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# Regularity of weak solutions of Maxwell's equations with mixed boundary conditions


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**Abstract**

In this paper global $H^s$ and $L^p$-regularity-results for the stationary and transient Maxwell-equations with mixed boundary-conditions in a bounded spatial domain are proved. First it is shown that certain elements belonging to the fractional-order domain of the Maxwell-operator belong to $H^s(\Omega)$ for sufficiently small $s > 0$. It follows from this regularity result that $H^s(\Omega)$ is an invariant subspace of the unitary group corresponding to the homogeneous Maxwell-equations with mixed boundary-conditions. In the case that a possibly nonlinear conductivity is present a $L^p$-regularity-theorem for the transient equations is proved.

# 1 Introduction

The subject of this paper are global $H^s$- and $L^p$-regularity theorems for the stationary and transient Maxwell equations in a bounded domain with mixed boundary-conditions describing the electromagnetic field, [10].

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with piecewise smooth boundary $\partial\Omega$, $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$. The initial-boundary-value problem

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{H}, \text{ and } \mu\partial_t\mathbf{H} = - \text{ curl } \mathbf{E}, \tag{1.1}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0,\infty) \times \Gamma_1, \quad \vec{n} \wedge \mathbf{H} = 0 \text{ on } (0,\infty) \times \Gamma_2, \tag{1.2}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{H}(0,x) = \mathbf{H}_0(x). \tag{1.3}$$

with $\mathbf{E}_0, \mathbf{H}_0 \in L^2(\Omega)$ is considered. Such boundary value problems arise for example in semiconductor modelling, see [6], [7], where $\Gamma_2$ is the insulating boundary and $\Gamma_1$ represents the electric contacts.

In (1.1) the variable matrices $\varepsilon, \mu \in L^\infty(\Omega, \mathbb{C}^{3\times 3})$ are assumed to be uniformly positive.

The following $H^s$-regularity-result will be proved.

There exist $\bar{s} \in (0, s_0)$ depending only on $\Omega, \Gamma_1, \varepsilon$ and $\mu$, such that for all $s \in [0, \bar{s}]$ and $\mathbf{E}_0, \mathbf{H}_0 \in H^s(\Omega)$ one has

$$(\mathbf{E}, \mathbf{H}) \in C([0,\infty), H^s(G)) \tag{1.4}$$

Here $H^s(\Omega)$ denotes the $L^2$-Sobolev space of fractional order s, see [18].
For this purpose it is assumed that $\varepsilon, \mu$ have the multiplier property

$$\varepsilon\mathbf{F} \in H^{s_0}(\Omega) \text{ and } \mu\mathbf{F} \in H^{s_0}(\Omega) \text{ for all vector-fields } \mathbf{F} \in H^{s_0}(\Omega)$$

for some $s_0 \in (0, 1/2)$.

This condition is fulfilled for $s_0 \in (0, 1/2)$ in the case that the coefficients are piecewise smooth, that means $\varepsilon, \mu$ may have jump discontinuities on finitely many 2 dimensional surfaces. In particular a piecewise constant $\varepsilon, \mu$ is admissible, which is important for many applications.
In general 1.4 does not hold for $s \geq 1/2$ under these general assumptions on $\Omega, \Gamma_1$ and the coefficients.

The proof of 1.4 relies on the following $H^s$-regularity-result for the stationary Maxwell-equations.

There exist $\bar{s} \in (0, s_0)$ depending only on $\Omega, \Gamma_1$ and $\varepsilon$, such that for all $s \in [0, \bar{s}]$ and $\mathbf{e} \in W^s(\Omega, \Gamma_1)$ with $\varepsilon\mathbf{e} \in X^s(\Omega, \Gamma_1)$ one has

$$\mathbf{e} \in H^s(\Omega). \tag{1.5}$$

Here $W^s(\Omega, \Gamma_1)$ and $X^s(\Omega, \Gamma_1)$ denote for $s \in [0, 1]$ the complex interpolation spaces $[L^2(\Omega), W(\Omega, \Gamma_1)]_s$ and $[L^2(\Omega), X(\Omega, \Gamma_1)]_s$, where $W(\Omega, \Gamma_1)$ denotes the space of all $\mathbf{E} \in L^2(\Omega)$ with curl $\mathbf{E} \in L^2(\Omega)$ and $\vec{n} \wedge \mathbf{E} = 0$ on $\Gamma_1$ and $X(\Omega, \Gamma_1)$ denotes the space of all $\mathbf{D} \in L^2(\Omega)$ with div $\mathbf{D} \in L^2(\Omega)$ and $\vec{n} \cdot \mathbf{D} = 0$ on $\Gamma_2$.

The regularity-results 1.4 and 1.5 have already been obtained in [7] for the case that the spatial domain $\Omega$ is two-dimensional using a $H^{1+s}$-regularity-result for mixed second-order elliptic boundary-value-problems similar to the $W^{1,p}$- result in [5]. However, in this paper the general three-dimensional case is considered.

1.5 implies that the solution $u \in H^1(\Omega)$ of the mixed elliptic boundary-value-problem

$$\text{div } (\varepsilon \nabla u) = F \in L^2(\Omega), \quad u = 0 \text{ on } \Gamma_1, \text{ and } \partial_n u = 0 \text{ on } \Gamma_2,$$

satisfies $\nabla u \in H^s(\Omega)$ for all $s \in [0, \bar{s}]$, see [2], [4], [5], [15], [16] and [17]. This follows from 1.5 using the fact that $\nabla u \in W(\Omega, \Gamma_1)$ and $\varepsilon \nabla u \in X(\Omega, \Gamma_1)$

A further consequence of 1.5 is that $W(\Omega, \Gamma_1) \cap \varepsilon^{-1}(X(\Omega, \Gamma_1))$ is compactly imbedded in $L^2(\Omega)$. This has already been proved in [8] and in [14], [19] without mixed boundary-conditions.

In section 6 a $L^p$-regularity-theorem for Maxwell's equations with conductivity

$$\varepsilon \partial_t \mathbf{E} = \text{ curl } \mathbf{H} - \sigma \mathbf{E}, \text{ and } \mu \partial_t \mathbf{H} = - \text{ curl } \mathbf{E}, \tag{1.6}$$

supplemented by the same initial-boundary-conditions as in1.1-1.3 is proved.

Here $\sigma \in L^\infty(\Omega)$ represents the electrical conductivity. It is shown that there exists some $\tilde{p} \in (2, \infty)$ depending only on $\Omega, \Gamma_1, \varepsilon$ and $\mu$, such that

$(\mathbf{E},\mathbf{H}) \in C([0,\infty), L^p(\Omega))$ for all $p \in [2,\tilde{p}]$ and initial-states $(\mathbf{E}_0,\mathbf{H}_0) \in L^p(\Omega)$ with curl $\mathbf{E}_0 \in L^2(\Omega)$, curl $\mathbf{H}_0 \in L^2(\Omega)$, $\vec{n} \wedge \mathbf{E}_0 = 0$ on $\Gamma_1$ and $\vec{n} \wedge \mathbf{H}_0 = 0$ on $\Gamma_2$.

Here the $H^s$-regularity result 1.5 and the $W^{1,p}$-result in [5] are used. The term $\sigma\mathbf{E}$ in 1.6 can also be replaced by certain nonlinear operators modelling for example a nonlinear resistor, see section 6.

# 2 Notation, assumptions and auxiliary lemmata

Suppose that $\Omega \subset I\!R^3$ is a bounded domain, $\Gamma_1 \subset \partial\Omega$ and let $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$.

Then the following function-spaces are intrduced.

For $s \in [0,1]$ the fractional-order Sobolev-space is denoted by $H^s(\Omega)$. It coincides with the complex interpolation space $[L^2(\Omega), H^1(\Omega)]_s$ between $L^2(\Omega)$ and $H^1(\Omega)$.

Let $Z(\Omega,\Gamma_1)$ be the closure of $C_0^\infty(I\!R^3 \setminus \overline{\Gamma_1})$ in $H^1(\Omega)$. Next, $H_{curl}(\Omega)$ denotes the space of all $\mathbf{E} \in L^2(\Omega)$ with curl $\mathbf{E} \in L^2(\Omega)$. The space of all $\mathbf{E} \in H_{curl}(\Omega)$ with $\vec{n} \wedge \mathbf{E} = 0$ on $\Gamma_1$ in the sense that

$$\int_\Omega (\mathbf{E} \text{ curl } \mathbf{h} - \mathbf{h} \text{ curl } \mathbf{E}) dx = 0 \text{ for all } \mathbf{h} \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2}) \tag{2.7}$$

in denoted by $W(\Omega,\Gamma_1)$.
Let $X(\Omega,\Gamma_1)$ be the space of all $\mathbf{D} \in L^2(\Omega)$ with div $\mathbf{D} \in L^2(\Omega)$ and $\vec{n}\mathbf{D} = 0$ on $\Gamma_2$ in the sense that

$$\int_\Omega \mathbf{D}\nabla\varphi dx = -\int_\Omega \text{div } \mathbf{D}\varphi dx \text{ for all } \varphi \in Z(\Omega,\Gamma_1).$$

Next, $W^s(\Omega,\Gamma_1)$ and $X^s(\Omega,\Gamma_1)$ denote for $s \in [0,1]$ the complex interpolation spaces $[L^2(\Omega), W(\Omega,\Gamma_1)]_s$ and $[L^2(\Omega), X(\Omega,\Gamma_1)]_s$.
Finally, let $W_0(\Omega,\Gamma_1)$ and $X_0(\Omega,\Gamma_1)$ be the space of all $\mathbf{E} \in W(\Omega,\Gamma_1)$ and $\mathbf{D} \in X(\Omega,\Gamma_1)$ with curl $\mathbf{E} = 0$ and div $\mathbf{D} = 0$ respectively.

In the sequel the following lemma will be used frequently, which says that piecewise smooth functions are $H^s$-multipliers for $s < 1/2$.

