SAMPLE PATH LARGE DEVIATIONS FOR A CLASS OF MARKOV CHAINS RELATED TO DISORDERED MEAN FIELD MODELS

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Abstract: We prove a large deviation principle on path space for a class of discrete time Markov processes whose state space is the intersection of a regular domain $\Lambda \subset \mathbb{R}^d$ with some lattice of spacing ϵ. Transitions from x to y are allowed if $\epsilon^{-1}(x-y) \in \Delta$ for some fixed set of vectors Δ. The transition probabilities $p_\epsilon(t,x,y)$, which themselves depend on ϵ, are allowed to depend on the starting point x and the time t in a sufficiently regular way, except near the boundaries, where some singular behaviour is allowed. The rate function is identified as an action functional which is given as the integral of a Lagrange function. Markov processes of this type arise in the study of mean field dynamics of disordered mean field models.

Keywords: Large deviations, stochastic dynamics, Markov chains, disordered systems, mean field models.

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1. Introduction.

In this paper we study a class of Markov processes with discrete state space which have the property that their transition probabilities vary slowly with time as the processes progresses (we will give a precise meaning to this later). Such processes occur in many applications and have been studied both in the physical and mathematical literature. For an extensive discussion, we refer e.g. to van Kampen's book [vK], Chapter IX. It has been shown by Kurtz [Ku], under suitable conditions, that these processes can be scaled in such a way that a law of large numbers holds that states that the rescaled process converges, almost surely, to the solution of a certain differential equation. He also established a central limit theorem showing that the deviations from the solution under proper scaling converges to a generalized Ornstein-Uhlenbeck process [Ku2]. The simplest example of such Markov processes are of course symmetric random walks (in $\mathbb{Z}^d$, say). In this case one the LLN scaling consists in considering the process (for $t \in \mathbb{R}_+$) $Z_n(t) = \frac{1}{n}\sum_{i=1}^{[nt]} X_i$, and one has the obvious result that as n tends to infinity, $Z_n(t)$ converges to 0, which solves the differential equation is $X'(t) = 0$. The corresponding central limit theorem is then nothing but Donsker's invariance principle [Do] which asserts that $\sqrt{n}Z_n(t)$ converges to Brownian motion. In this simple situation, the LLN and the CLT are accompanied by a large deviation principle, due to Mogulskii [Mo] that states that the family of laws of the processes $Z_n(t), t \in [0,T]$ satisfies a large deviation principle with some rate function of the form $S(x) = \int_0^T dt\mathcal{L}(\dot{x}(t))$. This LDP is the analog of Schilder's theorem for Brownian motion (in which case the function $\mathcal{L}$ is just the square). Generalizations of Mogulskii's theorem were studied in a series of paper by Wentzell [W1-4]. A partial account of this work is given in Section 5 of the book by Wentzell and Freidlin [WF]. The class of locally infinitely divisible processes studied there include Markov jump processes. Wentzell proved large deviation principles under some spatial regularity assumptions on the moment generating functions of the local jump-distributions and its Legendre transforms. The particular case of pure Markov jump processes is worked out in [SW]. This theory has been developed considerably in a large number of works principlly by Dupois, Ellis, and Weiss and co-workers (see e.g. [DEW,DE,DE1,DE2,DR,AD,SW] and references therein). The main thrust of this line of research was to weaken the spatial regularity hypothesis on the transition rates to include situations with boundaries and discontinuities. The main motivation was furnished by applications to queing systems. Given the variety of possibls situations, is not surprising that there is no complete theory availble, but rather a large set of examples satisfying particular hypothesis. Among the rare general results is an upper large deviation bound proven in [DEW] that holds under measurability assumptions only;

the question under which conditions these bounds are sharp remain open in general. The upper bounds in [DEW] are also stated for discrete time Markov processes. Needless to say, the bulk of the literature is concerned with the diffusion case, i.e. large deviations for solutions of stochastic differential equations driven by Wiener processes [WF,Az]. Questions of discontinuous statistics have been considered in this context in [BDE,CS]. For other related large deviation principles, see also [Ki1,Ki2].

In the present case we consider discrete time Markov chains depending on a small parameter ϵ defined on a state space $\Lambda_\epsilon \subset \mathbb{R}^d$ that have transition rates $p_\epsilon(x, y, t)$ of the form $p_\epsilon(x, x + \epsilon\delta, t) = \exp(f_\epsilon(x, \delta, t))$, for $\delta \in \Delta$ where Δ is some finite set and f_ϵ is required to satisfy some regularity conditions to be specified in detail later. The new feature of our results are

(i) The functions f_ϵ themselves are allowed to depend (in a controlled way) on the small parameter ϵ.

(ii) Regularity conditions are required in the interiors of the domains, but some singular behaviour near the boundary is allowed.

(iii) The transition rates are time-dependent.

Features (i) and (ii) are motivated from applications to stochastic dynamics in disordered mean-field models of statistical mechanics which we will not discuss here. See e.g. [BEGK,BG]. Let us mention that the large deviations results obtained in the present paper were needed (in the particular setting of time-homogeneous and reversible processes) in [BEGK] to show that a general transition between metastable states proceeds along a (asymptotically) deterministic sequence of so-called admissible transitions. The necessity to consider (i) arises mainly from the fact that in such systems, rather strong finite size effect due to the disorder are present and these effect the transition probabilities. Control of this dependence requires a certain amount of extra work.

The problem at boundaries (ii) is also intrinsic for most of the systems we are interested in. While for many application it would be sufficient to have an large deviation estimates for sets of paths that stay away from the boundary, we feel that it is more satisfactory to have a full LDP under conditions that are generally met in the systems we are interested in. The types of singularities we must deal with differ from those treated in the queing motivated literature cited above.

(iii) is motivated by our interest to study the behaviour of such systems under time dependent external variations of parameters, and in particular to study hysteresis phenomena. This causes no particular additional technical difficulties.

We have chosen to give complete and elementary proves of our results, even though the basic ideas are now standard in large deviation theory and any technical lemmata (mainly from convex analysis) are also served in similar situations in the past. But there are some subtle points, mainly in the dealing with boundary effects, and we feel that it is easier and more instructive to follow a complete line of argument using only the minimal amount of technical tools.

The remainder of the paper is organized as follows. In Section 2 we give precise formulation of our results. Section 3 states the basic large deviation upper and lower bounds and shows why they imply our theorems, Section 4 establishes some elementary fact from convex analysis that will be needed later, and in Section 5 the upper and lower bounds are proven.

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2. Statements of results

Let Γ denote some lattice in $\mathbb{R}^d$ and let $\Lambda \subset \mathbb{R}^d$ be a convex set (finite or infinite) that is complete w.r.t. the Euclidean metric. Define, for $\epsilon > 0$, the rescaled lattice $\epsilon\Gamma$ and its intersection with the set Λ, $\Gamma_\epsilon \equiv \Lambda \cap (\epsilon\Gamma)$. We consider discrete time Markov chains with state space Γ_ϵ. ϵ will play the rôle of a small parameter[3]. Let $\Delta \subset \Gamma$ denote a finite subset of lattice vectors.

The time t-to-$(t+1)$ transition probabilities, $(t,x,y) \in \mathbb{N} \times \Gamma_\epsilon \times \Gamma_\epsilon \mapsto p_\epsilon(t,x,y) \in [0,1]$ are of the form

$$p_\epsilon(t,x,y) \equiv \begin{cases} g_\epsilon\left(\epsilon t, x, \epsilon^{-1}(y-x)\right) & \text{if } \epsilon^{-1}(y-x) \in \Delta, x \in \Gamma_\epsilon, y \in \Gamma_\epsilon \\ \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where the functions $\{g_\epsilon, \epsilon > 0\}$, $g_\epsilon : \mathbb{R}^+ \times \mathbb{R}^d \times \Delta \to [-\infty, 0]$, are obviously required to meet the condition

$$\sum_{\delta \in \Delta} g_\epsilon(s,x,\delta) = 1, \qquad \forall s \in \mathbb{R}^+, \forall x \in \Lambda \tag{2.2}$$

We will set

$$f_\epsilon(t,x,\delta) \equiv \begin{cases} \ln g_\epsilon(t,x,\delta), & \text{if } g_\epsilon(t,x,\delta) > 0 \\ -\infty, & \text{if } g_\epsilon(t,x,\delta) = 0 \end{cases} \tag{2.3}$$

These functions will be assumed to verify a number of additional hypothesis; in order to state them we need some notation: For any set $\mathcal{S}$ in $\mathbb{R}^d$ the *convex hull* of $\mathcal{S}$ is denoted by $\operatorname{conv}\mathcal{S}$; the *closure, interior* and *boundary* of $\mathcal{S}$ are denoted by $\operatorname{cl}\mathcal{S}$, $\operatorname{int}\mathcal{S}$ and $\operatorname{bd}\mathcal{S} = (\operatorname{cl}\mathcal{S}) \setminus (\operatorname{int}\mathcal{S})$. For each $\epsilon > 0$ we define the ϵ*-interior* of $\mathcal{S}$, denoted by $\operatorname{int}_\epsilon\mathcal{S}$, to be:

$$\operatorname{int}_\epsilon\mathcal{S} = \{x \in \mathcal{S} \mid \forall \delta \in \Delta, x + \epsilon\delta \in \mathcal{S}\} \tag{2.4}$$

Note that $\operatorname{int}_\epsilon\mathcal{S}$ is not necessarily open. The ϵ*-boundary* of $\mathcal{S}$ is then defined by $\operatorname{bd}_\epsilon\mathcal{S} = (\operatorname{cl}\mathcal{S}) \setminus (\operatorname{int}_\epsilon\mathcal{S})$. For each $\epsilon > 0$ we set:

$$\begin{aligned} \Lambda^{(\delta,\epsilon)} &= \{x \in \Lambda \mid x + \epsilon\delta \in \Lambda\}, \quad \delta \in \Delta \\ \Lambda^{(\delta)} &= \{x \in \Lambda \mid \exists \epsilon > 0 \text{ s.t. } x + \epsilon\delta \in \Lambda\}, \quad \delta \in \Delta \end{aligned} \tag{2.5}$$

Obviously

$$\begin{aligned} \bigcup_{\delta \in \Delta} \Lambda^{(\delta,\epsilon)} = \Lambda \quad \text{and} \quad \bigcap_{\delta \in \Delta} \Lambda^{(\delta,\epsilon)} = \operatorname{int}_\epsilon\Lambda \\ \bigcup_{\delta \in \Delta} \Lambda^{(\delta)} = \Lambda \quad \text{and} \quad \bigcap_{\delta \in \Delta} \Lambda^{(\delta)} = \operatorname{int}\Lambda \end{aligned} \tag{2.6}$$

[3]In applications to dynamics of mean field models ϵ will enter as the the inverse of the system size N, hence only take discrete values. This will not be important here.

Moreover we have:

Lemma 2.1:

(i) $\mathrm{int}_{\epsilon'}\Lambda \subset \mathrm{int}_{\epsilon}\Lambda$ *for all* $\epsilon' > \epsilon > 0$*;*

(ii) $\mathrm{int}_{\epsilon}\Lambda \subset \mathrm{int}\Lambda$ *for all* $\epsilon > 0$*;*

(iii) $\mathrm{int}\Lambda = \{x \in \Lambda \mid \exists \epsilon > 0 \text{ s.t.} \, x \in \mathrm{int}_{\epsilon}\Lambda\}$.

Proof: (i) is immediate. Given $x \in \mathrm{int}_{\epsilon}\Lambda$ each of the points $x + \epsilon\delta$, $\delta \in \Delta$, belongs to Λ. Forming the convex hull of this set of points we have, by convexity of Λ: $\mathrm{conv}\{x + \epsilon\delta \mid \delta \in \Delta\} = x + \epsilon \, \mathrm{conv}\Delta \subset \Lambda$. Let B be the closed unit ball in $\mathbb{R}^d$ centered at the origin. Since by assumption $\mathrm{conv}\Delta$ is a d-dimensional set, there exists $r \equiv r(\mathrm{diam}\Delta) > 0$ such that $rB \subset \mathrm{conv}\Delta$. Hence $x + \epsilon r B \subset \Lambda$ and $\mathrm{int}_{\epsilon}\Lambda \subset \{x \in \Lambda \mid x + \epsilon r B \subset \Lambda\}$, proving (ii). Similarly we obtain that for any $x \in \bigcup_{\epsilon>0} \mathrm{int}_{\epsilon}\Lambda = \{x \in \Lambda \mid \exists \epsilon > 0 \text{ s.t.} \, \forall \delta \in \Delta, x + \epsilon\delta \in \Lambda\}$ there exists $\epsilon' > 0$ such that $x + \epsilon' B \subset \Lambda$, which yields (iii). The lemma is proven. $\diamondsuit$.

Hypothesis 2.2:[4] *For each* $\epsilon > 0$ *and each* $\delta \in \Delta$*,*

$$\begin{aligned} g_{\epsilon}(s, x, \delta) > 0, \quad \forall (s, x) \in \mathbb{R}^+ \times \Lambda^{(\delta,\epsilon)} \\ g_{\epsilon}(s, x, \delta) = 0, \quad \forall (s, x) \in \mathbb{R}^+ \times \Lambda \setminus \Lambda^{(\delta,\epsilon)} \end{aligned} \tag{2.7}$$

and

$$g_{\epsilon}(s, x, \delta) = 0, \quad \forall (s, x, \delta) \in \mathbb{R}^+ \times (\mathbb{R}^d \setminus \Lambda) \times \Delta \tag{2.8}$$

Moreover,

$\forall x \in \mathrm{int}\Lambda, \exists \epsilon' > 0$ *and* $c > 0$ *such that* $\forall 0 < \epsilon < \epsilon'$*,*

$$g_{\epsilon}(s, x, \delta) > c, \quad \forall (s, \delta) \in \mathbb{R}^+ \times \Delta \tag{2.9}$$

$\forall x \in \mathrm{bd}\Lambda, \exists \epsilon' > 0$ *and* $c > 0$ *such that* $\forall 0 < \epsilon < \epsilon'$*,*

$$g_{\epsilon}(s, x, \delta) > c, \quad \forall s \in \mathbb{R}^+, \forall \delta \in \{\delta' \in \Delta \mid \Lambda^{(\delta')} \ni x\} \tag{2.10}$$

and

$$g_{\epsilon}(s, x, \delta) = 0, \quad \forall s \in \mathbb{R}^+, \forall \delta \notin \{\delta' \in \Delta \mid \Lambda^{(\delta')} \ni x\} \tag{2.11}$$

[4] The statements "for each $\epsilon > 0$" should in fact be replaced by "for each $\epsilon > 0$ sufficiently small".

Remark: Hypothesis 2.2 implies in particular that for each $\epsilon > 0$,

$$f_\epsilon(s,x,\delta) > -\infty, \quad \forall (s,x,\delta) \in \mathbb{R}^+ \times \mathrm{int}_\epsilon\Lambda \times \Delta \tag{2.12}$$

and

$$\forall x \in \mathrm{bd}_\epsilon\Lambda, \exists \delta \in \Delta \quad \text{s.t.} \quad f_\epsilon(s,x,\delta) > -\infty \tag{2.13}$$

Remark: Lemma 2.1 and Hypothesis 2.2 also imply that for any $x \in \Lambda$, $\exists \epsilon' > 0$ s.t. $\forall 0 < \epsilon < \epsilon'$

$$\left\{\delta \in \Delta \,\middle|\, \Lambda^{(\delta,\epsilon)} \ni x\right\} = \left\{\delta \in \Delta \,\middle|\, \Lambda^{(\delta)} \ni x\right\} \tag{2.14}$$

Hypothesis 2.3: *There exist functions, $f_\epsilon^{(0)}$ and $f_\epsilon^{(1)}$ such that*

$$f_\epsilon = f_\epsilon^{(0)} + \epsilon f_\epsilon^{(1)}, \tag{2.15}$$

satisfying:

(H0) $f_\epsilon^{(0)}(s,x,\delta) = -\infty$ *if and only if* $f_\epsilon(s,x,\delta) = -\infty$.

(H1) For any closed bounded subset $\mathcal{S} \subset \mathrm{int}\Lambda$ there exists a positive constant $K \equiv K(\mathcal{S}) < \infty$ such that, for each $\epsilon > 0$,

$$\sup_{x\in\mathcal{S}} \sup_{\substack{\delta\in\Delta: \\ \mathcal{S}\cap\Lambda^{(\delta,\epsilon)}\ni x}} \left| f_\epsilon^{(1)}(s,x,\delta) \right| \leq K, \quad \forall s \in \mathbb{R}^+ \tag{2.16}$$

(H2) There exists a constant $0 < \theta < \infty$ such that, for each $\epsilon > 0$,

$$\sup_{x\in\Lambda} \sup_{\substack{\delta\in\Delta: \\ \Lambda^{(\delta,\epsilon)}\ni x}} \left| f_\epsilon^{(0)}(s,x,\delta) - f_\epsilon^{(0)}(s',x,\delta) \right| \leq \theta |s-s'|, \qquad \forall s \in \mathbb{R}^+, \forall s' \in \mathbb{R}^+ . \tag{2.17}$$

(H3) For any closed bounded subset $\mathcal{S} \subset \mathrm{int}\Lambda$ there exists a positive constant $\vartheta \equiv \vartheta(\mathcal{S}) < \infty$ such that, for each $\epsilon > 0$,

$$\sup_{s\in\mathbb{R}^+} \sup_{\substack{\delta\in\Delta: \\ \mathcal{S}\cap\Lambda^{(\delta,\epsilon)}\ni \{x,x'\}}} \left| f_\epsilon^{(0)}(s,x,\delta) - f_\epsilon^{(0)}(s,x',\delta) \right| \leq \vartheta |x-x'|, \qquad \forall x \in \mathcal{S}, \forall x' \in \mathcal{S} \tag{2.18}$$

Hypothesis 2.4: *The functions g_ϵ converge uniformly to a function g on the set $\mathbb{R}^+ \times \Lambda \times \Delta$. Moreover, for any $(s,x,\delta) \in \mathbb{R}^+ \times \Lambda \times \Delta$,*

$$\lim_{\epsilon\to 0} g_\epsilon(s,x,\delta) = \lim_{\epsilon\to 0} e^{f_\epsilon^{(0)}(s,x,\delta)} \tag{2.19}$$

Remark: Note that Hypothesis 2.4 together with Hypothesis 2.2 implies that the limits

$$\lim_{\epsilon\to 0} f_\epsilon(s,x,\delta) = \lim_{\epsilon\to 0} f_\epsilon^{(0)}(s,x,\delta) = f(s,x,\delta) \tag{2.20}$$

exist and are finite at every (s,x,δ) in the set defined by:

$$s \in \mathbb{R}^+, x \in \Lambda, \delta \in \{\delta' \in \Delta \mid \Lambda^{(\delta')} \ni x\} \tag{2.21}$$

We put $f(s,x,\delta) = -\infty$ on the complement of (2.21).

Remark: For $x \in \mathrm{int}\Lambda$ then $\{\delta' \in \Delta \mid \Lambda^{(\delta')} \ni x\} = \Delta$.

Remark: The limiting function f of course inherits the properties $(H2)$ and $(H3)$ of Hypothesis 2.3 with $\Lambda^{(\delta,\epsilon)}$ replaced by $\Lambda^{(\delta)}$.

As a consequence of Hypothesis 2.3 and 2.4 we have:

Lemma 2.5:

(i) For each $\epsilon > 0$ and each $\delta \in \Delta$, the function $(s,x) \mapsto f_\epsilon^{(0)}(s,x,\delta)$ is jointly continuous in s and x relative to $\mathbb{R}^+ \times \mathrm{int}(\mathrm{int}_\epsilon\Lambda)$.

(ii) For each $\delta \in \Delta$, the function $(s,x) \mapsto f(s,x,\delta)$ is jointly continuous in s and x relative to $\mathbb{R}^+ \times \mathrm{int}\Lambda$.

Proof: It follows from $(H2)$ of Hypothesis 2.3 that the collection of functions $\{f_\epsilon^{(0)}(\,.\,,x,\delta) \mid x \in \mathrm{int}_\epsilon\Lambda, \delta \in \Delta\}$ is equi-Lipshitzian on $\mathbb{R}^+$, implying that the function $s \mapsto f_\epsilon^{(0)}(s,x,\delta)$ is continuous relative to $\mathbb{R}^+$ for each $x \in \mathrm{int}_\epsilon\Lambda$ and $\delta \in \Delta$. Using Lemma 2.1, (ii), it follows from $(H3)$ of Hypothesis 2.3 that the collection of functions $\{f_\epsilon^{(0)}(s,\,.\,,\delta) \mid s \in \mathbb{R}^+, \delta \in \Delta\}$ is equi-Lipshitzian on all closed bounded subsets $\mathcal{S} \subset \mathrm{int}(\mathrm{int}_\epsilon\Lambda)$ and hence, in particular, the function $x \mapsto f_\epsilon^{(0)}(s,x,\delta)$ is continuous relative to $\mathrm{int}(\mathrm{int}_\epsilon\Lambda)$ for each $s \in \mathbb{R}^+$ and $\delta \in \Delta$. The joint continuity of $f_\epsilon^{(0)}(s,x,\delta)$ in s and x simply results from the fact that $\mathbb{R}^+$ and $\mathrm{int}(\mathrm{int}_\epsilon\Lambda)$ are locally compact topological space. This proves (i). In view of the remark following Hypothesis 2.4, the proof of (ii) is identical to that of (i). The lemma is proven.$\diamondsuit$

Each of the following functions are mapping $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ into $[-\infty, +\infty]$:

$$\mathcal{L}(t,u,v) = \log \sum_{\delta\in\Delta} e^{(v,\delta)+f(t,u,\delta)} \tag{2.22}$$

$$\mathcal{L}^*(t,u,v^*) = \sup_{v\in\mathbb{R}^d}\{(v,v^*) - \mathcal{L}(t,u,v)\} \tag{2.23}$$

$$L_\epsilon(t,u,v) = \log\sum_{\delta\in\Delta} e^{(v,\delta)+f_\epsilon(t,u,\delta)} \tag{2.24}$$

$$L^*_\epsilon(t,u,v^*) = \sup_{v\in\mathbb{R}^d}\{(v,v^*) - L_\epsilon(t,u,v)\} \tag{2.25}$$

We set

$$\mathcal{L}^{(r)*}_\epsilon(t,u,v^*) \equiv \inf_{t':|t'-s|\leq r}\ \inf_{u':|u'-u|\leq r} L^*_\epsilon(t',u',v^*), \quad r>0 \tag{2.26}$$

Finally, we set

$$\mathcal{L}^{(r)*}(t,u,v^*) \equiv \inf_{t':|t'-s|\leq r}\ \inf_{u':|u'-u|\leq r} \mathcal{L}^*(t,u,v^*), \quad r>0 \tag{2.27}$$

and

$$\overline{\mathcal{L}}^*(t,u,v^*) \equiv \lim_{r\downarrow 0}\mathcal{L}^{(r)*}(t,u,v^*) \tag{2.28}$$

The main function spaces appearing in the text are listed hereafter. All of them are spaces of $\mathbb{R}^d$-valued functions on some finite interval $[0,T]$. By $C([0,T])$ we denote the space of continuous functions equipped with the supremum norm: $\|\phi(.)\|_C = \max_{0\leq t\leq T}|\phi(t)|$, where $|\,.\,|$ denotes the Euclidean norm on $\mathbb{R}^d$ (i.e. $|x| = \sqrt{(x,x)}$). $L^p([0,T])$, $1\leq p<\infty$, is the familiar space of Lebesgue measurable functions for which $\int_0^T|\phi(t)|^p dt$ is finite and is equipped with the norm $\|\phi(.)\|_p = \left(\int_0^T|\phi(t)|^p dt\right)^{1/p}$. $W([0,T])$ denotes the Banach space of absolutely continuous functions and can be equipped, e.g., with the norm, $\|\phi(.)\|_W = |\phi(0)| + \|\dot\phi(.)\|_1$. Recall that

$$W([0,T]) = \left\{\phi\in C([0,T])\,\middle|\,\forall\epsilon>0\,\exists\delta>0\, s.t. \sum_{l=1}^k|t_l - t_{l-1}|<\delta \Rightarrow \sum_{l=1}^k|\phi(t_l)-\phi(t_{l-1})|<\epsilon\right\} \tag{2.29}$$

or, equivalently,

$$W([0,T]) = \left\{\phi\in C([0,T])\,\middle|\,\forall t'\in[0,T],\,\forall t\in[t',T],\ \phi(t)-\phi(t') = \int_{t'}^t\dot\phi(s)ds,\ \dot\phi\in L^1([0,T])\right\} \tag{2.30}$$

As a rule all spaces above are metrized with the norm-induced metric and are considered in the metric topology (i.e., the topology of uniform convergence).

