

Convergence results for a nonlinear parabolic control problem

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Abstract

We investigate a general control problem for a class of nonlinear parabolic evolution equations. Applications are related to solid–solid and solid–liquid phase transitions.

We prove compactness of the solution operator, existence of optimal controls and show convergence of the finite–dimensional approximate control problem to the original one.

1 Introduction

We consider the following distributed optimal control problem:

$$(P) \quad \text{Minimize } J(u) = \int_0^T (\psi_1(u) + \psi_2(y)) dt + \psi_3[y](T),$$

subject to the state equation

$$\mu(y)' + Ay + F[y] = \alpha(y)Bu, \quad \text{in } (0, T) \tag{1.1a}$$

$$y(0) = y_0, \tag{1.1b}$$

and the control constraint

$$u \in U_{ad}. \tag{1.2}$$

Here, U and H are real Hilbert spaces, V is a real reflexive Banach space densely embedded in H and we assume that $V \subset H \subset V^*$, where the inclusion from V into H is compact and H is identified with its dual space. We shall denote by (\cdot, \cdot) the inner product in H and also the duality pairing between V^* and V . The norms will be denoted by $\|\cdot\|$ with adequate subscripts. U_{ad} is a closed, convex and bounded subset of $\mathcal{U} = L^2(0, T; U)$.

$A : V \rightarrow V^*$ is a monotone, hemicontinuous, coercive and bounded operator. Moreover, we assume $A = \partial\phi$ is the subdifferential of the l.s.c. proper convex function $\phi : V \rightarrow (-\infty, +\infty]$. The realization of A in H , defined by $A_H y = Ay \cap H$ will also be denoted by A . Then A_H is maximal monotone in H . We also assume that $y_0 \in D(\phi) \cap V$.

$F : L^2(0, T; H) \rightarrow L^2(0, T; H)$ is a Lipschitz continuous, causal operator, $B : U \rightarrow H$ is linear and continuous. For α and μ we assume $\alpha \in C^{0,1}(\mathbb{R})$ and bounded, and $\mu \in C^1(\mathbb{R})$ satisfies for all $x \in \mathbb{R}$

$$\mu'(x) \geq \mu_0 > 0, \quad |\mu(x)| \leq c_1|x| + c_2.$$

Finally, we assume that $\psi_1 : U \rightarrow \mathbb{R}_+$ and $\psi_2 : H \rightarrow \mathbb{R}_+$ are continuous, convex mappings and $\psi_3 : L^2(0, T; H) \rightarrow C(0, T; H)$ is a continuous and bounded causal operator. Moreover, ψ_1 shall be bounded from below by a quadratic, i.e.

$$\psi_1(u) \geq c\|u\|_U^2, \quad \text{with a constant } c > 0. \quad (1.3)$$

Remark 1.1

- (1) ψ_3 describes a nonlinear, causal observation operator related to the application discussed in the next section.
- (2) The boundedness assumption for U_{ad} is also motivated from this application. However, one can do without this assumption by using (1.3) and assuming that $\psi_3[y](T)$ is uniformly bounded by a constant independent of y , which indeed is the case in the application.

Convergence results for discretized parabolic control problems have been considered e.g. in [12], [13].

Optimality systems for problems related to (P) have been investigated in [3] and [7]. The aim of the present paper is to investigate convergence properties of the discretized version of our optimization problem (P).

In the next section, we discuss applications of the abstract control problem, with special emphasis on solid–solid phase transitions. In Section 3 we demonstrate the compactness of the solution operator $\Gamma : U_{ad} \rightarrow L^2(0, T; H)$ to (1.1a,b) and prove that (P) admits a solution $u^* \in U_{ad}$. Section 4 is devoted to the investigation of the discretized control problem.

Numerical simulations for the applied control problem presented in Section 2 will be discussed in detail in a forthcoming paper [2].

2 An example: surface hardening of steel

In [3], [5], [7], a model for the surface hardening of steel has been investigated. It consists of a system of ODEs to describe the volume fractions of the occurring solid phases in steel coupled with the following nonlinear heat transfer equation:

$$\rho c_p(\theta)\theta_t - \operatorname{div} (k(\theta) \operatorname{grad} \theta) = -\rho L_1(\theta)F_1(\theta, a) + \rho L_2(\theta)F_2(\theta, a, m) + \alpha(\theta)u, \quad \text{in } Q, \quad (2.1a)$$

$$k(\theta)\frac{\partial \theta}{\partial \nu} + \gamma\theta = 0, \quad \text{in } \Sigma, \quad (2.1b)$$

$$\theta(0) = \theta_0, \quad \text{in } \Omega, \quad (2.1c)$$

where $\Omega \subset \mathbb{R}^3$ with smooth boundary, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$. Here, ρ is a positive constant, and $c_p, k, L_1, L_2, \alpha, \gamma$ are assumed to be positive, bounded and Lipschitz continuous data functions. In addition, c_p shall be bounded from below by a positive constant. The first two terms on the right-hand side of (2.1a) describe the recalcence effects caused by the phase transitions, the last one models a volumetric heat source, e.g. heating by a laser beam (cf. Mazhukin and Samarskii, [8]).