**Lemma 1** *Let $U \subset I\!R^N$ be a Lipschitz-domain and $s \in [0,1/2)$.*
*Assume further that the function $f : I\!R^N \to \mathbb{C}$ has the form $g = \sum_{k=1}^n \chi_{C_k} f_k$, where the bounded functions $f_k \in C^\alpha(I\!R^N)$, are Hölder-continuous for some $\alpha > s$ and $\chi_{C_k}$ are the characteristic functions of Lipschitz-domains $C_k \subset I\!R^N$.*
*Then $gf \in H^s(U)$ for all $f \in H^s(U)$.*

**Proof:**
For each Lipschitz-domain $G \subset I\!R^N$ and $s < 1/2$ one has

$$\chi_G F \in H^s(I\!R^N) \text{ with } ||\chi_G F||_{H^s} \leq c_{G,s}||F||_{H^s} \text{ for all } F \in H^s(I\!R^N) \tag{2.8}$$

with some $c_{G,s} \in (0,\infty)$ independent of $F$. This follows from the well-known fact that the extension $\tilde{\varphi} \in L^2(I\!R^N)$ of a function $\varphi \in H^s(U)$ by zero outside $U$ belongs to $H^s(I\!R^N)$, provided $s < 1/2$, see [11], chapter 11.3. Let $u \in H^s(U)$. Since $U$ is a Lipschitz-domain and $s < 1/2$, the extension $\tilde{u}$ of $u$ defined by $\tilde{u}(x) = u(x)$ if $x \in U$ and $\tilde{u}(x) = 0$ if $x \in I\!R^N \setminus U$ belongs to $H^s(I\!R^N)$. Moreover, (2.8) yields $\chi_{C_j}\tilde{u} \in H^s(I\!R^N)$. Next,

$$f_j\chi_{C_j}\tilde{u} \in H^s(I\!R^N) \text{ for all } j \in \{1,..,n\}. \tag{2.9}$$

Here the well known fact is used that bounded functions in $C^\alpha(I\!R^N)$ are $H^s$-multipliers, provided that $\alpha > s_0$. This follows for example easily from the representatation

$$||f||^2_{H^s} = ||f||^2_{L^2} + s\int_0^\infty t^{-(1+2s)}\sum_{k=1}^N ||f(te_k+\cdot) - f||^2_{L^2}dt$$

of the $H^s$-norm for $s \in (0,1)$, $f \in H^s$, where $e_k$ is the unit-vector in the $x_k$ direction, see [11], ch.1.10.2.
Finally, (2.9) yields $gu = \sum_{j=1}^n \left(f_j\chi_{C_j}\tilde{u}\right)|_U \in H^s(U)$.

**Lemma 2** *Let $U, V \subset I\!R^3$ be open sets, $p \in [1,\infty)$, $\mathbf{w} \in L^p_{loc}(U)$ with curl $\mathbf{w} \in L^p_{loc}(U)$. Moreover, let $T : V \to U$ be a Bi-Lipschitz transformation.*
*Define*

$$\mathbf{f}(y) \stackrel{\text{def}}{=} DT(y)^*\mathbf{w}(T(y)) \text{ for } y \in V.$$

*Then $\mathbf{f} \in L^p_{loc}(V)$ with curl $\mathbf{f} \in L^p_{loc}(V)$ and*

$$(\,curl\,\mathbf{f})(y) = M_T(y)(\,curl\,\mathbf{w})(T(y)) \text{ for } y \in V, \tag{2.10}$$

*where $M_T \in L^\infty_{loc}(V, I\!R^{3\times3})$ is defined by $M_T(y) \stackrel{\text{def}}{=} [det\ DT(y)]DT(y)^{-1}$.*

This can be found in the appendix of [9]. The main idea is to approximate $\mathbf{w}$ and $T$ by smooth functions.

# 3 The regularity-theorem for a rectangle

Througout this section let $G \subset I\!R^3$ be a rectangle, i.e. $G \stackrel{\text{def}}{=} (0,a) \times (0,b) \times (0,c)$ with $a,b,c \in (0,\infty)$. Let

$$\{(x_1,x_2,0) : x_1 \in (0,a), x_2 \in (0,b)\} \subset S_2 \subset \{(x_1,x_2,0) : x_1 \in [0,a], x_2 \in [0,b]\},$$

i. e. $S_2 \subset \partial G$ is one side of the boundary of $G$, and $S_1 \stackrel{\text{def}}{=} \partial G \setminus S_2$.
Recall that $Z(G,S_1)$ is the closure of $C_0^\infty(I\!R^3 \setminus \overline{S_1})$ in $H^1(G)$. It has been shown in [8], lemma 5i) that $W(G,S_1)$, which consists of all $\mathbf{E} \in H_{curl}(G)$ with $\vec{n} \wedge \mathbf{E} = 0$

on $S_1$ in the sense described in the previous section, coincides with the closure of $C_0^\infty(I\!R^3 \setminus \overline{S_1})$ in $H_{curl}(G)$. Since $G$ is a rectangle and $S_2$ is one side of it, this can also be shown directly by reflection at $S_2$ as in the proof of the subsequent lemma 3.

Next, let $A \in L^\infty(G, \mathbb{C}^{3\times 3})$ is assumed to be uniformly positve definite, i. e. re $(\xi A(y)\bar{\xi}) \geq c_0|\xi|^2$ for all $y \in G, \xi \in \mathbb{C}^N$ with some $c_0 > 0$ independent of $y, \xi$. It is assumed that $A$ has in addition the multiplier property

$$Af \in H^{s_0}(\Omega) \text{ for all } f \in H^{s_0}(\Omega) \text{ with some } s_0 \in (0, 1/2). \tag{3.11}$$

For example this assumption is fulfilled int he case that $A$ is pieceiwse Hölder continuous, i.e. if it has the form $A = \sum_{k=1}^n \chi_{U_k} f_k$, where $f_k \in C^\alpha(G)$, that means $f_k$ is Hölder-continuous for some $\alpha > s_0$. Here $\chi_{U_k}$ are the characteristic functions of Lipschitz-domains $U_k \subset I\!R^3$.
The aim of this section is to prove the following theorem.

**Theorem 1** *There exist $\bar{s} \in (0, s_0), c_0 \in (0, \infty)$ depending only on $A$, such that for all $s \in [0, \bar{s}]$ and $\mathbf{E} \in W^s(G, S_1)$ with $A\mathbf{E} \in X^s(G, S_1)$ one has*
*$\mathbf{E} \in H^s(G)$ and $||\mathbf{E}||_{H^s(G)} \leq c_0 \left(||A\mathbf{E}||_{X^s(G,S_1)} + ||\mathbf{E}||_{W^s(G,S_1)}\right)$*

For $\mathbf{E} \in L^2(G)$ we define $P_E\mathbf{E} \stackrel{\text{def}}{=} \mathbf{E} - \nabla\varphi \in X_0(G, S_1)$, where $\varphi \in Z(G, S_1)$ satisfies

$$\int_G \nabla\varphi\nabla\psi dx = \int_G \mathbf{E}\nabla\psi dx \text{ for all } \psi \in Z(G, S_1). \tag{3.12}$$

**Lemma 3** *i) $X(G, S_1) \cap W(G, S_1) \subset H^1(G)$.*
*ii) $P_E(W^s(G, S_1)) \subset H^s(G)$.*
*iii) $(1 - P_E)(X^s(G, S_1)) \subset H^s(G)$.*

**Proof:**
In order to prove i) assume $\mathbf{E} \in W(G, S_1) \cap X(G, S_1)$.
Let $\tilde{G} \stackrel{\text{def}}{=} \{x \in I\!R^3 : (x_1, x_2, -x_3) \in G \text{ or } x \in G\} = (0, a) \times (0, b) \times (-c, c)$ and define $\tilde{\mathbf{E}} \in L^2(\tilde{G})$ by reflection at the plane $\{x_3 = 0\}$, i.e. $\tilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} \mathbf{E}(x)$ if $x \in G$ and $\tilde{\mathbf{E}}(x) \stackrel{\text{def}}{=} (\mathbf{E}_1(x_1, x_2, -x_3), \mathbf{E}_2(x_1, x_2, -x_3), -\mathbf{E}_3(x_1, x_2, -x_3))$ if $x \in \tilde{G}$ with $x_3 < 0$.

Next it is shown that $\tilde{\mathbf{E}} \in \overset{0}{H}_{curl}(\tilde{G})$.
Suppose $\mathbf{f} \in C_0^\infty(I\!R^3)$ and set $\mathbf{g}(x) \stackrel{\text{def}}{=} (\mathbf{f}_1(x_1, x_2, -x_3), \mathbf{f}_2(x_1, x_2, -x_3), -\mathbf{f}_3(x_1, x_2, -x_3))$.
Then $\vec{n} \wedge (\mathbf{f} - \mathbf{g}) = 0$ on $S_2$ and since $\mathbf{E}$ belongs to the closure of $C_0^\infty(I\!R^3 \setminus \overline{S_1})$ in $H_{curl}(G)$ it follows easily that

$$\int_G ((\mathbf{f} - \mathbf{g}) \text{ curl } \mathbf{E} - \mathbf{E} \text{ curl } (\mathbf{f} - \mathbf{g}))\, dx = 0. \tag{3.13}$$

Now,

$$\int_{\tilde{G}} \tilde{\mathbf{E}} \text{ curl } \mathbf{f} dx = \int_G \mathbf{E} \text{ curl } (\mathbf{f} - \mathbf{g}) dx = \int_G (\mathbf{f} - \mathbf{g}) \text{ curl } \mathbf{E} dx = \int_{\tilde{G}} \mathbf{h}\mathbf{f} dx$$

where $\mathbf{h}(x) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{E})(x)$ if $x \in G$ and $(\mathbf{h}_1(x), \mathbf{h}_2(x), -\mathbf{h}_3(x)) \stackrel{\text{def}}{=} -(\text{ curl } \mathbf{E})(x_1, x_2, -x_3)$ if $x \in \tilde{G}$ with $x_3 < 0$. This means

$$\tilde{\mathbf{E}} \in \overset{0}{H}_{curl} (\tilde{G}) \text{ wth } \text{ curl } \tilde{\mathbf{E}} = \mathbf{h}. \tag{3.14}$$

From quite similar arguments it follows

$$\text{div } \tilde{\mathbf{E}} = \rho \in L^2(\tilde{G}) \tag{3.15}$$

where $\rho(x) \stackrel{\text{def}}{=} \text{ div } \mathbf{E}(x)$ if $x \in G$ and $\rho(x) \stackrel{\text{def}}{=} \text{ div } \mathbf{E}(x_1, x_2, -x_3)$ if $x \in \tilde{G}$ with $x_3 < 0$.
Now, 3.14 and 3.15 imply $\tilde{\mathbf{E}} \in H^1(\tilde{G})$, which can be shown for example by developing $\tilde{\mathbf{E}}$ in Fourier-series on the rectangle $\tilde{G}$.