We need to introduce some subsets of this space. Recall that the effective domain of a an extended-real-valued function g on X is the set $\text{dom}g \equiv \{x\in X \mid g(x)<\infty\}$. For each $(t,u)\in\mathbb{R}^+\times\Lambda$ define the extended-real-valued function $\overline{\Phi}^*_{t,u}$ through:

$$\overline{\Phi}^*_{t,u}(v^*) = \overline{\mathcal{L}}^*(t,u,v^*) \tag{2.31}$$

Setting

$$D_u = \mathrm{dom}\overline{\Phi}^*_{t,u}, \quad D = \mathrm{conv}\Delta \tag{2.32}$$

we define,

$$\overline{\mathcal{D}}([0,T]) \equiv \left\{\phi \in W([0,T]) \middle| \phi(t) \in \Lambda \text{ and } \dot{\phi}(t) \in D_{\phi(t)} \text{ for Lebesgue a.e. } t \in [0,T]\right\} \tag{2.33}$$

$$\mathcal{D}^\circ([0,T]) \equiv \left\{\phi \in W([0,T]) \middle| \phi(t) \in \mathrm{int}\Lambda \text{ and } \dot{\phi}(t) \in D \text{ for Lebesgue a.e. } t \in [0,T]\right\} \tag{2.34}$$

Our prime interest will be in the large deviation behaviour of a family of continuous time processes constructed from the Markov chains $\{X_\epsilon, \epsilon > 0\}$ by linear interpolation on the coordinate variables and rescaling of the time. More precisely, let $[0,T]$ be an arbitrary but finite interval and define the process Y_ϵ on sample path space $(C([0,T]), \mathcal{B}(C([0,T])))$ by setting, for each $t \in [0,T]$,

$$Y_\epsilon(t) = X_\epsilon\left(\left[\tfrac{t}{\epsilon}\right]\right) + \left(\tfrac{t}{\epsilon} - \left[\tfrac{t}{\epsilon}\right]\right)\left(X_\epsilon\left(\left[\tfrac{t}{\epsilon}\right]+1\right) - X_\epsilon\left(\left[\tfrac{t}{\epsilon}\right]\right)\right) \tag{2.35}$$

Let $\widetilde{\mathcal{P}}_{\epsilon,\phi_0} \equiv \mathcal{P}_{\epsilon,\phi_0} \circ Y_\epsilon^{-1}$ denote it's law. We are now in a position to state our main result.

Theorem 2.6: *Assume that the Hypothesis 2.2, 2.3, 2.4 are satisfied. If moreover*

(H4) For any convex set $\mathcal{A} \subset W([0,T])$

$$\inf_{\phi \subset \mathcal{A}\cap\overline{\mathcal{D}}([0,T])} \int_0^T \overline{\mathcal{L}}^*(t,\phi(t),\dot{\phi}(t))dt = \inf_{\phi \subset \mathcal{A}\cap\mathcal{D}^\circ([0,T])} \int_0^T \overline{\mathcal{L}}^*(t,\phi(t),\dot{\phi}(t))dt \tag{2.36}$$

then the family of measures $\left\{\widetilde{\mathcal{P}}_{\epsilon,\phi_0}, \epsilon > 0\right\}$ on $(C([0,T]), \mathcal{B}(C([0,T])))$ obeys a full large deviation principle with good rate function $\mathcal{I} : C([0,T]) \to \mathbb{R}^+$ given by

$$\mathcal{I}(\phi(.)) = \begin{cases} \displaystyle\int_0^T \overline{\mathcal{L}}^*(t,\phi(t),\dot{\phi}(t))dt & \text{if } \phi(.) \in \overline{\mathcal{D}}([0,T]) \text{ and } \phi(0) = \phi_0 \\ +\infty & \text{otherwise} \end{cases} \tag{2.37}$$

Proposition 2.7: *Condition (H4) is satisfied if the following two conditions hold:*

(i) At each $(t,u,v^*) \in \mathbb{R}^+ \times \Lambda \times \mathbb{R}^d$

$$\lim_{i\to\infty} \overline{\mathcal{L}}^*(t,u_i,v^*) \le \overline{\mathcal{L}}^*(t,u,v^*) \tag{2.38}$$

for every sequence $u_1, u_2, \ldots$ in $\mathrm{int}\Lambda$ converging to $u \in \Lambda$.

(ii) For some function $g:\mathbb{R}^+\to\mathbb{R}^+$ *satisfying* $\lim_{\alpha\downarrow 0}\alpha g(\alpha)=0$, *for all* $(s,u,v^*)\in\mathbb{R}^+\times$ $\text{int}\Lambda\times D$,

$$\overline{\mathcal{L}}^*(s,u,v^*)\leq g\left(\text{dist}(u,\Lambda^c)\right) \tag{2.39}$$

Remark: Since $\overline{\mathcal{L}}^*\leq\mathcal{L}^*$, it is of course enough to verify (2.39) for the more explicitly given function $\mathcal{L}^*$. This condition is realized in most examples of interest. Condition (H4) is of course always realized in situations where the process cannot reach the boundary of Λ in finite time, and in particular if $\Lambda=\mathbb{R}^d$.

Proposition 2.7 will be proven in Section 4.

For later reference the properties of $\mathcal{I}$ are given explicitly in the proposition below.

Proposition 2.8: *The function* $\mathcal{I}$ *defined in (2.37) verifies:*

(i) $0\leq\mathcal{I}(\phi(.))\leq\infty$ *and* $\text{dom}\mathcal{I}=\overline{\mathcal{D}}([0,T])$

(ii) $\mathcal{I}(\phi(.))$ *is lower semi continuous.*

(iii) For each $l<\infty$, *the set* $\{\phi(.)\mid\mathcal{I}(\phi(.))<l\}$ *is compact in* $C([0,T])$.

Proof: The proof of this proposition is in fact a more or less identical rerun of the proof given Section 9.1 of Ioffe and Tihomirov [IT] and we will not repeat it here. $\diamondsuit$

By definition (i) and (ii) are the standard properties of a rate function while goodness is imparted to it by property *(iii)*.

Remark: The LDP of Theorem 2.6 can easily be extended beyond the continuous setting arising from the definition of Y_ϵ in that, instead of Y_ϵ, we could consider the process Z_ϵ defined by,

$$Z_\epsilon(t)=X_\epsilon\left(\left[\tfrac{t}{\epsilon}\right]\right),\quad\text{for each } t\in[0,T] \tag{2.40}$$

Naturally the path space of Z_ϵ is now the space $D([0,T])$ of functions that are right continuous and have left limits which, equipped with the Skorohod topology, $\mathcal{S}$, is rendered Polish (we refer to the beautiful book by [Bi] for questions related to this space). It can then be shown that the family of measures $\left\{\widehat{\mathcal{P}}_{\epsilon,\phi_0},\epsilon>0\right\}$ on $(D([0,T]),\mathcal{S})$ obeys a full large deviation principle with good rate function $\mathcal{I}'$ where $\mathcal{I}'=\mathcal{I}$ on $C([0,T])$ and $\mathcal{I}'=\infty$ on $D([0,T])\setminus C([0,T])$. The basic step needed to extend the LDP of Theorem 2.6 to the present case is to establish that the measures $\widetilde{\mathcal{P}}_{\epsilon,\phi_0}$ and $\widehat{\mathcal{P}}_{\epsilon,\phi_0}$, both defined on $(D([0,T]),\mathcal{S})$, are exponentially equivalent. As will become clear in the next chapter (see Lemma 3.1), this property is very easily seen to hold.

Let us finally make some remarks on the large deviation principle we have obtained. The rate function (2.37) has the form of a classical action functional with $\overline{\Lambda}^*(t,x,v)$ being a (in general time dependent) Lagrangian. Note however that in contrast to the setting of classical mechanics, the function space is one of absolutely continuous function, rather than functions with absolutely continuous derivatives. Therefore the minimizers in the LDP need not be solutions of the corresponding Euler-Lagrange equations everywhere, but jumps between solutions can occur. A particular feature, that is due to the discrete-time nature of the process is the presence of a maximal velocity (i.e. a "speed of light"), due to the fact that the Lagrangian is infinite for $v \notin D$. In that respect one can consider the rate function as the action of a relativistic classical mechanics.

3. The basic large deviation estimates.

The aim of this short chapter is to bring into focus the basic large deviation estimates on which the proof of Theorem 2.6 relies. These estimates are established in a subset of the continuous paths space, chosen in such a way as to retain the underlying geometrical properties of the paths of Y. Assuming these estimates we then proceed to give the proof of Theorem 2.6.

More precisely set:

$$\mathcal{E}([0,T]) = \left\{ \phi \in C([0,T]) \,\middle|\, \frac{\phi(t)-\phi(t')}{t-t'} \in D \ \forall t \in [0,T], \forall t' \in [0,T], t \neq t' \right\} \tag{3.1}$$

Lemma 3.1: *$\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{E}([0,T])) = 1$ for all $\epsilon > 0$.*

Proof: Assume that $t > t'$ and set $t = (i+\gamma)\epsilon$, $t' = (j+\gamma')\epsilon$ where $i, j \in \mathbb{N}$, $\gamma, \gamma' \in [0,1)$. By (2.24),

$$\frac{Y(t)-Y(t')}{t-t'} = \frac{X(i)-X(j)+\gamma[X(i+1)-X(i)]-\gamma'[X(j+1)-X(j)]}{[(i+\gamma)-(j+\gamma')]\epsilon} \tag{3.2}$$

Using that all sample paths of X have increments of the form $X(k+1)-X(k) = \epsilon\delta_{k+1}$ with $\delta_k \in \Delta$, (3.2) yields

$$\frac{Y(t)-Y(t')}{t-t'} = \begin{cases} \delta_{i+1} & \text{if } i = j \\ \dfrac{(1-\gamma')\delta_{j+1} + \left(\sum_{k=j+2}^{i} \delta_k \mathbb{1}_{\{i>j+1\}}\right) + \gamma\delta_{i+1}}{(1-\gamma')+(i-j-1)+\gamma} & \text{if } i \geq j+1 \end{cases} \tag{3.3}$$

The last line in the r.h.s. of (3.3) is a convex combination of elements of Δ. Thus, remembering that $D = \operatorname{conv}\Delta$, the lemma is proven. $\diamondsuit$

Being a subset of a metric space, $\mathcal{E}([0,T])$ is itself a metric space with metric given by the supremum norm-derived metric, and thus, can be considered a topological space in it's own right in the metric topology. In addition, it inherits the topology induced by $C([0,T])$. But those two topologies are easily seen to coincide, i.e., $\mathcal{B}(\mathcal{E}([0,T])) = \{\mathcal{A} \cap \mathcal{E}([0,T]) : \mathcal{A} \in \mathcal{B}(C([0,T])\}$. From this and Lemma (3.1) it follows that $\left(\mathcal{E}([0,T]), \mathcal{B}(\mathcal{E}([0,T])), \widetilde{\mathcal{P}}_{\epsilon,\phi_0}\right)$ is a measure space.

Let $\mathcal{B}_\rho(\phi) \in \mathcal{E}([0,T])$ denote the open ball of radius ρ around ϕ and let $\bar{\mathcal{B}}_\rho(\phi)$ be it's closure. Our first result will be a pair of upper and lower bounds that hold under much weaker hypothesis than those of Theorem 2.6.

Proposition 3.2: *Assume that Hypothesis 2.2, 2.3 and 2.4 hold. Let* $\left\{\widetilde{\mathcal{P}}_{\epsilon,\phi_0}, \epsilon > 0\right\}$ *be defined on* $(\mathcal{E}([0,T]), \mathcal{B}(\mathcal{E}([0,T])))$. *Then, for any* $\rho > 0$ *and* $\phi \in \mathcal{E}([0,T])$,

$$\limsup_{\epsilon\to 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\bar{\mathcal{B}}_\rho(\phi)) \le - \inf_{\substack{\psi\in\bar{\mathcal{B}}_\rho(\phi)\cap\overline{\mathcal{D}}([0,T]):\\ \psi(0)=\phi_0}} \int_0^T dt\overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t)) \tag{3.4}$$

$$\liminf_{\epsilon\to 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi)) \ge - \inf_{\substack{\psi\in\mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T]):\\ \psi(0)=\phi_0}} \int_0^T dt\mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) \tag{3.5}$$

In Section 4 we will prove the following lemma:

Lemma 3.3: *Under the same hypothesis as Proposition 3.2, for all* $\psi \in \mathcal{D}^\circ([0,T])$,

$$\int_0^T dt\mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) = \int_0^T dt\overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t)) \tag{3.6}$$

This lemma together with hypothesis (H4) will in fact imply the stronger

Proposition 3.4: *If in addition to the assumptions of Proposition 3.2 condition (H4) is satisfied. Then, for any* $\rho > 0$ *and* $\phi \in \mathcal{E}([0,T])$,

$$\limsup_{\epsilon\to 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\bar{\mathcal{B}}_\rho(\phi)) \le - \inf_{\psi\in\bar{\mathcal{B}}_\rho(\phi)} \mathcal{J}(\psi) \tag{3.7}$$

$$\liminf_{\epsilon\to 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi)) \ge - \inf_{\psi\in\mathcal{B}_\rho(\phi)} \mathcal{J}(\psi) \tag{3.8}$$

where $\mathcal{J} : \mathcal{E}([0,T]) \ni \psi \mapsto \mathcal{J}(\psi) \equiv \mathcal{I}(\psi)$ *is the restriction of* $\mathcal{I}$ *to* $\mathcal{E}([0,T])$.

Proof: We prove the proposition assuming Proposition 3.2 and Lemma 3.3. Using first (H4) and then (ii) of Lemma 3.3, we get

$$\begin{aligned} &\inf_{\psi\in\bar{\mathcal{B}}_\rho(\phi)\cap\overline{\mathcal{D}}([0,T])} \int_0^T dt\overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t)) = \inf_{\psi\in\bar{\mathcal{B}}_\rho(\phi)\cap\mathcal{D}^\circ([0,T])} \int_0^T dt\overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t)) \\ &= \inf_{\psi\in\bar{\mathcal{B}}_\rho(\phi)\cap\mathcal{D}^\circ([0,T])} \int_0^T dt\mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) \ge \inf_{\psi\in\bar{\mathcal{B}}_\rho(\phi)\cap\overline{\mathcal{D}}([0,T])} \int_0^T dt\mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) \end{aligned} \tag{3.9}$$

which implies (3.7).

On the other hand, using first (ii) of Lemma 3.3, then (H4), and finally the fact that, since for any $r, \epsilon > 0$, $\mathcal{L}_\epsilon^{(r)*}(t,u,v^*) \le \mathcal{L}_\epsilon^*(t,u,v^*)$, we have $\overline{\mathcal{L}}(t,u,v^*) \le \mathcal{L}^*(t,u,v^*)$, we also get

$$\begin{aligned} &\inf_{\psi\in\mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T])} \int_0^T dt\mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) = \inf_{\psi\in\mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T])} \int_0^T dt\overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t)) \\ &= \inf_{\psi\in\mathcal{B}_\rho(\phi)\cap\overline{\mathcal{D}}([0,T])} \int_0^T dt\overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t)) \le \inf_{\psi\in\mathcal{B}_\rho(\phi)\cap\overline{\mathcal{D}}([0,T])} \int_0^T dt\mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) \end{aligned} \tag{3.10}$$

which implies (3.8). $\diamondsuit$

The proof of Theorem 2.6, assuming Proposition 3.4 and Proposition 2.8, is now classical.

Proof of Theorem 2.6: Assume Proposition 3.2 and Proposition 2.3 to hold. Then, on the one hand, since $C([0,T])$ is Polish, goodness of the rate function entails exponential tightness of the family $\left\{\widetilde{\mathcal{P}}_{\epsilon,\phi_0}, \epsilon > 0\right\}$, which implies that the full LDP obtains whenever it's weak version obtains. On the other hand, since $\mathcal{E}([0,T])$ is compact, it follows from Proposition 3.2 that the family $\left\{\widetilde{\mathcal{P}}_{\epsilon,\phi_0}, \epsilon > 0\right\}$ on $\mathcal{E}([0,T])$ obeys a weak LDP with rate function $\mathcal{J}$. The connection between these LDP's is made in through:

Lemma 3.5: ([DZ], Lemma 4.1.5) *Let S be a regular topological space and $\{\mu_\epsilon, \epsilon \geq 0\}$ a family of probability measures on S. Let $\mathcal{S}$ be a measurable subset of S such that $\mu_\epsilon(\mathcal{S}) = 1$ for all $\epsilon > 0$. Assume $\mathcal{S}$ equipped with the topology induced by S.*

(i) if $\mathcal{S}$ is a closed subset of S and $\{\mu_\epsilon\}$ satisfies the LDP in $\mathcal{S}$ with rate function $\mathcal{J}$, then $\{\mu_\epsilon\}$ satisfies the LDP in S with rate function $\mathcal{I} = \mathcal{J}$ on $\mathcal{S}$ and $\mathcal{I} = \infty$ on $S \setminus \mathcal{S}$.

(ii) If $\{\mu_\epsilon\}$ satisfies the LDP in S with rate function $\mathcal{I}$ and $\mathrm{dom}\mathcal{I} \subset \mathcal{S}$, *then the same LDP holds in $\mathcal{S}$.*

Remark: Lemma 3.5 holds for the weak as well as the full LDP.

Theorem 2.6 now follows from Lemma 3.5 together with Lemma 3.1 and the fact that being compact, $\mathcal{E}([0,T])$ is closed in $C([0,T])$ $\diamondsuit$

4. Convexity related results

This rather lengthy chapter establishes most of the basic analytic properties of the logarithmic moment generating functions and their Legendre transforms that will be needed to prove the upper and lower large deviation estimates in Section 5. We begin by fixing some notations.

Let $\mathcal{L}_\epsilon$ and $\mathcal{L}^*_\epsilon$ be the functions, mapping $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}$, defined by:

$$\mathcal{L}_\epsilon(s,u,v) = \log \sum_{\delta\in\Delta} e^{(v,\delta)+f^{(0)}_\epsilon(s,u,\delta)} \tag{4.1}$$

$$\mathcal{L}^*_\epsilon(s,u,v^*) = \sup_{v\in\mathbb{R}^d} \{(v,v^*) - \mathcal{L}_\epsilon(s,u,v)\} \tag{4.2}$$

It plainly follows from Hypothesis 2.2 and (H0) of Hypothesis 2.3 that on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \Lambda) \times \mathbb{R}^d$, $\mathcal{L}_\epsilon = -\infty$, $\mathcal{L}^*_\epsilon = +\infty$, $\mathcal{L} \equiv -\infty$, and $\mathcal{L}^* = +\infty$. We will thus limit our attention to the behaviour of these functions on $\mathbb{R}^+ \times \Lambda \times \mathbb{R}^d$.

Let $\mathcal{M}(\Delta)$ denote the set of all probability measures on Δ. The support of a measure $\nu \in \mathcal{M}(\Delta)$, denoted $\operatorname{supp}\nu$, is defined by $\operatorname{supp}\nu = \{\delta \in \Delta \mid \nu(\delta) > 0\}$. For any fixed $(s,u) \in \mathbb{R}^+ \times \Lambda$ and any $v \in \mathbb{R}^d$ let $\nu^v_{\epsilon,s,u}$ be the probability measure on $\mathcal{M}(\Delta)$ that assigns to $\delta \in \Delta$ the density

$$\nu^v_{\epsilon,s,u}(\delta) = \frac{e^{(v,\delta)+f^{(0)}_\epsilon(s,u,\delta)}}{\sum_{\delta\in\Delta} e^{(v,\delta)+f^{(0)}_\epsilon(s,u,\delta)}} \tag{4.3}$$

Similarly $\nu^v_{s,u} \in \mathcal{M}(\Delta)$ is defined by (4.3) with $f^{(0)}_\epsilon(s,u,\delta)$ replaced by $f(s,u,\delta)$.

Observe that if $u \in \Lambda$ then either $u \in \operatorname{int}_\epsilon\Lambda$ or $u \in \operatorname{bd}_\epsilon\Lambda$ and, according to the remark following Hypothesis 2.2,

$$\operatorname{supp}\nu^0_{\epsilon,s,u} = \Delta, \qquad \forall(s,u) \in \mathbb{R}^+ \times \operatorname{int}_\epsilon\Lambda \tag{4.4}$$

whereas

$$\emptyset \neq \operatorname{supp}\nu^0_{\epsilon,s,u} \subset \Delta, \qquad \forall(s,u) \in \mathbb{R}^+ \times \operatorname{bd}_\epsilon\Lambda \tag{4.5}$$

Moreover, for χ a random variable with law $\nu^v_{\epsilon,s,u}$,

$$\mathcal{L}_\epsilon(s,u,v) = \mathbb{E}_{\nu^0_{\epsilon,s,u}} e^{(v,\chi)} + \log \sum_{\delta\in\Delta} e^{f^{(0)}_\epsilon(s,u,\delta)} \tag{4.6}$$

where $\mathbb{E}_{\nu^v_{\epsilon,s,u}}$ denotes the expectation w.r.t. $\nu^v_{\epsilon,s,u}$. Thus, up to a small term (which goes to zero as $\epsilon \downarrow 0$) $\mathcal{L}_\epsilon$ is the logarithmic moment generating function of $\nu^v_{\epsilon,s,u}$, $\mathcal{L}^*_\epsilon$ being termed it's conjugate.