Now, we introduce the Kirchoff transform

$$y = \int_{\theta_0}^{\theta} k(x) dx =: K[\theta]. \quad (2.2)$$

Note that K and K^{-1} are strictly increasing functions. For any data function f , we define

$$\tilde{f}(y) = (f \circ K^{-1})(y).$$

Then (2.1 a-c) is replaced by

$$\begin{aligned} \rho \frac{\tilde{c}_p(y)}{\tilde{k}(y)} y_t - \Delta y &= -\rho \tilde{L}_1(y) F_1(K^{-1}(y), \tilde{a}) \\ &\quad + \rho \tilde{L}_2(y) F_2(K^{-1}(y), \tilde{a}, \tilde{m}) + \tilde{\alpha}(y) u, \quad \text{in } Q, \end{aligned} \quad (2.3a)$$

$$\frac{\partial y}{\partial \nu} + \gamma K^{-1}(y) = 0, \quad \text{in } \Sigma, \quad (2.3b)$$

$$y(0) = 0, \quad \text{in } \Omega. \quad (2.3c)$$

We define $V = H^1(\Omega)$, $H = L^2(\Omega) = U$, and

$$F[y] = -\rho \tilde{L}_1(y) F_1(K^{-1}(y), \tilde{a}) + \rho \tilde{L}_2(y) F_2(K^{-1}(y), \tilde{a}, \tilde{m}). \quad (2.4)$$

From Lemma 2.1 below we can infer that $F : L^2(0, T; H) \rightarrow L^2(0, T; H)$ is Lipschitz continuous and bounded. For μ we take the primitive of $\rho \tilde{c}_p(y)/\tilde{k}(y)$, i.e.

$$\mu(y) = \rho \int_0^y \frac{c_p(K^{-1}(x))}{k(K^{-1}(x))} dx. \quad (2.5)$$

Testing (2.3a) with $v \in V$ and using Green's formula, we obtain finally the system (1.1a,b), where $A : V \rightarrow V^*$ is defined by

$$\langle Ay, v \rangle = \int_{\Omega} \nabla y \cdot \nabla v dx + \gamma \int_{\partial\Omega} K^{-1}[y] v dx$$

It is easy to check that A is monotone and continuous. Moreover, A is the subdifferential (Gâteaux derivate) of the l.s.c. proper convex function

$$\phi(y) = \frac{1}{2} \int_{\Omega} |\nabla y|^2 dx + \gamma \int_{\partial\Omega} j(y) dx,$$

where j is the primitive of the monotone function K^{-1} (cf. Brezis [4]).

Remark 2.1

- (1) Adding a functional $F(v) = \int_{\partial\Omega} g v dx$ to the right-hand side of (1.1a), one could also allow for an inhomogeneity in the boundary condition (2.3b).
- (2) The choice of μ can also be interpreted as a regularization of the classical enthalpy formulation of the Stefan-Problem. Using the same approximation procedure as in Shi et al. [11], we could extend our results to the Stefan Problem.

In the simplest case, the phase transitions can be described by a system of two ODEs:

$$a_t = F_1(\theta, a), \quad \text{in } Q, \tag{2.6a}$$

$$m_t = F_2(\theta, a, m), \quad \text{in } Q, \tag{2.6b}$$

$$a(\cdot, 0) = m(\cdot, 0) = 0, \quad \text{in } \Omega, \tag{2.6c}$$

where

$$F_1(\theta, a) = \frac{1}{\tau_1(\theta)} (\bar{\alpha}(\theta) - a) \mathcal{H}(\bar{\alpha}(\theta) - a),$$

$$F_2(\theta, a, m) = \frac{1}{\tau_2(\theta)} (a \cdot \bar{m}(\theta) - m) \mathcal{H}(a \cdot \bar{m}(\theta) - m) \mathcal{H}(M_s - \theta).$$

Here, a and m are the volume fractions of the occurring phases, $\tau_1, \tau_2, \bar{\alpha}$ and \bar{m} are positive, Lipschitz continuous data functions, \mathcal{H} is a regularization of the Heaviside graph, and M_s is a threshold temperature.

In view of these assumptions, it is an easy application of Gronwall's lemma to prove (cf. [7], Lemma 3.1)

Lemma 2.1

- (1) For $\theta \in L^2(Q)$, (2.6 a-c) has an unique solution $(a, m) \in \left[W^{1,\infty} \left(0, T; L^\infty(\Omega) \right) \right]^2$

(2) Let $\theta_1, \theta_2 \in L^2(Q)$ and (a_i, m_i) be the corresponding solution to (2.6a-c), then there exists a constant $L > 0$ such that

$$\|a_1 - a_2\|_{H^1(0,T;L^2(\Omega))}^2 + \|m_1 - m_2\|_{H^1(0,T;L^2(\Omega))}^2 \leq L \|\theta_1 - \theta_2\|_{L^2(Q)}^2.$$

As an easy corollary we can infer the Lipschitz continuity and boundedness of the operator F defined in (2.4).

