,t)\,.$$

There exists a smooth function $\psi : [0,1] \to [\nu, T-\nu]$ where ν satisfies $0 < \nu < T$ such that

$$\varphi_1(x,\psi(x)) \equiv \varphi_2(x,\psi(x)) \quad \text{for} \quad 0 \le x \le 1\,, \tag{3.4}$$

$$\begin{aligned} \varphi_1(x,t) > \varphi_2(x,t) \quad &\text{for} \quad 0 \le t < \psi(x)\,, \\ \varphi_1(x,t) < \varphi_2(x,t) \quad &\text{for} \quad \psi(x) < t \le T\,. \end{aligned} \tag{3.5}$$

The relation (3.4) says that the surfaces $u = \varphi_1(x,t)$ and $u = \varphi_2(x,t)$ intersect in a curve whose projection into $\bar{D}$ has the representation $t = \psi(x)$.

(V_2) For $0 \le x \le 1$ it holds

$$f_u(\varphi_1(x,t),x,t,0) \begin{cases} < 0 & \text{for} \quad 0 \le t < \psi(x)\,, \\ > 0 & \text{for} \quad \psi(x) < t \le T\,, \end{cases}$$

$$f_u(\varphi_2(x,t),x,t,0) \begin{cases} > 0 & \text{for} \quad 0 \le t < \psi(x)\,, \\ < 0 & \text{for} \quad \psi(x) < t \le T\,. \end{cases}$$

We note that from (3.4) it follows

$$f_u(\varphi_i(x,t),x,t,0)|_{t=\psi(x)} \equiv 0\,. \tag{3.6}$$

Under assumption (V_2) the family of equilibria $v=\varphi_1(x,t)$ $\big(v=\varphi_2(x,t)\big)$ of the associated equation

$$\frac{dv}{d\tau} = f(v,x,t,0)\,, \quad \tau \geq 0\,, \tag{3.7}$$

where x and t are considered as parameters is asymptotically stable (unstable) for $0 \leq t < \psi(x)$ and unstable (asymtotically stable) for $\psi(x) < t \leq T$. Thus, on the curve $t=\psi(x)$, the exchange of stabilities of the families of equilibria takes place.

A simple example of a function $f(u,x,t,0)$ satisfying the assumption (V_1) and (V_2) is given by the quadratic function with respect to u

$$f(u,x,t,0) = -(u-\varphi_1(x,t)\,)(u-\varphi_2(x,t)\,)\,, \tag{3.8}$$

if φ_1 and φ_2 satisfy the conditions (3.4) and (3.5).

(V_3) *The initial function $u^0(x)$ belongs to the basin of attraction of the rest point $v=\varphi_1(x,0)$ of the associated equation (3.7) for $t=0$.*

Assumption (V_3) means that the solution $\Pi(x,\tau)$ of the initial problem (x is considered as parameter)

$$\frac{d\Pi}{d\tau} = f(\varphi_1(x,0)+\Pi,x,0,0)\,, \quad \tau \geq 0\,; \quad \Pi(x,0) = u^0(x) - \varphi_1(x,0) \tag{3.9}$$

exists for $\tau \geq 0$ and $\Pi(x,\tau) \to 0$ as $\tau \to \infty$.

By assumption (V_3), for small ε the solution $u(x,t,\varepsilon)$ of the problem (3.1), (3.2) has an exponentially fast change from the initial value $u^0(x)$ to values close to $\varphi_1(x,t)$ within a small time interval. After that the solution $u(x,t,\varepsilon)$ will be close to $\varphi_1(x,t)$ as long as the root $\varphi_1(x,t)$ will be stable. But for $t=\psi(x)$ the exchange of stability of the roots φ_1 and φ_2 takes place. The question arises about behavior of the solution $u(x,t,\varepsilon)$ near the curve $t=\psi(x)$ and for $\psi(x) < t \leq T$.

Form the composed stable solution of the degenerate equation

$$\hat{u}(x,t) = \begin{cases} \varphi_1(x,t)\,, & 0 \leq t \leq \psi(x)\,, \\ \varphi_2(x,t)\,, & \psi(x) \leq t \leq T\,, \end{cases} \quad 0 \leq x \leq 1\,.$$

We note that $\hat{u}(x,t)$ is a continuous function in $\bar{D}$, but not smooth on the curve $t=\psi(x)$.

We introduce the notation

$$\hat{f}_{uu}(x,t) \equiv f_{uu}(\hat{u}(x,t),x,t,0)\,, \quad \hat{f}_\varepsilon(x,t) \equiv f_\varepsilon(\hat{u}(x,t),x,t,0)$$

and assume:

(V$_4$) $\hat{f}_{uu}(x, \psi(x)) < 0$ *for* $0 \le x \le 1$.

Note that for the quadratic function (3.8) $f_{uu} = -2$, i. e. assumption (A$_4$) holds. Further we assume

(V$_5$) $\hat{f}_\varepsilon(x, \psi(x)) > 0$.

Theorem 2. *Under assumptions (V_0) – (V_5) and for sufficiently small* ε *, the initial-boundary value problem* (3.1) – (3.3) *has a solution* $u(x,t,\varepsilon)$ *satisfying*

$$u(x,t,\varepsilon) = \hat{u}(x,t) + \Pi(x, t/\varepsilon^2) + w(x,t,\varepsilon) \quad \text{in} \quad \bar{D}\,, \tag{3.10}$$

where $\Pi(x,\tau)$ *is defined by* (3.9) *,* $w(x,t,\varepsilon) = O(\varepsilon^{1/2})$ *in some small (but fixed as* $\varepsilon \to 0$ *)* δ *– neighborhood* D_δ *of the curve* $t = \psi(x)$*, and* $w(x,t,\varepsilon) = O(\varepsilon)$ *for* $(x,t) \in \overline{D \setminus D_\delta}$.