Since $W(G, S_1)$ is the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{S_1})$ in $H_{curl}(G)$ and $Z(G, S_1)$ is the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{S_1})$ in $H^1(G)$, it follows easily that $\nabla\varphi \in W_0(G, S_1)$ for all $\varphi \in Z(G, S_1)$ and hence

$$(1 - P_E)\mathbf{E} \in W_0(G, S_1) \text{ for all } \mathbf{E} \in L^2(G). \tag{3.16}$$

Suppose $\mathbf{E} \in W(G, S_1)$. Then 3.16 yields
$P_E\mathbf{E} \in X_0(G, S_1) \cap W(G, S_1) \subset H^1(G)$ by i).
Now, assertion ii) follows from interpolation.
Next, suppose $\mathbf{E} \in X(G, S_1)$. By the definition of $P_E$ it follows from 3.12 that

$$\int_G [(1 - P_E)\mathbf{E}]\nabla\psi dx = \int_G \mathbf{E}\nabla\psi dx = -\int_G (\text{ div } \mathbf{E})\psi dx \text{ for all } \psi \in Z(G, S_1),$$

which implies $(1 - P_E)\mathbf{E} \in X(G, S_1)$. By 3.16 and i) this yields
$(1 - P_E)\mathbf{E} \in W(G, S_1) \cap X(G, S_1) \subset H^1(G)$. Finally, assertion iii) follows for $s \in [0, 1]$ from interpolation.

**Lemma 4** $P_E(H^s(G)) \subset H^s(G)$ *for all* $s \in (0, 1/2)$ *and*
$||P_E||_{B(H^s(G),H^s(G))} \stackrel{s\to 0}{\longrightarrow} 1$ .

**Proof:**
Suppose $\mathbf{E} \in \overset{0}{H}{}^1 (G) \subset W(G, S_1)$. Then lemma 3 ii) yields

$$P_E\mathbf{E} \subset H^1(G). \tag{3.17}$$

For all $s_1 \in [0, 1/2)$ one has

$$H^s(G) = [L^2(G), \overset{0}{H}{}^1 (G)]_s, \tag{3.18}$$

see [11]. Since $||P_E||_{B(L^2(G),L^2(G))} \leq 1$, it follows from 3.17 and 3.18 by interpolation that

$$P_E\mathbf{E} \in [P_E(L^2(G)), P_E(\overset{0}{H}{}^1 (G))]_s \subset [L^2(G), H^1(G)]_s = H^s(G)$$

$$\text{and } ||P_E\mathbf{E}||_{H^s(G)} \leq c_2^s||\mathbf{E}||_{H^s(G)}$$

for all $s \in [0, s_1], \mathbf{E} \in H^s(G)$.

Now, the main result of this section can be proved.

**Proof of theorem 1:**
Choose $\lambda > 0$ with $L_0 \stackrel{\text{def}}{=} ||1 - \lambda A||_{L^\infty} < 1$. Then it follows from 3.11 that there exists some $C_1 > 0$ with

$$||1 - \lambda A^{-1}||_{B(H^s(G),H^s(G))} \leq C_1^s L_0 \text{ for all } s \in [0, s_0]. \tag{3.19}$$

By lemma 4 and 3.19 there exists $\bar{s} > 0$, such that for all $s \in [0, \bar{s}]$

$$||P_E||_{B(H^s(G),H^s(G))}||1 - \lambda A^{-1}||_{B(H^s(G),H^s(G))} \leq L_2 < 1 \tag{3.20}$$

Now, assume $s \in [0, \bar{s}]$ and $\mathbf{E} \in W^s(G, S_1)$ with $A\mathbf{E} \in X^s(G, S_1)$. Then it follows from lemma 3 iii) that

$$(1 - P_E)A\mathbf{E} \in H^s(G) \tag{3.21}$$

and therefore

$$P_E\mathbf{E} - P_E A^{-1} P_E A\mathbf{E} = P_E A^{-1}(1 - P_E)A\mathbf{E} \in H^s(G) \tag{3.22}$$

by 3.11 and lemma 4. Lemma 3 ii) yields $P_E\mathbf{E} \in H^s(G)$ and hence by 3.22

$$\mathbf{f} \stackrel{\text{def}}{=} P_E A^{-1} P_E A\mathbf{E} \in H^s(G) \cap X_0(G, S_1) \tag{3.23}$$

Let $U_s \stackrel{\text{def}}{=} X_0(G, S_1) \cap H^s(G)$ and $Q : U_s \to X_0$ by

$$Q\mathbf{u} \stackrel{\text{def}}{=} P_E(1 - \lambda A^{-1})\mathbf{u} + \lambda\mathbf{f} = \mathbf{u} - \lambda P_E A^{-1}\mathbf{u} + \lambda\mathbf{f} \tag{3.24}$$

Suppose $\mathbf{u} \in U_s$. By assumption 3.11 and lemma 4 one has $P_E A^{-1}\mathbf{u} \in H^s(G)$. Together with 3.23 this yields $Q\mathbf{u} \in U_s$. From 3.20 it follows that $Q$ is Lipschitz-continuous on $U_s$ (with respect to the $H^s$-topology) with Lipschitz-constant $L_2 < 1$. Hence $Q$ has a unique fixed-point $\mathbf{u}_0 \in U_s$, i.e.

$$\mathbf{u}_0 = Q\mathbf{u}_0 = \mathbf{u}_0 - \lambda P_E A^{-1}\mathbf{u}_0 + \lambda P_E A^{-1} P_E A\mathbf{E}$$

and thus $P_E A^{-1}[\mathbf{u}_0 - P_E A\mathbf{E}] = 0$.
Since $\mathbf{u}_0 - P_E A\mathbf{E} \in X_0(G, S_1)$, this yields

$$0 = \langle P_E A^{-1}[\mathbf{u}_0 - P_E A\mathbf{E}], \mathbf{u}_0 - P_E A\mathbf{E}\rangle_{L^2(G)}$$

$$= \langle A^{-1}[\mathbf{u}_0 - P_E A\mathbf{E}], \mathbf{u}_0 - P_E A\mathbf{E}\rangle_{L^2(G)} \geq c_0||\mathbf{u}_0 - P_E A\mathbf{E}||^2_{L^2(G)},$$

which implies

$$P_E A\mathbf{E} = \mathbf{u}_0 \in U_s \subset H^s(G) \tag{3.25}$$

Finally, 3.21, 3.25 yield $A\mathbf{E} \in H^s(G)$ and therefore $\mathbf{E} \in H^s(G)$ by 3.11.

# 4 Regularity-theorem for general domains

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz-domain, $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$.
Moreover, let $A \in L^\infty(\Omega)$ be a uniformly positive variable matrix with the $H^{s_0}$-multiplier-property for some $s_0 \in (0, 1/2)$, i.e.

$$A\mathbf{w} \in H^s(\Omega) \text{ for all } s \in [0, s_0] \text{ and } \mathbf{w} \in H^s(\Omega). \tag{4.26}$$

The aim of this section is to prove the following regularity-theorem

**Theorem 2** *There exist $\bar{s} \in (0, s_0), c_0 \in (0, \infty)$ depending only on $\Omega, \Gamma_1$ and $A$, such that for all $s \in [0, \bar{s}]$ and $\mathbf{E} \in W^s(\Omega, \Gamma_1)$ with $A\mathbf{E} \in X^s(\Omega, \Gamma_1)$ one has*

$$\mathbf{E} \in H^s(\Omega) \textit{ and } ||\mathbf{E}||_{H^s(\Omega)} \leq c_0 \left( ||\mathbf{E}||_{W^s(\Omega,\Gamma_1)} + ||A\mathbf{E}||_{X^s(\Omega,\Gamma_1)} \right)$$

For this purpose some technical but mild regularity-assumptions are imposed on $\Omega$ and the decomposition of its boundary.

It is assumed that there are open sets $U_1, .., U_M \subset \mathbb{R}^3$ and bi-Lipschitz mappings $T_k : Q = (-1, 1)^3 \to U_k$ (i.e. $T_k$ is bijective, $T_k, T_k^{-1}$ are globally Lipschitz-continuous and $\det DT_k$ is uniformly positive), such that $\bar{\Omega} \subset \cup_{k=1}^M U_k$ and $U_k \cap \Omega$ is a Lipschitz-domain.
The sets $U_k$ fall into four categories. In the first case $k \in \{1, .., M_1\}$ $U_k$ does not intersect $\Gamma_2$, i.e.

$$G_k \stackrel{\text{def}}{=} T_k^{-1}(U_k \cap \Omega) = \{x \in Q : x_3 > 0\}$$
$$\text{and } U_k \cap \Gamma_1 = U_k \cap \partial\Omega = T_k(\{x \in Q : x_3 = 0\})$$

In the second case $k \in \{M_1 + 1, .., M_2\}$ the same holds with $\Gamma_1$ replaced by $\Gamma_2$ and vice versa, that means that $U_k$ intersects only $\Gamma_2$.
The third category $k \in \{M_2 + 1, .., M_3\}$ consists of those sets, which intersect $\Gamma_1$ and $\Gamma_2$. Here $T_k^{-1}$ maps the two parts of the boundary onto orthogonal planes, more precisely

$$\{x \in Q : x_2 = 0, x_3 > 0\} \subset T_k^{-1}(U_k \cap \Gamma_1) \subset \{x \in Q : x_2 = 0, x_3 \geq 0\},$$

$$\{x \in Q : x_2 > 0, x_3 = 0\} \subset T_k^{-1}(U_k \cap \Gamma_2) \subset \{x \in Q : x_2 \geq 0, x_3 = 0\}$$

and

$$G_k = T_k^{-1}(U_k \cap \Omega) = \{x \in Q : x_2 > 0, x_3 > 0\}.$$

For the sake of generality it is not assumed that any part $\Gamma_j$ of the boundary is closed.
In the last case $k \in \{M_3 + 1, .., M\}$ $U_k$ does not intersect $\partial\Omega$ and $G_k = Q$.