To demonstrate the utilization of this model, we present some numerical simulations for laser surface hardening. In addition to the two phases *austenite* and *martensite* described in (2.6a-c), another phase called *bainite* has been included in these simulations. For details concerning the algorithm and the physical data as well as for further results we refer to [5].

Let the part of the workpiece surface to be hardened lie in the plane $z = 0$. Then the laser radiation penetrates into the workpiece according to the radiation transfer equation (cf. [8])

$$G = \kappa_1 G_f e^{\kappa_2 z}, \quad z \leq 0.$$

Here, G is the radiation intensity of the laser beam, G_f the radiation intensity in the focal plane, κ_2 the absorption coefficient and κ_1 the absorptivity of the surface, depending on the angle of incidence, the surface constitution (smoothness, cleanliness) and on the temperature.

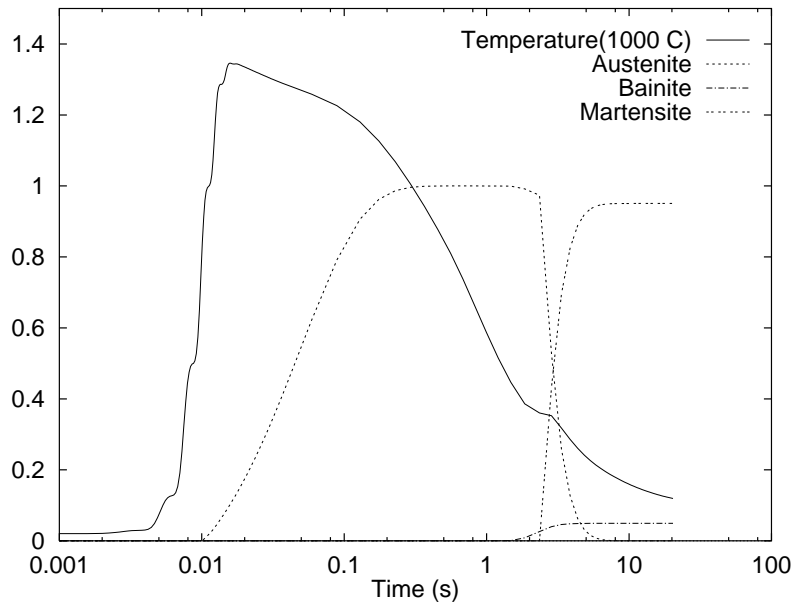


Figure 1: Time evolution of temperature, austenite, bainite and martensite fraction for $x = (0.0, 5.0, -0.01) \in \Omega$.

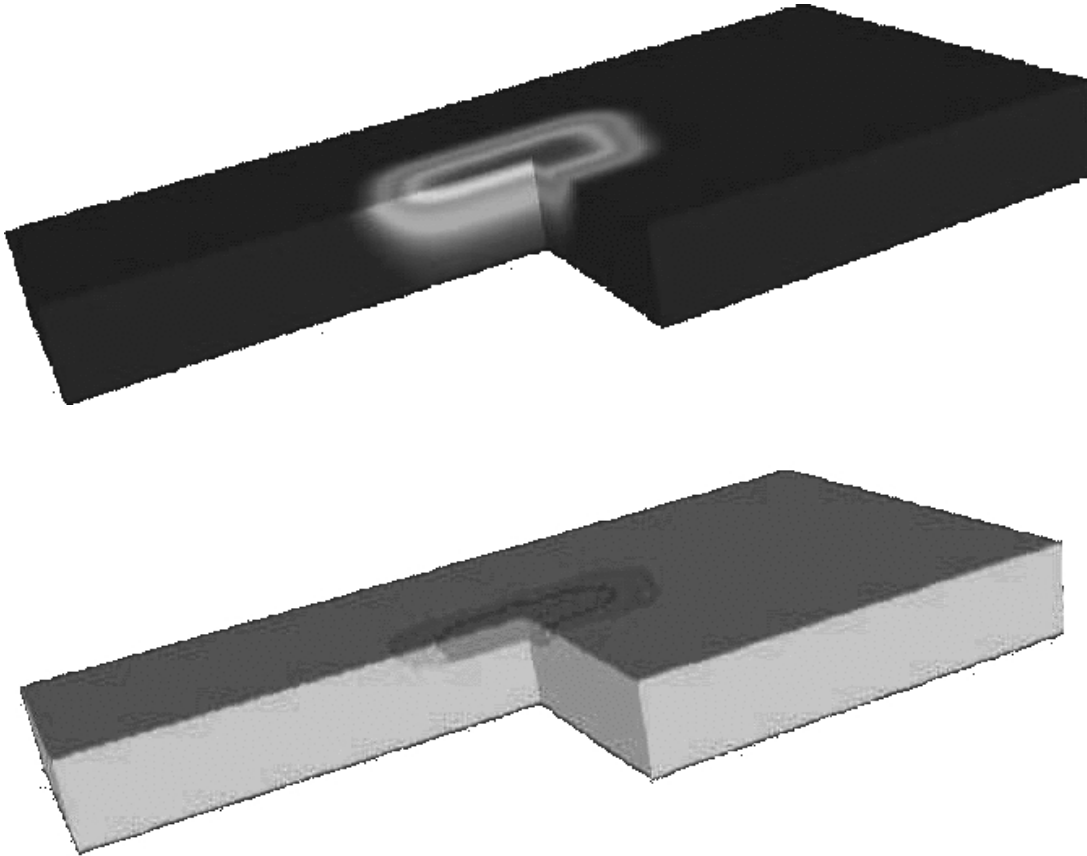


Figure 2: Temperature distribution inside Ω (above) and the resulting hardening profile (below).