The proof of this theorem can be found in [3].

3.2 Application: The nonstationary purely bimolecular reaction.

We consider system (1.3) under the assumptions: dim $x = 1, g_1 \equiv g_2 \equiv 0$ (pure bimolecular reaction), the inputs I_a and I_b depend on (x,t), $r(\bar{u},\bar{v}) \equiv k\bar{u}\bar{v}$ where k is a positive constant. In that case system (1.5) can be rewritten as

$$\begin{aligned} \varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\Big) &= -\varepsilon^2 I_a(x,t) + ku(u-v), \\ \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} &= I_b(x,t) - I_a(x,t), \qquad (x,t) \in D. \end{aligned} \tag{3.11}$$

Additionally we suppose that the corresponding initial and boundary conditions are such that the solution $v(x,t)$ of the second equation in (3.11) satisfies

$$\begin{aligned} v(x,t) = 0 &\quad \text{for} \quad t = \psi(x),\ 0 \le x \le 1, \\ v(x,t) < 0 &\quad \text{for} \quad 0 \le t < \psi(x), \\ v(x,t) > 0 &\quad \text{for} \quad \psi(x) < t \le T \end{aligned}$$

where $t = \psi(x)$ is a smooth curve having the properties as described in subsection 3.1. Hence, we have to solve an initial-boundary value problem for the equation

$$\varepsilon^2\Big(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t}\Big) = -\varepsilon^2 I_a(x,t) + ku\big(u - v(x,t)\big). \tag{3.12}$$

In this case the composed stable solution reads

$$\hat{u}(x,t)=\begin{cases}0\,, & 0\le t\le\psi(x)\,,\\ v(x,t)\,, & \psi(x)\le t\le T\,,\end{cases}\quad 0\le x\le 1\,. \tag{3.13}$$

Note that assumption (V_5) is not valid for the case under consideration. But Theorem 2 can be extended such that it can be applied to (3.12) (similar to the extension of Theorem 1). Therefore by the extended Theorem 2 we obtain that the initial-boundary value problem under consideration has a solution $u(x,t,\varepsilon)$ satisfying

$$\lim_{\varepsilon\to 0} u(x,t,\varepsilon)=\hat{u}(x,t)\quad \text{for } 0\le x\le 1,\quad 0<t\le T.$$

As in Corollary 1 we can prove that

$$\frac{\partial^2 u}{\partial x^2}-u_t=\frac{\partial^2\hat{u}}{\partial x^2}-\hat{u}_t+O(\varepsilon)\qquad\text{for}\quad (x,t)\in D\setminus(\Gamma_\delta\cup D_\delta) \tag{3.14}$$

where D_δ is a sufficiently small δ - neighborhood of the curve $t=\psi(x)$, Γ_δ denotes the subset of D defined by $0\le t\le\delta$ where δ is a sufficiently small positive number (see Fig. 2).

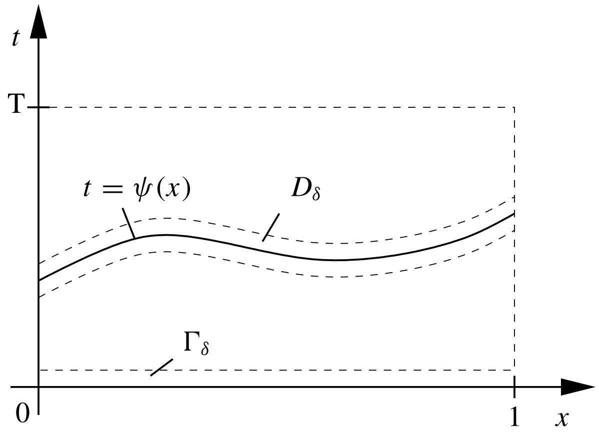

Fig. 2: Location of the subsets D_δ and Γ_δ of D

For the reaction rate $\tilde{r}(x,t,\varepsilon):=r(u(x,t,\varepsilon),u(x,t,\varepsilon)-v(x,t))/\varepsilon^2$ we have by (3.12)

$$\tilde{r}(x,t,\varepsilon)=ku(x,t,\varepsilon)(u(x,t,\varepsilon)-v(x,t))/\varepsilon^2=\ \tfrac{\partial^2 u}{\partial x^2}-\tfrac{\partial u}{\partial t}+I_a(x,t). \tag{3.15}$$

Now, by using (3.14),(3.13) and the second equation in (3.11) we obtain from (3.15)

$$\tilde{r}(x,t,\varepsilon)=\begin{cases} I_a(x,t)+O(\varepsilon) & \text{for}\quad (x,t)\in D_1\setminus(\Gamma_\delta\cup D_\delta),\\ I_b(x,t)+O(\varepsilon) & \text{for}\quad (x,t)\in D_2\setminus D_\delta.\end{cases}$$

Since δ is any small positive number we can conclude that the reaction rate $\tilde{r}(x,t,\varepsilon)$ has a jump (transition layer) near the curve $t=\psi(x)$ describing the exchange of stabilities.

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