In the sequel the following mild additional regularity-property will be imposed on $\partial\Omega$ and its decomposition into $\Gamma_1$ and $\Gamma_2$.
For each $k \in \{1, .., M\}$ there are bounded Lipschitz-domains $K_1^{(k)}, ..., K_n^{(k)} \subset \mathbb{R}^3$ and

$\tilde{K}_1^{(k)},...,\tilde{K}_n^{(k)} \subset \mathbb{R}^3$ and Hölder-continuous functions $f_1^{(k)},...,f_n^{(k)} \in C^{1/2}(\mathbb{R}^3, \mathbb{R}^{3\times 3})$ and $\tilde{f}_1^{(k)},...,\tilde{f}_n^{(k)} \in C^{1/2}(\mathbb{R}^3)$, such that

$$DT_k(y)^{-1} = \sum_{j=1}^{n} f_j^{(k)}(y)\chi_{K_j^{(k)}}(y) \tag{4.27}$$

$$\det\ DT_k(y) = \sum_{j=1}^{n} \tilde{f}_n^{(k)}(y)\chi_{\tilde{K}_j^{(k)}}(y) \text{ for all } y \in Q$$

This means in particular that these functions may be discontinuous on finitely many two-dimensional manifolds. The main purpose of this assumption is that the functions in 4.27 are $H^s$-multipliers for $s \in (0, 1/2)$.

In the sequel let $S_{2,k} \stackrel{\text{def}}{=} T_k^{-1}(U_k \cap \Gamma_2)$ and $S_{1,k} \stackrel{\text{def}}{=} (\partial G_k) \setminus S_{2,k}$.
Next, $A_k \in L^\infty(G_k, \mathbb{C}^{3\times 3})$ denotes for $k \in \{1, .., M\}$ the matrix-valued function defined by

$$A_k(y) = [\det DT_k(y)]DT_k(y)^{-1}A(T_k(y))(DT_k(y)^*)^{-1} \text{ for } y \in G_k \tag{4.28}$$

Let $\chi_k \in C_0^\infty(U^{(k)})$, $k \in \{1, .., M\}$ be a partition of unity subordinate to the covering $U^{(k)}$, $k \in \{1, .., M\}$ of $\overline{\Omega}$.
For $\mathbf{F} \in L^2(\Omega)$ define $\mathcal{T}_k\mathbf{F} \in L^2(G_k)$ and $\mathcal{S}_k\mathbf{F} \in L^2(G_k)$ by

$$(\mathcal{T}_k\mathbf{F})(y) \stackrel{\text{def}}{=} \chi_k(T_k(y))DT_k(y)^*\mathbf{F}(T_k(y))$$

and

$$(\mathcal{S}_k\mathbf{F})(y) \stackrel{\text{def}}{=} \chi_k(T_k(y))[\det DT_k(y)]DT_k(y)^{-1}\mathbf{F}(T_k(y)) \text{ for } y \in G_k.$$

**Lemma 5** *Suppose $s \in [0, 1]$. Then*

$$\mathcal{T}_k\mathbf{E} \in W^s(G_k, S_{1,k}) \textit{ for all } \mathbf{E} \in W^s(\Omega, \Gamma_1). \tag{4.29}$$

*and*

$$\textit{and } \mathcal{S}_k\mathbf{D} \in X^s(G_k, S_{1,k}) \textit{ for all } \mathbf{D} \in X^s(\Omega, \Gamma_1). \tag{4.30}$$

**Proof:**
Suppose $\mathbf{f} \in C_0^\infty(\mathbb{R}^3 \setminus \overline{S_{k,2}})$ and define $\mathbf{F} \stackrel{\text{def}}{=} D(T_k^{-1})^* \cdot (\mathbf{f} \circ T_k^{-1}) \in L^\infty(U_k)$. Then lemma 2 yields $\mathbf{F} \in H_{curl}(U_k)$ with

$$\text{curl } \mathbf{F} = [\ \det\ D(T_k^{-1})]\,[(DT_k) \cdot\ \text{curl } \mathbf{f}] \circ T_k^{-1} \in L^\infty(U_k) \subset L^2(U_k). \tag{4.31}$$

Since (supp $\mathbf{f}$)$\cap$ $T_k^{-1}$ (supp $\chi_k$) is a compact subset of $Q$ and
supp $\mathbf{f} \subset \mathbb{R}^3 \setminus \overline{S_{k,2}}$, it follows that the sets
$T_k(Q \cap \text{supp } \mathbf{f}) \cap$ supp $\chi_k$ and $T_k(S_{k,2}) = U_k \cap \Gamma_2$ have positive distance. Hence

$$\text{supp } (\chi_k\mathbf{F}) \subset \overline{T_k(Q \cap \text{supp } \mathbf{f}) \cap \text{supp } \chi_k} \subset U_k \setminus \overline{\Gamma_2}, \tag{4.32}$$

After extending $\chi_k\mathbf{F}$ by zero outside supp $\chi_k$ it follows from 4.31 and 4.32 using the usual mollifying-argument that

$$\chi_k\mathbf{F} \text{ belongs to the closure of } C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2}), \text{ in } H_{curl}(I\!R^3). \tag{4.33}$$

Now suppose $\mathbf{E} \in W(\Omega, \Gamma_1)$. Then 4.31 yield by the substution-formula

$$\int_{G_k} (\mathcal{T}_k\mathbf{E}) \text{ curl } \mathbf{f} dy = \int_{G_k} \chi_k(T_k(y))[DT_k(y)^*\mathbf{E}(T_k(y))] \text{ curl } \mathbf{f}(y) dy \tag{4.34}$$

$$= \int_{U_k\cap\Omega} \chi_k(x) \left[\det D(T_k^{-1})(x)\right] \mathbf{E}(x) \cdot \left[(DT_k)(T_k^{-1}(x)) \cdot (\text{ curl } \mathbf{f})(T_k^{-1}(x))\right] dx$$

$$= \int_{U_k\cap\Omega} \chi_k\mathbf{E} \text{ curl } \mathbf{F} dx = \int_\Omega \mathbf{E} \text{ curl } [\chi_k\mathbf{F}]\, dx - \int_{U_k\cap\Omega} \mathbf{E} \cdot (\nabla\chi_k) \wedge \mathbf{F} dx$$

Since $\mathbf{E} \in W(\Omega, \Gamma_1)$, it follows from 4.33 that

$$\int_{G_k} (\mathcal{T}_k\mathbf{E}) \text{ curl } \mathbf{f} dy = \int_{U_k\cap\Omega} \mathbf{F} \text{ curl } [\chi_k\mathbf{E}]\, dx \tag{4.35}$$

$$= \int_{G_k} [\det DT_k(y)]\mathbf{F}(T_k(y)) \cdot [(\text{ curl } (\chi_k\mathbf{E}))(T_k(y))]\, dy$$

$$= \int_{G_k} [\det DT_k(y)] \left(DT_k(y)^{-1} \cdot [(\text{ curl } (\chi_k\mathbf{E}))(T_k(y))]\right) \cdot \mathbf{f}(y) dy$$

for all $\mathbf{f} \in C_0^\infty(I\!R^3 \setminus \overline{S_{k,2}})$, which implies $\mathcal{T}_k\mathbf{E} \in W(G_k, S_{k,1})$ with

$$\text{curl } (\mathcal{T}_k\mathbf{E}) = (\text{ det } DT_k)(DT_k(\cdot))^{-1} [\text{ curl } (\chi_k\mathbf{E}) \circ T_k]\, . \tag{4.36}$$

Hence, 4.29 follows from interpolation.

To prove ii) suppose $\mathbf{D} \in X(\Omega, \Gamma_1)$.
Let $\varphi \in C_0^\infty(I\!R^3 \setminus \overline{S_{1,k}})$ and $\psi \overset{\text{def}}{=} \varphi \circ T_k^{-1} \in H^1(U_k)$.
As in the proof of i) (supp $\varphi$)$\cap$ $T_k^{-1}$ (supp $\chi_k$) is a compact subset of $Q$ and supp $\varphi \subset I\!R^3 \setminus \overline{S_{k,1}}$. Hence $T_k(Q\cap$ supp $\varphi)\cap$ supp $\chi_k$ has positive distance to $T_k(Q \cap S_{k,1})$ and therefore also to the set $U_k \cap \Gamma_1 = (U_k \cap \partial\Omega) \setminus (U_k \cap \Gamma_2) \subset T_k(Q \cap \partial G_k) \setminus T_k(S_{k,2}) \subset T_k(Q \cap S_{k,1})$. Thus,

$$\text{supp } (\chi_k\psi) \subset \overline{T_k(Q \cap \text{supp } \varphi) \cap \text{supp } \chi_k} \subset U_k \setminus \overline{\Gamma_1}, \tag{4.37}$$

After extending $\chi_k\psi$ by zero outside supp $\chi_k$ it follows from 4.37 that

$$\chi_k\psi \in \overset{0}{H^1} (I\!R^3 \setminus \overline{\Gamma_1}), \tag{4.38}$$

With 4.38 and $\mathbf{D} \in X(\Omega, \Gamma_1)$ one obtains

$$\int_{G_k} (\mathcal{S}_k\mathbf{D})\nabla\varphi dy = \int_{G_k} \chi_k(T_k(y))[\det DT_k(y)][DT_k(y)^{-1}\mathbf{D}(T_k(y))]\nabla\varphi(y) dy \tag{4.39}$$

$$= \int_{G_k} [\det DT_k(y)]\chi_k(T_k(y))\mathbf{D}(T_k(y)) \cdot (\nabla\psi)(T_k(y)) dy$$

$$= \int_{\Omega\cap U_k} \chi_k \mathbf{D}\nabla\psi dx = \int_\Omega \mathbf{D}\nabla[\chi_k\psi]dx - \int_{\Omega\cap U_k} (\nabla\chi_k)\mathbf{D}\psi dx$$
$$= - \int_{\Omega\cap U_k} \text{div } (\chi_k\mathbf{D})\psi dx = - \int_{G_k} [\det DT_k(y)][\text{ div } (\chi_k\mathbf{D})(T_k(y))]\varphi(y)dy$$