In applications, the laser beam moves along the workpiece surface according to a curve $t \rightarrow r(t) \in \mathbb{R}^2$, $t \in [0, T]$, hence we have

$$G_f(x, y, t) = G_0 e^{-\frac{(x-r_1(t))^2 + (y-r_2(t))^2}{2R^2}},$$

where R is the radius of the focusing spot and G_0 its intensity in the spot center. The heat source then takes the form

$$\alpha(\theta)u = \kappa_1 G.$$

We simulate the hardening along a strip around the y -axis on the upper face ($z = 0$) of the cube $\bar{\Omega} = [-2.5, 2.5] \times [0, 10.0] \times [-1.0, 0]$.

Figure 1 shows the time evolution at the point $x = (0.0, 5.0, -0.01) \in \Omega$. Owing to the oscillations of the laser beam, the point is heated by steps. Austenite is formed, and during cooling this austenite is transformed to martensite and a fairly small amount of bainite. In the course of martensite growth, the cooling process is slowed down by the release of latent heat. Finally, Fig. 2 depicts the temperature distribution inside the workpiece during the heating process and the resulting hardening profile.

Now we pass to the optimal control problem. The idea of surface hardening is to increase the volume fraction of martensite (m). To this end we introduce U_{ad} , a closed convex subset of $L^2(0, T; U)$ and the cost functional

$$J(u) = \frac{\beta_1}{2} \int_Q u^2 dxdt + \frac{\beta_2}{2} \int_Q (\theta_m - \theta)^2 \mathcal{H}(\theta_m - \theta) dxdt + \frac{\beta_3}{2} \int_{\Omega} (m(x, T) - m_f)^2 dx \quad (2.7)$$

where $\beta_1, \beta_2, \beta_3 > 0$ are given weights and m_f is a given smooth function (the desired volume fraction of martensite). The second term in (2.7) penalizes temperatures below melting temperature θ_m , since this would destroy the quality of the workpiece surface. Owing to Lemma 2.1, this choice for the cost functional fits in the framework of Section 1.

3 Existence of optimal controls

We are concerned first with the state system (1.1). Let us recall it as

$$\mu'(y(t))y'(t) + Ay(t) + F[y](t) = \alpha(y(t))Bu(t) \text{ a.e. } t \in (0, T), \quad (3.1a)$$

$$y(0) = y_0. \quad (3.1b)$$

We introduce the solution operator $\Gamma : L^2(0, T; U) \rightarrow L^2(0, T; H)$ defined by $\Gamma u = y$ where u is the control in the right-hand side of Eqn. (3.1) and y the corresponding solution.

Lemma 3.1

The operator Γ defined above is compact from $L^2(0, T; U)$ to $L^2(0, T; H)$ in the sense that for any sequence

$$u_n \rightarrow u \text{ weakly in } L^2(0, T; U)$$

the corresponding sequence $y_n = \Gamma u_n$ satisfies

$$\begin{aligned} y_n \rightarrow y & \quad \text{strongly in } C(0, T; H) \text{ and} \\ & \quad \text{weakly star in } L^\infty(0, T, V), \end{aligned}$$

where $y = \Gamma u$.

Proof.

Let $\{u_n\} \subset L^2(0, T; U)$ such that

$$u_n \rightarrow u \text{ weakly in } L^2(0, T; U),$$

and $\{y_n\} \subset L^2(0, T; H)$ defined by $y_n = \Gamma u_n$. We write the corresponding Eqn. (3.1), we multiply it by $y_n'(t)$ and we integrate from 0 to t to obtain

$$\begin{aligned} \int_0^t \mu'(y(s)) \|y_n'(s)\|_H^2 ds + \phi(y_n(t)) - \phi(y_0) + \\ \int_0^t (y_n'(s), F[y_n](s)) ds = \int_0^t (y_n'(s), \alpha(y_n(s)) B u_n(s)) ds, \end{aligned} \quad (3.2)$$

and therefore

$$\begin{aligned} \mu_0 \int_0^t \|y_n'(s)\|_H^2 ds + \phi(y_n(t)) \leq M_1 + M_2 \int_0^t \|y_n'(s)\|_H \cdot \|u_n(s)\|_U ds + \\ + \int_0^t (M_3 + M_4 \|y_n(s)\|_H) \|y_n'(s)\|_H ds. \end{aligned}$$

