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Some intriguing definite integrals

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Dedicated to Professor Dr. Herbert Gajewski
on the occasion of his sixtieth birthday

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Abstract

We calculate some definite integrals which (up to now) computer algebra systems like Maple or Mathematica are unable to evaluate. The first one is a simply looking integral involving *cos* and *log* , the others are some integrals containing polylogarithmic functions. It is shown that they can be evaluated by rational combinations of ζ–functions and products of ζ–functions at positive integers.

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1. Introduction

To justify our doing, we quote J.J. SYLVESTER (1814–1897):

> *"It seems to be expected of every pilgrim up the slopes of the mathematical Parnassus, that he will at some point or other of his journey sit down and invent a definite integral or two towards the increase of the common stock."* ([S])

The first integral we consider is

$$I(a) = \int_0^\pi (1+\cos x)\log(a+\cos x)dx, \quad a \geq 1.$$

We show that

$$I(a) = \pi\left\{a - \sqrt{a^2-1} + \log\left(\frac{a+\sqrt{a^2-1}}{2}\right)\right\}. \tag{1.1}$$

The other integrals contain polylogarithmic functions. The polylogarithmic function $\mathcal{L}_p$ is defined for any complex p and any complex $|z| < 1$ by the power series

$$\mathcal{L}_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p}. \tag{1.2}$$

We show that

$$I_m = \int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds, \quad J_m = \int_{-1}^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds$$

can be expressed for $m = 1, 2, ...$ in terms of ζ–functions at positive integers by

$$I_m = (-1)^{m-1}\left\{(m+1)\zeta(2m+1) - 2\sum_{k=1}^{[m/2]} \zeta(2k)\zeta(2m+1-2k)\right\} \tag{1.3}$$

and

$$\begin{aligned} J_m = \quad & (-1)^{m-1}\Big\{(2-2^{-2m})\zeta(2m+1) \\ & - 2^{2-2m}\sum_{k=1}^{[m/2]}[2^{2p-1}+2^{2m-2p}-1]\zeta(2k)\zeta(2m+1-2k)\Big\}. \end{aligned} \tag{1.4}$$

2. The proof of the identity (1.1)

Already Euler knew that

$$\int_0^\pi \log\sin x\,dx = 2\int_0^{\pi/2} \log\sin x\,dx = -\pi\log 2.$$

Consider

$$A = \int_0^{\pi/2} \cos^2 x \, \log \sin x \, dx, \quad B = \int_0^{\pi/2} \sin^2 x \, \log \sin x \, dx.$$

The preceding line shows that

$$A + B = \int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2,$$

and obviously holds

$$A - B = \int_0^{\pi/2} \cos 2x \, \log \sin x \, dx.$$

Integration by parts

$$\int_0^{\pi/2} f'(x) g(x) \, dx = f(x) g(x) \Big|_0^{\pi/2} - \int_0^{\pi/2} f(x) g'(x) \, dx$$

with

$$f' = \cos 2x, \quad g = \log \sin x, \quad f = \frac{1}{2} \sin 2x, \quad g' = \frac{\cos x}{\sin x}$$

gives

$$\begin{aligned} \int_0^{\pi/2} \cos 2x \, \log \sin x \, dx &= \frac{1}{2} \sin 2x \, \log \sin x \Big|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \sin 2x \, \frac{\cos x}{\sin x} \, dx \\ &= -\int_0^{\pi/2} \cos^2 x \, dx = -\frac{\pi}{4}, \end{aligned}$$