With $\mathcal{L}$ and $\mathcal{L}^*$ given by (2.22) and (2.23), for fixed $(s,u) \in \mathbb{R}^+ \times \Lambda$, we further define the functions, mapping $\mathbb{R}^d$ into $\mathbb{R}$:

$$\begin{aligned} \Phi_{\epsilon,s,u}(v) &= \mathcal{L}_\epsilon(s,u,v) \\ \Phi^*_{\epsilon,s,u}(v^*) &= \mathcal{L}^*_\epsilon(s,u,v^*) \\ \Phi_{s,u}(v) &= \mathcal{L}(s,u,v) \\ \Phi^*_{s,u}(v) &= \mathcal{L}^*(s,u,v^*) \end{aligned} \tag{4.7}$$

This chapter is divided into five subchapters. In the first subchapter we establish the properties of the functions Φ_ϵ, Φ, and their conjugates Φ^*_ϵ, Φ^*. Although most of them are well know folklore of the theory of convex analysis, it is more convenient to state them at once rather then laboriously recall them from the literature when we need to put them in use. The proofs are merely compilations of references, chiefly taken from the books by Rockafellar [Ro] and Ellis [E]. In the second subchapter we go back to the functions $\mathcal{L}_\epsilon$, $\mathcal{L}^*$, and their limits, and establish their topological properties. The third subchapter establishes some basic properties of semi-continuous regularisations of our functions, and in particular provides an important result on the uniform convergence of the regularised functions as $\epsilon \downarrow 0$. In the forth subsection we present a result, based on these topological properties, which shall be crucial in establishing the large deviation bounds, while the last subsection is devoted to the proof of Proposition 2.7.

Most of the results of this section will be established simultaneously for either the function $\mathcal{L}_\epsilon$ or $\mathcal{L}^*_\epsilon$ at fixed ϵ, and (what we shall see are their limits) $\mathcal{L}$ or $\mathcal{L}^*$. We stress here once for all that, according to the remark following Hypothesis 2.4, all results for $\mathcal{L}_\epsilon$ or $\mathcal{L}^*_\epsilon$ directly infered from Hypothesis 2.2 and 2.3 obviously carry through to the limiting functions. As a rule we systematically skip the proofs of results for $\mathcal{L}$ or $\mathcal{L}^*$ whenever they are simple repetitions of those for $\mathcal{L}_\epsilon$ or $\mathcal{L}^*_\epsilon$.

4.1. The functions Φ_ϵ, Φ, and their conjugates.

We begin with a short reminder of terminology and a few definitions. Recall that $\text{dom} g \equiv \{x \in X \mid g(x) < \infty\}$. All through this chapter we shall adopt the usual convention that consists in identifying a convex function g on $\text{dom} g$ with the convex function defined throughout $\mathbb{R}^d$ by setting $g(x) = +\infty$ for $x \notin \text{dom} g$. A real valued function g on a convex set C is said to be *strictly* convex on C if

$$g((1-\lambda)x + \lambda y) < (1-\lambda)g(x) + \lambda g(y), \quad 0 < \lambda < 1 \tag{4.8}$$

for any two different points x and y in C. It is called *proper* if $g(x) < \infty$ for at least one x and $g(x) > -\infty$ for every x. The *closure* of a convex function g is defined to be the lower semi-continuous hull of g if g nowhere has value $-\infty$, whereas the closure of g is defined to be the convex function $-\infty$ if g is an improper convex function such that $g(x) = -\infty$ for some x. Either way the closure of g is another convex function and is denoted $\mathrm{cl}\, g$. A convex function is said to be closed if $g = \mathrm{cl}\, g$. For a proper convex function closedness is thus the same as lower semi-continuity. A function g on R^d is said to be continuous *relative to* a subset $\mathcal{S}$ of $\mathbb{R}^d$ if the restriction of d to $\mathcal{S}$ is a continuous function.

For any set C in $\mathbb{R}^d$ we denote by $\mathrm{cl}\, C$, $\mathrm{int}\, C$ and by $\mathrm{bd}\, C = (\mathrm{cl}\, C) \setminus (\mathrm{int}\, C)$ the *closure, interior* and *boundary* of C. If C is convex, we denote by $\mathrm{ri}\, C$ and $\mathrm{rbd}\, C = (\mathrm{cl}\, C) \setminus (\mathrm{ri}\, C)$ it's *relative interior* and *relative boundary.*

Definition 4.1: *A proper convex function g on $\mathbb{R}^d$ is called essentially smooth if it satisfies the following three conditions for $C = \mathrm{int}(\mathrm{dom} g)$:*

(a) C is non empty;

(b) g is differentiable throughout C;

(c) $\lim_{i\to\infty} |\nabla g(x_i)| = +\infty$ whenever $x_1, x_2, \ldots,$ is a sequence in C converging to a boundary point x of C.

Note that a smooth convex function on $\mathbb{R}^d$ is in particular essentially smooth (since (c) holds vacuously).

Definition 4.2: *The conjugate g^* of an arbitrary function g on $\mathbb{R}^d$ is defined by*

$$g^*(x^*) = \sup_{x\in\mathbb{R}^d} \{(x, x^*) - g(x)\} \tag{4.9}$$

Note that both Φ_ϵ, Φ_ϵ^* and Φ, Φ^* are pairs of conjugate functions.

Lemma 4.3: *([Ro], Theorem 12.2) Let g be a convex function. The conjugate function g^* is then a closed convex function, proper if and only if g is proper. Moreover $(\mathrm{cl}\, g)^* = g^*$ and $g^{**} = \mathrm{cl}\, g$.*

Finally, for g an extended-real-valued function on $\mathbb{R}^d$ which is is finite and twice differentiable throughout $\mathbb{R}^d$, we denote by $\nabla g(x) \equiv \left(\frac{\partial g}{\partial x_1}(x), \ldots, \frac{\partial g}{\partial x_d}(x)\right)$, $\nabla^2 g(x) \equiv \left(\frac{\partial g}{\partial x_i \partial x_j}(x)\right)_{i,j=1,\ldots,d}$

and $\Delta g(x) = \sum_{i=1}^{d} \frac{\partial^2 g}{\partial^2 x_i}(x)$, respectively, the gradient, the Hessian, and the Laplacian of g at x.

In order to unburden formulas the indices s and u in (4.7) and (4.3) will systematically be dropped in the sequel. We start by listing some of the properties of Φ_ϵ and Φ.

Lemma 4.4: *For all $\epsilon > 0$ the following conclusions hold. For any fixed $(s,u) \in \mathbb{R}^+ \times \Lambda$,*

(i) $|\Phi_\epsilon(v)| < \infty$ *for all* $v \in \mathbb{R}^d$.

(ii) Φ_ϵ *is a closed, convex, and continuous function on* $\mathbb{R}^d$.

(iii) Φ_ϵ *has mixed partial derivatives of all order which can be calculated by differentiation under the sum sign. In particular, for all $v \in \mathbb{R}^d$, if $\chi = (\chi_1, \ldots, \chi_d)$ denotes a random vector with law $\nu^v_{\epsilon,s,u}$,*

$$\nabla \Phi_\epsilon(v) = \mathbb{E}_{\nu^v_\epsilon} \chi = \left(\mathbb{E}_{\nu^v_\epsilon} \chi_i \right)_{i=1,\ldots,d} \tag{4.10}$$

$$\nabla^2 \Phi_\epsilon(v) = \left(\mathbb{E}_{\nu^v_\epsilon} \chi_i \chi_j - \mathbb{E}_{\nu^v_\epsilon} \chi_i \mathbb{E}_{\nu^v_\epsilon} \chi_j \right)_{i,j=1,\ldots,d} \tag{4.11}$$

$$\Delta \Phi_\epsilon(v) = \mathbb{E}_{\nu^v_\epsilon} |\chi - \mathbb{E}_{\nu^v_\epsilon} \chi|^2 = \sum_{i=1}^{d} \mathbb{E}_{\nu^v_\epsilon} |\chi_i - \mathbb{E}_{\nu^v_\epsilon} \chi_i|^2 \tag{4.12}$$

Moreover, for any fixed $(s,u) \in \mathbb{R}^+ \times \mathrm{int}_\epsilon \Lambda$, Φ_ϵ is a strictly convex function on $\mathbb{R}^d$.

All assertions above hold with Φ_ϵ replaced by Φ and ν^v_ϵ replaced by ν^v.

Proof: If $u \in \Lambda$ then, by Hypothesis 2.2,

$$\left| \log \textstyle\sum_{\delta \in \Delta} e^{f^{(0)}_\epsilon (s,u,\delta)} \right| < \infty \tag{4.13}$$

Assertion (i) is then a consequence of (4.6). Given assertion (i), assertions (ii) and (iii) are proven, e.g., in [E] (see pp230 for the former and Theorem VII.5.1 for the latter); formulae (4.10), (4.11) and (4.12) may be found in [BG]. Finally, a necessary and sufficient condition for Φ_ϵ to be strictly convex (see e.g. [E], Proposition VIII.4.2) is that the affine hull of $\mathrm{supp}\,\nu^0_\epsilon$ coincides with $\mathbb{R}^d$; but by Hypothesis 2.1 this condition is fulfilled whenever $u \in \mathrm{int}_\epsilon \Lambda$. The lemma is proven. $\diamondsuit$

We next turn to the functions Φ^*_ϵ and Φ^*. We first state an important relationship between the support of ν^0_ϵ and the effective their effective domains.

Lemma 4.5: *Let $d \geq 1$, $\epsilon > 0$ and $(s,u) \in \mathbb{R}^+ \times \Lambda$. Then,*

$$\mathrm{dom} \Phi^*_\epsilon = \mathrm{conv}(\,\mathrm{supp}\,\nu^0_\epsilon) \tag{4.14}$$

In particular, if $(s,u)\in\mathbb{R}^+\times\operatorname{int}_\epsilon\Lambda$,

$$\mathrm{dom}\Phi_\epsilon^* = \operatorname{conv}\Delta \tag{4.15}$$

The same holds with Φ_ϵ *replaced by* Φ *and* $\operatorname{int}_\epsilon\Lambda$ *replaced by* $\operatorname{int}\Lambda$.

Remark: Since $\operatorname{supp}\nu^0_{\epsilon,s,u} = \left\{\delta\in\Delta \,\middle|\, f_\epsilon^{(0)}(s,u,\delta) > -\infty\right\}$, we have by the second remark following Hypothesis 2.2 and (H0) that $\exists\epsilon'=\epsilon'(u)>0$ s.t. $\forall 0<\epsilon\le\epsilon'$

$$\operatorname{supp}\nu^0_{\epsilon,s,u} = \left\{\delta\in\Delta \,\middle|\, \Lambda^{(\delta)}\ni u\right\} \tag{4.16}$$

and therefore

$$\mathrm{dom}\Phi^*_{\epsilon,s,u} = \mathrm{dom}\Phi^*_{s,u} \tag{4.17}$$

Proof: Obviously, if ν_ϵ^0 is the unit mass at δ^*, $\Phi_\epsilon^*(v^*)=0$ if $v^*=\delta^*$ whereas $\Phi_\epsilon^*(v^*)=+\infty$ if $v^*\neq\delta^*$, so that (4.14) and (4.15) hold true. Assume now that ν_ϵ^0 is non degenerate. The starting point to prove the lemma under this assumption is a theorem by Ellis ([E], Theorem VIII.4.3) which, rephrased in our setting and putting $S\equiv\operatorname{conv}(\operatorname{supp}\nu_\epsilon^0)$, states that,

$$\mathrm{dom}\Phi_\epsilon^*\subseteq S \quad \text{and} \quad \operatorname{int}(\mathrm{dom}\Phi_\epsilon^*) = \operatorname{int} S \tag{4.18}$$

From this (4.14) automatically follows if we can show that $\Phi_\epsilon^*(v^*)<\infty$ for $v^*\in\operatorname{bd}S$. The proof is built upon the fact that, since $\operatorname{supp}\nu_\epsilon^0\subseteq\Delta$, the set S is a polytope and hence is closed. Let $\{a_1,\dots,a_\kappa\}$ be the subset of Δ generating S that is, the smallest subset of Δ such that $\operatorname{conv}(\{a_1,\dots,a_\kappa\})=S$. Set $\kappa\equiv|\operatorname{supp}\nu_\epsilon^0|$. By assumption ν_ϵ^0 is non degenerate so that $\kappa>1$. All points v of $\operatorname{bd}S$ can then be expressed in the form $v^*=\sum_{i=1}^\kappa\lambda_i a_i$ where $\sum_{i=1}^\kappa\lambda_i=1$, $\lambda_i\ge 0$, the number of non zero coefficients λ_i being at most $\kappa-1$.

We now introduce a representation of Φ_ϵ^* due to Donsker and Varadhan ([DV], p. 425). For $\mu\in\mathcal{M}(\Delta)$ define the relative entropy of μ with respect to ν_ϵ^0 by

$$I(\mu)=\sum_{\delta\in\Delta}\mu(\delta)\log\left(\tfrac{\mu(\delta)}{\nu_\epsilon^0(\delta)}\right) \tag{4.19}$$

Then

$$\Phi_\epsilon^*(v^*) = \inf\left\{I(\mu)\,\middle|\,\mu\in\mathcal{M}(\Delta),\textstyle\sum_{\delta\in\Delta}\delta\mu(\delta)=v^*\right\} - \log\textstyle\sum_{\delta\in\Delta}e^{f_\epsilon^{(0)}(s,u,\delta)} \tag{4.20}$$

First, observe that for $v=a\in\{a_1,\dots,a_\kappa\}$ the set $\left\{\mu\in\mathcal{M}(\Delta),\sum_{\delta\in\Delta}\delta\mu(\delta)=a\right\}$ reduces to the unit mass at a, and, by (4.20) and (4.13),

$$I(\mu) = -\log(\nu_\epsilon^0(a)) - \log\textstyle\sum_{\delta\in\Delta}e^{f_\epsilon^{(0)}(s,u,\delta)} < \infty \tag{4.21}$$

Next, by Lemma 4.3, Φ_ϵ^* is convex so that

$$\Phi_\epsilon^*\left(\sum_{i=1}^{\kappa}\lambda_i a_i\right) \le \sum_{i=1}^{\kappa}\lambda_i \Phi_\epsilon^*(a_i) < \infty \tag{4.22}$$

proving that $\operatorname{bd} S \subset \operatorname{dom}\Phi_\epsilon^*$. The lemma is proven. ♢

We now list some of the properties of Φ_ϵ^* and Φ^*.

Lemma 4.6: *For all $\epsilon > 0$ the following conclusions hold. For any fixed $(s,u) \in \mathbb{R}^+ \times \Lambda$,*

(i) Φ_ϵ^* *is a closed convex function on* $\mathbb{R}^d$.

(ii) Φ_ϵ^* *has compact level sets.*

(iii) Let $v_0^* = \mathbb{E}_{\nu_\epsilon^v}\chi|_{v=0}$. *Then for any* $v^* \in \mathbb{R}^d$, $\Phi_\epsilon^*(v^*) \ge 0$ *and* $\Phi_\epsilon^*(v^*) = 0$ *if and only if* $v^* = v_0^*$.

(iv) For $d = 1$, Φ_ϵ^* *is strictly convex and for* $d \ge 2$, Φ_ϵ^* *is strictly convex on* $\operatorname{ri}(\operatorname{dom}\Phi_\epsilon^*)$.

(v) Φ_ϵ^* *is continuous relative to* $\operatorname{dom}\Phi_\epsilon^*$.

Moreover, for any $(s,u) \in \mathbb{R}^+ \times \operatorname{int}_\epsilon\Lambda$, Φ_ϵ^* *is essentially smooth.*

All assertions above hold with Φ_ϵ *replaced by* Φ *and* ν_ϵ^v *replaced by* ν^v.

Proof: Assertions (i) to (iv) are taken from [E], Theorem VII.5.5. Since by Lemma 4.6 Φ_ϵ^* is closed, and since by Lemma 4.5 $\operatorname{dom}\Phi_\epsilon^*$ is a polytope, then (v) is a special case of [Ro], Theorem 10.2. Finally, the essential smoothness of Φ_ϵ^* follows from the fact that, by Lemma 4.4 , Φ_ϵ is strictly convex for $(s,u) \in \mathbb{R}^+ \times \operatorname{int}_\epsilon\Lambda$ together with Theorem 26.3 of [Ro], implying that the conjugate of a proper and strictly convex function having effective domain $\mathbb{R}^d$ is essentially smooth. ♢

The following lemma finally relates the functions Φ_ϵ and Φ to their conjugates.

Lemma 4.7: *Let* $(s,u) \in \mathbb{R}^+ \times \Lambda$, $\epsilon > 0$. *For any* $v \in \mathbb{R}^d$, *the following three conditions on* v^* *are equivalent to each other:*

(i) $v^* = \nabla\Phi_\epsilon(v)$;

(ii) $(v', v^*) - \Phi_\epsilon(v')$ *achieves it's supremum in* v' *at* $v' = v$;

(iii) $(v, v^*) - \Phi_\epsilon(v) = \Phi_\epsilon^*(v^*)$.

If $(s,u) \in \mathbb{R}^+ \times \operatorname{int}_\epsilon\Lambda$, *two more conditions can be added to this list;*

(iv) $v = \nabla\Phi_\epsilon^*(v^*)$*;*

(v) $(v, v') - \Phi_\epsilon^*(v')$ *achieves it's supremum in* v' *at* $v' = v^*$*.*

The same holds when Φ_ϵ *and* Φ_ϵ^* *are replaced by* Φ *and* Φ^**.*

Proof: By lemma 4.4 and the definition of essential smoothness, Φ_ϵ and Φ are closed, proper, convex, essentially smooth functions and are differentiable throughout $\mathbb{R}^d$. By Lemma 4.5 and Lemma 4.6, for each $(s,u) \in \mathbb{R}^+ \times \mathrm{int}_\epsilon\Lambda$, Φ_ϵ^* and Φ^* are closed, proper, convex, essentially smooth functions with effective domain $\mathrm{conv}\Delta$; hence they are differentiable on $\mathrm{int}(\mathrm{conv}\Delta)$. Since for a closed, proper, convex, and essentially smooth function g on $\mathbb{R}^d$, the subgradient of g at x, denoted by $\partial g(x)$, reduces to the gradient mapping $\nabla g(x)$[5] (see [Ro], Theorem 26.1), then Lemma 4.5 is a special case of Theorem 23.5 of [Ro]. $\diamondsuit$

4.2. Topological properties of the functions $\mathcal{L}_\epsilon$, $\mathcal{L}_\epsilon^*$, and their limits.

We have so far gathered information on the collections of convex functions $v \mapsto \mathcal{L}_\epsilon(s,u,v)$, $v \mapsto \mathcal{L}_\epsilon^*(s,u,v^*)$, and their limits for $s \in \mathbb{R}^+$ and u in either Λ, $\mathrm{int}_\epsilon\Lambda$ or $\mathrm{int}\Lambda$. We saw in particular that $\mathcal{L}_\epsilon$ (respectively $\mathcal{L}$) is continuous in v throughout $\mathbb{R}^d$ and that if $u \in \mathrm{int}_\epsilon\Lambda$ (respectively $u \in \mathrm{int}\Lambda$) then $\mathcal{L}_\epsilon^*$ (respectively $\mathcal{L}^*$) is continuous in v^* relative to $\mathrm{conv}\Delta$. In order to complete this picture we shall devote this subchapter to establishing the continuity properties of these functions in the variables t and u.

Lemma 4.8: *For all* $\epsilon > 0$*,*

(i) There exists a constant $0 < \theta < \infty$ *such that:*

$$\sup_{\substack{u\in\Lambda \\ v\in\mathbb{R}^d}} |\mathcal{L}_\epsilon(s,u,v) - \mathcal{L}_\epsilon(s',u,v)| \le \theta|s-s'|, \qquad \forall s \in \mathbb{R}^+, \forall s' \in \mathbb{R}^+ \tag{4.23}$$

(ii) For any closed bounded subset $\mathcal{S} \subset \mathrm{int}_\epsilon\Lambda$*, there exists a positive constant* $\vartheta \equiv \vartheta(\mathcal{S}) < \infty$ *such that:*

$$\sup_{\substack{s\in\mathbb{R}^+ \\ v\in\mathbb{R}^d}} |\mathcal{L}_\epsilon(s,u,v) - \mathcal{L}_\epsilon(s,u',v)| \le \vartheta|u-u'|, \qquad \forall u \in \mathcal{S}, \forall u' \in \mathcal{S} \tag{4.24}$$

(iii) The function $\mathcal{L}_\epsilon(s,u,v)$ *is jointly continuous in* s*,* u *and* v *relative to* $\mathbb{R}^+ \times \mathrm{int}(\mathrm{int}_\epsilon\Lambda) \times \mathbb{R}^d$*.*

[5] that is, $\partial g(x)$ consists of the vector $\nabla g(x)$ alone when $x \in \mathrm{int}(\mathrm{dom}\, g)$, while $\partial g(x) = \emptyset$ when $x \notin \mathrm{int}(\mathrm{dom}\, g)$.

Assertions (i)-(iii) hold with $\mathcal{L}_\epsilon$ replaced by $\mathcal{L}$ and $\mathrm{int}_\epsilon\Lambda$ replaced by $\mathrm{int}\Lambda$.

In addition:

(iv) For any $u \in \Lambda$, $s \in \mathbb{R}^+$, the function $\mathcal{L}_\epsilon(s,u,\cdot)$ converges uniformly to $\mathcal{L}(s,u,\cdot)$ on $\mathbb{R}^d$.

(v) For any closed bounded $\mathcal{S} \subset \mathrm{int}\Lambda$, $\mathcal{L}_\epsilon$ converges uniformly to $\mathcal{L}$ on $\mathbb{R}^+ \times \mathcal{S} \times \mathbb{R}^d$.

Proof: By Lemma 4.4, both $\mathcal{L}_\epsilon$ and $\mathcal{L}$ are finite on $\mathbb{R}^+ \times \Lambda \times \mathbb{R}^d$. Using Hypothesis 2.2 and (H2) of Hypothesis 2.3 we may write, for any $s \in \mathbb{R}^+$, $s' \in \mathbb{R}^+$, and any $(u,v) \in \Lambda \times \mathbb{R}^d$,

$$|\mathcal{L}_\epsilon(s,u,v) - \mathcal{L}_\epsilon(s',u,v)| \le \sup_{\substack{\delta\in\Delta: \\ \Lambda^{(\delta,\epsilon)}\ni u}} |f_\epsilon^{(0)}(s,u,\delta) - f_\epsilon^{(0)}(s',u,\delta)| \le \theta|s-s'| \tag{4.25}$$

This proves (i). Assertions (ii) and (iv) are likewise deduced from $(H3)$ of Hypothesis 2.3 and Hypothesis 2.4. Knowing (i), (ii), and (ii) of Lemma 4.4, the proof of assertion (iii) is similar to that of Lemma 2.5. Assertion (iv) is an immediate consequence of Hypothesis (H4).

To prove (iv), by the second remark following Hypothesis 2.2, for any $(s,u) \in \mathbb{R}^+ \times \Lambda$, there exists $\epsilon' = \epsilon'(u) > 0$ such that for all $\epsilon \le \epsilon'$ such that

$$\mathcal{L}_\epsilon(s,u,v) = \log \sum_{\delta\in\Delta:\Lambda^{(\delta)}\ni u} e^{(v,\delta)+f_\epsilon^{(0)}(s,u,\delta)} \tag{4.26}$$

This implies that

$$|\mathcal{L}_\epsilon(s,u,v) - \mathcal{L}(s,u,v)| \le \sup_{\delta\in\Delta:\Lambda^{(\delta)}\ni u} |f_\epsilon^{(0)}(s,u,\delta) - f(s,u,\delta)| \tag{4.27}$$

where the right hand side is independent of v and, by Hypothesis 2.4, converges to zero. This yields (iv).

