On the volume of the supercritical super-Brownian sausage conditioned on survival

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Abstract

Let α and β be positive constants. Let X be the supercritical super-Brownian motion corresponding to the evolution equation $u_t = \frac{1}{2}\Delta + \beta u - \alpha u^2$ in $\mathbb{R}^d$ and let Z be the binary branching Brownian-motion with branching rate β. For $t \geq 0$, let $R(t) = \overline{\bigcup_{s=0}^{t} \mathrm{supp}(X(s))}$, that is $R(t)$ is the (accumulated) support of X up to time t. For $t \geq 0$ and $a > 0$, let $R^a(t) = \bigcup_{x \in R(t)} \bar{B}(x,a)$. We call $R^a(t)$ the *super-Brownian sausage* corresponding to the supercritical super-Brownian motion X. For $t \geq 0$, let $\hat{R}(t) = \overline{\bigcup_{s=0}^{t} \mathrm{supp}(Z(s))}$, that is $\hat{R}(t)$ is the (accumulated) support of Z up to time t. For $t \geq 0$ and $a > 0$, let $\hat{R}^a(t) = \bigcup_{x \in R(t)} \bar{B}(x,a)$. We call $\hat{R}^a(t)$ the *branching Brownian sausage* corresponding to Z. In this paper we prove that

$$\lim_{t\to\infty} \frac{1}{t} \log E_{\delta_0}[\exp(-\nu|R^a(t)|) | X \text{ survives}] = \lim_{t\to\infty} \frac{1}{t} \log \hat{E}_{\delta_0} \exp(-\nu|\hat{R}^a(t)|) = -\beta,$$

for all $d \geq 2$ and all $a, \alpha, \nu > 0$.

1 Introduction and main results.

In the classic paper [DV 75], Donsker and Varadhan described the asymptotic behaviour of the volume of the so-called 'Wiener-sausage'. If W denotes the Wiener-process in d-dimension, then for $t > 0$,

$$W_t^a = \bigcup_{0 \le s \le t} \bar{B}(W(s), a) \tag{1.1}$$

is called the *Wiener-sausage* up to time t, where $\bar{B}(x, a)$ denotes the closed ball with radius a centered at $x \in \mathbb{R}^d$. Let us denote by $|W_t^a|$ the d-dimensional volume of W_t^a. By the classical result of Donsker and Varadhan, the Laplace-transform of $|W_t^a|$ obeys the following asymptotics:

$$\lim_{t \to \infty} t^{-d/(d+2)} \log E_0 \exp(-\nu |W_t^a|) = -c(d, \nu), \;\; \nu > 0 \tag{1.2}$$

for any $a > 0$ where

$$c(d, \nu) = \nu^{2/(d+2)} \left(\frac{d+2}{2} \right) \left(\frac{2\gamma_d}{d} \right)^{d/(d+2)},$$

and γ_d is the lowest eigenvalue of $-\frac{1}{2}\Delta$ for the d-dimensional sphere of unit volume with Dirichlet boundary condition.

The lower estimate for (1.2) had been known by Kac and Luttinger (see [KL 74]), and in fact the upper estimate turned out to be much harder. This latter one was obtained in [DV 75] by using techniques from the theory of large deviations.

Note, that if $\mathbb{P}$ denotes the law of the Poisson point process ω on $\mathbb{R}^d$ with intensity $\nu\, dl$ (l is the Lebesgue-measure), $\nu > 0$ and with expectation $\mathbb{E}$ (in the notation we suppress the dependence on ν), and

$$T_0 := \inf \left\{ s \ge 0, \; W(s) \in \bigcup_{x_i \in \text{supp}(\omega)} \bar{B}(x_i, a) \right\},$$

then

$$E_0 \exp(-\nu |W_t^a|) = \mathbb{E} \times P_0(T_0 > t), \;\text{ for } t > 0. \tag{1.3}$$

Definition 1 The random set

$$K := \bigcup_{x_i \in \text{supp}(\omega)} \bar{B}(x_i, a)$$

is called a *trap configuration* (or hard obstacle) attached to ω.

Remark 1 In the sequel we will identify ω with K, that is an ω-wise statement will mean that it is true for all trap configurations (with a fixed).

By (1.3), the law of $|W_t^a|$ can be expressed in terms of the 'annealed' or 'averaged' probabilities that the Wiener-process avoids the Poissonian traps of size a up to time t. Using this interpretation of the problem, Sznitman [Sz 98] presented an alternative proof for (1.2). His method, called the 'enlargement of obstacles' turned out to be extremely useful and resulted in a whole array of results concerning similar questions (see [Sz 98], and references therein).

Another research area, the behavior of a class of measure-valued processes called *superprocesses* has also been extensively studied by numerous authors in the past three decades (see e.g. [D 93], [Dy 93], and references therein).

Notation 1 In the sequel $C_b^+(\mathbb{R}^d)$ and $C_c^+(\mathbb{R}^d)$ will denote the space of nonnegative, bounded, continuous functions on $\mathbb{R}^d$ and the space of nonnegative, continuous functions with compact support in $\mathbb{R}^d$ respectively. $\mathcal{M}_F(\mathbb{R}^d)$ and $\mathcal{N}(\mathbb{R}^d)$ will denote the space of finite measures and the space of discrete measures on $\mathbb{R}^d$ respectively.

Definition 2 Let $\alpha > 0$ and β be constants. One can define a unique continuous measure-valued Markov process X which satisfies for any $\mu \in \mathcal{M}_F(\mathbb{R}^d)$ and $g \in C_b^+(\mathbb{R}^d)$,

$$E_\mu \exp(- < X(t), g >) = \exp\left(-\int_{\mathbb{R}^d} u(x,t)\,\mu(dx)\right),$$

where $u(x,t)$ is the minimal nonnegative solution to

$$\begin{aligned} u_t &= \frac{1}{2}\Delta u + \beta u - \alpha u^2,\ (x,t) \in \mathbb{R}^d \times (0,\infty), \\ \lim\nolimits_{t\to 0} u(x,t) &= g(x),\ x \in \mathbb{R}^d. \end{aligned} \tag{1.4}$$

We call X an $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-*superprocess* (see [P 96], [EP 97]).

X is also called a supercritical (resp. critical, subcritical) *super-Brownian motion* (SBM) for positive (resp. zero, negative) β.