We shall denote by M_i several positive constants independent of indices like n, h , etc. Let $\delta > 0$ be an arbitrary constant, then we get from Young's inequality and the inequality above

$$\begin{aligned} \mu_0 \int_0^t \|y_n'(s)\|_H^2 ds + \phi(y_n(t)) \leq M_1 + M_2 \left(\delta \int_0^t \|y_n'(s)\|_H^2 ds + \right. \\ \left. \delta^{-1} \int_0^t \|u_n(s)\|_U^2 ds \right) + \delta \int_0^t \|y_n'(s)\|_H^2 ds + \\ \delta^{-1} \int_0^T (2M_3^2 + 2M_4^2 \|y_n(s)\|_H^2) ds, \end{aligned}$$

which yields

$$\left[\mu_0 - (M_2 + 1)\delta \right] \int_0^t \|y_n'(s)\|_H^2 ds \leq M_{5,\delta} + M_{6,\delta} \int_0^t \|u_n(s)\|_U^2 ds + M_{7,\delta} \int_0^t \|y_n(s)\|_H^2 ds.$$

We now choose $\delta > 0$ such that $\mu_0 > (M_2 + 1)\delta$ and apply Gronwall's lemma to infer that $\{y_n'\}$ is bounded in $L^2(0, T; H)$ and $\{y_n\}$ is equally uniformly continuous in $C(0, T; H)$.

Moreover, the above estimate yields

$$\phi(y_n(t)) \leq M_8.$$

Hence $\{\phi(y_n(t))\}$ is uniformly bounded and since the coerciveness of the operator A implies the one of ϕ , we infer that $\{y_n\}$ is uniformly bounded in $L^\infty(0, T; V)$. By Corollary 4 in [10], we have, extracting a subsequence

$$y_n \longrightarrow y \begin{cases} \text{strongly in } & C(0, T; H) \text{ and} \\ \text{weakly star in } & L^\infty(0, T; V). \end{cases} \quad (3.3)$$

Using Lebesgue's convergence theorem we obtain

$$\begin{aligned} F(y_n) &\rightarrow F(y) \text{ strongly in } L^2(0, T; H), \\ \alpha(y_n) &\rightarrow \alpha(y) \text{ strongly in } L^2(0, T; H). \end{aligned}$$

Now from (3.1) it follows that

$$Ay_n = \partial\phi(y_n) \rightarrow w \text{ weakly in } L^2(0, T; H).$$

Since the operator $\partial\phi$ from $L^2(0, T; H)$ into itself is strongly-weakly closed, using (3.3) we may conclude that $w = \partial\phi(y)$.

Finally, we also have, extracting a subsequence

$$Bu_n \rightarrow Bu \text{ weakly in } L^2(0, T; H),$$

and hence $y = \Gamma u$.

The convergence holds for the whole sequence $\{u_n\}$, since the solution to (1.1) is uniquely defined. \square

We can now give

Theorem 3.2 *Under the above assumptions, Problem (P) has at least one optimal pair*

$$[u^*, y^*] \in L^2(0, T; U) \times L^2(0, T; H).$$

Proof.

Problem (P) may be written as $\inf \{J(u); u \in U_{ad}\}$.

Let us denote by l the infimum above and let $\{u_n\} \subset U_{ad}$ be a minimizing sequence, that is

$$J(u_n) \rightarrow l \text{ in } \mathbb{R}. \quad (3.4)$$

Since the set U_{ad} is bounded (cf. Remark 1.1), it follows that $\{u_n\}$ is bounded and, extracting a subsequence, we have

$$u_n \rightarrow u^* \text{ weakly in } \mathcal{U} = L^2(0, T; U).$$

The set U_{ad} is convex and closed and therefore it is weakly closed. Hence $u^* \in U_{ad}$. Since ψ_1 is a convex integrand it follows that it is weakly l.s.c. Using Lemma 3.1 and the properties of ψ_2 and ψ_3 we get that J is also weakly l.s.c. and from (3.4) we obtain that $J(u^*) \leq l$, which means that u^* is a solution of Problem (P). \square

4 Finite Element approximation

Let $h > 0$ be the discretization parameter destined to converge to 0. For any $h > 0$ we introduce the finite dimensional linear subspace $V_h \subset V$ and the linear and continuous operator $r_h : H \rightarrow V_h$ (some interpolation or projection operator), the finite dimensional linear subspace $U_h \subset U$ and the corresponding linear and continuous operator $s_h : U \rightarrow U_h$. We make the following assumption:

(H1)

(i) There exist constants $c_1, c_2 > 0$ independent of h such that

$$\begin{aligned} \|r_h\|_{L(H, V)} &\leq c_1 \text{ for any } h > 0, \\ \|s_h\|_{L(U, U)} &\leq c_2 \text{ for any } h > 0. \end{aligned}$$

(ii) $r_h v \rightarrow v$ strongly in H , for any $v \in H$,
 $s_h u \rightarrow u$ strongly in U , for any $u \in U$.

The approximation of (3.1) is

$$\mu'_h(y_h(t))y'_h(t) + A_h y_h(t) + F_h[y_h](t) = \alpha_h(y_h(t))B_h u_h(t) \text{ a.e. } t \in (0, T), \quad (4.1a)$$

$$y_h(0) = r_h y_0. \quad (4.1b)$$

The operator $A_h : V_h \rightarrow V_h$ is defined by $(A_h y_h, v_h) = (A y_h, v_h)$ for any $y_h, v_h \in V_h$.