Finally, the prove of (v) is almost identical to that of (iv). We only need to observe that the $\epsilon'(u)$ can be chosen uniform for $u \in \mathcal{S}$ if $\mathcal{S}$ is a compact subset of the interior of Λ, and that as indicated in the remark following Hypothesis 2.4, the right hand side of (4.27) converges to zero uniformly on $\mathbb{R}^+ \times \mathcal{S}$. $\diamondsuit$

Lemma 4.9: *For all $\epsilon > 0$,*

(i) There exists a constant $0 < \theta < \infty$ such that:

$$\sup_{\substack{u\in\Lambda \\ v^*\in \mathrm{conv}\,\Delta}} |\mathcal{L}_\epsilon^*(s,u,v^*) - \mathcal{L}_\epsilon^*(s',u,v^*)| \le \theta|s-s'|, \quad \forall s \in \mathbb{R}^+, \forall s' \in \mathbb{R}^+ \tag{4.28}$$

(ii) For any closed bounded subset $\mathcal{S} \subset \mathrm{int}_\epsilon\Lambda$, there exists a positive constant $\vartheta \equiv \vartheta(\mathcal{S}) < \infty$ such that:

$$\sup_{\substack{s\in\mathbb{R}^+ \\ v^*\in \,\mathrm{conv}\,\Delta}} |\mathcal{L}_\epsilon^*(s,u,v^*) - \mathcal{L}_\epsilon^*(s,u',v^*)| \leq \vartheta|u-u'|, \quad \forall u \in \mathcal{S}, \forall u' \in \mathcal{S} \tag{4.29}$$

(iii) The function $\mathcal{L}_\epsilon^*(s,u,v^*)$ is jointly continuous in s, u and v^* relative to $\mathbb{R}^+ \times \mathrm{int}(\,\mathrm{int}_\epsilon\Lambda) \times \mathrm{conv}\Delta$.

Moreover (i)-(iii) hold with $\mathcal{L}_\epsilon^*$ replaced by $\mathcal{L}^*$ and $\mathrm{int}_\epsilon\Lambda$ replaced by $\mathrm{int}\Lambda$.

In addition:

(iv) For each $(s,u,v^*) \in \mathbb{R}^+ \times \Lambda \times \,\mathrm{conv}\,\Delta$,

$$\lim_{\epsilon\to 0} \mathcal{L}_\epsilon^*(s,u,v^*) = \mathcal{L}^*(s,u,v^*) \tag{4.30}$$

exists and is finite for all (s,u,v^*) such that $s \in \mathbb{R}^+$, $u \in \Lambda$, $v^* \in \mathrm{dom}\Phi^*_{s,u}$.

(v) For every closed bounded set $\mathcal{S} \subset \mathrm{int}\Lambda$, $\mathcal{L}_\epsilon^*$ converges uniformly to $\mathcal{L}^*$ on $\mathbb{R}^+ \times \mathcal{S} \times \mathrm{conv}\,\Delta$.

Proof: By Lemma 4.5, both $\mathcal{L}_\epsilon^*$ and $\mathcal{L}^*$ are finite on $\mathbb{R}^+ \times \mathrm{int}\Lambda \times \mathrm{conv}\,\Delta$. To prove (i) we write that for any $s \in \mathbb{R}^+$, $s' \in \mathbb{R}^+$, and any $(u,v^*) \in \Lambda \times \mathrm{conv}\Delta$,

$$\begin{aligned}
\mathcal{L}_\epsilon^*(s,u,v^*) &= \sup_{v\in\mathbb{R}^d} \{(v,v^*) - \mathcal{L}_\epsilon(s',u,v) + (\mathcal{L}_\epsilon(s',u,v) - \mathcal{L}_\epsilon(s,u,v))\} \\
&\leq \sup_{v\in\mathbb{R}^d} \left\{(v,v^*) - \mathcal{L}_\epsilon(s',u,v) + \sup_{v\in\mathbb{R}^d} |\mathcal{L}_\epsilon(s',u,v) - \mathcal{L}_\epsilon(s,u,v))|\right\} \\
&= \mathcal{L}_\epsilon^*(s',u,v^*) + \sup_{v\in\mathbb{R}^d} |\mathcal{L}_\epsilon(s',u,v) - \mathcal{L}_\epsilon(s,u,v))|
\end{aligned} \tag{4.31}$$

and by (4.23) of Lemma 4.8,

$$\mathcal{L}_\epsilon^*(s,u,v^*) - \mathcal{L}_\epsilon^*(s',u,v^*) \leq \theta|s-s'| \tag{4.32}$$

Similarly we can show that

$$\mathcal{L}_\epsilon^*(s,u,v^*) - \mathcal{L}_\epsilon^*(s',u,v^*) \geq -\theta|s-s'| \tag{4.33}$$

Thus (i) is proven. Assertions (ii) is obtained in the same way on the basis of assertion (ii) of Lemma 4.8. whereas (iii) is deduced from Lemma 4.8, (iii), together with Lemma 4.6, (v).

To prove (iv), note that using the remark following Lemma 4.5, there exists $\epsilon' = \epsilon'(u) > 0$ such that for $\epsilon < \epsilon'(u)$, for any $v^* \in \mathrm{dom}\Phi^*_{s,u}$

$$|\mathcal{L}^*_\epsilon(s,u,v^*) - \mathcal{L}^*(s,u,v^*)| \leq \sup_{v\in\mathbb{R}^d} |\mathcal{L}_\epsilon(s,u,v) - \mathcal{L}(s,u,v)| \tag{4.34}$$

and the right hand side converges to zero by Lemma 4.8 (iv). Note that the convergence is even uniform in v^*. (v) now follows by the same arguments that were used in the proof of (v) of Lemma 4.8. The proof is done. $\diamondsuit$

4.3. Some properties of semi-continuous regularisations.

The results established in the previous sub-section will be mainly used for the lower bounds. For these the use of the functions $\mathcal{L}_\epsilon$, $\mathcal{L}^*_\epsilon$, defined in terms of the functions $f^{(0)}_\epsilon$ will be convenient. The upper bounds will rely on the use of (upper-, resp. lower) semi-continuous regularisations of the functions L_ϵ, resp. L^*_ϵ. Let us first note that all results of in 4.2 that did not rely to the Lipshitz continuity of $f^{(0)}_\epsilon$ are also valid for L_ϵ and L^*_ϵ.

For $r > 0$ we define:

$$\mathcal{L}^{(r)}_\epsilon(s,u,v) \equiv \sup_{s':|s-s'|\leq r}\ \sup_{u':|u-u'|\leq r} L_\epsilon(s',u',v) \tag{4.35}$$

Set $\Lambda(r) \equiv \{u \in \mathbb{R}^d \mid \mathrm{dist}(u,\Lambda) \leq r\}$. The following lemma establishes some simple properties of $\mathcal{L}^{(r)}_\epsilon$ we will need later.

Lemma 4.10:

(i) *On* $\mathbb{R}^+ \times (\mathbb{R}^d \backslash \Lambda(r)) \times \mathbb{R}^d$, $\mathcal{L}^{(r)}_\epsilon = -\infty$.

(ii) *For all* $(s,v) \in \mathbb{R}^+ \times \mathbb{R}^d$, *and all* $e > 0, r > 0$ *the function* $u \to \mathcal{L}^{(r)}_\epsilon(s,u,v)$ *is upper semi-continuous (u.s.c.) at each* $u \in \Lambda(r)$.

(iii) *For all* $(s,u) \in \mathbb{R}^+ \times \Lambda(r)$, *the function* $\Phi^{(r)}_{\epsilon,s,u}$ *is convex and* $\mathrm{dom}\Phi^{(r)}_{\epsilon,s,u} = \mathbb{R}^d$.

Proof: The proof is trivial and is left to the reader.$\diamondsuit$

The next Lemma relates the function $\mathcal{L}^{(r)}_\epsilon$ to the function $\mathcal{L}^{(r)}_\epsilon$ defined in (2.26).

Lemma 4.11: *For any* $(s,u,v^*) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\left(\mathcal{L}^{(r)}_\epsilon\right)^*(s,u,v^*) = \mathcal{L}^{(r)*}_\epsilon(s,u,v^*) \tag{4.36}$$

Proof: We first prove that $\left(\mathcal{L}_\epsilon^{(r)}\right)^*(s,u,v^*) \geq \mathcal{L}_\epsilon^{(r)*}(s,u,v^*)$. For any $\widetilde{v} \in \mathbb{R}^d$,

$$\begin{aligned}\left(\mathcal{L}_\epsilon^{(r)}\right)^*(s,u,v^*) &\geq (\widetilde{v},v^*) - \mathcal{L}_\epsilon^{(r)}(s,u,\widetilde{v}) \\ &= \inf_{s':|s-s'|\leq r}\ \inf_{u':|u-u'|\leq r} \{(\widetilde{v},v^*) - L_\epsilon(s',u',\widetilde{v})\}\end{aligned} \tag{4.37}$$

Now we choose for $\widetilde{v}$ the value s.t.

$$\sup_{v\in\mathbb{R}^d}\{(v,v^*) - L_\epsilon(s',u',v)\} = (\widetilde{v},v^*) - L_\epsilon(s',u',\widetilde{v}) \tag{4.38}$$

With this choice (4.37) becomes indeed

$$\left(\mathcal{L}_\epsilon^{(r)}\right)^*(s,u,v^*) \geq \inf_{s':|s-s'|\leq r}\ \inf_{u':|u-u'|\leq r} L_\epsilon^*(s',u',v^*) = \mathcal{L}_\epsilon^{(r)*}(s,u,v^*) \tag{4.39}$$

Next we show the converse inequality. Note that for any $\widetilde{s},\widetilde{u}$ s.t. $|s-\widetilde{s}|\leq r, |u-\widetilde{u}|\leq r$, and any $v \in \mathbb{R}^d$,

$$\sup_{s':|s-s'|\leq r}\ \sup_{u':|u-u'|\leq r} L_\epsilon(s',u',v) \geq L_\epsilon(\widetilde{s},\widetilde{u},v) \tag{4.40}$$

Hence

$$\begin{aligned}&\left(\mathcal{L}_\epsilon^{(r)}\right)^*(s,u,v^*) = \sup_{v\in\mathbb{R}^d}\left\{(v,v^*) - \sup_{s':|s-s'|\leq r}\ \sup_{u':|u-u'|\leq r} L_\epsilon(s',u',v)\right\} \\ &\leq \sup_{v\in\mathbb{R}^d}\{(v,v^*) - L_\epsilon(\widetilde{s},\widetilde{u},v)\} = L_\epsilon^*(\widetilde{s},\widetilde{u},v^*)\end{aligned} \tag{4.41}$$

Since (4.41) holds for all $\widetilde{s},\widetilde{u}$ in the given sets, it follows that

$$\left(\mathcal{L}_\epsilon^{(r)}\right)^*(s,u,v^*) \leq \inf_{\widetilde{s}:|\widetilde{s}-s|\leq r}\ \inf_{\widetilde{u}:|\widetilde{u}-u|\leq r} L_\epsilon^*(\widetilde{s},\widetilde{u},v^*) = \mathcal{L}_\epsilon^{(r)*}(s,u,v^*) \tag{4.42}$$

we obtain the desired inequality. The two inequalities imply (4.36). $\diamondsuit$

The previous Lemma allows to deduce the following analog of Lemma 4.10:

Lemma 4.12:

(i) On $\mathbb{R}^+ \times (\mathbb{R}^d\backslash\Lambda(r)) \times \mathbb{R}^d$, $\mathcal{L}_\epsilon^{(r)*} = +\infty$.

(ii) For all $(s,v^*) \in \mathbb{R}^+ \times \mathbb{R}^d$, and all $e > 0, r > 0$ the function $u \to \mathcal{L}_\epsilon^{(r)*}(s,u,v^*)$ is lower semi-continuous (l.s.c.) at each $u \in \Lambda(r)$.

(iii) For all $(s,u)\in\mathbb{R}^+\times\Lambda(r)$*, the function* $\Phi^{(r)*}_{\epsilon,s,u}$ *is convex and for* $(s,u)\in\mathbb{R}^+\times\mathrm{int}_\epsilon\Lambda(r)$*,* $\mathrm{dom}\Phi^{(r)*}_{\epsilon,s,u}=\mathrm{conv}\Delta$.

Finally we come to the central result of this sub-section.

Lemma 4.13: *For any* $r>0$ *and for any closed bounded* $\mathcal{S}\subset\mathrm{int}\Lambda(r)$ *the following holds:*

(i) $\mathcal{L}^{(r)}_\epsilon$ *converges uniformly to* $\mathcal{L}^{(r)}$ *on* $\mathbb{R}^+\times\mathcal{S}\times\mathbb{R}^d$.

(ii) $\mathcal{L}^{(r)*}_\epsilon$ *converges uniformly to* $\mathcal{L}^{(r)*}$ *on* $\mathbb{R}^+\times\mathcal{S}\times\mathrm{conv}\Delta$.

Proof: Since (ii) follows from (i) in the same way as Lemma 4.9 follows from Lemma 4.8, we concentrate on the proof of (i). Fix $r>0$. Define the sets

$$\begin{aligned}A_\epsilon\equiv\Big\{(s^*,u^*,v)\in\mathbb{R}^+\times\Lambda(r)\times\mathbb{R}^d\,\Big|\,\exists(s,u):|s-s^*|\leq r,|u-u^*|\leq r:\\ \mathcal{L}_\epsilon(s^*,u^*,v)=\sup_{s':|s-s'|\leq r}\sup_{u':|u-u'|\leq r}L_\epsilon(s',u',v)\Big\}\end{aligned}\tag{4.43}$$

and put

$$A^\epsilon\equiv\cup_{0\leq\epsilon'\leq\epsilon}A_{\epsilon'}\tag{4.44}$$

Define

$$\mathcal{L}^{(r)}_{\epsilon,\epsilon_0}(s,u,v)\equiv\lim_{\epsilon\downarrow0}\sup_{\substack{s':|s-s'|\leq r,u':|u-u'|\leq r\\(s',u',v)\in A}}L_\epsilon(s',u',v)\tag{4.45}$$

Write

$$\begin{aligned}&\left|\mathcal{L}^{(r)}_\epsilon(s,u,v)-\mathcal{L}^{(r)}(s,u,v)\right|\\ &\leq\left|\mathcal{L}^{(r)}_\epsilon(s,u,v)-\mathcal{L}^{(r)}_{\epsilon,\epsilon_0}(s,u,v)\right|+\left|\mathcal{L}^{(r)}_{\epsilon,\epsilon_0}(s,u,v)\mathcal{L}^{(r)}(s,u,v)\right|\end{aligned}\tag{4.46}$$

By definition of the set $\mathcal{A}^\epsilon$, for $\epsilon_0\geq\epsilon$,

$$\left|\mathcal{L}^{(r)}_\epsilon(s,u,v)-\mathcal{L}^{(r)}_{\epsilon,\epsilon_0}(s,u,v)\right|=0\tag{4.47}$$

On the other hand, for $(s^*,u^*,v)\in A^{\epsilon_0}$, $\exists\epsilon'\leq\epsilon_0$ and (s,u) with $|s-s^*|\leq r,|u-u^*|\leq r$, such that for all (s',u') with $|s-s'|\leq r,|u-u'|\leq r$,

$$L_{\epsilon'}(s^*,u^*,v)\geq L_{\epsilon'}(s',u',v)\tag{4.48}$$

Recalling the definition of $L_{\epsilon'}$, (4.48) implies that for any $\gamma>0$,

$$\begin{aligned}&\sum_{\substack{\delta\in\Delta\\g_{\epsilon'}(s^*,u^*,\delta)\geq\gamma}}e^{(\delta,v)}g_{\epsilon'}(s^*,u^*,\delta)+\sum_{\substack{\delta\in\Delta\\g_{\epsilon'}(s^*,u^*,\delta)<\gamma}}e^{(\delta,v)}g_{\epsilon'}(s^*,u^*,\delta)\\ \geq&\sum_{\substack{\delta\in\Delta\\g_{\epsilon'}(s^*,u^*,\delta)\geq\gamma}}e^{(\delta,v)}g_{\epsilon'}(s',u',\delta)+\sum_{\substack{\delta\in\Delta\\g_{\epsilon'}(s^*,u^*,\delta)<\gamma}}e^{(\delta,v)}g_{\epsilon'}(s',u',\delta)\end{aligned}\tag{4.49}$$

The important point is now that since $\mathcal{S} \subset \mathrm{int}\Lambda(r)$, no matter what $u^* \in \mathcal{S}$, there exists a $q = \mathrm{dist}(\mathcal{S}, \Lambda(r)^c) > 0$, such that for some u' with $|u' - u| \le r$. By Hypothesis 2.2, and the continuity assumptions of Hypothesis 2.3, one has that there exists a constant $c_q > 0$ such that for all these points, and for all $\delta \in \Delta$, $g_{\epsilon'}(s', u', \delta) > c_q$. Choosing such u' and $s' = s^*$, (4.49) implies that

$$(c_q - \gamma) \sum_{\substack{\delta\in\Delta \\ g_{\epsilon'}(s^*,u^*,\delta)<\gamma}} e^{(\delta,v)} \le \sum_{\substack{\delta\in\Delta \\ g_{\epsilon'}(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g_{\epsilon'}(s^*, u^*, \delta) \tag{4.50}$$

By Hypothesis 2.4, g_ϵ converges uniformly. Therefore, for any $\eta > 0$, there exists $\epsilon_0 > 0$, such that for all $\epsilon, \epsilon' \le \epsilon_0$, and all $(s^*, u^*, \delta) \in \mathbb{R}^+ \times (\mathcal{S} \cap \Lambda) \times \mathbb{R}^d$,

$$|g_{\epsilon'}(s^*, u^*, \delta) - g_{\epsilon'}(s^*, u^*, \delta)| \le \eta \tag{4.51}$$

Given η, let now ϵ_0 be such that (4.51) holds. Then (4.50) implies that for all $\epsilon \le \epsilon_0$,

$$\begin{aligned}(c_q - \gamma) \sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)<\gamma-\eta}} e^{(\delta,v)} &\le \sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma-\eta}} e^{(\delta,v)} (g_\epsilon(s^*, u^*, \delta) + \eta) \\ &\le (1 + \frac{\eta}{\gamma}) \sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma+\eta}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta)\end{aligned} \tag{4.52}$$

Therefore, for all $\epsilon \le \epsilon_0$, and $(s^*, u^*, v) \in \mathcal{A}^{\epsilon_0}$,

$$\begin{aligned}&\mathcal{L}_\epsilon(s^*, u^*, v) \\ &= \ln\left(\sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta) \left(1 + \frac{\sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)<\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta)}{\sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta)}\right)\right) \\ &= \ln \sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta) \\ &+ \ln\left(1 + \frac{\sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)<\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta)}{\sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta)}\right)\end{aligned} \tag{4.53}$$

The last term in (4.53) is bounded by

$$\ln\left(1 + \frac{\gamma}{c_q - \gamma - \eta}\right) \le \frac{\gamma}{c_q - \gamma - \eta} \tag{4.54}$$

which will be made small by choosing γ small enough. On the other hand,

$$\left| \ln \sum_{\substack{\delta\in\Delta \\ g_{\epsilon}(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g_\epsilon(s^*, u^*, \delta) - \ln \sum_{\substack{\delta\in\Delta \\ g_(s^*,u^*,\delta)\ge\gamma}} e^{(\delta,v)} g(s^*, u^*, \delta) \right| \le \frac{\eta}{\gamma} \tag{4.55}$$

Therefore, choosing $\gamma = \sqrt{c_q \eta}$, we see that for all $\epsilon \le \epsilon_0$,

$$\left| \mathcal{L}^{(r)}_{\epsilon,\epsilon_0}(s,u,v) - \mathcal{L}^{(r)}(s,u,v) \right| \le 3\sqrt{\eta / c_q} \tag{4.56}$$

Combining both observations, we see that with $\epsilon = \epsilon_0$, we get in fact that

$$\left| \mathcal{L}^{(r)}_{\epsilon_0}(s,u,v) - \mathcal{L}^{(r)}(s,u,v) \right| \le 3\sqrt{\eta c_q} \tag{4.57}$$

which implies the desired uniform convergence and proves (i). (ii) follows easily in the same way as the convergence result in Lemma 4.9 (v) follows from Lemma 4.8 (v).$\diamondsuit$

Proof of Lemma 3.3: By definition, for any $(s,u,v^*) \in \mathbb{R}^+ \times \Lambda \times \operatorname{conv}\Delta$,

$$\overline{\mathcal{L}}^*(s,u,v^*) = \liminf_{\substack{s' \to s \\ u' \to u}} \mathcal{L}^*(s',u',v^*) \tag{4.58}$$

But by Lemma 4.11, the function $\mathcal{L}^*(s,u,v^*)$ is jointly continuous in the variables s,u at any $(s,u,v^*) \in \mathbb{R}^+ \times \operatorname{int}\Lambda \times \operatorname{conv}\Delta$ so that on this set the right hand side of (4.58) coincides with $\mathcal{L}^*(s,u,v^*)$. This proves Lemma 3.3.$\diamondsuit$

4.4. A continuity derived result.

We shall here be interested in the case $u \in \operatorname{int}\Lambda$ only. As seen in Lemma 4.7 the conjugacy correspondence between Φ and Φ^* is closely connected to their differentiability properties. To this we may add:

Lemma 4.14: *Let $(s,u) \in \mathbb{R}^+ \times \operatorname{int}\Lambda$. Then $\nabla\Phi^*(v^*)$ is bounded if and only if $v^* \in$* $\operatorname{ri}(\operatorname{dom}\Delta)$.

Proof: We know from Lemma 4.5 and Lemma 4.6 that for each $(s,u) \in \mathbb{R}^+ \times \operatorname{int}\Lambda$, Φ^* is a proper, closed, and strictly convex function having effective domain $\operatorname{conv}\Delta$. Moreover, we saw in the proof of lemma 4.5 that the subgradient of Φ^* reduces to the gradient mapping. Finally, invoking Theorem 23.4 of [Ro], the subgradient of Φ^* at v^* is a non empty and bounded set if and only if $v^* \in \operatorname{ri}(\operatorname{dom}\Delta)$. The lemma is proven. $\diamondsuit$

Now boundedness of $\nabla\Phi^*$ turns out to be an essential ingredient of the proof of the large deviations estimates of Chapter 5. The particular place where it is needed appears in the context of the minimisation problem of Lemma 4.15 below. There, we shall see that the continuity property of Φ^*, which in contrast with it's differentiability properties hold up to $\operatorname{rbd}(\operatorname{dom}\Delta)$, enables us to restrict ourselves to situations where $\nabla\Phi^*$ is bounded.