Now, 4.39 yields $\mathcal{S}_k\mathbf{D} \in X(G_k, S_{1,k})$ with

$$\text{div } (\mathcal{S}_k\mathbf{D}) = [\det DT_k][(\text{ div } (\chi_k\mathbf{D})) \circ T_k]$$

Finally, 4.30 follows for all $s \in [0,1]$ by interpolation

**Lemma 6** *The $A_k$ are $H^{s_0}$-multipliers, i.e. $A_k\mathbf{f} \in H^{s_0}(G_k)$ for all $\mathbf{f} \in H^{s_0}(G_k)$.*

**Proof:**
By the assumption 4.27 the functions $|\det DT_k(\cdot)|$ and $DT_k(\cdot)^{-1}$ are $H^s$-multipliers for $s \in (0,1/2)$. Hence, it remains to show that $A \circ T_k$ is a $H^{s_0}$-multiplier, i.e.

$$(A \circ T_k)\mathbf{f} \in H^{s_0}(G_k) \text{ for all } \mathbf{f} \in H^{s_0}(G_k). \tag{4.40}$$

For $\mathbf{f} \in H^1(G_k)$ we have $\mathbf{f} \circ T_k^{-1} \in H^1(U_k \cap \Omega)$, since $T_k$ is a bi-Lipschitz mapping. Therefore it follows from interpolation

$$\mathbf{f} \circ T_k^{-1} \in H^s(U_k \cap \Omega) \text{ for all } s \in [0,1] \text{ and } \mathbf{f} \in H^s(G_k) \tag{4.41}$$

Now, it follows from 4.26 and 4.41 that

$$\mathbf{f} \circ T_k^{-1}A \in H^{s_0}(U_k \cap \Omega) \text{ for all } \mathbf{f} \in H^{s_0}(G_k). \tag{4.42}$$

In anologogy to 4.41 one has

$$\mathbf{g} \circ T_k \in H^s(G_k) \text{ for all } s \in [0,1] \text{ and } \mathbf{g} \in H^s(U_k \cap \Omega) \tag{4.43}$$

Finally 4.40 follows from 4.42 and 4.43.

Now, the proof of theorem 2 can be completed.

**Proof of theorem 2:**
By theorem 1 and lemma 6 there exists some $\bar{s} \in (0,1/2), c_0 \in (0,\infty)$ depending only on $\Omega, \Gamma_1$, such that for all $s \in [0,\bar{s}]$ and $k \in \{1,..,M\}$ one has

$$\mathbf{F} \in H^s(G_k) \text{ for all } \mathbf{F} \in W^s(G_k, S_{1,k}) \text{ with } A_k\mathbf{F} \in X^s(G_k, S_{1,k}) \tag{4.44}$$

This follows from theorem 1 directly in the case $k \in \{M_2+1,..,M_3\}$. Obvious modifications of the proof of theorem 1 shows that assertion 4.44 also holds in the remaining, even easier cases.
Now, suppose $\mathbf{E} \in W^s(\Omega, \Gamma_1)$ with $A\mathbf{E} \in X^s(\Omega, \Gamma_1)$ for $s \in [0,\bar{s}]$.
Then lemma 5 yields $\mathcal{T}\mathbf{E} \in W^s(G_k, S_1)$ and $A_k\mathcal{T}_k\mathbf{E} = \mathcal{S}_k(A\mathbf{E}) \in X^s(G_k, S_1)$. With 4.44 one obtains $\mathcal{T}_k\mathbf{E} \in H^s(G_k)$ and hence

$$(\chi_k\mathbf{E}) \circ T_k = (DT_k(y)^*)^{-1} (\mathcal{T}_k\mathbf{E}) \in H^s(G_k), \tag{4.45}$$

since $(DT_k(\cdot)^*)^{-1}$ is a $H^s$-mulitiplier by the assumptions 4.27 on $T_k$.
Finally 4.45 and 4.41 yield $\mathbf{E} \in H^s(\Omega)$, since $\sum_{k=1}^M \chi_k = 1$ on $\Omega$.

# 5 $H^s$-regularity-results for ME

Let $\Omega, \Gamma_1 \subset \partial\Omega$ as in the previous section. Suppose $\varepsilon \in L^\infty(\Omega)$ and $\mu \in L^\infty(\Omega)$ are uniformly positive variable matrices in the sense that

$$(y^T \varepsilon(x) y) \geq m|y|^2 \text{ for all } x \in \Omega \text{ and all vectors } y \in \mathbb{C}^3 \text{ with some } m > 0.$$

In the sequel the operator $B$ is defined by

$$B(\mathbf{E}, \mathbf{h}) \stackrel{\text{def}}{=} (\varepsilon^{-1} \text{ curl } \mathbf{h}, -\mu^{-1} \text{ curl } \mathbf{E})$$

for $(\mathbf{E}, \mathbf{h}) \in D(B) \stackrel{\text{def}}{=} W(\Omega, \Gamma_1) \times \tilde{W}(\Omega, \Gamma_2)$.
Here $\tilde{W}(\Omega, \Gamma_2)$ is defined as the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2})$ in $H_{curl}(\Omega)$.
Therefore $B$ has the form $D(B) = D(\mathcal{A}^*) \times D(\mathcal{A})$ and

$$B(\mathbf{E}, \mathbf{h}) = (\varepsilon^{-1} \mathcal{A}\mathbf{h}, -\mu^{-1} \mathcal{A}^* \mathbf{E}) \text{ for all } \mathbf{E} \in D(\mathcal{A}^*) \text{ and } \mathbf{h} \in D(\mathcal{A}),$$

where $D(\mathcal{A})$ is the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2})$ in $H_{curl}(\Omega)$ and $\mathcal{A}\mathbf{h} \stackrel{\text{def}}{=}$ curl $\mathbf{h}$. Since $\mathcal{A}$ is densely defined and closed, it follows that $B$ is a densely defined skew self-adjoint operator in the Hilbert-space $\mathcal{X}_0 \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the scalar-product $< (\mathbf{E}, \mathbf{h}), (\mathbf{F}, \mathbf{g}) >_{\mathcal{X}_0} \stackrel{\text{def}}{=} \int_\Omega (\varepsilon \mathbf{E}\overline{\mathbf{F}} + \mu \mathbf{h}\overline{\mathbf{g}}) dx$ .
Hence, $-B^2$ is a positive, self-adjoint operator, and by the spectral-theorem

$$|B|^s \stackrel{\text{def}}{=} (-B^2)^{s/2} = \int_{\mathbb{R}} |\lambda|^s dE_\lambda \tag{5.46}$$

can be defined as a positive self-adjoint operator in $\mathcal{X}_0$ for $s \in [0,1]$. Here $(E_\lambda)_{t \in \mathbb{R}}$ denotes the spectral-family of the self-adjoint operator $iB$ in $\mathcal{X}_0$. The domain $D(|B|^s)$ of $|B|^s$ can be characterized as the interpolation space $[\mathcal{X}_0, D(B)]_s$, see [18], and will be denoted by $\mathcal{X}_s$ in the sequel.
With $D(B) = W(\Omega, \Gamma_1) \times \tilde{W}(\Omega, \Gamma_2)$ it follows easily by interpolation that

$$\mathcal{X}_s = W^s(\Omega, \Gamma_1) \times \tilde{W}^s(\Omega, \Gamma_2), \tag{5.47}$$

where $\tilde{W}^s(\Omega, \Gamma_k) \stackrel{\text{def}}{=} [L^2(\Omega), \tilde{W}(\Omega, \Gamma_k)]_s$.
Since $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}) \subset W(\Omega, \Gamma_2)$, one has

$$\tilde{W}^s(\Omega, \Gamma_2) \subset W^s(\Omega, \Gamma_2). \tag{5.48}$$

**Remark 1** *It has been shown in [8], lemma 5i) that under the present assumptions on $\Omega$ and the partition of its boundary the space $W(\Omega, \Gamma_2)$ coincides with the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2})$ in $H_{curl}(\Omega)$, i.e.*

$$\tilde{W}^s(\Omega, \Gamma_2) = W^s(\Omega, \Gamma_2).$$

*But this fact is not necessary for the following considerations.*

Recall that $Z(\Omega,\Gamma_k)$ is defined as the closure of $C_0^\infty(I\!R^3 \setminus \overline{\Gamma_k})$ in $H^1(\Omega)$.
Let $\varphi \in Z(\Omega,\Gamma_1)$ and $\psi \in Z(\Omega,\Gamma_2)$.
Then $\nabla\varphi \in W_0(\Omega,\Gamma_1)$ and $\nabla\psi \in \tilde{W}(\Omega,\Gamma_2)$, see [6], and thus

$$(\nabla\varphi, \nabla\psi) \in ker B. \tag{5.49}$$

In the sequel $P$ denotes the orthogonal-projecor on $(ker B)^\perp = \overline{ran B}$ in $\mathcal{X}_0$.
Let $(\exp(tB))_{t\in I\!R}$ be the unitary group generated by $B$.
Then $(\mathbf{E}(t), \mathbf{h}(t)) = \mathbf{w}(t) \stackrel{\text{def}}{=} \exp(tB)\mathbf{w}_0$ solves the homogeneous Maxwell equations

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{h}, \text{ and } \mu\partial_t\mathbf{h} = -\text{ curl } \mathbf{E}, \tag{5.50}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0,\infty) \times \Gamma_1, \quad \vec{n} \wedge \mathbf{h} = 0 \text{ on } (0,\infty) \times \Gamma_2, \tag{5.51}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{h}(0,x) = \mathbf{h}_0(x). \tag{5.52}$$

for $\mathbf{w}_0 = (\mathbf{E}_0, \mathbf{h}_0) \in \mathcal{X}_0$. The aim of this section is to prove a $H^s$-regularity-theorem for Maxwell's equations. For this purpose it is assumed that $\varepsilon, \mu$ have the $H^{s_0}$-multiplier-property 4.26 for some $s_0 \in (0, 1/2)$.