The size of the support for super-Brownian motions has been investigated in a number of research papers (see e.g. [I 88], [P 95b], [De 99], [T 94]). Results has been obtained concerning the radius of the support (that is the radius of the smallest ball centered at the origin and containing the support) in [I 88] and [P 95b]; the volume of the ϵ-neighborhood of the range up to $t > 0$ is studied in [De 99] and [T 94]. Note that in [I 88], [De 99] and [T 94] the super-Brownian motion is critical; [P 95b] is one of the few articles in the literature which treat the supercritical case.

In the light of these two mathematical topics, it seems quite natural to ask whether a similar asymptotics to (1.2) can be obtained for superprocesses. In this paper we will prove an analogous result to (1.2) replacing W by a supercritical super-Brownian motion, X. In the sequel X will denote the $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-superprocess and β will be assumed to be positive. We will denote the corresponding probabilities by $\{P_\mu, \mu \in \mathcal{M}_F(\mathbb{R}^d)\}$. Note that X will become extinct with positive probability and in fact (see formula 1.4 in [P 96]),

$$P_{\delta_x}(X(t) = 0 \text{ for all } t \text{ large}) = e^{-\beta/\alpha}. \tag{1.5}$$

Notation 2 In the sequel we will use the notations $S = \{X(t, \mathbb{R}^d) > 0 \text{ for all } t > 0\}$ and $E = S^c$, and will call the events S and E *survival* and *extinction*.

We will need the following result concerning extinction (for the proof see [EP 97, proof of Theorem 3.1]).

Proposition 1 *If* $\{P_\mu, \mu \in \mathcal{M}_F(\mathbb{R}^d)\}$ *correspond to the* $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-*superprocess, then* $\{P_\mu(\cdot \mid E), \mu \in \mathcal{M}_F(\mathbb{R}^d)\}$ *correspond to the* $(\frac{1}{2}\Delta, -\beta, \alpha; \mathbb{R}^d)$-*superprocess.*

Analogously to (1.1) we make the following definition.

Definition 3 For $t \geq 0$, let $R(t) = \overline{\bigcup_{s=0}^{t} \mathrm{supp}(X(s))}$, that is $R(t)$ is the (accumulated) support of X up to time t. Note that $R(t)$ is a.s. compact whenever $\mathrm{supp}(X(0))$ is compact (see [EP 97]). For $t \geq 0$ and $a > 0$, let

$$R^a(t) = \bigcup_{x \in R(t)} \bar{B}(x, a) = \{x \in \mathbb{R}^d \mid \mathrm{dist}(x, R(t)) \leq a\}. \tag{1.6}$$

We call $R^a(t)$ the *super-Brownian sausage* corresponding to the supercritical super-Brownian motion X.

By the rightmost expression in (1.6) it is clear that also $R^a(t)$ is compact P_μ-a.s. whenever $\mathrm{supp}(\mu)$ is compact and thus it has a finite volume (Lebesgue measure) in $\mathbb{R}^d$ P_μ-a.s. Let us denote this (random) volume by $|R^a(t)|$. Note, that if τ denotes extinction time, that is $\tau = \inf\{t > 0 \mid X(t, \mathbb{R}^d) = 0\}$, then

$$\exists l := \lim_{t \to \infty} E_{\delta_0}(\exp(-\nu |R^a(t)|) \mid E) = E_{\delta_0} \exp(-\nu |R^a(\tau)|),$$

and $l \in (0, 1)$. Therefore , we are interested in the long-time behaviour of

$$E_{\delta_0}[\exp(-\nu |R^a(t)|) \mid S].$$

Our main result is as follows.

Theorem 1 *Let $\beta > 0$ be fixed. Then*

$$\lim_{t \to \infty} \frac{1}{t} \log E_{\delta_0}[\exp(-\nu |R^a(t)|) \mid S] = -\beta, \tag{1.7}$$

for all $d \geq 2$ and all $a, \alpha, \nu > 0$.

It is interesting that, unlike in (1.2), the scaling is independent of d and the limit is independent of d and ν. One should interpret this phenomenon as the overwhelming influence of the branching over the spatial motion. (Recall that the measure-valued process X can also be obtained as a high density limit of approximating branching particle systems — see e.g. [D 93].) The fact that the limit does not depend on α as well could also be surprising at the first sight. An explanation for this is as follows. Using (1.4) it is easy to check that if P corresponds to the $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-superprocess, and P' correponds to the $(\frac{1}{2}\Delta, \beta, \beta; \mathbb{R}^d)$-superprocess, then

$$P_{\delta_0}\left(\frac{\beta}{\alpha} X \in \cdot\right) = P'_{\frac{\beta}{\alpha} \cdot \delta_0}(X \in \cdot).$$

Since the support process $t \mapsto \mathrm{supp}(X(t))$ is the same for X and $\beta/\alpha \cdot X$, one only has to show that the starting measures δ_0 and $\beta/\alpha \cdot \delta_0$ give the same limit in (1.7) if such a limit exists at all. This is not hard and we omit the details.

We now give a brief description on the strategy of the proof. The proof of Theorem 1 will be 'atypical' in a sense. In similar problems, like in the proof of (1.2), the upper estimate is usually obtained by using either large deviation techniques or Sznitman's method; the lower estimate is relatively simple in general. For the upper estimate in (1.7) we will use a comparison theorem based on a result due to Evans and O'Connel (see [EO 94]) and some elementary estimates. The lower estimate in (1.7) however will be quite tedious — it will be obtained by analyzing certain pde's related to the superprocess. In the analysis we will also utilize some estimates on the solutions of these equations obtained by Pinsky (see [P 95a]).

The difficulty in proving (1.7) is, of course, compounded in the fact that although, by Proposition 1, $X(\cdot \mid E)$ is a superprocess itself, this does not hold for $X(\cdot \mid S)$. However we will be able to show that the long-term behaviour of $E_{\delta_0}[\exp(-\nu|R^a(t)|) \mid S]$ remains the same if we replace X by a certain branching process, which survives with probability one. Let Z_λ denote the branching Brownian-motion (BBM) with branching term $\beta(z - z^2)$ (that is, binary branching at rate β), and starting with initial measure $\lambda \in \mathcal{N}(\mathbb{R}^d)$. The corresponding expectation will be denoted by $\hat{E}_\lambda$. Note that Z_λ can be considered as a measure-valued process where the state space of the process is $\mathcal{N}(\mathbb{R}^d)$. One can define the 'branching Brownian sausage', $\{\hat{R}^a(t), t \geq 0\}$ analogously to (1.6) as follows.