Lemma 4.15: *Let $\mathcal{F} \subset \mathcal{D}^\circ([0,T])$ be a convex subset of $\mathcal{D}^\circ([0,T])$ and set*

$$\mathcal{G} \equiv \left\{ \psi \in \mathcal{F} \,\middle|\, \dot{\psi}(t) \in \mathrm{ri}(\,\mathrm{conv}\Delta),\ 0 \le t \le T \right\} \tag{4.59}$$

Then,

$$\inf_{\psi \in \mathcal{F}} \int_0^T dt \mathcal{L}^*(t, \psi(t), \dot{\psi}(t)) = \inf_{\psi \in \mathcal{G}} \int_0^T dt \mathcal{L}^*(t, \psi(t), \dot{\psi}(t)) \tag{4.60}$$

Proof: With $\mathcal{D}^\circ([0,T])$ defined in (2.34) recall that $\Phi^* t, \psi(t)(\cdot) = \mathcal{L}^*(t, \psi(t), \cdot)$. As seen in the proof of Lemma 4.14, for $\psi \in \mathcal{D}^\circ([0,T])$, $\Phi^*_{t,\psi(t)}$ is a proper, closed, strictly convex, and positive function having effective domain $\mathrm{conv}\Delta$. This in particular ensures that both sides of (4.60) are finite. Since $\mathcal{F} \supseteq \mathcal{G}$,

$$\inf_{\psi \in \mathcal{F}} \int_0^T dt \mathcal{L}^*(t, \psi(t), \dot{\psi}(t)) \le \inf_{\psi \in \mathcal{G}} \int_0^T dt \mathcal{L}^*(t, \psi(t), \dot{\psi}(t)) \tag{4.61}$$

and we only have to prove the reverse inequality. To do so we will use that for any $\psi_1 \in \mathcal{G}$ and any $\psi_2 \in \mathcal{F}$ the path $\alpha\psi_1 + (1-\alpha)\psi_2$ belongs to $\mathcal{G}$ for each $0 < \alpha \le 1$: obviously, by the convexity assumption on $\mathcal{F}$, $\alpha\psi_1 + (1-\alpha)\psi_2 \in \mathcal{F}$; but since for each $t \le 0 \le 1$ $\dot{\psi}_1(t)$ is a point in $\mathrm{ri}(\,\mathrm{conv}\Delta)$ and $\dot{\psi}_2(t)$ a point in $\mathrm{conv}\Delta$, the point $\alpha\dot{\psi}_1(t) + (1-\alpha)\dot{\psi}_2(t)$ lies in $\mathrm{ri}(\,\mathrm{conv}\Delta)$ for each $0 < \alpha \le 1$ (see [Ro], Theorem 6.1) so that $\alpha\psi_1 + (1-\alpha)\psi_2$ lies in $\mathcal{G}$. Thus, given $\psi_1 \in \mathcal{G}$ and $\psi_2 \in \mathcal{F}$ we have, for each $0 < \alpha \le 1$,

$$\begin{aligned} &\inf_{\psi \in \mathcal{G}} \int_0^T dt \mathcal{L}^*(t, \psi(t), \dot{\psi}(t)) \\ &\le \int_0^T dt \mathcal{L}^*(t, \psi_2(t) + \alpha[\psi_1(t) - \psi_2(t)], \dot{\psi}_2(t) + \alpha[\dot{\psi}_1(t) - \dot{\psi}_2(t)]) \end{aligned} \tag{4.62}$$

where the integrand in the last line is positive and bounded for each $0 < \alpha \le 1$. Thus, taking the limit $\alpha \downarrow 0$, we may write, using Lebesgue's dominated convergence Theorem,

$$\begin{aligned} &\inf_{\psi \in \mathcal{G}} \int_0^T dt \mathcal{L}^*(t, \psi(t), \dot{\psi}(t)) \\ &\le \lim_{\alpha \downarrow 0} \int_0^T dt \mathcal{L}^*(t, \psi_2(t) + \alpha[\psi_1(t) - \psi_2(t)], \dot{\psi}_2(t) + \alpha[\dot{\psi}_1(t) - \dot{\psi}_2(t)]) \\ &= \int_0^T dt \lim_{\alpha \downarrow 0} \mathcal{L}^*(t, \psi_2(t) + \alpha[\psi_1(t) - \psi_2(t)], \dot{\psi}_2(t) + \alpha[\dot{\psi}_1(t) - \dot{\psi}_2(t)]) \\ &= \int_0^T dt \mathcal{L}^*(t, \psi_2(t), \dot{\psi}_2(t)) \end{aligned} \tag{4.63}$$

where in the last line we used that $\mathcal{L}^*(s,u,v^*)$ is jointly continuous in the variables s, u, and v^* relative to $\mathcal{D}^\circ([0,T])$ (see Lemma 4.9, last line and assertion (iii)). Finally, since (4.63) is true for any $\psi_2 \in \mathcal{F}$,

$$\inf_{\psi \in \mathcal{G}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) \leq \inf_{\psi \in \mathcal{F}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot{\psi}(t)) \tag{4.64}$$

which concludes the proof of the lemma.$\diamondsuit$

4.5. Proof of Proposition 2.7.

The proof of Proposition 2.7 goes along the same lines as that of Lemma 4.14.

Let ψ_1 be any path in $\mathcal{A} \cap \mathcal{D}^\circ([0,T])$ and let ψ_2 be any path in $\mathcal{A} \cap \overline{\mathcal{D}}([0,T])$. It follows from the convexity of $\mathcal{A}$ together with the definitions of $\mathcal{D}^\circ([0,T])$ and $\overline{\mathcal{D}}([0,T])$ that the path $\alpha\dot{\psi}_1(t) + (1-\alpha)\dot{\psi}_2(t)$ lies in $\mathcal{A} \cap \mathcal{D}^\circ([0,T])$ for each $0 < \alpha \leq 1$. Hence, for each such α,

$$\begin{aligned}
&\inf_{\psi \subset \mathcal{A}\cap\mathcal{D}^\circ([0,T])} \int_0^T \overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t))dt \\
&\leq \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \alpha\dot{\psi}_1(t)+(1-\alpha)\dot{\psi}_2(t))dt \\
&\leq \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \dot{\psi}_2(t))dt \\
&+ \alpha\Big\{ \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \dot{\psi}_1(t))dt \\
&\quad - \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \dot{\psi}_2(t))dt \Big\}
\end{aligned} \tag{4.65}$$

Now condition (i) implies that

$$\lim_{\alpha\downarrow 0} \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \dot{\psi}_2(t))dt \leq \int_0^T \overline{\mathcal{L}}^*(t,\psi_2(t),\dot{\psi}_2(t))dt \tag{4.66}$$

while condition (ii) guarantees that

$$\begin{aligned}
&\lim_{\alpha\downarrow 0} \Big\{ \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \dot{\psi}_1(t))dt \\
&\qquad - \int_0^T \overline{\mathcal{L}}^*(t,\alpha\psi_1(t)+(1-\alpha)\psi_2(t), \dot{\psi}_2(t))dt \Big\} = 0
\end{aligned} \tag{4.67}$$

Since this is true for all $\psi_2 \in \mathcal{A} \cap \overline{\mathcal{D}}([0,T])$, we have

$$\inf_{\psi \subset \mathcal{A}\cap\mathcal{D}^\circ([0,T])} \int_0^T \overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t))dt \leq \inf_{\psi \subset \mathcal{A}\cap\overline{\mathcal{D}}([0,T])} \int_0^T \overline{\mathcal{L}}^*(t,\psi(t),\dot{\psi}(t))dt \tag{4.68}$$

As the reverse inequality trivially holds, the proposition is proven. $\diamondsuit$

5. Proof of Proposition 3.2

We are now ready to prove the main estimates of the paper. Basically, the idea of the proof is simple and consist of exploiting the "almost-independence" of consecutive jumps over length scales large compared to 1 but small compared to $1/\epsilon$, as in Wentzell's work. The source of this almost independence are of course the regularity properties of the transition probabilities. On the basis of this independence, we bring to bear classical Cramér type-techniques. The main difficulties arise from the non-uniformity of our regularity assumptions near the boundaries.

The chapter is divided in three subchapters. We will first get equipped with some preparatory tools. Armed with these, the basic upper and lower bounds are next derived. Lastly, using results from Chapter 4, the proof is brought to a close. From now on the letter t will be used exclusively for time parameters taking value in $[0,T]$ (that is, on 'macroscopic scale' 1) while k will be reserved for discrete time parameters (on 'microscopic scale' ϵ^{-1}).

5.1: Preparatory steps.

Lemma 5.1 below provides a covering of the ball $\mathcal{B}_\rho(\phi)$ into basic 'tubes'.

Λ^c denotes the complement of Λ in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$, $\text{dist}(x,A) \equiv \inf_{y\in A}|x-y|$. Recall that given $\rho > 0$ and $\phi \in \mathcal{E}([0,T])$, $\mathcal{B}_\rho(\phi) = \left\{\psi \in \mathcal{E}([0,T]) \,\middle|\, \max_{0\leq t\leq T}|\psi(t)-\phi(t)| < \rho\right\}$.

Lemma 5.1: *Let $0 = t_0 < t_1 < \cdots < t_n = T$ be any partition of $[0,T]$ into n intervals and set*

$$\tau \equiv \max_{0\leq i\leq n} \epsilon^{-1}|t_{i+1}-t_i| \tag{5.1}$$

For $\eta > 0$ and $\psi \in \mathcal{E}([0,T])$ define,

$$\mathcal{A}_\eta(\psi) = \left\{\psi' \in \mathcal{E}([0,T]) \,\middle|\, \max_{0\leq i\leq n}|\psi'(t_i)-\psi(t_i)| \leq 2\eta\right\} \tag{5.2}$$

and for $\gamma \geq 0$ define,

$$\begin{aligned}
\mathcal{B}_{\rho,\gamma}(\phi) &= \left\{\psi' \in \mathcal{B}_\rho(\phi) \,\middle|\, \inf_{0\leq t\leq T} \text{dist}(\psi'(t),\Lambda^c) \geq \gamma\right\} \\
\bar{\mathcal{B}}_{\rho,\gamma}(\phi) &= \left\{\psi' \in \bar{\mathcal{B}}_\rho(\phi) \,\middle|\, \inf_{0\leq t\leq T} \text{dist}(\psi'(t),\Lambda^c) \geq \gamma\right\}
\end{aligned} \tag{5.3}$$

the restrictions of $B_\rho(x)$ and its closure to the γ-interior of Λ.

(i) *For any $\gamma \geq 0$ and $\eta > 0$ such that $\rho > 2\eta$, there exists a subset $\mathcal{R}_{\rho,\eta,\gamma}(\phi)$ of $\mathcal{E}([0,T])$ such that:*

$$\mathcal{R}_{\rho,\eta,\gamma}(\phi) \subset \bar{\mathcal{B}}_{\rho,\gamma}(\phi) \subset \bigcup_{\psi \in \mathcal{R}_{\rho,\eta,\gamma}(\phi)} \mathcal{A}_\eta(\psi) \tag{5.4}$$

$$|\mathcal{R}_{\rho,\eta,\gamma}(\phi)| \leq e^{dn\left(\log\left(\frac{\rho}{\eta}\right)+2\right)}, \quad \forall\gamma \geq 0 \tag{5.5}$$

(ii) *For any $\gamma \geq 0$ and $\eta > 0$ such that $\rho > 2(\eta + \epsilon\tau \operatorname{diam}\Delta)$,*

$$\bigcup_{\psi \in \mathcal{B}_{\rho-2(\eta+\epsilon\tau \operatorname{diam}\Delta),\gamma}(\phi)} \mathcal{A}_\eta(\psi) \subset \mathcal{B}_\rho(\phi) \tag{5.6}$$

Proof: The proof of (5.4) relies on the following construction. Given $\eta > 0$ let $\mathcal{W}_\eta$ be the Cartesian lattice in $\mathbb{R}^d$ with spacing $\frac{\eta}{\sqrt{d}}$. For $y \in \mathbb{R}^d$ set $\mathcal{W}_{\rho,\eta}(y) = \mathcal{W}_\eta \cap \{y' \in \mathbb{R}^d \mid |y'-y| \leq \rho\}$ and for $\phi \in \mathcal{E}([0,T])$, $\mathcal{V}_{\rho,\eta}(\phi) = \times_{i=0}^n \mathcal{W}_{\rho,\eta}(\phi(t_i))$. Next, for $x = (x_0, \ldots, x_n) \in \mathcal{V}_{\rho,\eta}(\phi)$, define

$$A_{\rho,\eta,\gamma}(x) = \left\{ \psi' \in \bar{\mathcal{B}}_{\rho,\gamma}(\phi) \,\middle|\, \max_{0 \leq i \leq n} |\psi'(t_i) - x_i| \leq \eta, \right\} \tag{5.7}$$

Thus $A_{\rho,\eta,\gamma}(x)$ is the set of paths in $\bar{\mathcal{B}}_{\rho,\gamma}(\phi)$ which at time t_i are within a distance η of the lattice point x_i. Obviously, the collection of all (not necessarily disjoint and possibly empty sets) $A_{\rho,\eta,\gamma}(x)$ form a covering of $\bar{\mathcal{B}}_{\rho,\gamma}(\phi)$:

$$\bar{\mathcal{B}}_{\rho,\gamma}(\phi) = \bigcup_{x \in \mathcal{V}_{\rho,\eta}(\phi)} A_{\rho,\eta,\gamma}(x) \tag{5.8}$$

In each of those sets $A_{\rho,\eta,\gamma}(x)$ that are non empty pick one element arbitrarily and label it ψ_x. Clearly $\psi_x \in \bar{\mathcal{B}}_{\rho,\gamma}(\phi)$. Moreover for all $\psi' \in A_{\rho,\eta,\gamma}(x)$, $|\psi'(t_i) - \psi_x(t_i)| \leq 2\eta$ for all $i = 0, \ldots, n$, and hence $A_{\rho,\eta,\gamma}(x) \subset \mathcal{A}_\eta(\psi_x)$. Putting these information together with (5.8) and taking $\mathcal{R}_{\rho,\eta,\gamma}(\phi) = \{\psi_x \mid x \in \mathcal{V}_{\rho,\eta}(\phi)\}$ yields (5.4). Finally (5.5) follows from the bound $|\mathcal{R}_{\rho,\eta,\gamma}(\phi)| \leq |\mathcal{V}_{\rho,\eta}(\phi)| \leq (\max_i |\mathcal{W}_{\rho,\eta}(\phi(t_i))|)^n$ together with the estimate $|\mathcal{W}_{\rho,\eta}(y)| \leq \exp\left\{d\left(\log\left(\frac{\rho}{\eta}\right) + 2\right)\right\}$, $y \in \mathbb{R}^d$, whose (simple) proof can be found e.g. in [BG5].

We now prove (5.6). Set $\bar{\rho} \equiv \rho - 2(\eta + \epsilon\tau \operatorname{diam}\Delta)$. Let $\psi' \in \bigcup_{\psi \in \mathcal{B}_{\bar{\rho},\gamma}(\phi)} \mathcal{A}_\eta(\psi)$. Then $\psi' \in \mathcal{A}_\eta(\psi)$ for some $\psi \in \mathcal{B}_{\bar{\rho},\gamma}(\phi)$. Hence,

$$\begin{aligned}
\max_{0 \leq t \leq T} |\psi(t) - \phi(t)| &\leq \max_{0 \leq t \leq T} (|\psi'(t) - \psi(t)| + |\psi(t) - \phi(t)|) \\
&< \max_{0 \leq t \leq T} |\psi'(t) - \psi(t)| + \bar{\rho} \\
&< \max_{0 \leq i \leq n} \max_{t_i \leq t \leq t_{i+1}} (|\psi'(t) - \psi'(t_i)| + |\psi'(t_i) - \psi(t_i)| + |\psi(t) - \psi(t_i)|) + \bar{\rho} \\
&< \max_{0 \leq i \leq n} \max_{t_i \leq t \leq t_{i+1}} (|\psi'(t) - \psi'(t_i)| + |\psi(t) - \psi(t_i)|) + 2\eta + \bar{\rho}
\end{aligned} \tag{5.9}$$

Thus, using that for $\psi'' \in \mathcal{E}([0,T])$,

$$\max_{0\leq i\leq n} \max_{t_i\leq t\leq t_{i+1}} |\psi''(t) - \psi''(t_i)| \leq \max_{0\leq i\leq n} |t_{i+1} - t_i| \text{ diam}\Delta \leq \epsilon\tau \text{ diam}\Delta \tag{5.10}$$

(5.9) entails $\psi' \in \mathcal{B}_{\bar\rho+2(\eta+\epsilon\tau \,\mathrm{diam}\Delta)}(\phi)$, proving (5.6). Lemma 5.1 is proven.$\diamondsuit$

Remark: Note that in general $B_{\rho,0}(x) \neq B_\rho(x)$. However, due to Lemma 3.1, it is true that

$$\widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(B_\rho(\phi)\right) = \widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(B_{\rho,\gamma}(\phi)\right) \tag{5.11}$$

and the same holds true for the closed balls. Thus it will suffice to get upper and lower bounds for the set $B_{\rho,\gamma}$, for all $\gamma \geq 0$. Therefore the following Lemma will be a sufficient starting point.

Lemma 5.1 allows us to control the probabilities in path space by the probabilities of some discrete time observations of the chain. This is the content of the next lemma.

Lemma 5.2: *With the notation of Lemma 5.1, the following holds for any* $0 = t_0 < t_1 < \dots < t_n = T$, $t_i \in \mathbb{R}$, $n \in \mathbb{N}$.

(i) For any $\gamma \geq 0$ *and* $\eta > 0$ *such that* $\rho > 2\eta$,

$$\begin{aligned}\log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\bar{\mathcal{B}}_{\rho,\gamma}(\phi)) \leq \sup_{\psi\in\bar{\mathcal{B}}_{\rho,\gamma}(\phi)} \log \mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\left[\tfrac{t_i}{\epsilon}\right]) - \psi(\epsilon\left[\tfrac{t_i}{\epsilon}\right])\right| \leq 2\eta + 2\epsilon \text{ diam}\Delta\right)\\ + dn\left(\log\left(\frac{\rho}{\eta}\right) + 2\right)\end{aligned} \tag{5.12}$$

(ii) For any $\gamma \geq 0$, *any* η *such that* $\eta > \epsilon\,\mathrm{diam}\Delta$ *and* $\rho > 2(\eta + \epsilon\tau\,\mathrm{diam}\Delta)$, *and any* $\psi \in \mathcal{B}_{\rho-2(\eta+\epsilon\tau\,\mathrm{diam}\Delta),\gamma}(\phi)$,

$$\log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi)) \geq \log \mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\left[\tfrac{t_i}{\epsilon}\right]) - \psi(\epsilon\left[\tfrac{t_i}{\epsilon}\right])\right| < 2\eta - 2\epsilon \text{ diam}\Delta\right) \tag{5.13}$$

Proof: We first prove assertion (i). Assume that η, ρ and γ satisfy the conditions of Lemma 5.1, (i). Then, by (5.4),

$$\begin{aligned}\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\bar{\mathcal{B}}_{\rho,\gamma}(\phi)) &\leq |\mathcal{R}_{\rho,\eta,\gamma}(\phi)| \exp\left\{\sup_{\psi\in\mathcal{R}_{\rho,\eta,\gamma}(\phi)} \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{A}_\eta(\psi))\right\}\\ &\leq |\mathcal{R}_{\rho,\eta,\gamma}(\phi)| \exp\left\{\sup_{\psi\in\bar{\mathcal{B}}_{\rho,\gamma}(\phi)} \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{A}_\eta(\psi))\right\}\end{aligned} \tag{5.14}$$

Now

$$\begin{aligned}\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{A}_\eta(\psi)) =&\widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}|Z(t_i)-\psi(t_i)|\le 2\eta\right)\\ \le&\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X(\left[\tfrac{t_i}{\epsilon}\right])-\psi(\epsilon\left[\tfrac{t_i}{\epsilon}\right])\right|\le 2\eta+\epsilon\ \mathrm{diam}\Delta\right)\end{aligned}\tag{5.15}$$

where we used that, (for $Z\in\mathcal{E}([0,T])$),

$$\begin{aligned}|Z(t)-\psi(t)|\ge&\left|Z(\epsilon\left[\tfrac{t}{\epsilon}\right])-\psi(\epsilon\left[\tfrac{t}{\epsilon}\right])\right|-\left|Z(t)-Z(\epsilon\left[\tfrac{t}{\epsilon}\right])\right|-\left|\psi(t)-\psi(\epsilon\left[\tfrac{t}{\epsilon}\right])\right|\\ \ge&\left|X(\left[\tfrac{t}{\epsilon}\right])-\psi(t)\right|-2\left|t-\epsilon\left[\tfrac{t}{\epsilon}\right]\right|\ \mathrm{diam}\Delta\\ \ge&\left|X(\left[\tfrac{t}{\epsilon}\right])-\psi(t)\right|-2\epsilon\ \mathrm{diam}\Delta\end{aligned}\tag{5.16}$$

Inserting (5.5) and (5.15) in (5.14) gives (5.12). Similarly we derive assertion (ii) of Lemma 5.2 from assertion (ii) of Lemma 5.1, writing first that by (5.6), for any $\psi\in\mathcal{B}_{\rho-2(\eta+\epsilon\tau\,\mathrm{diam}\Delta),\gamma}(\phi)$,

$$\log\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi))\ge\log\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{A}_\eta(\psi))\tag{5.17}$$

and using next that, since $Z\in\mathcal{E}([0,T])$, analogous to (5.16),

$$|Z(t)-\psi(t)|\le 2\epsilon\ \mathrm{diam}\Delta+\left|X(\left[\tfrac{t}{\epsilon}\right])-\psi(\epsilon\left[\tfrac{t}{\epsilon}\right])\right|\tag{5.18}$$

so that

$$\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{A}_\eta(\psi))\ge\mathcal{P}_{\epsilon,\psi_0}\left(\max_{0\le i\le n}\left|X(\left[\tfrac{t_i}{\epsilon}\right])-\psi(\epsilon\left[\tfrac{t_i}{\epsilon}\right])\right|\le 2\eta-\epsilon\,\mathrm{diam}\Delta\right)\tag{5.19}$$

This concludes the proof of Lemma 5.2. $\diamondsuit$

Remark: We could arrange to use Lemma 5.2 with t_i that are multiples of ϵ only, except that $t_n=T$ has to be allowed to be what it wants to be. Thus we prefer to write the more homogeneous form above.

In view of Lemma 5.2 the problem is reduced to estimating the probability that the chain $X(t)$ be pinned in a small neighbourhood of a prescribed point $\psi(t_i)$ at each time t_i. As explained earlier we will do this by comparing the chain in each time interval $[t_{i-1},t_i)$ with a random walk whose steps, on microscopic time scale, take value in Δ and are distributed according to $p_\epsilon([t_{i-1}/\epsilon],\psi(t_{i-1}),\cdot)$. Let $P_{\epsilon,k}=(p_\epsilon(k,x,y))_{y\in\Gamma_\epsilon,x\in\Gamma_\epsilon}$ denote the transition matrix of the chain at time k and, for $\ell\ge 1$, let $P_{\epsilon,k}^{(k,k+\ell)}=\left(P_{\epsilon,k}^{(k,k+\ell)}(x,y)\right)_{y\in\Gamma_\epsilon,x\in\Gamma_\epsilon}$ denote the matrix product

$$P_{\epsilon,k}^{(k,k+\ell)}=\prod_{l=1}^{k}P_{\epsilon,k+l-1}\tag{5.20}$$

Set $k_i \equiv \left[\frac{t_i}{\epsilon}\right]$. By the Markov property, for $\zeta > 0$,

$$
\begin{aligned}
&\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X(\left[\tfrac{t_i}{\epsilon}\right]) - \psi(t_i)\right| \le \zeta\right) \\
&= \sum_{x(k_0)\in\Gamma_\epsilon} \pi_{\epsilon,\phi_0}(x(k_0))\mathbb{1}_{\{|x(k_0)-\psi(t_0)|\le\zeta\}} \sum_{x(k_1)\in\Gamma_\epsilon} \mathbb{1}_{\{|x(k_1)-\psi(t_1)|\le\zeta\}} P_{\epsilon,k_0}^{(k_0,k_1)}(x(k_0),x(k_1))\dots \\
&\qquad\qquad \dots\sum_{x(k_n)\in\Gamma_\epsilon} \mathbb{1}_{\{|x(k_n)-\psi(t_n)|\le\zeta\}} P_{\epsilon,k_{n-1}}^{(k_{n-1},k_n)}(x(k_{n-1}),x(k_n))
\end{aligned}
\tag{5.21}
$$

The following lemma provides estimates for terms of the form $P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i))$.