The operator $F_h : L^2(0, T; V_h) \rightarrow L^2(0, T; H)$ is defined as the restriction of F to $L^2(0, T; V_h)$, α_h is just α restricted to V_h . The same is valid for μ_h and μ'_h .

The operator $B_h : U_h \rightarrow H$ is the restriction of B to U_h , i.e.

$$(B_h u_h, v) = (B u_h, v) \text{ for any } u_h \in U_h \text{ and any } v \in H.$$

Remark 4.1 *In the applications discussed in Section 2, we have $H = U = L^2(\Omega)$. In the FE approximation we can then even take $V_h = U_h \subset L^2(\Omega)$ and $r_h = s_h$. In such a case $B_h = B$ is of course the identity operator and $B(V_h) = V_h$.*

Now, (4.1) takes the form

$$\mu'(y_h(t))y_h'(t) + Ay_h(t) + F[y_h](t) = \alpha(y_h(t))Bu_h(t) \text{ a.e. } t \in (0, T), \quad (4.2a)$$

$$y_h(0) = r_h y_0. \quad (4.2b)$$

For any $h > 0$ fixed, we introduce the operator $\Gamma_h : L^2(0, T; U_h) \rightarrow L^2(0, T; V_h)$ defined by $\Gamma_h u_h = y_h$, where u_h is the control in the right-hand side of (4.2) and y_h is the corresponding solution.

The equivalent of Lemma 3.1 is given by

Lemma 4.1

The operator Γ_h defined above is compact from $L^2(0, T; U_h)$ to $L^2(0, T; V_h)$.

Now, we pass to the approximate optimal control problem, which is

$$(P_h) \quad \text{Minimize } J_h(u_h)$$

over the set of all functions $u_h \in \mathcal{U}_h = L^2(0, T; U_h)$ subject to the state equation (4.2) and to the control constraint

$$u_h \in U_{ad}^h. \quad (4.3)$$

The cost functional J_h is defined by

$$J_h(u_h) = \int_0^T (\psi_1(u_h) + \psi_2(y_h)) dt + \psi_3[y_h](T),$$

where $y_h = \Gamma_h u_h$ is the corresponding solution to (4.2).

Here $U_{ad}^h \subset \mathcal{U}_h$ should be an adequate approximation of U_{ad} . To this end, let us recall the concept of convergence in the sense of Mosco ([9], p. 595, see also [6], p. 41).

Definition 4.1

We say that $\lim_{h \rightarrow 0} U_{ad}^h = U_{ad}$ (in the sense of Mosco), if and only if the following conditions are satisfied :

- (i) *For any $u \in U_{ad}$ there exists a sequence $\{u_h\}$ such that $u_h \in U_{ad}^h$ for any $h > 0$ and $u_h \rightarrow u$ strongly in \mathcal{U} .*
- (ii) *If $\{u_h\}$ is a sequence such that $u_h \rightarrow u$ weakly in \mathcal{U} and $u_h \in U_{ad}^h$ for any $h > 0$, then $u \in U_{ad}$.*

One obvious way to define U_{ad}^h is to take

$$U_{ad}^h = s_h U_{ad} \quad (4.4)$$

It is easy to see that in such a case U_{ad}^h is a closed, convex and bounded set. We make the following hypothesis to be satisfied by the FE approximation:

(H2) $U_{ad}^h \subset U_{ad}$ for any $h > 0$.

Lemma 4.2

If the hypotheses (H1) and (H2) are valid, together with (4.4), then $\lim_{h \rightarrow 0} U_{ad}^h = U_{ad}$ in the sense of Definition 4.1.

Proof.

(i) For any $u \in U_{ad}$, we consider the sequence $\{u_h\}$, where $u_h = s_h u$. According to (4.4) $u_h \in U_{ad}^h$ for any h and (H1)(ii) ensures that $u_h \rightarrow u$ strongly in \mathcal{U} .

(ii) Let $\{u_h\}$ be a sequence such that $u_h \in U_{ad}^h$ for every h and $u_h \rightarrow u$ weakly in \mathcal{U} . Since, by (H2), $U_{ad}^h \subset U_{ad}$ for any h , it follows that $\{u_h\} \subset U_{ad}$. But U_{ad} is a closed convex subset of \mathcal{U} and therefore U_{ad} is weakly closed in \mathcal{U} and we infer that $u \in U_{ad}$. \square

In view of the FE convergence established in Theorem 4.2 below, we make also the following hypothesis (see also (3.4)):

(H3) $\psi_1(s_h u) \rightarrow \psi_1(u)$ in $L^1(0, T)$, for any $u \in L^2(0, T; U)$.

Coming back to Problem (P_h) , we have as in the case of Problem (P) the existence result

Theorem 4.1 *Problem (P_h) has at least one optimal pair $[u_h^*, y_h^*] \in L^2(0, T; U_h) \times L^2(0, T; V_h)$.*

We pass now to the convergence result for a sequence of solutions to Problems (P_h) .