Definition 4 For $t \geq 0$, let $\hat{R}(t) = \overline{\bigcup_{s=0}^{t} \operatorname{supp}(Z_\lambda(s))}$, that is $\hat{R}(t)$ is the (accumulated) support of Z_λ up to time t. For $t \geq 0$ and $a > 0$, let

$$\hat{R}^a(t) = \bigcup_{x \in R(t)} \bar{B}(x, a). \tag{1.8}$$

We call $\hat{R}^a(t)$ the *branching Brownian sausage* corresponding to the branching Brownian motion Z_λ.

Again, it is easy to see that $\hat{R}^a(t)$ is compact for any compactly supported starting measure $\lambda \in \mathcal{N}(\mathbb{R}^d)$ (and thus has a finite volume) with probability one.

A crucial ingredient of the proof will be the demonstration of a comparison result for $E_{\delta_0}[\exp(-\nu|R^a(t)|)|\, S]$ and $\hat{E}_{\delta_0} \exp(-\nu|\hat{R}^a(t)|)$. We will prove the following proposition.

Proposition 2 (Comparison) *Let $\beta > 0$ be fixed. Then*

$$E_{\delta_0}[\exp(-\nu|R^a(t)|)|\, S] \leq \hat{E}_{\delta_0} \exp(-\nu|\hat{R}^a(t)|), \tag{1.9}$$

for all $d \geq 1$ and all $a, \alpha, \nu > 0$.

After having the above comparison, we will prove the following two estimates.

Proposition 3 (Upper estimate on BBM) *Let $\beta > 0$ be fixed. Then*

$$\limsup_{t \to \infty} \frac{1}{t} \log \hat{E}_{\delta_0} \exp(-\nu|\hat{R}^a(t)|) \leq -\beta, \tag{1.10}$$

for all $d \geq 2$ and all $a, \nu > 0$.

Proposition 4 (Lower estimate on SBM) *Let $\beta > 0$ be fixed. Then*

$$\liminf_{t \to \infty} \frac{1}{t} \log \hat{E}_{\delta_0}[\exp(-\nu|R^a(t)|)|\, S] \geq -\beta, \tag{1.11}$$

for all $d \geq 1$ and all $a, \alpha, \nu > 0$.

Theorem 1 will then follow from (1.9)-(1.11). Also, as a by-product of our method, we will obtain the following result for the branching Brownian motion.

Theorem 2 *Let $\beta > 0$ be fixed. Then*

$$\lim_{t \to \infty} \frac{1}{t} \log \hat{E}_{\delta_0} \exp(-\nu|\hat{R}^a(t)|) = -\beta, \tag{1.12}$$

for all $d \geq 2$ and all $a, \nu > 0$.

The proof of Theorem 2 is immediate from (1.9)-(1.11).

We end this section with four further problems.

Problem 1 (higher order asymptotics) In the light of the fact that the limit under (1.7) does not depend on $\nu > 0$ and $d \geq 2$, it would be desirable to refine the results

$$E_{\delta_0}[\exp(-\nu|R^\alpha(t)|) \mid S] = \exp(-\beta t + o(t)) \text{ as } t \to \infty$$

and

$$\hat{E}_{\delta_0} \exp(-\nu|\hat{R}^\alpha(t)|) = \exp(-\beta t + o(t)) \text{ as } t \to \infty,$$

by obtaining higher order asymptotics for the above Laplace-transforms.

Problem 2 (one-dimensional case) What are the corresponding asymptotics for $d = 1$?

Problem 3 (zero or negative β) According to (1.5), the event S has zero probability if $\beta \leq 0$ (critical or subcritical super-Brownian motion). However, replacing S by $\{\tau > t\}$ (τ was defined in the paragraph preceding Theorem 1), the asymptotic behavior of $E_{\delta_0}[\exp(-\nu|R^\alpha(t)|) \mid \tau > t]$ can still be studied. How can the result of Theorem 1 be extended for this case?

Problem 4 (general superprocesses) The concept of the $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-superprocess has been generalized for spatially dependent α and β and for a general domain $D \subset \mathbb{R}^d$ with the elliptic operator L on it (see [EP 97]). (See also the remark after Theorem 3 in the next section.) How can the results of this paper be generalized for such a general setting?

Remark 2 Problem 3 was suggested by A.-S. Sznitman.

2 Proofs.

It will be more convenient to rewrite the Laplace-transforms in (1.9)-(1.11) using a 'trap-avoiding' setting similarly to (1.3). We record the basic correspondences. Let $\mathbb{E}$ be as in (1.3) and K as in Definition 1, and recall that Z denotes the branching Brownian motion with branching rate $\beta > 0$. The corresponding probabilities will be denoted by $\hat{P}_\lambda$. Let

$$T := \inf\{t \geq 0 \mid X(t, K) > 0\},$$

and

$$\hat{T} := \inf\{t \geq 0 \mid Z(t, K) > 0\},$$

where $Z(t, B)$ denotes the number of particles of Z located in B at time t for $B \subset \mathbb{R}^d$ Borel and $t > 0$. Then

$$E_{\delta_0}[\exp(-\nu|R^\alpha(t)|) \mid S] = \mathbb{E} \times P_{\delta_0}(T > t \mid S), \tag{2.13}$$

and

$$\hat{E}_{\delta_0} \exp(-\nu|\hat{R}^\alpha(t)|) = \mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t). \tag{2.14}$$

The proof of Proposition 2 will be based on a result on the decomposition of superprocesses with immigration which was proved in [EO 94]. Let X be the $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$-superprocess with corresponding probabilities $\{P_\mu\}$.

Theorem 3 (Evans and O'Connell) *Let $\tilde{X}$ be the $(\frac{1}{2}\Delta, -\beta \ , \alpha; \mathbb{R}^d)$-superprocess. By Proposition 1, $\tilde{X}$ corresponds to $P_\mu(\cdot \mid E)$. Let $Z_\lambda^* = 2\alpha Z_\lambda$. Conditional on $\{Z_\lambda^*(s)\}_{s=0}^\infty$, let $(R, \mathbb{P}^{\mu,\lambda})$ be the superprocess obtained by taking the process $\tilde{X}$ with starting measure μ, and adding immigration, where the immigration at time t is according to the measure $Z_\lambda^*(t)$. This is described mathematically by the conditional Laplace functional*

$$\mathbb{E}^{\mu,\lambda}(\exp(- < R(t), g > - < Z^*(t), k >) \mid Z_\lambda^*(s), s \geq 0) =$$

$$\exp\left(- < \mu, \tilde{u}_g(\cdot, t) > - \int_0^t ds < Z_\lambda^*(s), \tilde{u}_g(\cdot, t-s) > - < Z_\lambda^*(t), k > \right), \qquad (2.15)$$

for $g, k \in C_c^+(\mathbb{R}^d)$, where $\tilde{u}_g$ denotes the minimal nonnegative solution to (1.4), with β replaced by $-\beta$. Denote by N_μ the law of the Poisson random measure on $\mathbb{R}^d$ with intensity $\frac{\beta}{\alpha}\mu$ and define the random initial measure η by

$$\mathcal{L}(\eta) \equiv \delta_\mu \times N_\mu.$$

Then the law of R under $\mathbb{P}^\eta$ is P_μ.