Lemma 5.3: *Let $\mathcal{S}$ be any closed bounded subset of $\mathrm{int}\Lambda$. Let $\mathcal{S}'$ be an open subset of $\mathcal{S}$ and, for ℓ an integer, assume that the following condition is satisfied: for each $\ell \ge 1$ and $\epsilon > 0$ small enough,*

$$\inf_{x\in\mathcal{S}'} \mathrm{dist}(x,\mathcal{S}^c) > \epsilon\ell\,\mathrm{diam}\Delta \tag{5.22}$$

For $r \ge 0$ set

$$q(\ell,r) = \epsilon\frac{\ell^2}{2}(\theta + \vartheta(\mathcal{S})\,\mathrm{diam}\Delta) + \ell(r + 2\epsilon K(\mathcal{S})) \tag{5.23}$$

with θ, $\vartheta(\mathcal{S})$ and $K(\mathcal{S})$ as in Hypothesis 2.3. Then, for any $x \in \mathcal{S}'$ and any $z \in \mathcal{S}'$,

$$P_{\epsilon,k}^{(k,k+\ell)}(x,y) \mathrel{\substack{<\\>}} e^{\pm q(\ell,|x-z|)} \sum_{\delta(1)\in\Delta}\cdots\sum_{\delta(\ell-1)\in\Delta}\prod_{l=1}^{\ell} e^{f_\epsilon^{(0)}(\epsilon k,z,\delta(l))}\mathbb{1}_{\{\epsilon^{-1}(y-x)-\sum_{m=1}^{\ell-1}\delta(m)\in\Delta\}} \tag{5.24}$$

Proof: First note that if y is such that $P_{\epsilon,k}^{(k,k+\ell)}(x,y) = 0$ then $\mathbb{1}_{\{\epsilon^{-1}(y-x)-\sum_{m=1}^{\ell-1}\delta(m)\in\Delta\}} = 0$ for all sequences $(\delta(1),\dots,\delta(\ell-1)) \in \times_{l=1}^{\ell-1}\Delta$, and hence (5.24) holds true. Assume that y is such that $P_{\epsilon,k}^{(k,k+\ell)}(x,y) \ne 0$ and set $x(k) \equiv x$, $x(k+\ell) \equiv y$, and

$$
\begin{aligned}
&\delta(0) \equiv 0 \\
&\delta(\ell) \equiv \epsilon^{-1}(x(k+\ell) - x(k)) - \sum_{m=1}^{\ell-1}\delta(m)
\end{aligned}
\tag{5.25}
$$

(We slightly abuse the notation in that $\delta(0)$ and $\delta(\ell)$ do not necessarily belong to Δ). By

(5.20),

$$
\begin{aligned}
&P_{\epsilon,k}^{(k,k+\ell)}(x(k),x(k+\ell)) \\
&=\sum_{x(k+1)\in\Gamma_\epsilon}\cdots\sum_{x(k+\ell-1)\in\Gamma_\epsilon}\prod_{l=1}^{\ell}p_\epsilon(k+l-1,x(k+l-1),x(k+l)) \\
&=\sum_{\delta(1)\in\Delta}\cdots\sum_{\delta(\ell-1)\in\Delta}\prod_{l=1}^{\ell}p_\epsilon\left(k+l-1,x(k)+\epsilon\sum_{m=0}^{l-1}\delta(m),x(k)+\epsilon\sum_{m=1}^{l}\delta(m)\right)\mathbb{1}_{\{\delta(\ell)\in\Delta\}}
\end{aligned}
\tag{5.26}
$$

Note that since

$$
\epsilon\sup|\sum_{m=1}^{\ell}\delta(m)|\leq\epsilon\ell\,\mathrm{diam}\Delta \tag{5.27}
$$

it follows from (5.22) that

$$
\inf_{x\in\mathcal{S}'}\mathrm{dist}(x,\mathcal{S}^c)>\epsilon\sup|\sum_{m=1}^{\ell}\delta(m)| \tag{5.28}
$$

so that the chain starting at $x(k)\in\mathcal{S}'$ at time k cannot reach the boundary of $\mathcal{S}$ by time $k+\ell$. This in particular implies that for each $x(k)\in\mathcal{S}'$, each sequence $(\delta(1),\ldots,\delta(\ell-1))\in\times_{l=1}^{\ell-1}\Delta$, and each $l=1,\ldots,\ell-1$,

$$
x(k)+\epsilon\sum_{m=1}^{l}\delta(m)\in\mathrm{int}_\epsilon\mathcal{S}\subset\mathrm{int}_\epsilon\Lambda \tag{5.29}
$$

Thus by (2.1) and Hypothesis 2.2 (see e.g. (2.12)), each of the probabilities in the last line of (5.26) is strictly positive. In addition, under our assumption on z, by $(H0)$ of Hypothesis 2.3, $e^{f_\epsilon^{(0)}(\epsilon k,z,\delta(l))}>0$. We may thus write

$$
P_{\epsilon,k}^{(k,k+\ell)}(x(k),x(k+\ell))=\sum_{\delta(1)\in\Delta}\cdots\sum_{\delta(\ell-1)\in\Delta}\prod_{l'=1}^{\ell}R_{l'}\prod_{l=1}^{\ell}e^{f_\epsilon^{(0)}(\epsilon k,z,\delta(l))}\mathbb{1}_{\{\delta(\ell)\in\Delta\}} \tag{5.30}
$$

where

$$
R_l\equiv p_\epsilon\left(k+l-1,x(k)+\epsilon\sum_{m=0}^{l-1}\delta(m),x(k)+\epsilon\sum_{m=1}^{l}\delta(m)\right)e^{-f_\epsilon^{(0)}(\epsilon k,z,\delta(l))},\quad\forall l=1,\ldots,\ell \tag{5.31}
$$

Setting $k'=k+l-1$ and $x'=x(k)+\epsilon\sum_{m=0}^{l-1}\delta(m)$ and using (2.1) and (2.15), we have

$$
\begin{aligned}
|\log R_l|&=\left|f_\epsilon(\epsilon k',x',\delta(l))-f_\epsilon^{(0)}(\epsilon k,z,\delta(l))\right| \\
&\leq\epsilon\left|f_\epsilon^{(1)}(\epsilon k',x',\delta)\right|+\left|f_\epsilon^{(0)}(\epsilon k',x',\delta)-f_\epsilon^{(1)}(\epsilon k,z,\delta(l))\right|
\end{aligned}
\tag{5.32}
$$

where by $(H1)$ of Hypothesis 2.3, $\left|f_\epsilon^{(1)}(\epsilon k', x', \delta(l))\right| \leq K(\mathcal{S})$ and by $(H2)$ and $(H3)$ of Hypothesis 2.3,

$$\begin{aligned}
&\left|f_\epsilon^{(0)}(\epsilon k', x', \delta(l)) - f_\epsilon^{(0)}(\epsilon k, z, \delta(l))\right| \\
\leq &\left|f_\epsilon^{(0)}(\epsilon k', x', \delta(l)) - f_\epsilon^{(0)}(\epsilon k, x', \delta(l))\right| + \left|f_\epsilon^{(0)}(\epsilon k, x', \delta(l)) - f_\epsilon^{(0)}(\epsilon k, z, \delta(l))\right| \\
\leq &\epsilon\theta|k - k'| + \vartheta(\mathcal{S})|z - x'|
\end{aligned} \tag{5.33}$$

Thus

$$|\log R_l| \leq \epsilon\theta l + \vartheta(\mathcal{S}) \left|(x(k) - z) + \epsilon \textstyle\sum_{m=1}^{l-1} \delta(m)\right| + \epsilon K(\mathcal{S}) \tag{5.34}$$

and for $\delta(\ell) \in \Delta$, we have

$$\begin{aligned}
\left|\log\left(\prod_{l=1}^{\ell} R_l\right)\right| &\leq \sum_{l=1}^{\ell} \left(\epsilon\theta l + \vartheta(\mathcal{S}) \left|(x(k) - z) + \epsilon \textstyle\sum_{m=1}^{l-1} \delta(m)\right| + \epsilon K(\mathcal{S})\right) \\
&\leq \epsilon\theta \tfrac{\ell(\ell-1)}{2} + \epsilon\vartheta(\mathcal{S}) \,\mathrm{diam}\Delta \tfrac{\ell(\ell-1)}{2} + \vartheta(\mathcal{S})\ell|x(k) - z| + \epsilon\ell K(\mathcal{S})
\end{aligned} \tag{5.35}$$

Inserting the bound (5.35) in (5.30) yields (5.24). This concludes the proof of the lemma. $\diamondsuit$

5.2: Basic upper and lower large deviation estimates.

We define the following sets:

$$\bar{\Lambda}_{\rho,\gamma}(\phi) = \left\{x \in \Lambda \mid \exists \psi \in \bar{\mathcal{B}}_{\rho,\gamma}(\phi),\, \exists t \in [0,T] \text{ } s.t. \text{ } \psi(t) = x\right\} \tag{5.36}$$

$$\bar{\mathcal{S}}_{\rho,r}(\phi) = \mathrm{cl}\left(\left\{x \in \Lambda \mid \mathrm{dist}\left(x, \bar{\Lambda}_{\rho,\gamma}(\phi)\right) \leq r\right\}\right), \quad r \geq 0 \tag{5.37}$$

Observe that for $r < \gamma$, $\bar{\mathcal{S}}_{\rho,r}(\phi)$ is a closed bounded subset of $\mathrm{int}\Lambda$.

$$\mathcal{T}(\phi_0) = \phi_0 + \left[-(T + \epsilon\sqrt{d}) \,\mathrm{diam}\Delta, (T + \epsilon\sqrt{d}) \,\mathrm{diam}\Delta\right]^d \tag{5.38}$$

(this definition has to do with the fact that the initial condition π_{ϵ,ϕ_0} of the chain has support in $\{x \in \Gamma_\epsilon \mid |x - \phi_0| \leq \epsilon\sqrt{d}\}$). Finally,

$$\mathcal{S}_{\gamma/2}(\phi_0) = \mathrm{cl}\left(\{x \in \Lambda \mid \mathrm{dist}\left(x, (\mathcal{T}(\phi_0) \cap \Lambda)^c\right) \geq \gamma/2\}\right) \tag{5.39}$$

The upper bound we will prove is analogous to that of [DEW].

Lemma 5.4: *Let* $0 = t_0 < t_1 < \cdots < t_n = \epsilon\left[\frac{T}{\epsilon}\right]$ *be such that for all* $0 \leq i \leq n-1$, $t_i = \epsilon\left[\frac{t_i}{\epsilon}\right] \equiv \epsilon k_i$, $k_i \in \mathbb{N}$. *Assume that the conditions of Lemma 5.2, (i), are verified and set* $\bar{\zeta} = 2\eta + 2\epsilon \,\mathrm{diam}\Delta$. *For any fixed* $r > 0$ *assume that* η, ϵ *and* τ *are such that*

$$r > 2\bar{\zeta} + \epsilon\tau \,\mathrm{diam}\Delta \tag{5.40}$$

Then the following conclusions hold for any ψ in $\bar{\mathcal{B}}_{\rho,0}(\phi)$.

(i) If $|\psi(t_0)-\phi_0| \le \bar\zeta + \epsilon\sqrt{d}$ then,

$$
\begin{aligned}
&\epsilon \log \mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X(\tfrac{t_i}{\epsilon}) - \psi(t_i)\right| \le \bar\zeta\right)\\
&\le \sup_{\psi':\forall_{i=0}^{n}|\psi'(t_i)-\psi(t_i)|\le\bar\zeta}\left(-\sum_{i=1}^{n}(t_i-t_{i-1})\mathcal{L}_\epsilon^{(r)*}\left(t_{i-1},\psi'(t_i),\tfrac{\psi'(t_i)-\psi'(t_{i-1})}{t_i-t_{i-1}}\right)\right)
\end{aligned}
\tag{5.41}
$$

(ii) If $|\psi(t_0)-\phi_0| > \bar\zeta + \epsilon\sqrt{d}$ then,

$$
\epsilon \log \mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X(\tfrac{t_i}{\epsilon}) - \psi(t_i)\right| \le \bar\zeta\right) = -\infty \tag{5.42}
$$

Proof: The proof starts from equation (5.21), replacing ζ by $\bar\zeta$. We follow the procedure used by Varadhan [Va] for the multidimensional Cramér theorem[6] and write

$$
\begin{aligned}
&\prod_{i=0}^{n}\mathbb{1}_{\{|x(k_i)-\psi(t_i)|\le\bar\zeta\}} \le \inf_{\bar\lambda_1,\ldots,\bar\lambda_n\in\mathbb{R}^d}\ \sup_{\substack{\psi'(t_1),\ldots,\psi'(t_n)\\ \forall_i|\psi'(t_i)-\psi(t_i)|\le\bar\zeta}} e^{\sum_{i=1}^{n}\left(\bar\lambda_i, x(k_i)-\psi'(t_i)\right)}\\
&\quad\times\prod_{i=0}^{n}\mathbb{1}_{\{|x(k_i)-\psi(t_i)|\le\bar\zeta\}}\\
&= \inf_{\bar\lambda_1,\ldots,\bar\lambda_n\in\mathbb{R}^d}\ \sup_{\substack{\psi'(t_1),\ldots,\psi'(t_n)\\ \forall_i|\psi'(t_i)-\psi(t_i)|\le\bar\zeta}}\prod_{i=0}^{n}\mathbb{1}_{\{|x(k_i)-\psi(t_i)|\le\bar\zeta\}}\\
&\quad\times e^{\sum_{i=1}^{n}\left(\left(\sum_{j=i}^{n}\bar\lambda_j\right),\left(x(k_i)-x(k_{i-1})-\psi'(t_i)+\psi'(t_{i-1})\right)\right)}\\
&\quad\times e^{\left(\left(\sum_{j=1}^{n}\bar\lambda_j\right),x(k_0)-\psi(t_0)\right)}\\
&\le \inf_{\lambda_1,\ldots,\lambda_n\in\mathbb{R}^d}\ \sup_{\substack{\psi'(t_1),\ldots,\psi'(t_n)\\ \forall_i|\psi'(t_i)-\psi(t_i)|\le\bar\zeta}}\prod_{i=0}^{n}\mathbb{1}_{\{|x(k_i)-\psi(t_i)|\le\bar\zeta\}}\\
&\quad\times e^{\sum_{i=2}^{n}(\lambda_i,x(k_i)-x(k_{i-1}))-\left(\lambda_i,\psi'(t_i)-\psi'(t_{i-1})\right)}\\
&\quad\times e^{(\lambda_1,x(k_1)-x(k_0))-\left(\lambda_1,\psi'(t_1)-\psi(t_0)+x_0-\psi(t_0)\right)}
\end{aligned}
\tag{5.43}
$$

We now insert (5.43) into (5.21). Relaxing all constraints on the endpoints of summations

[6] This allows us to avoid Wentzell's assumptions of boundedness of the derivatives of the Lagrangian function $\mathcal{L}^*$ with respect to the velocities.

(this is reasonable since we already assume that $\psi(t)$ remains in Λ) we obtain, using (5.26),

$$
\begin{aligned}
&\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X\left(\left[\tfrac{t_i}{\epsilon}\right]\right)-\psi(t_i)\right|\le\bar\zeta\right)\\
&\le\sum_{x(k_0)\in\Gamma_\epsilon}\pi_{\epsilon,\phi_0}(x(k_0))\mathbb{1}_{\{|x(k_0)-\psi(t_0)|\le\bar\zeta\}}\\
&\times\inf_{\lambda_1,\dots,\lambda_n\in\mathbb{R}^d}\sup_{\substack{\psi'(t_1),\dots,\psi'(t_n)\\ \forall_i|\psi'(t_i)-\psi(t_i)|\le\bar\zeta}}\prod_{i=2}^n\left(e^{-\left(\lambda_i,\psi'(t_i)-\psi'(t_{i-1})\right)}\right.\\
&\left.\times\sup_{x(k_i)\in\Gamma_\epsilon:|x(k_{i-1})-\psi(t_{i-1})|\le\bar\zeta}\sum_{\delta(1),\dots,\delta(\ell_i)}\prod_{l=1}^{\ell_i}e^{f_\epsilon\left(t_{i-1}+l-1,x(k_{i-1})+\epsilon\sum_{k=1}^{l-1}\delta(k),\delta(l)\right)}e^{(\epsilon\lambda_i,\delta(l))}\right)\\
&\times\left(e^{-\left(\lambda_1,\psi'(t_1)-x(k_0)\right)}\sum_{\delta(1),\dots,\delta(\ell_1)}\prod_{l=1}^{\ell_1}e^{f_\epsilon\left(t_0+l-1,x(k_0)+\epsilon\sum_{k=1}^{l-1}\delta(k),\delta(l)\right)}e^{(\epsilon\lambda_1,\delta(l))}\right)
\end{aligned}
\tag{5.44}
$$

where $\ell_i\equiv k_{i+1}-k_i$. Taking into account the constraints on the suprema over the $x(k_i)$ and the $\psi(t_i)$, we see that all terms $x(k_i)+\epsilon\sum_{k=1}^{l-1}\delta(k)$ appearing satisfy $|x(k_i)+\epsilon\sum_{k=1}^{l-1}\delta(k)-\psi(t_i)|\le 2\zeta+\epsilon\tau\,\mathrm{diam}\Delta$. Therefore, for $r>2\bar\zeta+\epsilon\tau\,\mathrm{diam}\Delta$,

$$
\begin{aligned}
&\sup_{x(k_i)\in\Gamma_\epsilon:|x(k_{i-1})-\psi(t_{i-1})|\le\bar\zeta}\sum_{\delta(l)}e^{f_\epsilon\left(t_{i-1}+\epsilon(l-1),x(k_{i-1})+\epsilon\sum_{k=1}^{l-1}\delta(k),\delta(l)\right)}e^{(\epsilon\lambda_i,\delta(l))}\\
&\qquad\le\sup_{t':|t'-t_{i-1}|\le r}\sup_{u:|u-\psi(t_{i-1})|\le r}\sum_{\delta(l)}e^{f_\epsilon(t',u,\delta(l))}e^{(\epsilon\lambda_i,\delta(l))}
\end{aligned}
\tag{5.45}
$$

to bound all the summations over the $\delta(l)$ successively. This leads with the above notation to the bound

$$
\begin{aligned}
&\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X\left(\tfrac{t_i}{\epsilon}\right)-\psi(t_i)\right|\le\bar\zeta\right)\\
&\le\sum_{x(k_0)\in\Gamma_\epsilon}\pi_{\epsilon,\phi_0}(x(k_0))\mathbb{1}_{\{|x(k_0)-\psi(t_0)|\le\bar\zeta\}}\\
&\times\inf_{\lambda_1,\dots,\lambda_n\in\mathbb{R}^d}\sup_{\substack{\psi'(t_1),\dots,\psi'(t_n)\\ \forall_i|\psi'(t_i)-\psi(t_i)|\le\bar\zeta}}\prod_{i=2}^n e^{-\left(\lambda_i,\psi'(t_i)-\psi'(t_{i-1})\right)+\ell_i\mathcal{L}_\epsilon^{(r)}(t_{i-1},\psi(t_{i-1}),\epsilon\lambda_i)}\\
&\times e^{-\left(\lambda_1,\psi'(t_1)-x(k_0)\right)+\ell_1\mathcal{L}_\epsilon^{(r)}(t_0,\psi(t_0),\epsilon\lambda_1)}
\end{aligned}
\tag{5.46}
$$

Using that for $|\psi-\psi'|\le\bar\zeta$, $\sup_{u:|u-\psi|\le r}L_\epsilon(t,u,v)\le\sup_{u:|u-\psi'|\le r+\bar\zeta}L_\epsilon(t,u,v)$, we can replace $\psi(t_{i-1})$ by $\psi'(t_{i-1})$ in the second argument of $\mathcal{L}_\epsilon^{(r)}$ at the expense of increasing r by $\bar\zeta$ (which will lead to the condition $r>2\bar\zeta+\epsilon\tau\,\mathrm{diam}\Delta$). The argument in the inf sup is convex in the variables λ_i and concave (since linear) in the $\psi'(t_i)$ and verifies the assumptions of the minimax theorem (see [Ro], Section 37 Corollary 37.3.1.) so that we may interchange the order in which they are taken. Thus we obtain

$$
\begin{aligned}
\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\tfrac{t_i}{\epsilon})-\psi(t_i)\right|\leq\bar\zeta\right) &\leq \sum_{x(k_0)\in\Gamma_\epsilon}\pi_{\epsilon,\phi_0}(x(k_0))\mathbb{1}_{\{|x(k_0)-\psi(t_0)|\leq\bar\zeta\}}\\
&\times \sup_{\substack{\psi'(t_0),\dots,\psi'(t_n)\\ \forall_i|\psi'(t_i)-\psi(t_i)|\leq\bar\zeta}}\exp\left(-\epsilon^{-1}\sum_{i=1}^n(t_i-t_{i-1})\mathcal{L}_\epsilon^{(r)*}\left(t_{i-1},\psi'(t_{i-1}),\tfrac{\psi'(t_i)-\psi'(t_{i-1})}{t_i-t_{i-1}}\right)\right)
\end{aligned}
\tag{5.47}
$$

The first factor in the last line is always less than one which implies (i) and is zero if $|\psi(t_0)-\phi(0)|>\bar\zeta+\epsilon\sqrt{d}$. This implies (ii).$\diamondsuit$

We now turn to the lower bound. Recall from (4.7) that $\Phi_{\epsilon,t_{i-1},\psi(t_{i-1})}(\,\cdot\,)=\mathcal{L}_\epsilon(t_{i-1},\psi(t_{i-1}),\,\cdot\,)$.