and, consequently,

$$A = -\frac{\pi}{8} (1 + \log 4), \quad B = \frac{\pi}{8} (1 - \log 4).$$

Next consider

$$C = \int_0^{\pi} (1 + \cos x) \, \log(1 + \cos x) \, dx.$$

Using $1 + \cos x = 2 \cos^2 \frac{x}{2}$ and some obvious substitutions we get

$$\begin{aligned} C &= 2 \int_0^{\pi} \cos^2 \frac{x}{2} \log \left(2 \cos^2 \frac{x}{2} \right) \, dx = 4 \int_0^{\pi/2} \cos^2 z \, \log \left(2 \cos^2 z \right) dz \\ &= 4 \int_0^{\pi/2} \sin^2 y \, \log \left(2 \sin^2 y \right) dy = 4 \int_0^{\pi/2} \sin^2 y \, [\log 2 + 2 \log \sin y] dy \\ &= 4 \log 2 \int_0^{\pi/2} \sin^2 y \, dy + 8 \int_0^{\pi/2} \sin^2 y \, \log \sin y \, dy = \pi \log 2 + 8B, \end{aligned}$$

hence

$$C = \pi(1 - \log 2).$$

This result is known to Maple and Mathematica. To evaluate

$$I(a) = \int_0^\pi (1 + \cos x) \log(a + \cos x) dx, \quad a \geq 1,$$

differentiate with respect to a and obtain

$$I'(a) = \int_0^\pi \frac{1 + \cos x}{a + \cos x} dx = \pi - (a-1) \int_0^\pi \frac{dx}{a + \cos x}.$$

The classical substitution $t = \tan(x/2)$ (or computer algebra, or a classical table of integrals like [RG]) shows that

$$\int_0^\pi \frac{dx}{a + \cos x} = \frac{\pi}{\sqrt{a^2 - 1}}.$$

So we have

$$I'(a) = \pi - \pi \sqrt{\frac{a-1}{a+1}}.$$

Integration yields

$$I(a) - I(1) = \int_1^a I'(s)\, ds = \pi(a-1) - \pi \int_1^a \sqrt{\frac{s-1}{s+1}}\, ds.$$

The remark that $I(1) = C$ and the elementary integral

$$\int_1^a \sqrt{\frac{s-1}{s+1}}\, ds = \sqrt{a^2-1} - \log(a + \sqrt{a^2 - 1})$$

(simply checked by differentiating both sides) proves (1.1).

3. The proof of the identities (1.3), (1.4)

A standard reference for the properties of polylogarithmic functions is the book of L. Lewin ([L]). According to A.B. Goncharov ([G]) the history of these functions can be traced back to Leibniz and J.Bernoulli. In the last time there seems to be an growing interest in these functions ([G],[M],[Z]). For the index $m = 1$ obviously holds

$$\mathcal{L}_1(z) = -\log(1 - z).$$

The computer algebra systems Maple and Mathematica know that

$$I_1 = \int_0^1 \frac{(\mathcal{L}_1(s))^2}{s} ds = \int_0^1 \frac{(\log(1-s))^2}{s} ds = 2\zeta(3)$$

and

$$J_1 = \int_{-1}^{1} \frac{(\mathcal{L}_1(s))^2}{s}\, ds = \int_{-1}^{1} \frac{(\log(1-s))^2}{s}\, ds = \frac{7}{4}\zeta(3).$$

The corresponding integrals for polylogarithmic functions of higher indices are unknown to these computer algebra systems. We evaluate these integrals using formulas proved in [BBG] (some of them go back to Euler [E]).

For $Re(p) > 1$ the Riemann zeta function is defined by

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

Since the series (1.2) for $Re(p) > 1$ converges still on $|z| = 1$ we have

$$\mathcal{L}_p(1) = \zeta(p), \quad \mathcal{L}_p(-1) = (2^{1-p} - 1)\zeta(p) \ \text{ for } \ Re(p) > 1.$$

Moreover, $\mathcal{L}_1(-1) = -\log 2$ and differentiation of (1.2) with respect to z yields the well–known relation

$$\mathcal{L}_p'(z) = \frac{\mathcal{L}_{p-1}(z)}{z}.$$

For positive integers $l \geq 2$ define

$$S_l(z) = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} \frac{z^k}{k} \right), \quad A_l = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} \frac{1}{k} \right), \quad B_l = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} \frac{(-1)^k}{k} \right).$$