Remark 3 We note that Theorem 3 has been generalized ([EP 97], Theorem 6.1) for the more general setting mentioned in Problem 4.

We now prove Propositions 2-4. In the proofs we will use the following notations:

Notation 3 *The notation $f \asymp g$ will mean that $\frac{f(x)}{g(x)} \to 1$. The notation $f = o(g)$ will mean that $\frac{f(x)}{g(x)} \to 0$.*

Proof of Proposition 2. By (2.13) and (2.14) we must prove that

$$\mathbb{E} \times P_{\delta_0}(T > t \,|\, S) \leq \mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t). \qquad (2.16)$$

We will in fact prove the stronger result that (2.16) holds ω-wise, that is

$$P_{\delta_0}(T > t \,|\, S) \leq \hat{P}_{\delta_0}(\hat{T} > t), \text{ for every } \omega. \qquad (2.17)$$

Let $\mathbb{Q}$ be the law of the Poisson distribution with parameter $\frac{\beta}{\alpha}$ conditioned on positive integers, that is

$$\mathbb{Q}(m = i) = \frac{e^{-\beta/\alpha}}{1 - e^{-\beta/\alpha}} \cdot \frac{(\beta/\alpha)^i}{i!}, \ i = 1, 2, \ldots.$$

Let $\mathbb{M}$ denote the corresponding expectation. Finally, let K be a fixed trap configuration. By Theorem 3 it is intuitively clear that

$$P_{\delta_0}(T > t \,|\, S) \leq \mathbb{M} \times \hat{P}_{m\delta_0}(\hat{T} > t). \qquad (2.18)$$

The rigorous proof of (2.18) however is a bit tedious. By standard probabilistic considerations, (2.18) will follow from the lemma below.

Lemma 1 *Fix an $x_0 \in \mathbb{R}^d$ and let $B = B(x_0, a)$. Fix also a $t > 0$ and an $\epsilon \in (0, t)$. Then for any $m \geq 1$,*

$$\mathbb{P}^{\delta_0, m\delta_0}(R(t, B) > 0 \mid Z_{m\delta_0}(s, B(x_0, a)) > 0 \ \forall \ t - \epsilon \leq s \leq t) = 1.$$

Obviously, $Z_{m\delta_0}$ can be replaced by $Z^*_{m\delta_0}$ in Lemma 1. Thus, taking $k = 0$ in (2.15), using the test functions

$$g_n(x) = \begin{cases} n & \text{if } |x - x_0| \le a \\ 0 & \text{if } |x - x_0| > a, \end{cases}$$

and finally letting $n \to \infty$, the proof of the lemma reduces to the proof of the following claim.

Claim 1 *Let U_a be the minimal nonnegative solution to (1.4) with β replaced by $-\beta$ and*

$$g(x) := \begin{cases} \infty & \text{if } |x - x_0| \le a \\ 0 & \text{if } |x - x_0| > a. \end{cases}$$

(In fact U_a can be obtained as a pointwise limit (as $n \to \infty$) of solutions to (1.4) with β replaced by $-\beta$ and with the above g_n's.) Then

$$\int_0^\epsilon U_a(l(t), t)\, dt = \infty,$$

for any $\epsilon > 0$ and any continuous function $l : [0, \epsilon] \to B(x_0, a)$.

Proof. We may assume that $x_0 = 0$. Let U_a^1 be the minimal nonnegative solution to (1.4) with β replaced by $-\beta$ and

$$g(x) := \begin{cases} 0 & \text{if } |x| \le a \\ \infty & \text{if } |x| > a, \end{cases}$$

and let V be the minimal nonnegative solution to (1.4) with β replaced by $-\beta$ and $g(x) \equiv \infty$. We claim that

$$V(x, t) = V(t) = \frac{\beta}{\alpha}[(1 - e^{-\beta t})^{-1} - 1]. \tag{2.19}$$

Note that by [EP 97, Theorem 3.1] and Proposition 1,

$$\tilde{P}_{\delta_x}(< 1, X(t) >= 0) = e^{-V(t)}, \text{ for all } x \in \mathbb{R}^d, \tag{2.20}$$

where $\tilde{P} = P(\cdot \mid E)$ and P corresponds to $(\frac{1}{2}\Delta, \beta, \alpha; \mathbb{R}^d)$. It is not hard to show that

$$P_{\delta_x}(< 1, X(t) >= 0) = \exp\left[-\frac{\beta}{\alpha}(1 - e^{-\beta t})^{-1}\right]$$

(see for example [EP 97, Proposition 3.1]). Using (1.5) it follows that

$$\tilde{P}_{\delta_x}(< 1, X(t) >= 0) = \exp\left[-\frac{\beta}{\alpha}\left((1 - e^{-\beta t})^{-1} - 1\right)\right].$$

Then (2.19) follows from this and (2.20).

By the minimality of the solution and the parabolic maximum principle (see [EP 97]), it is easy to show that

$$U_a + U_a^1 \ge V. \tag{2.21}$$

Since we know V explicitly it will suffice to have an upper estimate on U_a^1. Similarly as in the proof of [P 95a, Theorem 2(i)], one can show that

$$U_a^1(x, t) \le V(t) \exp(-(k_1 \cdot a^2/t - t - k_2)^+) \text{ for } x \in B(0, a) \tag{2.22}$$

with some $k_1, k_2 > 0$. (In [P 95a, Theorem 1(i)] the nonlinearity $u - u^2$ is considered and the above estimate is proved with the function $(1-e^{-t})^{-1}$ in place of $V(t)$. The same proof goes through in the recent case.) It then follows that for t small

$$U_a^1(x,t) \le V(t)\exp(-k \cdot a^2/t) \text{ for } x \in B(0,a) \tag{2.23}$$

with some $k > 0$. From (2.19), (2.21) and (2.23), one obtains

$$U_a(x,t) \ge V(t)(1 - \exp(-k \cdot a^2/t)) \text{ for } x \in B(0,a). \tag{2.24}$$

Using (2.19),

$$V(t)(1-\exp(-k \cdot a^2/t)) \asymp V(t) \asymp \frac{1}{\alpha t} \text{ as } t \to 0, \tag{2.25}$$

which completes the proof.