Theorem 4.2 *For any $h > 0$ let $[u_h^*, y_h^*]$ be an optimal pair for Problem (P_h) . Under the above assumptions, for $h \rightarrow 0$, we have*

$$\begin{aligned} u_h^* &\rightarrow \bar{u} \text{ weakly in } L^2(0, T; U), \\ y_h^* &\rightarrow \bar{y} \text{ strongly in } C(0, T; H), \end{aligned}$$

where $[\bar{u}, \bar{y}]$ is an optimal pair for Problem (P) .

Proof.

The proof will be done in 3 steps:

Step 1 – The sequences $\{u_h^*\}$ and $\{y_h^*\}$ are convergent to \bar{u} and \bar{y} respectively.

Step 2 – $[\bar{u}, \bar{y}]$ is an admissible pair for Problem (P).

Step 3 – $[\bar{u}, \bar{y}]$ is an optimal pair for Problem (P).

We begin with

Step 1. Let u_h^* be an optimal control for (P_h) . Since $u_h^* \in U_{ad}^h$ for any $h > 0$, using also the hypothesis (H2), we find that $\{u_h^*\} \subset U_{ad}$ which is a bounded set in $L^2(0, T; U)$. Hence $\{u_h^*\}$ is bounded in $L^2(0, T; U)$ (cf. Remark 1.1) and, extracting a subsequence, we have

$$u_h^* \rightarrow \bar{u} \text{ weakly in } L^2(0, T; U). \quad (4.5)$$

Moreover U_{ad} is a convex closed set and therefore it is weakly closed. Hence $\bar{u} \in U_{ad}$, that is \bar{u} satisfies the control constraint (1.2) of Problem (P).

We multiply now Eqn. (4.2) by $y_h^{* \prime}$, we integrate from 0 to $t \in (0, T]$ and we obtain (we omit the optimality upper index $*$)

$$\begin{aligned} & \int_0^t \mu'(y_h(s)) \|y_h'(s)\|_H^2 ds + \phi(y_h(t)) - \phi(r_h y_0) \\ &= \int_0^t (\alpha(y_h(s)) B u_h(s), y_h'(s)) ds - \int_0^t (F[y_h](s), y_h'(s)) ds, \end{aligned}$$

and therefore

$$\begin{aligned} & \mu_0 \int_0^t \|y_h'(s)\|_H^2 ds + \phi(y_h(t)) \\ & \leq M_1 + M_2 \int_0^t \|y_h'(s)\|_H \cdot \|u_h(s)\|_U + \int_0^t (M_3 + M_4 \|y_h(s)\|_H) \cdot \|y_h'(s)\|_H ds \\ & \leq M_1 + M_2 \left(\delta \int_0^t \|y_h'(s)\|_H^2 ds + \delta^{-1} \int_0^t \|u_h(s)\|_U^2 ds \right) \\ & \quad + \delta \int_0^t \|y_h'(s)\|_h^2 ds + \delta^{-1} \int_0^t (2M_3^2 + 2M_4^2 \|y_h(s)\|_H^2) ds. \end{aligned} \quad (4.6)$$

Finally this leads to

$$[\mu_0 - (M_2 + 1)\delta] \int_0^t \|y_h'(s)\|_H^2 ds \leq M_{5,\delta} + M_{6,\delta} \int_0^t \|u_h(s)\|_U^2 ds + M_{7,\delta} \int_0^t \|y_h(s)\|_H^2 ds.$$

If we choose $\delta > 0$ such that $\mu_0 > (M_2 + 1)\delta$ and if we take also into account the boundedness of $\{u_h^*\}$ in $L^2(0, T; U)$, applying Gronwall's lemma we get that $\{(y_h^*)'\}$ is bounded in $L^2(0, T; H)$.

Inequality (4.6) also implies

$$\phi(y_h^*(t)) \leq M_8 \text{ for any } t \in [0, T].$$

Since ϕ is coercive we infer that $\{y_h^*\}$ is bounded in $L^\infty(0, T; V)$. Applying again Corollary 4 of [10], we obtain

$$y_h^* \rightarrow \bar{y} \text{ strongly in } C(0, T; H) \text{ and weakly star in } L^\infty(0, T; V), \quad (4.7)$$

which finishes the proof of Step 1.

Step 2. Using (4.7) and the properties of α we get readily

$$\alpha(y_h^*) \rightarrow \alpha(\bar{y}) \text{ in } L^2(0, T; H). \quad (4.8)$$

Since $\{u_h^*\}$ is bounded in $L^2(0, T; U)$ it follows that $\{Bu_h^*\}$ is bounded in $L^2(0, T; H)$ and using also (4.5) we get that

$$Bu_h^* \rightarrow B\bar{u} \text{ weakly in } L^2(0, T; H). \quad (4.9)$$

Owing to the Lipschitz continuity of F , we have in the norm of $L^2(0, T; H)$

$$\|F[y_h^*] - F[\bar{y}]\| \leq M_9 \|y_h^* - \bar{y}\|. \quad (4.10)$$

In view of (4.7) we conclude that

$$F(y_h^*) \rightarrow F(\bar{y}) \text{ strongly in } L^2(0, T; H). \quad (4.11)$$