Obviously $A_l = S_l(1)$, $B_l = S_l(-1)$, and differentiation gives for $|z| < 1$

$$S_l'(z) = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\sum_{k=1}^{n} z^{k-1} \right) \quad \text{and with} \quad \sum_{k=1}^{n} z^{k-1} = \frac{1-z^n}{1-z}$$

follows

$$S_l'(z) = \sum_{n=1}^{\infty} \frac{1}{n^l} \left(\frac{1-z^n}{1-z} \right) = \frac{1}{(1-z)} \sum_{n=1}^{\infty} \frac{1}{n^l} - \frac{1}{(1-z)} \sum_{n=1}^{\infty} \frac{z^n}{n^l}$$

or

$$S_l'(z) = \frac{\zeta(l)}{(1-z)} - \frac{\mathcal{L}_l(z)}{(1-z)}.$$

By integration over $[0, x]$ for real $x = z$, $|x| < 1$ we obtain with $S_l(0) = 0$

$$S_l(x) = -\zeta(l)\log(1-x) - \int_0^x \frac{\mathcal{L}_l(s)}{(1-s)}\, ds.$$

Correspondingly, integration over $[-1, x]$ gives with $S_l(-1) = B_l$

$$S_l(x) = B_l - \zeta(l)\{\log(1-x) - \log 2\} - \int_{-1}^x \frac{\mathcal{L}_l(s)}{(1-s)}\, ds.$$

Integration by parts

$$\int_0^x f'(s)g(s)ds = f(s)g(s)|_0^x - \int_0^x f(s)g'(s)ds$$

on the right hand side with

$$f'(s) = \frac{1}{1-s}, \quad g(s) = \mathcal{L}_l(s), \quad f(s) = -\log(1-s), \quad g'(s) = \mathcal{L}_l'(s) = \frac{\mathcal{L}_{l-1}(s)}{s}$$

shows that

$$S_l(x) = \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) - \int_0^x \frac{\mathcal{L}_{l-1}(s)\log(1-s)}{s}\,ds$$

or

$$S_l(x) = \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) + \int_0^x \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds.$$

The same manipulation on the interval $[-1, x]$ gives

$$S_l(x) = B_l + \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) + \log 2\{\zeta(l) - \mathcal{L}_l(-1)\} + \int_{-1}^x \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds.$$

For $l \geq 2$, we have

$$\lim_{x\to 1-} \{\mathcal{L}_l(x) - \zeta(l)\}\log(1-x) = 0.$$

This can be seen using l'Hopital's rule

$$\lim_{x\to 1-} \frac{[\mathcal{L}_l(x) - \zeta(l)]}{1/\log(1-x)} = \lim_{x\to 1-} (1-x)\,(\log(1-x))^2 \mathcal{L}_l'(x) = 0.$$

So we obtain

$$A_l = \int_0^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds \tag{3.1}$$

and

$$A_l = B_l + \log 2\{\zeta(l) - \mathcal{L}_l(-1)\} + \int_{-1}^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds \tag{3.2}$$

Repeating integration by parts for $l \geq 3$

$$\int_0^1 f'(s)g(s)ds = f(s)g(s)|_0^1 - \int_0^1 f(s)g'(s)ds$$

with

$$f' = \frac{\mathcal{L}_1(s)}{s}, \quad g = \mathcal{L}_{l-1}(s), \quad f = \mathcal{L}_2(s) \quad g' = \mathcal{L}_{l-1}'(s) = \frac{\mathcal{L}_{l-2}(s)}{s}$$

gives

$$\begin{aligned}\int_0^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds &= \mathcal{L}_2(s)\mathcal{L}_{l-1}(s)|_0^1 - \int_0^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds \\ &= \zeta(2)\,\zeta(l-1) - \int_0^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds.\end{aligned}$$

In the same way we obtain by partial integration on the interval $[-1, 1]$

$$\int_{-1}^1 \frac{\mathcal{L}_{l-1}(s)\mathcal{L}_1(s)}{s}\,ds = \zeta(2)\,\zeta(l-1) - \mathcal{L}_2(-1)\mathcal{L}_{l-2}(-1) - \int_{-1}^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds$$

We use this in (3.1) and find

$$A_l = \zeta(2)\,\zeta(l-1) - \int_0^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds,$$