The proof of (2.18) has thus been completed. In order to complete the proof of the proposition, we must show how (2.18) implies (2.17). Let m be the random number of particles with expectation $\mathbb{M}$ as in (2.18). Obviously, it is enough to show that

$$\mathbb{M} \times \hat{P}_{m\delta_0}(\hat{T} > t) \le \hat{P}_{\delta_0}(\hat{T} > t).$$

For this, denote

$$p_{i,t} := \hat{P}_{i\delta_0}(\hat{T} > t), \; i = 1, 2, \ldots$$

where $\hat{T}$ is as in (2.14). Using the independence of the particles,

$$\begin{aligned}\mathbb{M} \times \hat{P}_{m\delta_0}(\hat{T} > t) &= \frac{e^{-\beta/\alpha}}{1-e^{-\beta/\alpha}} \sum_{i=1}^{\infty} \frac{(\beta/\alpha)^i}{i!} p_{i,t} \\ &= \frac{e^{-\beta/\alpha}}{1-e^{-\beta/\alpha}} \sum_{i=1}^{\infty} \frac{(\beta/\alpha \cdot p_{1,t})^i}{i!} = \frac{e^{-\beta/\alpha}}{1-e^{-\beta/\alpha}} (\exp(\beta/\alpha \cdot p_{1,t}) - 1).\end{aligned}$$

By the convexity of the exponential function,

$$e^{\beta/\alpha \cdot x} - 1 \le (e^{\beta/\alpha} - 1)x, \text{ for } 0 \le x \le 1,$$

and thus

$$\mathbb{M} \times \hat{P}_{m\delta_0}(\hat{T} > t) \le \frac{e^{-\beta/\alpha}}{1-e^{-\beta/\alpha}} (e^{\beta/\alpha} - 1) p_{1,t} = p_{1,t}.$$

This completes the proof of Proposition 2. □

Proof of Proposition 3. By (2.14) we must show that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t) \le -\beta. \tag{2.26}$$

The strategy for proving (2.26) is as follows. We will split the time interval $[0,t]$ into two parts : $[0, \theta t]$ and $(\theta t, t]$. In the first part we will use that 'many' particles are produced, in the second one, that each particle hits the traps with 'large' probability. Then the probability of avoiding traps in $[0,t]$ will be 'small'. To make things more precise, let T_0 be as in the paragraph preceding (1.3) and denote $Y_x^t = P_x(T_0 > t)$, where P_x corresponds to W. Then for any $x \in \mathbb{R}$, $t > 0$ fixed, $Y_x^t = Y_x^t(\omega)$ is a random variable. As before, $\hat{E}$ denote now the expectations corresponding to Z. Let $Z(t) = (Z_1(t), Z_2(t), ..., Z_k(t)), t > 0, k \ge 1$, for $|Z(t)| = k$, where $|Z(t)|$ denotes the number of particles alive at t. Using a well known formula (see e.g. [KT, formula 8.11.5] and the discussion afterwards) for the distribution of $|Z(t)|$, one has

$$q_{k,s} := \hat{P}_{\delta_0}(|Z(s)| = k) = e^{-\beta s}(1-e^{-\beta s})^{k-1} = c_{\beta s}(1-e^{-\beta s})^k,$$

where

$$c_{\beta s} := \frac{e^{-\beta s}}{1-e^{-\beta s}}.$$

Fix $\theta \in (0,1)$. By the Markov property,

$$\mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t) \leq \mathbb{E} \times \hat{E}_{\delta_0} \sum_{k=1}^{\infty} q_{k,\theta t} \cdot Y^{(1-\theta)t}_{Z_1(\theta t)} \dots Y^{(1-\theta)t}_{Z_k(\theta t)}. \tag{2.27}$$

At this point we note that we are going to work with expectations of nonnegative series like the right-hand side of (2.27) which in principle might sum up to $+\infty$. After a number of upper estimates however we will show that the last one is in fact finite for t large enough. Using the inequality between the arithmetic and the geometric means,

$$\mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t) \leq \sum_{k=1}^{\infty} q_{k,\theta t} \frac{1}{k} \hat{E}_{\delta_0} \times \mathbb{E} \left[\left(Y^{(1-\theta)t}_{Z_1(\theta t)} \right)^k + \dots + \left(Y^{(1-\theta)t}_{Z_k(\theta t)} \right)^k \right].$$

In fact, the right-hand expression can be much simplified since the spatial homogeneity of ω implies that

$$\hat{E}_{\delta_0} \times \mathbb{E} \left(Y^{(1-\theta)t}_{Z_i(\theta t)} \right)^k = \mathbb{E} \left(Y^{(1-\theta)t}_0 \right)^k, \ \forall \ 1 \leq i \leq k.$$

Denote $p_t := Y_0^t, t > 0$ and recall that p_t is the probability that a single Brownian particle starting at zero avoids the traps up to t. We have

$$\mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t) \leq \sum_{k=1}^{\infty} q_{k,\theta t} \, \mathbb{E} \, p^k_{(1-\theta)t} = \mathbb{E} \sum_{k=1}^{\infty} c_{\beta\theta t} \left[\ (1-e^{-\beta\theta t}) \cdot p_{(1-\theta)t} \right]^k$$

$$\leq c_{\beta\theta t} \, \mathbb{E} \sum_{k=0}^{\infty} p^k_{(1-\theta)t} = c_{\beta\theta t} \, \mathbb{E} \, \frac{1}{1-p_{(1-\theta)t}}.$$

We are going to show that there exists a $L > 0$ and a $t_0 > 0$ such that

$$k_t := \mathbb{E} \, \frac{1}{1-p_t} < L \text{ for } t > t_0. \tag{2.28}$$

If (2.28) holds then we are done because from (2.28) we obtain that

$$\mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t) \leq c_{\beta\theta t} \cdot k_{(1-\theta)t} < c_{\beta\theta t} \cdot L \text{ if } (1-\theta)t > t_0,$$

that is if $t > \frac{t_0}{1-\theta}$. Then

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E} \times \hat{P}_{\delta_0}(\hat{T} > t)$$

$$\leq \limsup_{t\to\infty} \tfrac{1}{t} \log(c_{\beta\theta t} \cdot L) = \quad \limsup_{t\to\infty} \frac{1}{t} \log c_{\beta\theta t} = -\beta\theta,$$

and (2.26) follows by letting $\theta \to 1$.