The following theorem will be proved in this section.

**Theorem 3** *$(\exp(tB))_{t\in I\!R}$ is a strongly continuous group in $H^s(\Omega)$ for all $s \in [0,\bar{s})$, i.e.*
*$\exp(\cdot B)\mathbf{w} \in C(I\!R, H^s(\Omega)) \cap L^\infty(I\!R, H^s(\Omega))$ for all $\mathbf{w} \in H^s(\Omega)$. Here $\bar{s} > 0$ as in theorem 2.*

This theorem says that the initial-boundary-value-problem 5.50-5.52 is well-posed in $H^s(\Omega)$ for all $s \in [0,\bar{s}]$. In the case that $\Omega$ is two-dimensional this result can be found in [7].

**Lemma 7** *Let $s \in [0,\bar{s}]$ with $\bar{s} > 0$ as in theorem 2.*

*i) $\mathcal{X}_s \cap (ker B)^\perp \subset H^s(\Omega)$, in particular $P(\mathcal{X}_s) \subset H^s(\Omega)$.*

*ii) $P(H^s(\Omega)) \subset H^s(\Omega)$.*

*iii) $H^s(\Omega) \subset \mathcal{X}_s$.*

**Proof:** Let $\mathbf{w} \stackrel{\text{def}}{=} (\mathbf{E}, \mathbf{h}) \in \mathcal{X}_s \cap (ker B)^\perp$. For $\varphi \in Z(\Omega,\Gamma_1)$ one has by 5.49

$$0 = \langle \mathbf{w}, (\nabla\varphi, 0) \rangle_{\mathcal{X}_0} = \int_\Omega \varepsilon\mathbf{E}\nabla\varphi dx,$$

i.e.

$$\varepsilon\mathbf{E} \in X_0(\Omega,\Gamma_1) \subset X^s(\Omega,\Gamma_1) \tag{5.53}$$

Now, 5.47, 5.53 and theorem 2 yield $\mathbf{E} \in W^s(\Omega, \Gamma_1) \cap \varepsilon^{-1}(X^s(\Omega, \Gamma_1)) \subset H^s(\Omega)$. By replacing $\Gamma_1$ by $\Gamma_2$ the same argument using 5.48 yields $\mathbf{h} \in H^s(\Omega)$, which completes the proof of i).

Proof of ii) and iii): As in the proof of theorem 4 one has $\overset{0}{H^1}(\Omega, \mathbb{C}^3) \subset X(\Omega, \Gamma_k) \cap \tilde{W}(\Omega, \Gamma_k) \subset X(\Omega, \Gamma_k) \cap W(\Omega, \Gamma_k)$ and therefore by interpolation

$$H^s(\Omega, \mathbb{C}^3) = [L^2(\Omega, \mathbb{C}^3), \overset{0}{H^1}(\Omega, \mathbb{C}^3)]_s \subset X^s(\Omega, \Gamma_1) \cap W^s(\Omega, \Gamma_1)$$

and

$$H^s(\Omega, \mathbb{C}^3) = [L^2(\Omega, \mathbb{C}^3), \overset{0}{H^1}(\Omega, \mathbb{C}^3)]_s \subset X^s(\Omega, \Gamma_2) \cap \tilde{W}^s(\Omega, \Gamma_2)$$

By 5.47 this implies iii). Moreover, it follows from i) and iii) that

$$P(H^s(\Omega, \mathbb{C}^3)) \subset P(\mathcal{X}_s) = \mathcal{X}_s \cap (ker B)^\perp \subset H^s(\Omega, \mathbb{C}^6).$$

Now, theorem 3 can be proved.

**Proof of theorem 3:** Let $\mathbf{w} \in H^s(\Omega)$. Since ran $(1 - P) =$ ker $B$, one has

$$\exp(tB)\mathbf{w} = (1 - P)\mathbf{w} + P \exp(tB)\mathbf{w} \tag{5.54}$$

Now, lemma 7 ii) yields

$$(1 - P)\mathbf{w} \in H^s(\Omega) \text{ and } ||(1 - P)\mathbf{w}||_{H^s} \leq C_1||\mathbf{w}||_{H^s} \tag{5.55}$$

It follows from lemma 7 iii) that $\mathbf{w} \in \mathcal{X}_s$ and thus $\exp(\cdot B)\mathbf{w} \in C(\mathbb{R}, \mathcal{X}_s) \cap L^\infty(\mathbb{R}, \mathcal{X}_s)$. Next, lemma 7 i) yields

$$P \exp(\cdot B)\mathbf{w} \in C(\mathbb{R}, \mathcal{X}_s \cap (ker B)^\perp) \subset C(\mathbb{R}, H^s(\Omega)) \tag{5.56}$$

and $||P \exp(tB)\mathbf{w}||_{H^s} \leq C_2||\mathbf{w}||_{H^s}$ with some $C_1, C_2 \in (0, \infty)$ independent of $t, \mathbf{w}$. Finally, the desired result follows from 5.54 - 5.56.

# 6 $L^p$-regularity for solutions of ME

Let $\Omega, \Gamma_1, \varepsilon$ and $\mu$ as in the previous section. Only the $H^{s_0}$-multiplier-property 4.26 of the coefficients $\varepsilon, \mu \in L^\infty(\Omega)$ is not necessary now.

In this section Maxwell's equations with nonlinear conductivity are considered.

$$\varepsilon\partial_t\mathbf{E} = \text{ curl } \mathbf{h} - \mathbf{S}(\mathbf{E}), \tag{6.57}$$

$$\mu\partial_t \mathbf{h} = -\ \mathrm{curl}\ \mathbf{E}, \tag{6.58}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0,\infty) \times \Gamma_1, \quad \vec{n} \wedge \mathbf{h} = 0 \text{ on } (0,\infty) \times \Gamma_2, \tag{6.59}$$

$$\mathbf{E}(0,x) = \mathbf{E}_0(x), \mathbf{h}(0,x) = \mathbf{h}_0(x). \tag{6.60}$$

Here $\mathbf{S} : L^2(\Omega, I\!R^3) \to L^2(\Omega, I\!R^3)$ is a generally nonlinear operator, which represent the electric current caused by the electric field. It is assumed that

$$||\mathbf{S}(\mathbf{u}) - \mathbf{S}(\mathbf{v})||_{L^2} \leq L||\mathbf{u} - \mathbf{u}||_{L^2} \text{ for all } \mathbf{u}, \mathbf{v} \in L^2(\Omega) \tag{6.61}$$

and

$$\mathbf{S}(\mathbf{E}) \in L^p(\Omega) \text{ and } ||\mathbf{S}(\mathbf{u})||_{L^p} \leq K\,(1 + ||\mathbf{u}||_{L^p}) \tag{6.62}$$

for all $p \in [2,\infty)$ and $\mathbf{u} \in L^p(\Omega)$ with constants $L \in (0,\infty)$ and $K \in (0,\infty)$.
In particular the linear case $\mathbf{S}(\mathbf{E}) = \sigma\mathbf{E}$ with an electric conductivity $\sigma \in L^\infty(\Omega)$ is possible.

For the definition of the notion of weak solutions of 6.57-6.60 see [6].
Setting $\mathbf{u} \stackrel{\text{def}}{=} (\mathbf{E}, \mathbf{h})$ 6.57-6.60 reads as

$$\partial_t \mathbf{u} = B\mathbf{u} + F_\sigma(\mathbf{u}), \quad \mathbf{u}(0) = \mathbf{w}_0 \stackrel{\text{def}}{=} (\mathbf{E}_0, \mathbf{h}_0) \tag{6.63}$$

where $F_\sigma : L^2(\Omega, I\!R^6) \to L^2(\Omega, I\!R^6) \subset \mathcal{X}_0$ is defined by

$$(F_\sigma(\mathbf{w})) \stackrel{\text{def}}{=} -\varepsilon^{-1}\,(\mathbf{S}(\mathbf{E}), 0) \text{ for } \mathbf{w} = (\mathbf{E}, \mathbf{h}) \in L^2(\Omega, I\!R^6).$$

A function $\mathbf{u} \in C([0,\infty), \mathcal{X}_0)$ is called a weak solution to 6.63, if for all $\mathbf{a} \in D(B)$

$$\frac{d}{dt} < \mathbf{u}(t), \mathbf{a} >_{\mathcal{X}_0} = - < \mathbf{u}(t), B\mathbf{a} >_{\mathcal{X}_0} + < F_\sigma(\mathbf{u}(t)), \mathbf{a} >_{\mathcal{X}_0} \tag{6.64}$$

This is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp(tB)\mathbf{w}_0 + \int_0^t \exp((t-s)B)F_\sigma(\mathbf{u}(s))ds, \tag{6.65}$$

where $B$ is defined as in the previous section and $\exp(tB), t \in I\!R$ is the unitary group generated by $B$. Since $F_\sigma$ is Lipschitz-continuous with respect to $\mathbf{E} \in L^2(\Omega)$ by assumption 6.61, it follows from a standard result that this integral equation has a unique solution $\mathbf{u} = (\mathbf{E}, \mathbf{h}) \in C([0,\infty), \mathcal{X}_0)$, see [6], chapter 6.
The main result of this section is the following $L^p$-regularity-theorem.