With (3.2) we get, respectively,

$$\begin{aligned} A_l = B_l \quad &+ \quad \log 2\{\zeta(l) - \mathcal{L}_l(-1)\} \\ &+ \quad \zeta(2)\,\zeta(l-1) - \mathcal{L}_{l-1}(-1)\mathcal{L}_2(-1) - \int_{-1}^1 \frac{\mathcal{L}_2(s)\mathcal{L}_{l-2}(s)}{s}\,ds. \end{aligned}$$

Going on this way one shows by induction that for $j = 1, ..., [l/2]$

$$A_l = \sum_{k=1}^{j-1}(-1)^{k-1}\zeta(k+1)\zeta(l-k) + (-1)^{j-1}\int_0^1 \frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds, \tag{3.3}$$

and, especially for $l = 2m,\ j = m$:

$$A_{2m} = \sum_{k=1}^{m-1}(-1)^{k-1}\zeta(k+1)\zeta(2m-k) + (-1)^{m-1}\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds. \tag{3.4}$$

Analogously we obtain for the interval $[-1, 1]$

$$\begin{aligned} A_l = \quad & B_l + \log 2\{\zeta(l) - \mathcal{L}_l(-1)\} \\ & + \sum_{k=1}^{j-1}(-1)^{k-1}\zeta(k+1)\zeta(l-k) \\ & - \sum_{k=1}^{j-1}(-1)^{k-1}\mathcal{L}_{k+1}(-1)\mathcal{L}_{l-k}(-1) + (-1)^{j-1}\int_{-1}^1 \frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds \end{aligned}$$

and, especially for $l = 2m,\ j = m$:

$$\begin{aligned} A_{2m} = \quad & B_{2m} + \log 2\{\zeta(2m) - \mathcal{L}_{2m}(-1)\} \\ & + \sum_{k=1}^{m-1}(-1)^{k-1}\zeta(k+1)\zeta(2m-k) \\ & - \sum_{k=1}^{m-1}(-1)^{k-1}\mathcal{L}_{k+1}(-1)\mathcal{L}_{2m-k}(-1) + (-1)^{m-1}\int_{-1}^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds \end{aligned}$$

The quantities $A_l,\ B_l$ are related to the Euler sums

$$\sigma_h(s,t) = \sum_{n=1}^{\infty}\frac{1}{n^t}\left(\sum_{k=1}^{n-1}\frac{1}{k^s}\right), \quad s = 1, 2, ..., \quad t = 2, 3, ...,$$

and

$$\sigma_a(s,t) = \sum_{n=1}^{\infty}\frac{1}{n^t}\left(\sum_{k=1}^{n-1}\frac{(-1)^{k-1}}{k^s}\right), \quad s = 1, 2, ..., \quad t = 2, 3, ...,$$

considered in [BBG]. Indeed, we have

$$A_l = \sigma_h(1,l) + \zeta(l+1), \quad B_l = -\sigma_a(1,l) + \mathcal{L}_{l+1}(-1). \tag{3.5}$$

We quote from [BBG] (p.278) (also proved in [N])

$$2\sigma_h(1,l) = l\zeta(l+1) - \sum_{k=1}^{l-2} \zeta(k+1)\,\zeta(l-k)$$

and (p.290)

$$2\sigma_a(1,l) = 2\eta(1)\,\zeta(l) - l\,\zeta(l+1) + \sum_{k=1}^{l} \eta(k)\,\eta(l+1-k),$$

where

$$\eta(p) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = (1-2^{1-p})\,\zeta(p) \quad \text{for } Re(p) > 1, \quad \eta(1) = \log 2.$$

With (3.5) we find

$$A_l = \left(\frac{l}{2}+1\right)\zeta(l+1) - \frac{1}{2}\sum_{k=1}^{l-2} \zeta(k+1)\,\zeta(l-k), \tag{3.6}$$

and, using $\mathcal{L}_l(-1) = -\eta(l)$,

$$B_l = -\zeta(l)\log 2 + \frac{l}{2}\,\zeta(l+1) - \frac{1}{2}\sum_{k=1}^{l} \eta(k)\,\eta(l+1-k) - \eta(l+1).$$