It remains to show (2.28). Let τ_n denote the first exit time of W from the d-dimensional n-box,

$$\tau_n := \inf\{t \geq 0 \mid W(t) \notin [-n,n]^d\}.$$

Let P_0^d denote the probability corresponding to W in d-dimension and starting from the origin. Then, as is well known [KT, formula 7.3.3], for $d = 1$

$$P_0^1(\tau_n \leq t) = \sqrt{\frac{2}{\pi}} \int_{n/\sqrt{t}}^{\infty} e^{-\frac{y^2}{2}}\, dy,$$

and thus by independence

$$P_0^d(\tau_n \leq t) = 1 - [1 - P_0^1(\tau_n \leq t)]^d = 1 - \left[1 - \sqrt{\frac{2}{\pi}} \int_{n/\sqrt{t}}^{\infty} e^{-\frac{y^2}{2}}\, dy\right]^d.$$

It follows from the binomial theorem that for $t > 0$ fixed,

$$P_0^d(\tau_n \leq t) \asymp d \cdot \sqrt{\frac{2}{\pi}} \int_{n/\sqrt{t}}^{\infty} e^{-\frac{y^2}{2}}\, dy \text{ as } n \to \infty.$$

Using L'Hospital's rule,

$$\int_{n/\sqrt{t}}^{\infty} e^{-\frac{y^2}{2}}\, dy \asymp \sqrt{t}\, n^{-1} e^{-n^2/2t} \text{ as } n \to \infty.$$

Therefore,

$$P_0^d(\tau_n \leq t) \asymp d \cdot \sqrt{\frac{2}{\pi}} \sqrt{t}\, n^{-1} e^{-n^2/2t} \text{ as } n \to \infty. \tag{2.29}$$

Let v_a denote the volume of the a-ball in d-dimension and let C_n denote the event that there is no point of ω in the n-box, that is $C_n := \{\omega([-n,n]^d) = 0\}$. Let C_n^c denote the complement of C_n. Since $\mathbb{P}(\cup_{n=1}^{\infty} C_n^c) = 1$,

$$\begin{aligned}
&\mathbb{E}\frac{1}{1-p_t} \leq \\
&\quad \sum_{n=1}^{\infty} \mathbb{P}(C_{n-1} \setminus C_n) \cdot \frac{1}{P_0^d(\tau_n \leq t) \cdot \mathbb{E} \times P_0^d(W(\tau_n) \in K)} \leq \\
&\quad \sum_{n=1}^{\infty} \mathbb{P}(C_{n-1}) \cdot \frac{1}{P_0^d(\tau_n \leq t) \cdot e^{-\nu v_a}} = \\
&\quad \sum_{n=1}^{\infty} \exp(-\nu[2(n-1)^d + v_a]) \cdot \frac{1}{P_0^d(\tau_n \leq t)}.
\end{aligned}$$

Putting this together with (2.29), we obtain that a sufficient condition for $\mathbb{E}(1-p_t)^{-1} < \infty$ is that

$$\sum_{n=1}^{\infty} n \exp\left(\frac{n^2}{2t} - 2\nu(n-1)^d\right) < \infty. \tag{2.30}$$

Now, if $d \geq 3$ then (2.30) holds for any $t > 0$ and thus $\mathbb{E}(1-p_t)^{-1} < \infty$ for *all* $t > 0$. If $d = 2$ then (2.30) holds whenever $t > t_0 := \frac{1}{4\nu}$. This proves (2.28) and completes the proof of Proposition 3. □

Proof of Proposition 4. We will need the following lemma.

Lemma 2 *Denote*

$$A_{n,t} := \{X(s, B^c(0,n)) = 0,\ s \le t\}. \tag{2.31}$$

(i) For each $n \in \mathbb{N}$ there exists a minimal positive solution u_n to

$$\begin{aligned}&u_t = \frac{1}{2}\Delta u + \beta u - \alpha u^2,\ (x,t) \in B(0,n) \times (0,\infty),\\ &\lim_{x\to\partial B(0,n)} u(x,t) = \infty,\ t > 0,\\ &\lim_{t\to 0} u(x,t) = 0,\ x \in B(0,n).\end{aligned} \tag{2.32}$$

Moreover,

$$P_{\delta_x}(A_{n,t}) = e^{-u_n(x,t)}. \tag{2.33}$$

(ii) For each $n \in \mathbb{N}$ there exists a minimal positive solution $\hat{u}_n$ to

$$\begin{aligned}&u_t = \frac{1}{2}\Delta u - \beta u - \alpha u^2,\ (x,t) \in B(0,n) \times (0,\infty),\\ &\lim_{x\to\partial B(0,n)} u(x,t) = \infty,\ t > 0,\\ &\lim_{t\to 0} u(x,t) = 0,\ x \in B(0,n).\end{aligned} \tag{2.34}$$

Moreover,

$$P_{\delta_x}(A_{n,t} \mid E) = e^{-\hat{u}_n(x,t)}. \tag{2.35}$$

For the proof of (i), see the proof of [P 95a, formula 1.3] and [P 95b, formula 6]. The proof of (ii) is similar.

Let $A_{n,t}$ be as in (2.31). Our goal is to show that if $t \to \infty$ and $n \to \infty$ simultaneously and in such a way that $n^d = o(t)$, then

$$\liminf_{t\to\infty} \frac{1}{t} \log P_{\delta_x}(A_{n,t} \mid S) \ge -\beta. \tag{2.36}$$

Indeed, if (2.36) is true, then using the notation v_1 for the volume of the d-dimensional unit ball (and suppressing the dependence of n on t in the notation),

$$\begin{aligned}&\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \times P_{\delta_0}(T > t \mid S) \ge\\ &\quad \liminf_{t\to\infty} \frac{1}{t} \log[P_{\delta_x}(A_{n,t} \mid S) \cdot \mathbb{P}\{\text{There are no traps in } B(0,n)\}] =\\ &\quad \liminf_{t\to\infty} \frac{1}{t} \log \left[P_{\delta_0}(A_{n,t} \mid S) \cdot \exp\left(-\nu v_1 n^d\right)\right] =\\ &\quad \liminf_{t\to\infty} \frac{1}{t} \log P_{\delta_0}(A_{n,t} \mid S) \ge -\beta,\end{aligned}$$

and (1.11) follows from this and (2.13).

In the rest of this paper the following assumption will be in force.

Assumption 1 $n(t) \to \infty$ and $n^d(t) = o(t)$ as $t \to \infty$.