**Theorem 4** *There exists some $\tilde{p} > 2$ depending only on $\Omega, \Gamma_1, \varepsilon$ and $\mu$, such that for all $p \in [2, \tilde{p}]$ and $\mathbf{w}_0 \in D(B) \cap L^p(\Omega)$ one has*
$\mathbf{u} \in L^\infty_{loc}([0,\infty), L^p(\Omega)) \cap C([0,\infty), L^r(\Omega))$ *for all $r \in [2,p)$.*

In the sequel $Y_p$ denotes for $p \in [2, \infty)$ the set of all $\mathbf{w} = (\mathbf{E}, \mathbf{h}) \in \mathcal{X}_0$, such that the semi-norm

$$||\mathbf{w}||_{Y_p} \stackrel{\text{def}}{=} \sup\{|\int_\Omega \varepsilon \mathbf{E} \nabla \varphi dx| : \varphi \in Z(\Omega, \Gamma_1), ||\varphi||_{W^{1,p^*}(\Omega)} \leq 1\}$$

$$+ \sup\{|\int_\Omega \mu \mathbf{h} \nabla \psi dx| : \psi \in Z(\Omega, \Gamma_2), ||\psi||_{W^{1,p^*}(\Omega)} \leq 1\}$$

is finite. Here

$$||\psi||_{W^{1,q}(\Omega)} \stackrel{\text{def}}{=} ||\psi||_{L^q(\Omega)} + ||\nabla \psi||_{L^q(\Omega)} \text{ for } q \in [1, \infty), \psi \in W^{1,q}(\Omega)$$

Obviously Hölder's inequality yields

$$L^p(\Omega) \subset Y_p \text{ and } ||\mathbf{w}||_{Y_p} \leq \max\{||\varepsilon||_{L^\infty}, ||\mu||_{L^\infty}\} ||\mathbf{w}||_{L^p} \text{ for all } \mathbf{w} \in L^p(\Omega). \quad (6.66)$$

It follows from 5.49 that for $\mathbf{w}_0 = (\mathbf{E}_0, \mathbf{h}_0) \in \mathcal{X}_0$, $(\mathbf{E}(t), \mathbf{h}(t)) \stackrel{\text{def}}{=} \exp(tB)\mathbf{w}_0$ and $\varphi \in Z(\Omega, \Gamma_1)$ and $\psi \in Z(\Omega, \Gamma_2)$ one has

$$\int_\Omega \mu \mathbf{E}(t) \nabla \varphi dx + \int_\Omega \mu \mathbf{h}(t) \nabla \psi dx = \langle \exp(tB)\mathbf{w}_0, (\nabla \varphi, \nabla \psi) \rangle_{\mathcal{X}}$$

$$= \langle \mathbf{w}_0, \exp(-tB)(\nabla \varphi, \nabla \psi) \rangle_{\mathcal{X}} = \langle \mathbf{w}_0, (\nabla \varphi, \nabla \psi) \rangle_{\mathcal{X}}$$

$$= \int_\Omega \mu \mathbf{E}_0 \nabla \varphi dx + \int_\Omega \mu \mathbf{h}_0 \nabla \psi dx$$

This implies

$$\exp(tB)(Y_p) \subset Y_p \text{ and } ||\exp(tB)\mathbf{w}||_{Y_p} = ||\mathbf{w}||_{Y_p} \text{ for all } \mathbf{w} \in Y_p. \quad (6.67)$$

Next, a $L^p$-regularity-theorem for elements belonging to $\mathcal{X}_{3/2-3/p} \cap Y_p$ is proved.

**Theorem 5** *There exists $\tilde{p} \in (2, 6/(3 - 2\bar{s}))$, such that for all $p \in [2, \tilde{p}]$ and $\mathbf{w} \in \mathcal{X}_{3/2-3/p} \cap Y_p$ one has $\mathbf{w} \in L^p(\Omega)$ and*

$$||\mathbf{w}||_{L^p} \leq C_3 \left( ||\mathbf{w}||_{\mathcal{X}_{3/2-3/p}} + ||\mathbf{w}||_{Y_p} \right)$$

*with some $C_3 \in (0, \infty)$ independent of $\mathbf{w}$. Here $\bar{s} > 0$ as in theorem 2 in the case $A = 1$.*

**Proof:**
Let $p \in (2, 6/(3 - 2\bar{s}))$ and $\mathbf{w} = (\mathbf{E}, \mathbf{h}) \in \mathcal{X}_{3/2-3/p} \cap Y_p$ and define $f \in Z(\Omega, \Gamma_1)$ and $g \in Z(\Omega, \Gamma_2)$ by

$$\int_\Omega \nabla f \nabla \varphi dx = \int_\Omega \mathbf{E} \nabla \varphi dx \text{ for all } \varphi \in Z(\Omega, \Gamma_1) \quad (6.68)$$

$$\text{and } \int_\Omega \nabla g \nabla \psi dx = \int_\Omega \mathbf{h} \nabla \psi dx \text{ for all } \psi \in Z(\Omega, \Gamma_2)$$

Then $\mathbf{E} - \nabla f \in X_0(\Omega, \Gamma_1)$ and also $\mathbf{E} - \nabla f \in W^{3/2-3/p}(\Omega, \Gamma_1)$ by 5.47, since $\nabla f \in W_0(\Omega, \Gamma_1) \subset W^{3/2-3/p}(\Omega, \Gamma_1)$. With $3/2 - 3/p \leq \bar{s}$ we have by Sobolev's embedding-theorem for fractional-order spaces and the $H^s$-regularity-theorem 2 (in the case $A = 1$)

$$\mathbf{E} - \nabla f \in H^{3/2-3/p}(\Omega) \subset L^p(\Omega) \text{ with} \tag{6.69}$$

$$||\mathbf{E} - \nabla f||_{L^p} \leq C_1 ||\mathbf{E} - \nabla f||_{W^{3/2-3/p}} \leq C_2 ||\mathbf{w}||_{\mathcal{X}_{3/2-3/p}}$$

with $C_2 > 0$ independent of $\mathbf{w}$. By the definition of $||\cdot||_{Y_p}$ Hölder's inequality yields for all $\varphi \in Z(\Omega, \Gamma_1)$ the estimate

$$|\int_\Omega \varepsilon \nabla f \nabla \varphi dx| \leq ||\varepsilon(\mathbf{E} - \nabla f)||_{L^p} ||\nabla \varphi||_{L^{p^*}} + |\int_\Omega \varepsilon \mathbf{E} \nabla \varphi dx| \tag{6.70}$$

$$\leq C_2(||\mathbf{w}||_{\mathcal{X}_{3/2-3/p}} + ||\mathbf{w}||_{Y_p}) ||\varphi||_{W^{1,p^*}}$$

It follows from 6.70 and the $W^{1,p}$-result in [5] that

$$f \in W^{1,p}(\Omega) \text{ , i.e. } \nabla f \in L^p(\Omega) \text{ with} \tag{6.71}$$

$$||\nabla f||_{L^p} \leq C_3(||\mathbf{w}||_{\mathcal{X}_{3/2-3/p}} + ||\mathbf{w}||_{Y_p})$$

provided that $p$ is sufficiently close to 2, that means $p \leq \bar{p}$ where $\bar{p} > 2$ depends on $\Omega, \Gamma_1$ and $\varepsilon$. Now, 6.69 and 6.71 yield $\mathbf{E} \in L^p(\Omega)$. Analogously one obtains $\mathbf{h} \in L^p(\Omega)$ and the lemma is proved with $\tilde{p} \stackrel{\text{def}}{=} \min\{6/(3 - 2\bar{s}), \bar{p}\}$.

**Remark 2** *The previous theorem does not follow immediately from the $H^s$-regularity-theorem 2, since the coefficients are not assumed to be $H^s$-multipliers in this section.*

**Corollary 1** *For all $p \in [2, \tilde{p}]$ and $\mathbf{E} \in L^2(\Omega)$ with*

$$\textit{curl } \mathbf{E} \in L^{p^*}(\Omega) \textit{ and } \vec{n} \wedge \mathbf{E} = 0 \textit{ on } \Gamma_1 \tag{6.72}$$

*and*

$$\sup\{|\int_\Omega \varepsilon \mathbf{E} \nabla \varphi dx| : \varphi \in Z(\Omega, \Gamma_1), ||\varphi||_{W^{1,p^*}(\Omega)} \leq 1\} < \infty \tag{6.73}$$

*one has $\mathbf{E} \in L^p(\Omega)$.*
*6.72 is understood in the sense that*

$$\int_\Omega (\mathbf{E} \textit{ curl } \mathbf{h} - \mathbf{h} \textit{ curl } \mathbf{E}) dx = 0 \textit{ for all } \mathbf{h} \in L^p(\Omega) \cap W(\Omega, \Gamma_2)$$

**Proof:**
Let $\mathbf{E} \in L^2(\Omega)$ satisfy 6.72 and 6.73.
Then

$$(\mathbf{E}, 0) \in Y_p \tag{6.74}$$

The aim of the following considerations is to show that $(\mathbf{E}, 0) \in D((1 + |B|)^{1/2}) = \mathcal{X}_{1/2}$.

Suppose $\mathbf{w} = (\underline{\mathbf{w}}_1, \underline{\mathbf{w}}_2) \in \mathcal{X}_1 = D(B)$ and define $\mathbf{u} \stackrel{\text{def}}{=} \mathbf{w} - (\nabla f, \nabla g)$, where $f \in Z(\Omega, \Gamma_1)$ and $g \in Z(\Omega, \Gamma_2)$ are defined by

$$\int_\Omega \varepsilon \nabla f \nabla \varphi dx = \int_\Omega \varepsilon \underline{\mathbf{w}}_1 \nabla \varphi dx \text{ for all } \psi \in Z(\Omega, \Gamma_1)$$

$$\int_\Omega \mu \nabla g \nabla \psi dx = \int_\Omega \mu \underline{\mathbf{w}}_2 \nabla \psi dx \text{ for all } \psi \in Z(\Omega, \Gamma_2)$$