Especially, after some transformation,

$$A_{2m} = (m+1)\zeta(2m+1) - \sum_{k=1}^{m-1} \zeta(k+1)\,\zeta(2m-k),$$

$$B_{2m} = -\{\zeta(2m) + \eta(2m)\}\log 2 + m\zeta(2m+1) - \sum_{k=1}^{m-1} \eta(k+1)\,\eta(2m-k) - \eta(2m+1).$$

We use the expression for A_{2m} in (3.4) and find

$$(-1)^{m-1}\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds = (m+1)\zeta(2m+1) - \sum_{k=1}^{m-1}[1+(-1)^{k-1}]\zeta(k+1)\,\zeta(2m-k).$$

In the sum the terms with k even cancel and we obtain, changing the sense of the summation index k ,

$$(-1)^{m-1}\int_0^1 \frac{(\mathcal{L}_m(s))^2}{s}\,ds = (m+1)\zeta(2m+1) - 2\sum_{k=1}^{[m/2]} \zeta(2k)\,\zeta(2m+1-2k).$$

So the identity (1.3) for I_m is proved.

To prove (1.4), we use the expressions for A_{2m}, B_{2m} in the identity containing J_m . Several terms cancel, and we obtain after some transformations the intermediate result

$$(-1)^{m-1}\int_{-1}^{1}\frac{(\mathcal{L}_m(s))^2}{s}\,ds = (2-2^{-2m})\,\zeta(2m+1) - \sum_{k=1}^{m-1}[1+(-1)^{k-1}]\zeta(k+1)\,\zeta(2m-k) + \sum_{k=1}^{m-1}[1+(-1)^{k-1}]\eta(k+1)\,\eta(2m-k).$$

Again the terms with k even cancel and with $\eta(j) = (1-2^{1-j})\zeta(j)$ this identity simplifies to

$$(-1)^{m-1}\int_{-1}^{1}\frac{(\mathcal{L}_m(s))^2}{s}\,ds = (2-2^{-2m})\,\zeta(2m+1) - 2^{2-2m}\sum_{k=1}^{[m/2]}\{2^{2p-1}+2^{2m-2p}-1\}\zeta(2k)\,\zeta(2m+1-2k).$$

This is identity (1.4) for J_m .

In the same way comparison of (3.6) and (3.3) yields the "mixed" integrals

$$I_{j,l} = \int_{0}^{1}\frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds, \quad l\geq 2, \quad j=1,...[l/2]$$

as

$$I_{j,l} = (-1)^{j+1}\Bigg\{\left(\frac{l}{2}+1\right)\zeta(l+1) - \sum_{k=1}^{j-1}\left((-1)^{k-1}+\frac{1}{2}\right)\zeta(k+1)\,\zeta(l-k) - \frac{1}{2}\sum_{k=j}^{l-2}\zeta(k+1)\,\zeta(l-k)\Bigg\}.$$

We could also give formulas for the corresponding integrals

$$J_{j,l} = \int_{-1}^{1}\frac{\mathcal{L}_j(s)\mathcal{L}_{l-j}(s)}{s}\,ds, \quad l\geq 2, \quad j=1,...[l/2]$$

over the interval $[-1,1]$. One gets the somewhat clumsy expression

$$J_{j,l} = (-1)^{j+1}\Bigg\{(2-2^{-l})\zeta(l+1) - \sum_{k=1}^{j-1}\left((-1)^{k-1}+\frac{1}{2}\right)\{2^{1-l+k}+2^{-k}-2^{1-l}\}\,\zeta(k+1)\,\zeta(l-k) - \frac{1}{2}\sum_{k=j}^{l-2}\{2^{1-l+k}+2^{-k}-2^{1-l}\}\zeta(k+1)\,\zeta(l-k)\Bigg\}.$$

It is obvious that the identities for $m=1$ mentioned in the introduction are included.

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