Notation 4 In the sequel we will mostly suppress the dependence of n on t in the notation and simply write n instead of $n(t)$; we will use $n(t)$ only when it will be important to remind the reader that n is a function of t.

We have

$$P_{\delta_x}(A_{n,t} \mid S) = \frac{P_{\delta_x}(A_{n,t})}{P_{\delta_x}(S)} \cdot P_{\delta_x}(S \mid A_{n,t}) = \frac{P_{\delta_x}(A_{n,t})}{P_{\delta_x}(S)} \cdot [1 - P_{\delta_x}(E \mid A_{n,t})]$$

$$= \frac{P_{\delta_x}(A_{n,t})}{P_{\delta_x}(S)} \cdot \left[1 - P_{\delta_x}(A_{n,t} \mid E) \cdot \frac{P_{\delta_x}(E)}{P_{\delta_x}(A_{n,t})}\right].$$

Using Lemma 2, it follows,

$$P_{\delta_x}(A_{n,t} \mid S) = \frac{P_{\delta_x}(A_{n,t})}{P_{\delta_x}(S)} \cdot (1 - \exp -[\hat{u}_n(x,t) + \beta/\alpha - u_n(x,t)]). \tag{2.37}$$

Before proving (2.36), first note that obviously

$$\lim_{t\to\infty} P_{\delta_0}(A_{n,t} \mid E) = 1, \text{ whenever } \lim_{t\to\infty} n(t) = \infty.$$

Therefore, by Lemma 2(ii),

$$\lim_{t\to\infty} \hat{u}_n(0,t) = 0, \text{ whenever } \lim_{t\to\infty} n(t) = \infty. \tag{2.38}$$

By Theorem A(a)(i-ii) in [P 95b] it follows that

$$\lim_{t\to\infty} u_n(0,t) = \beta/\alpha, \tag{2.39}$$

whenever $\lim_{t\to\infty} n(t) = \infty$ and $\lim_{t\to\infty} \frac{n(t)}{t} = 0$ hold. We conclude that

$$\lim_{t\to\infty} [\hat{u}_n(0,t) + \beta/\alpha - u_n(0,t)] = 0, \tag{2.40}$$

whenever $\lim_{t\to\infty} n(t) = \infty$ and $\lim_{t\to\infty} \frac{n(t)}{t} = 0$ hold.

By Lemma 2(i) and (2.39), $P_{\delta_0}(A_{n,t}) \to e^{-\beta/\alpha}$ under Assumption 1. From this and (2.37) we obtain that

$$P_{\delta_0}(A_{n,t} \mid S) \asymp \text{const } [\hat{u}_n(0,t) + \beta/\alpha - u_n(0,t)], \text{ as } t \to \infty. \tag{2.41}$$

whenever $\lim_{t\to\infty} n(t) = \infty$ and $\lim_{t\to\infty} \frac{n(t)}{t} = 0$ hold. Therefore, in order to prove (2.36), it will suffice to prove that

$$\liminf_{t\to\infty} \frac{1}{t} \log g_n(0,t) \geq -\beta, \tag{2.42}$$

where

$$g_n(x,t) := \hat{u}_n(x,t) + \beta/\alpha - u_n(x,t).$$

Note that $g_n \geq 0$ by (2.37).

The next observation is that we may assume that $\beta = \alpha$. To see this, note that if v_n and $\hat{v}_n$ denote the minimal positive solutions to (2.32) and (2.34) respectively, but with α replaced by β, then in fact (as a simple computation shows), $v_n = \frac{\alpha}{\beta}u_n$ and $\hat{v}_n = \frac{\alpha}{\beta}\hat{u}_n$. Therefore, if $g_n^*(x,t) := \hat{v}_n(x,t) + 1 - v_n(x,t)$, then $g^* = \frac{\alpha}{\beta}g_n$. This means that if (2.36) holds for $\beta = \alpha$, then it also holds for $\beta \neq \alpha$. So, from now on, for simplicity, we will work with $\beta = \alpha$ and

$$g_n = \hat{u}_n + 1 - u_n.$$

A key step in the proof is to observe that — as a direct computation reveals — $0 \leq g_n$ satisfies the following parabolic equation :

$$\begin{aligned} &g_t = \frac{1}{2}\Delta g - \beta(1+2\hat{u})g + \beta g^2, \ (x,t) \in B(0,n) \times (0,\infty), \\ &\lim_{t\to 0} u(\cdot,t) = 1. \end{aligned} \tag{2.43}$$

In order to analyze (2.43), we will need some more facts concerning u_n and $\hat{u}_n$. First, note that by (2.33) and (2.35), u_n and $\hat{u}_n$ are monotone nondecreasing in t. Let

$$\phi_n(\cdot) := \lim_{t\to\infty} u_n(\cdot,t), \ \text{and } \hat{\phi}_n(\cdot) := \lim_{t\to\infty} \hat{u}_n(\cdot,t).$$

Then, using (2.33) and (2.35), it follows that

$$P_{\delta_x}(X(t, B^c(0,n)) = 0, \ \forall t \geq 0) = e^{-\phi(x)}, \ x \in B(0,n) \tag{2.44}$$

and

$$P_{\delta_x}(X(t, B^c(0,n)) = 0, \ \forall t \geq 0 \mid E) = e^{-\hat{\phi}(x)}, \ x \in B(0,n). \tag{2.45}$$

Using this, it follows from Lemma 7.1 in [EP 97] that ϕ and $\hat{\phi}$ are finite. We claim that in fact

$$\hat{\phi} = \phi - 1. \tag{2.46}$$

To see this denote

$$A_n = \{X(t, B^c(0,n)) = 0, \ \forall t \geq 0\}.$$

By [EP 97, formula 3.8],

$$P_{\delta_x}(A_n \cap S) = 0, \ x \in B(0,n).$$

Therefore

$$P_{\delta_x}(A_n \cap E) = P_{\delta_x}(A_n),$$

and thus

$$P_{\delta_x}(A_n \mid E) = P_{\delta_x}(A_n) \cdot \frac{1}{P_{\delta_x}(E)}, \ x \in B(0,n).$$

By (1.5),(2.44) and (2.45) then,

$$e^{-\hat{\phi}} = e^{-\phi} \cdot e.$$

Now let us return to (2.43). In order to prove (2.42), first we will compare g_n to the unique nonnegative solution v_n of the *linear* equation

$$\begin{aligned} &v_t = \frac{1}{2}\Delta v - \beta(1+2\hat{\phi}_n)v, \quad (x,t) \in B(0,n) \times (0,\infty), \\ &\lim_{t\to 0} v(\cdot,t) = 1, \\ &\lim_{|x|\to n} v(x,t) = 0, \quad \forall t > 0. \end{aligned} \tag{2.47}$$