Lemma 5.5: *The notation is the same as in Lemma 5.4. Assume that the conditions of Lemma 5.2, (ii), are verified and set* $\zeta\equiv 2\eta-2\epsilon\,\mathrm{diam}\Delta$. *Define the set*

$$
\mathcal{E}^\circ([0,T])=\left\{\psi\in\mathcal{E}([0,T])\;\middle|\;\tfrac{\psi(t)-\psi(t')}{t-t'}\in\mathrm{ri}(\,\mathrm{conv}\Delta)\ \forall t\in[0,T],\forall t'\in[0,T],t\neq t'\right\}
\tag{5.48}
$$

Then, for any ψ *in*

$$
\mathcal{B}_{\rho-2(\eta+\epsilon\tau\,\mathrm{diam}\Delta),\gamma}(\phi)\cap\mathcal{E}^\circ([0,T])
\tag{5.49}
$$

there exist positive constants $c_0\equiv c_0(\psi)<\infty$ *such that, if* η, ϵ, *and* τ *are such that*

$$
\tfrac{\gamma}{2}\geq\zeta+\epsilon\tau\,\mathrm{diam}\Delta\quad\textit{and}\quad\sqrt{2\epsilon T}\,\mathrm{diam}\Delta+\epsilon\sqrt{d}<\zeta,
\tag{5.50}
$$

the following holds:

$$
\begin{aligned}
&\epsilon\log\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\tfrac{t_i}{\epsilon})-\psi(t_i)\right|\leq\zeta\right)\\
&\geq\begin{cases}
-\sum_{i=1}^n(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)-Q\left(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi),\zeta,c_0\right) & \textit{if } |\psi(t_0)-\phi_0|\leq\epsilon\sqrt{d}\\
-\infty & \textit{otherwise}
\end{cases}
\end{aligned}
\tag{5.51}
$$

where

$$
Q(\mathcal{S},\zeta,c_0)\equiv 3n(\epsilon\tau)^2(\theta+\vartheta(\mathcal{S})\,\mathrm{diam}\Delta)+3T\left(\zeta+2\epsilon K(\mathcal{S})\right)+4n\zeta c_0+\epsilon\log(8d^2+4)
\tag{5.52}
$$

Proof: Obviously, for any $\varrho\leq\zeta$,

$$
\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\tfrac{t_i}{\epsilon})-\psi(t_i)\right|\leq\zeta\right)\geq\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\tfrac{t_i}{\epsilon})-\psi(t_i)\right|\leq\varrho\right)
\tag{5.53}
$$

As will turn out, the generic term for which we shall want a lower bound is of the form:

$$\mathcal{T}'_i \equiv 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}} \sum_{x(k_i)\in\Gamma_\epsilon} \prod_{j=i}^{n} 1\!\!\mathrm{I}_{\{|(x(k_i)-\psi(t_i))+a_{i,j}|\leq\varrho\}} P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i)) \tag{5.54}$$

where, for each $j=i,\dots,n$, $a_{i,j}\in\mathbb{R}^d$ is independent of $\{x(k_j)\}_{i\leq j\leq n}$. We shall however only treat the term

$$\mathcal{T}_i \equiv 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}} \sum_{x(k_i)\in\Gamma_\epsilon} 1\!\!\mathrm{I}_{\{|(x(k_i)-\psi(t_i))+a|\leq\varrho\}} P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i)) \tag{5.55}$$

for $a\in\mathbb{R}^d$ an arbitrary constant, the extension of the resulting bound to $\mathcal{T}'_i$ being straightforward. Naturally our bound on $\mathcal{T}_i$ will be derived by means of Lemma 5.3. Let $\mathcal{G}$ denote the set (5.49). Since ψ belongs to $\mathcal{G}$ it belongs in particular to $\mathcal{B}_{\rho-2(\eta+\epsilon\tau\,\mathrm{diam}\Delta),\gamma}$ and hence to $\bar{\mathcal{B}}_{\rho,\gamma}$. Thus, under the assumptions (5.50), we may apply Lemma 5.3 with $\ell\equiv\tau$, $\mathcal{S}\equiv\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi)$, $\mathcal{S}'\equiv\bar{\mathcal{S}}_{\rho,\zeta}(\phi)$, and, in each time interval (k_{i-1},k_i), choose $z\equiv\psi(t_{i-1})$ in (5.24).

Following the classical pattern of Cramer's type techniques, the lower bound will come from 'centering the variables' (i.e. introducing a Radon-Nikodym factor). For a given $\psi\in\mathcal{G}$ let $\lambda_i^*\equiv\lambda_i^*\left(\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)$, $1\leq i\leq n$, be defined through:

$$\left(\epsilon\lambda_i^*,\tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)-\mathcal{L}_\epsilon(t_{i-1},\psi(t_{i-1}),\epsilon\lambda_i^*)=\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right) \tag{5.56}$$

Obviously the conditions in (5.50) imply that $\psi(t_i)\in\ \mathrm{int}(\,\mathrm{int}_\epsilon\Lambda)$ for all $1\leq i\leq n$. The point is that from this, Corollary 4.10, and the equivalence $(ii)\Leftrightarrow(iv)$ of Lemma 4.7 we can conclude that there exists a positive constant $c_0\equiv c_0(\psi)<\infty$ such that:

$$\max_{1\leq i\leq n}|\lambda_i^*|<c_0 \tag{5.57}$$

We then rewrite $\mathcal{T}_i$ as

$$\mathcal{T}_i=\mathcal{T}_{i,1}\mathcal{T}_{i,2} \tag{5.58}$$

where

$$\mathcal{T}_{i,1}\equiv 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}}\sum_{x(k_i)\in\Gamma_\epsilon} e^{(\lambda_i^*,x(k_i)-\psi(t_i))}P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i)) \tag{5.59}$$

$$\begin{aligned}\mathcal{T}_{i,2}&\equiv 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}}\\ &\times\sum_{x(k_i)\in\Gamma_\epsilon}\frac{e^{(\lambda_i^*,x(k_i)-\psi(t_i))}P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i))\,1\!\!\mathrm{I}_{\{|(x(k_i)-\psi(t_i))+a|\leq\varrho\}}}{\sum_{x(k_i)\in\Gamma_\epsilon}e^{(\lambda_i^*,x(k_i)-\psi(t_i))}P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i))}e^{-(\lambda_i^*,x(k_i)-\psi(t_i))}\end{aligned} \tag{5.60}$$

We first prove a lower bound for the term

$$\begin{aligned}\mathcal{T}_{i,3} \equiv & \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} \\ & \times \sum_{x(k_i)\in\Gamma_\epsilon} \mathbb{1}_{\{|(x(k_i)-\psi(t_i))+a|\le\varrho\}} e^{(\lambda_i^*, x(k_i)-\psi(t_i))} P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i))\end{aligned} \tag{5.61}$$

Setting $\ell_i \equiv k_i - k_{i-1}$ and using (5.24),

$$\begin{aligned}\mathcal{T}_{i,3} \ge & e^{-q(\ell_i,|x(k_{i-1})-\psi(t_{i-1})|)} \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} \\ & \times \sum_{x(k_i)\in\Gamma_\epsilon} \mathbb{1}_{\{|(x(k_i)-\psi(t_i))+a|\le\varrho\}} e^{(\lambda_i^*, x(k_i)-\psi(t_i))} \\ & \times \sum_{\delta(1)\in\Delta}\cdots\sum_{\delta(\ell_i-1)\in\Delta}\prod_{l=1}^{\ell_i} e^{f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta(l))} \mathbb{1}_{\{\delta(\ell_i)\in\Delta\}} \mathbb{1}_{\left\{x(k_i)-x(k_{i-1})=\epsilon\sum_{m=1}^{\ell_i}\delta(m)\right\}}\end{aligned} \tag{5.62}$$

We have,

$$\begin{aligned} & \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} \mathbb{1}_{\left\{x(k_i)-x(k_{i-1})=\epsilon\sum_{m=1}^{\ell_i}\delta(m)\right\}} e^{(\lambda_i^*, x(k_i)-\psi(t_i))} \\ \ge & \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} \mathbb{1}_{\left\{x(k_i)-x(k_{i-1})=\epsilon\sum_{m=1}^{\ell_i}\delta(m)\right\}} e^{-\varrho|\lambda_i^*|+(\epsilon\lambda_i^*,\sum_{m=1}^{\ell_i}\delta(m))-(\lambda_i^*,\psi(t_i)-\psi(t_{i-1}))}\end{aligned} \tag{5.63}$$

Consequently,

$$\begin{aligned}\mathcal{T}_{i,3} \ge & e^{-q(\ell_i,\varrho)-\varrho|\lambda_i^*|} \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} \\ & \times \sum_{\delta(1)\in\Delta}\cdots\sum_{\delta(\ell_i)\in\Delta}\prod_{l=1}^{\ell_i} e^{(\epsilon\lambda_i^*,\delta(l))} e^{f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta(l))} \\ & \times \mathbb{1}_{\left\{\left|\epsilon\sum_{m=1}^{\ell_i}\delta(m)-(\psi(t_i)-\psi(t_{i-1}))+(x(k_{i-1})-\psi(t_{i-1}))+a\right|\le\varrho\right\}}\end{aligned} \tag{5.64}$$

The same arguments applied to $\mathcal{T}_{i,1}$ give

$$\begin{aligned}\mathcal{T}_{i,1} \ge & e^{-q(\ell_i,\varrho)-\varrho|\lambda_i^*|} \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} \\ & \times e^{-(\lambda_i^*,\psi(t_i)-\psi(t_{i-1}))} \prod_{l=1}^{\ell_i}\sum_{\delta(l)\in\Delta} e^{(\epsilon\lambda_i^*,\delta(l))} e^{f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta(l))} \\ = & e^{-q(\ell_i,\varrho)-\varrho|\lambda_i^*|} \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} e^{-\ell_i\left\{\left(\epsilon\lambda_i^*,\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)-\mathcal{L}_\epsilon(t_{i-1},\psi(t_{i-1}),\epsilon\lambda_i^*)\right\}}\end{aligned} \tag{5.65}$$

and, by definition of λ_i^*,

$$\mathcal{T}_{i,1} \ge e^{-q(\ell_i,\varrho)-\varrho|\lambda_i^*|} \mathbb{1}_{\{|x(k_{i-1})-\psi(t_{i-1})|\le\varrho\}} e^{-\epsilon^{-1}(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)} \tag{5.66}$$

which is precisely the form of the bound we need.

We now turn to the term $\mathcal{T}_{i,2}$ and first write

$$\begin{aligned}\mathcal{T}_{i,2} \geq & 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}} e^{-\varrho|\lambda_i^*|} \\ & \times \frac{\sum_{x(k_i)\in\Gamma_\epsilon} e^{(\lambda_i^*, x(k_i)-\psi(t_i))} P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i)) 1\!\!\mathrm{I}_{\{|(x(k_i)-\psi(t_i))+a|\leq\varrho\}}}{\sum_{x(k_i)\in\Gamma_\epsilon} e^{(\lambda_i^*, x(k_i)-\psi(t_i))} P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i))}\end{aligned} \tag{5.67}$$

(5.64) allows to bound the numerator in (5.67) from above. Virtually the same arguments allow to bound the denominator from above:

$$\begin{aligned}&\sum_{x(k_i)\in\Gamma_\epsilon} e^{(\lambda_i^*, x(k_i)-\psi(t_i))} P_{\epsilon,k_{i-1}}^{(k_{i-1},k_i)}(x(k_{i-1}),x(k_i)) \\ &\leq e^{\{q(\ell_i,\varrho)+\varrho|\lambda_i^*|\}} \prod_{l=1}^{\ell_i} \sum_{\delta(l)\in\Delta} e^{(\epsilon\lambda_i^*,\delta(l))+f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta(l))}\end{aligned} \tag{5.68}$$

Combining these yields

$$\begin{aligned}\mathcal{T}_{i,2} \geq & e^{-\{2q(\ell_i,\varrho)+3\varrho|\lambda_i^*|\}} 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}} \\ & \times \sum_{\delta(1)\in\Delta} \cdots \sum_{\delta(\ell_i)\in\Delta} \prod_{l=1}^{\ell_i} \frac{e^{(\epsilon\lambda_i^*,\delta(l))+f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta(l))}}{\sum_{\delta(l)\in\Delta} e^{(\epsilon\lambda_i^*,\delta(l))+f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta(l))}} \\ & \times 1\!\!\mathrm{I}_{\left\{\left|\epsilon\sum_{m=1}^{\ell_i}\delta(m)-(\psi(t_i)-\psi(t_{i-1}))+(x(k_{i-1})-\psi(t_{i-1}))+a\right|\leq\varrho\right\}}\end{aligned} \tag{5.69}$$

At this point (5.69) may be recast in the following form: let $\{\chi_{m,i}\}_{1\leq m\leq\ell_i}$ be a family of i.i.d. r.v.'s taking values in Δ with law, ν_i, defined through (see (4.3))

$$\nu_i(\delta) \equiv \nu_{\epsilon,t_{i-1},\psi(t_{i-1})}^{\lambda_i^*}(\delta) = \frac{e^{(\epsilon\lambda_i^*,\delta)+f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta)}}{\sum_{\delta\in\Delta} e^{(\epsilon\lambda_i^*,\delta)+f_\epsilon^{(0)}(t_{i-1},\psi(t_{i-1}),\delta)}}, \quad \forall\delta\in\Delta \tag{5.70}$$

Set

$$\overline{\chi}_{m,i} = \chi_{m,i} - \tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}} \tag{5.71}$$

$$S_i = \sum_{m=1}^{\ell_i} \overline{\chi}_{m,i} \tag{5.72}$$

and let $\mathbb{E}_{\{\nu_i\}}$ denote the expectation w.r.t. $\{\chi_{m,i}\}$. Then (5.69) reads,

$$\mathcal{T}_{i,2} \geq e^{-\{2q(\ell_i,\varrho)+3\varrho|\lambda_i^*|\}} 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}} \mathbb{E}_{\{\nu_i\}} 1\!\!\mathrm{I}_{\{|\epsilon S_i+(x(k_{i-1})-\psi(t_{i-1}))+a|\leq\varrho\}} \tag{5.73}$$

Collecting (5.58), (5.66) and (5.73) we thus obtain

$$\begin{aligned}\mathcal{T}_i \geq & e^{-\varsigma_i-\epsilon^{-1}(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)} \\ & \times 1\!\!\mathrm{I}_{\{|x(k_{i-1})-\psi(t_{i-1})|\leq\varrho\}} \mathbb{E}_{\{\nu_i\}} 1\!\!\mathrm{I}_{\{|\epsilon S_i+(x(k_{i-1})-\psi(t_{i-1}))+a|\leq\varrho\}}\end{aligned} \tag{5.74}$$

where

$$\varsigma_i \equiv 3q(\ell_i, \varrho) + 4\varrho|\lambda_i^*| \tag{5.75}$$

We are now in a position to deal with the r.h.s. of (5.21). Applying (5.74) to $\mathcal{T}_n$ gives rise to a term of the form $\mathcal{T}'_{n-1}$ (see definition (5.54)) with $a_{n-1,n-1} = 0$ and $a_{n-1,n} = \epsilon S_n$. The second iteration step thus yields

$$\begin{aligned} &\mathbb{1}_{\{|x(k_{n-2})-\psi(t_{n-2})|\le\varrho\}} \sum_{x(k_{n-1})\in\Gamma_\epsilon} \mathbb{1}_{\{|x(k_{n-1})-\psi(t_{n-1})|\le\varrho\}} P_{\epsilon,k_{n-2}}^{(k_{n-2},k_{n-1})}(x(k_{n-2}),x(k_{n-1})) \\ &\quad\times \sum_{x(k_n)\in\Gamma_\epsilon} \mathbb{1}_{\{|x(k_n)-\psi(t_n)|\le\varrho\}} P_{\epsilon,k_{n-1}}^{(k_{n-1},k_n)}(x(k_{n-1}),x(k_n)) \\ &\ge e^{-(\varsigma_n+\varsigma_{n-1})-\epsilon^{-1}\sum_{i=n-1}^{n}(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)} \mathbb{1}_{\{|x(k_{n-2})-\psi(t_{n-2})|\le\varrho\}} \\ &\quad\times \mathbb{E}_{\{\nu_{n-1}\}} \mathbb{1}_{\{|\epsilon S_{n-1}+(x(k_{n-2})-\psi(t_{n-2}))|\le\varrho\}} \mathbb{E}_{\{\nu_n\}} \mathbb{1}_{\{|\epsilon(S_{n-1}+S_n)+(x(k_{n-2})-\psi(t_{n-2}))|\le\varrho\}} \end{aligned} \tag{5.76}$$

and gradually, setting

$$a_{i,j} = \begin{cases} 0 & \text{if } j = i \\ \epsilon(S_{j+1}+\cdots+S_n) & \text{if } i+1 \le j \le n \end{cases} \tag{5.77}$$

in (5.54) at step i, we obtain,

$$\begin{aligned} &\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\le i\le n}\left|X(\tfrac{t_i}{\epsilon})-\psi(t_i)\right| \le \varrho\right) \\ &\ge e^{-\frac{1}{\epsilon}\widetilde{Q}-\frac{1}{\epsilon}\sum_{i=1}^{n}(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)} \\ &\quad\times \sum_{x(t_0)\in\Gamma_\epsilon} \pi_{\epsilon,\phi_0}(x(k_0)) \mathbb{1}_{\{|x(k_0)-\psi(t_0)|\le\varrho\}} \\ &\quad\times \mathbb{E}_{\{\nu_1\}} \mathbb{1}_{\{|\epsilon S_1+(x(k_0)-\psi(t_0))|\le\varrho\}} \cdots \mathbb{E}_{\{\nu_n\}} \mathbb{1}_{\{|\epsilon(S_1+\cdots+S_n)+(x(k_0)-\psi(t_0))|\le\varrho\}} \\ &= \mathcal{R} e^{-\frac{1}{\epsilon}\widetilde{Q}-\frac{1}{\epsilon}\sum_{i=1}^{n}(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)} \end{aligned} \tag{5.78}$$

where

$$\widetilde{Q} \equiv \epsilon \sum_{i=1}^{n} \varsigma_i \tag{5.79}$$

$$\mathcal{R} \equiv \sum_{x(t_0)\in\Gamma_\epsilon} \pi_{\epsilon,\phi_0}(x(k_0)) \mathbb{1}_{\{|x(k_0)-\psi(t_0)|\le\varrho\}} \mathbb{E}_{\{\nu\}} \mathbb{1}_{\{|\epsilon(S_1+\cdots+S_n)+(x(k_0)-\psi(t_0))|\le\varrho\}} \tag{5.80}$$

and $\mathbb{E}_{\{\nu\}}$ denotes the expectation w.r.t. the joint law of $\{S_i\}_{1\le i\le n}$. We are left to estimate

$\mathcal{R}$. Assume that $\varrho \geq \epsilon\sqrt{d}$. Then

$$\begin{aligned}
\mathcal{R} &\geq \sum_{x(t_0)\in\Gamma_\epsilon} \pi_{\epsilon,\phi_0}(x(k_0)) 1\!\!1_{\{|x(k_0)-\psi(t_0)|\leq\epsilon\sqrt{d}\}} \mathbb{E}_{\{\nu\}} 1\!\!1_{\left\{|S_1+\dots+S_n|\leq\epsilon^{-1}(\varrho-\epsilon\sqrt{d})\right\}} \\
&= \frac{1}{|\{x\in\Gamma_\epsilon \,|\, |x-\phi_0|\leq\epsilon\sqrt{d}\}|} \sum_{x(t_0):|x(k_0)-\phi_0|\leq\epsilon\sqrt{d}} 1\!\!1_{\{|x(k_0)-\psi(t_0)|\leq\epsilon\sqrt{d}\}} \mathbb{E}_{\{\nu\}} 1\!\!1_{\left\{|S_1+\dots+S_n|\leq\epsilon^{-1}(\varrho-\epsilon\sqrt{d})\right\}} \\
&\geq \frac{1}{4d^2+1} 1\!\!1_{\{|x(k_0)-\psi(t_0)|\leq\epsilon\sqrt{d}\}} \mathbb{E}_{\{\nu\}} 1\!\!1_{\left\{|S_1+\dots+S_n|\leq\epsilon^{-1}(\varrho-\epsilon\sqrt{d})\right\}}
\end{aligned} \tag{5.81}$$

for any $x(k_0) \in \{x \in \Gamma_\epsilon \mid |x-\phi_0| \leq \epsilon\sqrt{d}\}$. Since

$$\bigcup_{x(k_0)\in\{x\in\Gamma_\epsilon \,|\, |x-\phi_0|\leq\epsilon\sqrt{d}\}} \{y\in\mathbb{R}^d \mid |y-x(k_0)|\leq\epsilon\sqrt{d}\} \supset \{y\in\mathbb{R}^d \mid |y-\phi_0|\leq\epsilon\sqrt{d}\} \tag{5.82}$$

then

$$\mathcal{R} \geq \begin{cases} \frac{1}{4d^2+1}\mathbb{E}_{\{\nu\}} 1\!\!1_{\left\{|S_1+\dots+S_n|\leq\epsilon^{-1}(\varrho-\epsilon\sqrt{d})\right\}} & \text{if } |\psi(t_0)-\phi_0|\leq\epsilon\sqrt{d} \\ \\ 0 & \text{otherwise} \end{cases} \tag{5.83}$$

and it remains to estimate the expectation. But this is immediate once observed that, recalling (5.56) and combining Lemma 4.4, (iii), together with the equivalence $(i) \Leftrightarrow (iii)$ of Lemma 4.7 we have, for all $1 \leq m \leq \ell_i$,

$$\mathbb{E}_{\nu_i}\chi_{m,i} = \nabla\Phi_{\epsilon,t_{i-1},\psi(t_{i-1})}(v)|_{v=\epsilon\lambda_i^*} = \frac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}} \tag{5.84}$$

and

$$\begin{aligned}
&\mathbb{E}_{\nu_i}\overline{\chi}_{m,i} = 0 \\
&\mathbb{E}_{\nu_i}\left|\overline{\chi}_{m,i}\right|^2 = \Delta\Phi_{\epsilon,t_{i-1},\psi(t_{i-1})}(v)|_{v=\epsilon\lambda_i^*}
\end{aligned} \tag{5.85}$$

Defining

$$\sigma^2 \equiv \sigma^2(\{\psi(t_i)\},\{\lambda_i^*\}) = T\max_{1\leq i\leq n}\Delta\Phi_{\epsilon,t_{i-1},\psi(t_{i-1})}(v)|_{v=\epsilon\lambda_i^*} \tag{5.86}$$

Moreover,

$$\sigma^2 \leq T(\operatorname{diam}\Delta)^2 \tag{5.87}$$

Hence, by independence and Chebyshev's inequality

$$\begin{aligned}
\mathbb{E}_{\{\nu\}} 1\!\!1_{\left\{|S_1+\dots+S_n|\leq\epsilon^{-1}(\varrho-\epsilon\sqrt{d})\right\}} &= 1 - \mathbb{E}_{\{\nu\}} 1\!\!1_{\left\{|S_1+\dots+S_n|>\epsilon^{-1}(\varrho-\epsilon\sqrt{d})\right\}} \\
&\geq 1 - \left(\epsilon(\varrho-\epsilon\sqrt{d})^{-1}\right)^2 \mathbb{E}_{\{\nu\}}(S_1+\dots+S_n)^2 \\
&\geq 1 - \left(\epsilon(\varrho-\epsilon\sqrt{d})^{-1}\right)^2 \sum_{i=1}^n \ell_i\Delta\Phi_{\epsilon,t_{i-1},\psi(t_{i-1})}(v)|_{v=\epsilon\lambda_i^*} \\
&\geq 1 - \epsilon\left(\varrho-\epsilon\sqrt{d}\right)^{-2}\sigma^2(\{\psi(t_i)\},\{\lambda_i^*\}) \\
&\geq 1 - \epsilon T(\operatorname{diam}\Delta)^2\left(\varrho-\epsilon\sqrt{d}\right)^{-2} \\
&\geq \frac{1}{2}
\end{aligned} \tag{5.88}$$

whenever $\varrho \geq \sqrt{2\epsilon T}\,\mathrm{diam}\Delta + \epsilon\sqrt{d}$. For such a ϱ, inserting (5.88) in (5.83) and combining with (5.78) proves Lemma 5.5 since $\widetilde{Q} \leq Q\left(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi), \zeta, \max_{1\leq i\leq n}|\epsilon\lambda_i^*|\right)$ and since by (5.57),

$$\sup_{\psi\in\mathcal{G}} Q\left(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi), \zeta, \max_{1\leq i\leq n}|\epsilon\lambda_i^*|\right) \leq Q(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi), \zeta, c_0) \tag{5.89}$$

(see definitions (5.1), (5.23), and (5.57) as well as (5.75) and (5.79) for the first of the last two inequalities). $\diamondsuit$

5.3: Proof of Proposition 3.2 (concluded).