By comparison in (4.2) we see that $\{Ay_h^*\}$ is bounded in $L^2(0, T; H)$. Hence

$$Ay_h^* \rightarrow \xi \text{ weakly in } L^2(0, T; H). \quad (4.12)$$

By the monotonicity of A we have

$$\int_0^T (Ay_h^*(t) - Av, y_h^*(t) - v) dt \geq 0 \quad \text{for any } v \in V.$$

Passing to the limit in the above inequality with $h \rightarrow 0$, using (4.7) and (4.12), we get

$$\int_0^T (\xi(t) - Av, \bar{y}(t) - v) dt \geq 0 \quad \text{for any } v \in V.$$

Since A is maximal monotone it follows that $\xi(t) = A\bar{y}(t)$ a.e. $t \in (0, T)$ and therefore

$$Ay_h^* \rightarrow A\bar{y} \text{ weakly in } L^2(0, T; H). \quad (4.13)$$

From the convergence properties already established ((4.8), (4.9), (4.11), (4.13)), it follows that $\bar{y} = \Gamma\bar{u}$.

Since we have also demonstrated (see Step 1) that $\bar{u} \in U_{ad}$, it is clear that $[\bar{u}, \bar{y}]$ is an admissible pair for Problem (P).

Step 3. Let u^* be an optimal control for Problem (P). We consider equation (4.2) in which the control is taken to be $s_h u^*$, i.e.

$$\mu'(y_h(t))y_h'(t) + Ay_h(t) + F[y_h](t) = \alpha(y_h(t))B(s_h u^*(t)) \text{ a.e. } t \in (0, T), \quad (4.14a)$$

$$y_h(0) = r_h y_0. \quad (4.14b)$$

According to (H1) we have

$$s_h u^*(t) \rightarrow u^*(t) \text{ strongly in } U \text{ for any } t \in [0, T], \quad (4.15)$$

and by Lebesgue's convergence theorem it follows that

$$s_h u^* \rightarrow u^* \text{ strongly in } L^2(0, T; U).$$

On the other hand we have in the $L^2(0, T; U)$ norm

$$\|s_h u^*\| \leq \|s_h u^* - u^*\| + \|u^*\|$$

and hence, for h sufficiently small, the following estimate is valid

$$\|s_h u^*\|_{L^2(0, T; U)} \leq M_9.$$

Since $B \in L(U, H)$, this yields

$$\|Bs_h u^*\|_{L^2(0, T; H)} \leq M_{10}.$$

From (4.15) we also get

$$Bs_h u^*(t) \rightarrow Bu^*(t) \text{ strongly in } H \text{ for any } t \in [0, T],$$

and using again Lebesgue's theorem we have

$$Bs_h u^* \rightarrow Bu^* \text{ strongly in } L^2(0, T; H).$$

Arguing as in the previous steps, we find that the sequence $\{y_h\}$, where for every h fixed y_h is the solution of Eqn. (4.14a,b), satisfies

$$\begin{aligned} y'_h &\rightarrow y' \text{ weakly in } L^2(0, T; H), \\ y_h &\rightarrow y \text{ strongly in } C(0, T; H) \text{ and weakly star in } L^\infty(0, T; V), \\ y_h(t) &\rightarrow y(t) \text{ strongly in } H, \text{ a.e. } t \in (0, T), \\ Ay_h &\rightarrow Ay \text{ weakly in } L^2(0, T; H), \\ F[y_h] &\rightarrow F[y] \text{ strongly in } L^2(0, T; H), \\ \alpha(y_h) &\rightarrow \alpha(y) \text{ strongly in } L^2(0, T; H). \end{aligned}$$

Passing to the limit in (4.14) with $h \rightarrow 0$ yields $y = y^*$, where y^* is the solution of Eqn. (3.1) corresponding to u^* , i.e. $y^* = \Gamma u^*$. Therefore

$$y_h(t) \rightarrow y^*(t) \text{ strongly in } H, \text{ a.e. } t \in (0, T)$$

and

$$y_h \rightarrow y^* \text{ strongly in } C(0, T; H) \text{ and weakly star in } L^\infty(0, T; V). \quad (4.16)$$

Let $[u_h^*, y_h^*]$ be an optimal pair for Problem (P_h) . Then

$$J(u_h^*) \leq J(u_h) \text{ for any } u_h \in L^2(0, T; U_h).$$

We take $u_h := s_h u^*$ and we get

$$J(u_h^*) \leq J(s_h u^*),$$

where the corresponding solutions to the state equation (4.2) are $y_h = \Gamma_h(s_h u^*)$ and $y_h^* = \Gamma_h u_h^*$.

From Step 1 we have the converge properties (4.5) and (4.7). Using also (4.16) and (H3) we pass to the limit in (4.18) with $h \rightarrow 0$ and we obtain

$$J(\bar{u}) \leq J(u^*).$$

Since, by Step 2, $[\bar{u}, \bar{y}]$ is an admissible pair for Problem (P), it follows now that it is also an optimal pair, thereby completing the proof of the theorem. \square

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