Then $\mathbf{u} \in \mathcal{X}_1$ by 5.49 and $\mathbf{u} \in Y_p$ with $||\mathbf{u}||_{Y_p} = 0$. With $3/2 - 3/p \leq 3/2 - 3/\tilde{p} \leq \bar{s} < 1/2$ one has by theorem 5

$$\mathbf{u} \in L^p(\Omega) \text{ with } ||\mathbf{u}||_{L^p} \leq C_1 ||\mathbf{w}||_{\mathcal{X}_{3/2-3/p}} \leq C_1 ||\mathbf{w}||_{\mathcal{X}_{1/2}} \tag{6.75}$$

with $C_1 > 0$ independent of $\mathbf{u}$. By 5.49 we obtain from 6.72 and 6.75

$$| < (\mathbf{E}, 0), B\mathbf{w} >_{\mathcal{X}_0} | = | < (\mathbf{E}, 0), B\mathbf{u} >_{\mathcal{X}_0} | = | \int_\Omega \mathbf{E} \text{ curl } \underline{\mathbf{u}}_2 dx|$$

$$= | \int_\Omega (\text{ curl } \mathbf{E}) \underline{\mathbf{u}}_2 dx| \leq || \text{ curl } \mathbf{E}||_{L^{p^*}} ||\mathbf{u}||_{L^p} \leq C_1 || \text{ curl } \mathbf{E}||_{L^{p^*}} ||\mathbf{w}||_{\mathcal{X}_{1/2}}$$

and hence

$$| < (\mathbf{E}, 0), B\mathbf{w} >_{\mathcal{X}_0} | \leq C_1 || \text{ curl } \mathbf{E}||_{L^{p^*}} ||\mathbf{w}||_{\mathcal{X}_{1/2}} \tag{6.76}$$

for all $\mathbf{w} \in D(B) = \mathcal{X}_1$.
Now, let $\mathbf{u} \in X_{1/2}$ and $\mathbf{w} = (1 + |B|)^{-1/2} \mathbf{u} \in X_1 = D(B)$.
Then 6.76 yields

$$| < (\mathbf{E}, 0), B(1 + |B|)^{-1/2} \mathbf{u} >_{\mathcal{X}_0} | = | < (\mathbf{E}, 0), B\mathbf{w} >_{\mathcal{X}_0} |$$

$$\leq C_1 || \text{ curl } \mathbf{E}||_{L^{p^*}} ||\mathbf{w}||_{\mathcal{X}_{1/2}} \leq C_1 || \text{ curl } \mathbf{E}||_{L^{p^*}} ||\mathbf{u}||_{\mathcal{X}_0}$$

Hence, $(\mathbf{E}, 0) \in D(B(1 + |B|)^{-1/2}) = D((1 + |B|)^{1/2}) = \mathcal{X}_{1/2}$, which implies by 6.74 and theorem 5 that $\mathbf{E} \in L^p(\Omega)$.

Now, the $L^p$-regularity-theorem for Maxwell's equations 6.57-6.60 can be proved.

**Proof of theorem 4:**
Let $\tilde{p} > 2$ as in theorem 5. Define $\mathcal{T} : C([0, T], \mathcal{X}_0) \to C([0, T], \mathcal{X}_0)$ by

$$(\mathcal{T}\mathbf{u})(t) = \exp(tB)\mathbf{w}_0 + \int_0^t \exp((t - s)B) F_\sigma(\mathbf{u}(s)) ds$$

Since $\mathbf{w}_0 \in D(B) \cap L^p(\Omega) \subset \mathcal{X}_1 \cap Y_p$, it follows from 6.67 and theorem 5 that

$$\exp(tB)\mathbf{w}_0 \in D(B) \cap Y_p \subset \mathcal{X}_1 \cap L^p(\Omega)$$

and

$$||\frac{d}{dt}(\exp(tB)\mathbf{w}_0)||_{\mathcal{X}_0} + ||\exp(tB)\mathbf{w}_0||_{L^p} \leq K_0 \text{ for all } t \in I\!R. \tag{6.77}$$

Suppose $\mathbf{u} \in W^{1,\infty}([0,T],\mathcal{X}_0)$, i.e. $\mathbf{u} : [0,T] \to \mathcal{X}_0$) is Lipschitz-continuous. Then one has by assumption 6.61

$$||(\mathcal{T}\mathbf{u})(t+h) - (\mathcal{T}\mathbf{u})(t)||_{\mathcal{X}_0} \leq ||(\exp(\tau B) - 1)\mathbf{w}_0||_{\mathcal{X}_0}$$

$$+||\int_0^{t+h} \exp(rB)F_\sigma(\mathbf{u}(t+h-r))dr - \int_0^t \exp(rB)F_\sigma(\mathbf{u}(t-r))dr||_{\mathcal{X}_0}$$

$$\leq C_1 h + h \sup_{s\leq h} ||F_\sigma(\mathbf{u}(r))||_{\mathcal{X}_0} + \int_0^t ||F_\sigma(\mathbf{u}(t+h-r)) - F_\sigma(\mathbf{u}(t-r))||_{\mathcal{X}_0} dr$$

$$\leq C_2(1 + \sup_{s\leq h} ||F_\sigma(\mathbf{u}(r))||_{\mathcal{X}_0})h + L\int_0^t ||\mathbf{u}(t+h-r) - \mathbf{u}(t-r)||_{\mathcal{X}_0} dr$$

and hence

$$\mathcal{T}(\mathbf{u}) \in W^{1,\infty}([0,T],\mathcal{X}_0) \text{ and} \tag{6.78}$$

$$||\partial_t \mathcal{T}(\mathbf{u})(t)||_{\mathcal{X}_0} \leq \limsup_{h\to 0} \left[h^{-1}||(\mathcal{T}\mathbf{u})(t+h) - (\mathcal{T}\mathbf{u})(t)||_{\mathcal{X}_0}\right]$$

$$\leq C_3 + L\int_0^t \limsup_{h\to 0} \left[h^{-1}||\mathbf{u}(s+h) - \mathbf{u}(s)||_{\mathcal{X}_0}\right] ds$$

$$\leq C_3 + L\int_0^t ||\partial_t \mathbf{u}(s)||_{\mathcal{X}_0} ds$$

Set

$$|\mathbf{u}|_{1,\infty} \stackrel{\text{def}}{=} \sup_{t\in[0,T]} (\exp(-2Lt)||\partial_t \mathbf{u}(s)||_{\mathcal{X}_0})$$

for $\mathbf{u} \in W^{1,\infty}([0,T],\mathcal{X}_0)$. Then 6.78 yields $\mathcal{T}(\mathbf{u}) \in W^{1,\infty}([0,T],\mathcal{X}_0)$ and

$$|\mathcal{T}\mathbf{u}|_{1,\infty} \leq C_3 + 1/2|\mathbf{u}|_{1,\infty} \text{ for all } \mathbf{u} \in W^{1,\infty}([0,T],\mathcal{X}_0). \tag{6.79}$$

Since $\frac{d}{dt}(\mathcal{T}(\mathbf{u}))(t) = B(\mathcal{T}(\mathbf{u}))(t) - F_\sigma(\mathbf{u}(t))$ weakly, it follows easily from 6.79 that $\mathcal{T}(\mathbf{u}) \in L^\infty([0,T],D(B)) = L^\infty([0,T],\mathcal{X}_1)$ and

$$||\mathcal{T}\mathbf{u}||_{L^\infty(0,T,\mathcal{X}_1)} \leq C_4\left(1 + ||\mathcal{T}\mathbf{u}||_{W^{1,\infty}(0,T,\mathcal{X}_0)}\right) \tag{6.80}$$

for all $\mathbf{u} \in W^{1,\infty}([0,T],\mathcal{X}_0)$.

Now let $\mathbf{u}_0 \in C([0,T],\mathcal{X}_0)$ the unique solution of 6.65 and consider the Picard-iteration $\mathbf{u}^{(n)} \stackrel{\text{def}}{=} \mathcal{T}^n(\mathbf{w}_0) \in C([0,T],\mathcal{X}_0)$. Then

$$\mathbf{u}^{(n)} \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{u}_0 \text{ in } C([0,T],\mathcal{X}_0) \text{ strongly.} \tag{6.81}$$

It follows inductively from 6.79 that $\mathbf{u}^{(n)} \in W^{1,\infty}((0,T),\mathcal{X}_0)$ with $|\mathbf{u}^{(n)}|_{1,\infty} \leq 2C_3$ and hence

$$\sup_{n\in\mathbb{N}} ||\mathbf{u}^{(n)}||_{W^{1,\infty}(0,T,\mathcal{X}_0)} < \infty \tag{6.82}$$

6.80 and 6.82 yield

$$\sup_{n\in\mathbb{N}} ||\mathbf{u}^{(n)}||_{L^\infty(0,T,D(B))} = \sup_{n\in\mathbb{N}} ||\mathbf{u}^{(n)}||_{L^\infty(0,T,\mathcal{X}_1)} < \infty \tag{6.83}$$

Next, it is shown inductively that $\mathbf{u}^{(n)}(t) \in D(B) \cap L^p(\Omega) \subset \mathcal{X}_1 \cap Y_p$.
Recall that

$$\mathbf{u}^{(n+1)} = (\mathcal{T}(\mathbf{u}^{(n)})(t) = \exp(tB)\mathbf{w}_0 + \int_0^t \exp((t-s)B)F_\sigma(\mathbf{u}^{(n)}(s))ds. \qquad (6.84)$$

It follows from 6.62 and the induction-hypothesis that

$$F_\sigma(\mathbf{u}^{(n)}(\cdot)) \in L^\infty((0,T), L^p(\Omega)) \subset L^\infty((0,T), Y_p)$$

and hence 6.67, 6.77 and 6.84 yield $\mathbf{u}^{(n+1)}(t) \in Y_p$.
By 6.83 and theorem 5 one has $\mathbf{u}^{(n+1)}(t) \in \mathcal{X}_1 \cap Y_p \subset L^p(\Omega)$ and

$$||\mathbf{u}^{(n+1)}(t)||_{L^p} \le C_5(||\mathbf{u}^{(n+1)}(t)||_{D(B)} + ||\mathbf{u}^{(n+1)}(t)||_{Y_p}) \qquad (6.85)$$

$$\le C_6(1 + ||\mathbf{u}^{(n+1)}(t)||_{Y_p}) \le C_6\left(1 + ||\mathbf{w}_0||_{Y_p} + \int_0^t ||F_\sigma(\mathbf{u}^{(n)}(s))||_{Y_p} ds\right).$$

$$\le C_7\left(1 + \int_0^t ||\mathbf{u}^{(n)}(s)||_{L^p} ds\right).$$

Using a weighted $L^\infty((0,T), L^p(\Omega))$-norm as in 6.79 one obtains $\sup_{n\in\mathbb{N}} ||\mathbf{u}^{(n)}||_{L^\infty(0,T,L^p(\Omega))} < \infty$ and hence together with 6.81

$$\mathbf{u}_0 \in L^\infty((0,T), L^p(\Omega)). \qquad (6.86)$$

Finally, the assertion follows from $\mathbf{u}_0 \in C([0,T], L^2(\Omega))$ and 6.86.

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