(As is well known, the existence of the solution to (2.47) is guaranteed by the negative sign of the zeroth order term in the right hand side of the first equation of (2.47); the uniqueness and the non-negativity follows by the parabolic maximum principle — see e.g. [PW 84] for more elaboration.) Using (2.43) along with the fact that $\hat{u}_n(\cdot,t) \le \hat{\phi}_n(\cdot)\ \forall t > 0,\ \forall n \ge 1$, we have

$$\begin{aligned} &\frac{1}{2}\Delta g_n - \beta(1+2\hat{\phi}_n)g_n - (g_n)_t \le 0, \quad (x,t) \in B(0,n) \times (0,\infty), \\ &\lim_{t\to 0} g_n(\cdot,t) = 1. \end{aligned} \tag{2.48}$$

Therefore, by the parabolic maximum principle again,

$$g_n \ge v_n,\ \forall n \ge 1,$$

and consequently it will suffice to prove (2.42) with g_n replaced by v_n. To do this, it will be useful to rescale (2.47). Let

$$w_n(x,t) := v_n(nx, n^2 t),\ \text{for } (x,t) \in B(0,1) \times [0,\infty).$$

Then a simple computation shows that $0 \le w_n$ satisfies the rescaled equation,

$$\begin{aligned} &w_t(x,t) = \frac{1}{2}\Delta w(x,t) - n^2\beta(1+2\hat{\phi}_n(nx))w(x,t), \quad (x,t) \in B(0,1) \times (0,\infty), \\ &\lim_{t\to 0} v(\cdot,t) = 1, \\ &\lim_{|x|\to 1} w(x,t) = 0, \quad \forall t > 0. \end{aligned} \tag{2.49}$$

We note that ϕ_n is in fact the minimal positive solution to the elliptic boundary-value problem,

$$\begin{aligned} &\frac{1}{2}\Delta\phi + \beta\phi - \beta\phi^2 = 0, \text{in } B(0,n), \\ &\lim_{|x|\to n} \phi(x) = \infty, \end{aligned} \tag{2.50}$$

and satisfies the upper estimate:

$$\phi_n(x) \le 1 + \frac{cn^2}{(n^2 - r^2)^2},$$

with some $c = c(n,d,\beta)$, where $r := |x|$. (See Proposition 1 in [P 95a] and its proof.) It then follows from (2.46) that

$$\hat{\phi}_n(nx) = \phi_n(nx) - 1 \le \frac{cn^2}{(n^2 - n^2 r^2)^2} = \frac{c}{n^2(1-r^2)^2}. \tag{2.51}$$

By (2.49) and the Feynman-Kac formula ([F 85], Theorem 2.2),

$$w_n(x,t) = \mathbb{E}_x\left[\exp\left\{-n^2\beta\int_0^t \left[1 + 2\hat{\phi}(nW(s))\right]\,ds\right\};\ \tau_1 > t\right],$$

where W is the d-dimensional Wiener process and

$$\tau_1 := \inf\{t \ge 0 \mid W(t) \notin B(0,1)\}.$$

Therefore,

$$\begin{aligned} v_n(0,t) &= w_n(0,t/n^2) = \\ &\mathbb{E}_x\left[\exp\left\{-n^2\beta\int_0^{t/n^2}\left[1 + 2\hat{\phi}(nW(s))\right]\,ds\right\};\ \tau_1 > \frac{t}{n^2}\right] = \\ &e^{-\beta t}\cdot\mathbb{E}_x\left[\exp\left\{-2\int_0^{t/n^2} n^2\beta\hat{\phi}(nW(s))\,ds\right\};\ \tau_1 > \frac{t}{n^2}\right]. \end{aligned}$$

Using this along with (2.51), we obtain

$$v_n(0,t) \geq e^{-\beta t} \cdot \mathbb{E}_x \left[\exp\left\{ -\int_0^{t/n^2} \frac{2\beta c}{(1-|W(s)|^2)^2}\, ds \right\};\ \tau_1 > \frac{t}{n^2} \right].$$

Using the notation

$$V(x) := -\frac{2\beta c}{(1-|x|^2)^2},$$

it follows

$$\liminf_{t\to\infty} \frac{1}{t} \log v_n(0,t) \geq -\beta + \tag{2.52}$$
$$\liminf_{t\to\infty} \frac{1}{n^2} \cdot \frac{n^2}{t} \log \mathbb{E}_x \left[\exp\left\{ -\int_0^{t/n^2} V(W(s))\, ds \right\};\ \tau_1 > \frac{t}{n^2} \right].$$

First consider $d \geq 2$. Then by assumption, $t/n^2(t) \to \infty$ as $t \to \infty$. Let $\lambda^{(1-\epsilon)}$ denote the leading eigenvalue of the operator $\frac{1}{2}\Delta + V$ on $B(0, 1-\epsilon)$ with Dirichlet boundary data. Since V is smooth on $\bar{B}(0, 1-\epsilon)$, by a well-known theorem ([P 95c], Theorem 3.6.1),

$$\lim_{t/n^2\to\infty} \frac{n^2}{t} \log \mathbb{E}_x \left[\exp\left\{ -\int_0^{t/n^2} V(W(s))\, ds \right\};\ \tau_{1-\epsilon} > \frac{t}{n^2} \right] = \lambda^{(1-\epsilon)},$$

where $\tau_{1-\epsilon}$ is defined analogously to τ_1. Since $\lambda^{(1-\epsilon)} > -\infty$, then, *a fortiori,*

$$\liminf_{t/n^2\to\infty} \frac{n^2}{t} \log \mathbb{E}_x \left[\exp\left\{ -\int_0^{t/n^2} V(W(s))\, ds \right\};\ \tau_1 > \frac{t}{n^2} \right] > -\infty.$$

Putting this together with (2.52) and with the fact that $\lim_{t\to\infty} n(t) = \infty$, we obtain that

$$\liminf_{t\to\infty} \frac{1}{t} \log v_n(0,t) \geq -\beta.$$

This completes the proof for $d \geq 2$.

Finally, even if $d = 1$ and $t/n^2(t) \to \infty$ is not satisfied as $t \to \infty$, the above argument still works because the expectation in the right hand side of (2.52) is clearly monotone nonincreasing with t/n^2. □

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