To conclude the proofs of the upper and lower bounds, we need the following two lemmata that will permit to replace the sums over t_i by integrals.

Lemma 5.6: *Recall that $D = \mathrm{conv}\Delta$ and define the sets*

$$\begin{aligned}
\mathcal{K}([0,T]) &= \left\{\psi \in W([0,T]) \,\middle|\, \dot\phi(t) \in D, \text{ for Lebesgue a.e. } t\in[0,T]\right\} \\
\mathcal{K}^\circ([0,T]) &= \left\{\psi \in W([0,T]) \,\middle|\, \dot\phi(t) \in \mathrm{ri}\, D, \text{ for Lebesgue a.e. } t\in[0,T]\right\}
\end{aligned} \tag{5.90}$$

With $\mathcal{E}([0,T])$ and $\mathcal{E}^\circ([0,T])$ defined respectively in (3.1) and (5.48) we have:

$$\begin{aligned}
\mathcal{K}([0,T]) &= \mathcal{E}([0,T]) \\
\mathcal{K}^\circ([0,T]) &\subset \mathcal{E}^\circ([0,T])
\end{aligned} \tag{5.91}$$

Proof: The proof is elementary. Recall that by assumption D is a bounded closed and convex subset of $\mathbb{R}^d$. For any bounded convex subset A in $\mathbb{R}^d$ and any $\psi \in \mathcal{C}([0,T])$ consider the following three conditions:

(i) $\psi \in L^1([0,T])$ and $\dot\psi(t) \in A$ for Lebesgue a.e. $t \in [0,T]$.

(ii) $\psi \in L^1([0,T])$ and $\frac{1}{t-t'}\int_{t'}^t ds\dot\psi(s) \in A$ $\forall t \in [0,T]$, $\forall t' \in [0,T]$, $t \neq t'$.

(iii) $\frac{\psi(t)-\psi(t')}{t-t'} \in A$ $\forall t \in [0,T]$, $\forall t' \in [0,T]$, $t \neq t'$.

Then the following conclusions hold:

(iv) If $A = D$ or if $A = \mathrm{ri}\, D$ then $(ii) \Leftrightarrow (iii)$

(v) If $A = D$ or if $A = \mathrm{ri}\, D$ then $(i) \Rightarrow (ii)$

(vi) If $A = D$ then $(ii) \Leftrightarrow (i)$

We first prove (iv): that $(ii) \Rightarrow (iii)$ is immediate whereas since A is bounded ψ is Lipshitz and, in particular, absolutely continuous, yielding $(iii) \Rightarrow (ii)$. Whenever A is a closed or opened set, the implication $(i) \Rightarrow (ii)$ results from it's convexity and the integrability of $\dot{\psi}$: this proves (v). If in addition A is closed then, by a standard result of real analysis, $(ii) \Rightarrow (i)$ (see e.g. [Ru], Theorem 1.40); this together with (v) yields (vi). Now (iv) together with (vi) implies the first relation in (5.91) while (iv) together with (v) implies the second. The proof is done.$\diamondsuit$

Lemma 5.7: *Let $\mathcal{S}$ be any closed bounded subset of* $\operatorname{int}(\operatorname{int}_\epsilon\Lambda)$, *and let t_i, $i=1,\ldots,n$ be as in Lemma 5.4.*

(i) If ψ is in

$$\left\{\psi \in \mathcal{E}([0,T]) \,\middle|\, \psi(t)\in\mathcal{S}, \quad \forall t\in[0,T]\right\} \tag{5.92}$$

then, for each $\varepsilon_0>0$ there corresponds $\varepsilon_1>0$ such that if $\epsilon\tau<\varepsilon_1$,

$$\begin{aligned}&\left|\sum_{i=1}^{n}(t_i-t_{i-1})\mathcal{L}^*_\epsilon\left(t_{i-1},\psi(t_{i-1}),\tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)-\int_0^T dt\mathcal{L}^*_\epsilon(t,\psi(t),\dot{\psi}(t))\right|\\&\le\varepsilon_0T+(\theta+\vartheta(\mathcal{S})\,\mathrm{diam}\Delta)n\tfrac{(\epsilon\tau)^2}{2}\end{aligned} \tag{5.93}$$

(ii) Let t_i, $i=0,\ldots,n$, n, $\bar{\zeta}$ and r be given as in Lemma 5.4. Assume that $\psi'(t_i)\in\mathbb{R}^d$ are such that

$$|\psi'(t_i)-\psi'(t_{i-1})|\le|t_i-t_{i-1}|C, \quad \forall i=1,\ldots,n \tag{5.94}$$

for some constant $0<C<\infty$ and

$$\mathrm{dist}\,(\psi'(t_i),\Lambda)\le\bar{\zeta} \tag{5.95}$$

Let $\widetilde{\psi}(t)$, $t\in[0,T]$ be the linear interpolation of the points $\psi'(t_i)$. Then, for each $\varepsilon_0>0$ there exists $\varepsilon_1>0$ (depending on r and C) such that, if $\epsilon\tau<\varepsilon_1$,

$$\sum_{i=1}^{n}(t_i-t_{i-1})\mathcal{L}^{(r)*}_\epsilon\left(t_{i-1},\psi'(t_{i-1}),\tfrac{\psi'(t_i)-\psi'(t_{i-1})}{t_i-t_{i-1}}\right)-\int_0^T dt\mathcal{L}^{(r)*}_\epsilon(t,\psi'(t),\dot{\psi}'(t))\ge-3\varepsilon_0T \tag{5.96}$$

Proof: We first prove (i). Recall that $\Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}(\cdot)=\mathcal{L}^*_\epsilon(t_{i-1},\psi(t_{i-1}),\cdot)$ and $\tau\equiv$

$\max_{0\leq i\leq n}\epsilon^{-1}|t_{i+1}-t_i|$ as defined in (4.7) and (5.1). Let us write:

$$\begin{aligned}
&(t_i-t_{i-1})\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right)\\
=&\int_{t_{i-1}}^{t_i} ds\mathcal{L}_\epsilon^*(s,\psi(s),\dot\psi(s))\\
&+\left[\int_{t_{i-1}}^{t_i} ds\left(\Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}\left(\tfrac{1}{t_i-t_{i-1}}\textstyle\int_{t_{i-1}}^{t_i}ds'\dot\psi(s')\right)-\Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}\left(\dot\psi(s)\right)\right)\right]\\
&+\left\{\int_{t_{i-1}}^{t_i} ds\left(\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\dot\psi(s)\right)-\mathcal{L}_\epsilon^*\left(s,\psi(s),\dot\psi(s)\right)\right)\right\}\\
=&\int_{t_{i-1}}^{t_i} ds\mathcal{L}_\epsilon^*(s,\psi(s),\dot\psi(s))+[I_i]+\{J_i\}
\end{aligned}\tag{5.97}$$

where the last line defines the terms I_i and J_i. In order to bound J_i we use the decomposition

$$\begin{aligned}
&\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\dot\psi(s)\right)-\mathcal{L}_\epsilon^*\left(s,\psi(s),\dot\psi(s)\right)\\
=&\left[\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\dot\psi(s)\right)-\mathcal{L}_\epsilon^*\left(s,\psi(t_{i-1}),\dot\psi(s)\right)\right]\\
+&\left[\mathcal{L}_\epsilon^*\left(s,\psi(t_{i-1}),\dot\psi(s)\right)-\mathcal{L}_\epsilon^*\left(s,\psi(s),\dot\psi(s)\right)\right]
\end{aligned}\tag{5.98}$$

and, applying Lemma 4.9, obtain

$$\begin{aligned}
\left|\mathcal{L}_\epsilon^*\left(t_{i-1},\psi(t_{i-1}),\dot\psi(s)\right)-\mathcal{L}_\epsilon^*\left(s,\psi(s),\dot\psi(s)\right)\right|&\leq\theta|s-t_{i-1}|+\vartheta|\psi(s)-\psi(t_{i-1})|\\
&\leq(\theta+\vartheta\,\mathrm{diam}\Delta)|s-t_{i-1}|
\end{aligned}\tag{5.99}$$

where $\vartheta\equiv\vartheta(\mathcal{S})$. Thus,

$$|J_i|\leq(\theta+\vartheta\,\mathrm{diam}\Delta)\int_{t_{i-1}}^{t_i}ds|s-t_{i-1}|=(\theta+\vartheta\,\mathrm{diam}\Delta)\tfrac{(t_i-t_{i-1})^2}{2}\leq(\theta+\vartheta\,\mathrm{diam}\Delta)\tfrac{(\epsilon\tau)^2}{2}\tag{5.100}$$

We now bound I_i. By Lemma 4.6, (i), $\Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}$ is convex and lower semi-continuous. Convexity implies $I_i\leq 0$. For an upper bound note first that by Lebesgue's Theorem: to each $\varepsilon_2>0$ there corresponds $\varepsilon_1>0$ such that, for Lebesgue almost every $s\in[t',t]$,

$$\left|\int_{t'}^{t}ds'\dot\psi(s')-\dot\psi(s)\right|<\varepsilon_2|t'-t|\tag{5.101}$$

for all $[t',t]\subset[0,T]$ verifying $s\in[t',t]$ and $|t-t'|<\varepsilon_1$[7]. Next, by definition of lower semi-continuity, for any $x\in\mathbb{R}^d$ we have: to each $\varepsilon_0>0$ there corresponds $\varepsilon_2>0$ such that if

[7]the set of s's for which (5.99) holds is usually called the Lebesgue set of ψ.

$|x-y| < \varepsilon_2$, then $\Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}(x) \geq \Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}(y) - \varepsilon_0$. Thus, for each $\varepsilon_0 > 0$, if ϵ is sufficiently small so that $\epsilon\tau < \varepsilon_1$ we have, on the Lebesgue set of ψ:

$$\Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}\left(\tfrac{1}{t_i-t_{i-1}}\int_{t_{i-1}}^{t_i} ds'\dot{\psi}(s')\right) \geq \Phi^*_{\epsilon,t_{i-1},\psi(t_{i-1})}\left(\dot{\psi}(s)\right) - \varepsilon_0 \tag{5.102}$$

and

$$I_i \geq -(t_i - t_{i-1})\varepsilon_0 \tag{5.103}$$

Inserting our bounds on I_i and J_i in (5.97) and adding up yields

$$\begin{aligned}&\left|\sum_{i=1}^{n}(t_i-t_{i-1})\mathcal{L}^*_\epsilon\left(t_{i-1},\psi(t_{i-1}),\tfrac{\psi(t_i)-\psi(t_{i-1})}{t_i-t_{i-1}}\right) - \int_0^{\epsilon\left[\frac{T}{\epsilon}\right]} dt\mathcal{L}^*_\epsilon(t,\psi(t),\dot{\psi}(t))\right| \\ &\leq \varepsilon_0 T + (\theta+\vartheta(\mathcal{S})\operatorname{diam}\Delta)n\tfrac{(\epsilon\tau)^2}{2}\end{aligned} \tag{5.104}$$

But $\left|\int_{\epsilon\left[\frac{T}{\epsilon}\right]}^{T} dt\mathcal{L}^*_\epsilon(t,\psi(t),\dot{\psi}(t))\right| \leq \epsilon const(\mathcal{S})$ so that (5.93) obtains upon minor modification of ε_0.

To prove (ii) we note that since $\widetilde{\psi}$ is linear between the points t_i, in the analogue of (5.97) the term corresponding to $[I_i]$ is absent, i.e. we have

$$\begin{aligned}&(t_i-t_{i-1})\mathcal{L}^{(r)*}_\epsilon\left(t_{i-1},\psi'(t_{i-1}),\tfrac{\psi'(t_i)-\psi'(t_{i-1})}{t_i-t_{i-1}}\right) \\ =&\int_{t_{i-1}}^{t_i} ds\mathcal{L}^{(r)*}_\epsilon(s,\widetilde{\psi}(s),\dot{\widetilde{\psi}}(s)) \\ &+\int_{t_{i-1}}^{t_i} ds\left(\mathcal{L}^{(r)*}_\epsilon\left(t_{i-1},\widetilde{\psi}(t_{i-1}),\dot{\widetilde{\psi}}(s)\right) - \mathcal{L}^*_\epsilon\left(s,\widetilde{\psi}(s),\dot{\widetilde{\psi}}(s)\right)\right)\end{aligned} \tag{5.105}$$

To bound the second term in (5.105) we use the same decomposition as in (5.98). However, instead of the Lipshitz bounds (5.99) we use the lower semi-continuity property of $\mathcal{L}^{(r)*}_\epsilon$ (see Lemma 4.12) together with the fact that $\widetilde{\psi}$ is Lipshitz by (5.94), it follows from the decomposition (5.98) that: for each ε_0 there corresponds $\varepsilon_1' > 0$ such that if $\epsilon\tau < \varepsilon_1'$,

$$\mathcal{L}^{(r)*}_\epsilon\left(t_{i-1},\psi(t_{i-1}),\dot{\psi}(s)\right) - \mathcal{L}^{(r)*}_\epsilon\left(s,\psi(s),\dot{\psi}(s)\right) \geq -2\varepsilon_0 \tag{5.106}$$

The lemma is proven. $\diamondsuit$

Proof of the lower bound (3.5): : Given any $\gamma > 0$ we may choose ζ and τ depending on ϵ in such a way that firstly, both $\zeta \downarrow 0$ and $\epsilon\tau \downarrow 0$ as $\epsilon \downarrow 0$ (hence $\eta \downarrow 0$ as $\epsilon \downarrow 0$), and secondly, that the conditions (5.50) of Lemma 5.5 as well as those of Lemma 5.2, (ii), are satisfied. It then easily follows from the first relation of Lemma 5.6 that

$$\bigcup_{\gamma>0}\bigcup_{\epsilon>0}\mathcal{B}_{\rho-2(\eta+\epsilon\tau\operatorname{diam}\Delta),\gamma}(\phi) = \mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T]) \tag{5.107}$$

Setting

$$\begin{aligned}\widetilde{\mathcal{G}} &\equiv \mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T])\cap\mathcal{E}^\circ([0,T])\\ \mathcal{G} &\equiv \mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T])\cap\mathcal{K}^\circ([0,T])\end{aligned}\tag{5.108}$$

and using now the second relation of Lemma 5.6, we moreover have $\mathcal{G}\subset\widetilde{\mathcal{G}}$. Let ψ be any path in $\widetilde{\mathcal{G}}$. Then obviously, $\exists\gamma_0>0$ s.t. $\forall 0<\gamma<\gamma_0$ $\exists 0<\epsilon_0$ s.t. $\forall\epsilon<\epsilon_0$, $\psi\in\mathcal{B}_{\rho-2(\eta+\epsilon\tau\,\mathrm{diam}\Delta),\gamma}(\phi)\cap\mathcal{E}^\circ([0,T])$. Thus, given $\gamma<\gamma_0$ and $\epsilon<\epsilon_0$ we may combine the bound (5.51) of Lemma 5.5 and Lemma 5.7, (i), to write, under the assumptions of Lemma 5.7, (i), and choosing $\mathcal{S}\equiv\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi)$ therein,

$$\epsilon\log\mathcal{P}_{\epsilon,\phi_0}\left(\max_{0\leq i\leq n}\left|X(\tfrac{t_i}{\epsilon})-\psi(t_i)\right|\leq\zeta\right)\geq-\int_0^T dt\mathcal{L}^*_\epsilon(t,\psi(t),\dot\psi(t))-\widetilde{Q}\left(\varepsilon_0,\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi),\zeta,c_0\right)\tag{5.109}$$

where

$$\widetilde{Q}\left(\varepsilon_0,\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi),\zeta,c_0\right)\equiv Q\left(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi),\zeta,c_0\right)+\varepsilon_0T+(\theta+\vartheta(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi))\,\mathrm{diam}\Delta)n\tfrac{(\epsilon\tau)^2}{2}\tag{5.110}$$

Making use of Lemma 5.2, (ii), (5.109) entails

$$\epsilon\log\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi))\geq-\int_0^T dt\mathcal{L}^*_\epsilon(t,\psi(t),\dot\psi(t))-\widetilde{Q}\left(\varepsilon_0,\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi),\zeta,c_0\right)\tag{5.111}$$

The next step consists in taking the limit as $\epsilon\downarrow 0$. This will be done with the help of the following two observations. On the one hand, by Lemma 4.5, $\mathcal{L}^*_\epsilon$ is positive and bounded on $\mathbb{R}^+\times\mathrm{int}_\epsilon\Lambda\times(\mathrm{conv}\Delta)$. Since, for all ϵ sufficiently small, $\psi(t)$ is contained for all $0\leq t\leq T$ in a compact subset of $\mathrm{int}(\mathrm{int}_\epsilon\Lambda)$, we have, by Lemma 4.9 (v) that $\mathcal{L}^*_\epsilon(t,\psi(t),\dot\psi(t))$ converges uniformly in $t\in[0,T]$. Hence, (for each $0<\gamma<\gamma_0$) ,

$$\lim_{\epsilon\to0}\int_0^T dt\mathcal{L}^*_\epsilon(t,\psi(t),\dot\psi(t))=\int_0^T dt\lim_{\epsilon\to0}\mathcal{L}^*_\epsilon(t,\psi(t),\dot\psi(t))=\int_0^T dt\mathcal{L}^*(t,\psi(t),\dot\psi(t))\tag{5.112}$$

On the other hand, for any $\psi\in\widetilde{\mathcal{G}}$ and any $\gamma<\gamma_0$, $c_1\equiv c_1(\psi)<\infty$ and $\vartheta(\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi))<\infty$. Thus, given our choice of the parameters ζ and τ, $\widetilde{Q}\left(\varepsilon_0,\bar{\mathcal{S}}_{\rho,\gamma/2}(\phi),\zeta,c_0\right)$ converges to zero when taking the limit $\epsilon\downarrow0$ first and the limit $\varepsilon_0\downarrow0$ next.

Combining the previous two observations and passing to the limit $\epsilon\downarrow0$ in (5.111) we obtain that

$$\liminf_{\epsilon\to0}\epsilon\log\widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi))\geq\begin{cases}-\displaystyle\int_0^T dt\mathcal{L}^*(t,\psi(t),\dot\psi(t)) & \text{if } \psi(t_0)=\phi_0\\ -\infty & \text{otherwise}\end{cases}\tag{5.113}$$

and since this is true for any $\psi \in \widetilde{\mathcal{G}}$,

$$\begin{aligned}\liminf_{\epsilon\to 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi)) &\geq - \inf_{\substack{\psi\in\widetilde{\mathcal{G}}:\\ \psi(t_0)=\phi_0}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot\psi(t)) \\ &\geq - \inf_{\substack{\psi\in\mathcal{G}:\\ \psi(t_0)=\phi_0}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot\psi(t))\end{aligned} \tag{5.114}$$

where we used that $\mathcal{G} \subset \widetilde{\mathcal{G}}$ in the last line and where the infimum is $+\infty$ vacuously. But by Lemma 4.15, taking $\mathcal{F} = \mathcal{B}_\rho(\phi) \cap \mathcal{D}^\circ([0,T])$ therein,

$$\inf_{\substack{\psi\in\mathcal{G}:\\ \psi(t_0)=\phi_0}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot\psi(t)) = \inf_{\substack{\psi\in\mathcal{F}:\\ \psi(t_0)=\phi_0}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot\psi(t)) \tag{5.115}$$

and so

$$\liminf_{\epsilon\to 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}(\mathcal{B}_\rho(\phi)) \geq - \inf_{\substack{\psi\in\mathcal{B}_\rho(\phi)\cap\mathcal{D}^\circ([0,T]):\\ \psi(t_0)=\phi_0}} \int_0^T dt \mathcal{L}^*(t,\psi(t),\dot\psi(t)) \tag{5.116}$$

The lower bound is proven. $\diamondsuit$

Proof of the upper bound (3.4): To prove the upper bound we first combine Lemmata 5.2 and 5.4. to get (with the notation of Lemma 5.4)

$$\begin{aligned}&\epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(\bar{\mathcal{B}}_\rho(\phi)\right) \\ &\leq - \inf_{\substack{\psi\in\bar{\mathcal{B}}_{\rho,0}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}}} \inf_{\psi'(t):\forall_{i=0}^n|\psi'(t_i)-\psi(t_i)|\leq\bar\zeta} \sum_{i=1}^n (t_i-t_{i-1})\mathcal{L}_\epsilon^{(r)*}\left(t_{i-1},\psi'(t_{i-1}),\tfrac{\psi'(t_i)-\psi'(t_{i-1})}{t_i-t_{i-1}}\right)\end{aligned} \tag{5.117}$$

Next we want to use Lemma 5.7 (ii) to replace sum in the right hand side by an integral. Before doing this, we observe, however, that the second infimum in (5.117) will always be realized for $\psi'(t_i)$'s for which $\frac{\psi'(t_i)-\psi'(t_{i-1})}{t_i-t_{i-1}} \in D$ (otherwise the infimum takes the value $+\infty$). Thus not only can we use Lemma 5.7 (ii) with $C = \mathrm{diam}\Delta$, but we actually have that $\widetilde\psi \in \mathcal{E}([0,T])$. Therefore we may first use (5.96) and then replace the infimum over the values $\psi(t_i)$ by an infimum over functions $\widetilde\psi(t) \in \mathcal{E}([0,T])$ that are piecewise linear (p.l.) between the times t_i , i.e. if $\epsilon\tau < \varepsilon_1$,

$$\begin{aligned}&\epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(\bar{\mathcal{B}}_\rho(\phi)\right) \\ &\leq - \inf_{\substack{\psi\in\bar{\mathcal{B}}_{\rho,0}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}}} \inf_{\substack{\widetilde\psi(t)\in\mathcal{E}([0,T]),\mathrm{p.l.}\\ \forall_{i=0}^n|\widetilde\psi(t_i)-\psi(t_i)|\leq\bar\zeta}} \int_0^T dt \mathcal{L}_\epsilon^{(r)*}\left(t,\widetilde\psi(t),\dot{\widetilde\psi}(t)\right) - 3\varepsilon_0 T\end{aligned} \tag{5.118}$$

Finally (using convexity arguments), the two infima can be combined to a single infimum over a slightly enlarged set:

$$\epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(\bar{\mathcal{B}}_\rho(\phi)\right) \leq - \inf_{\substack{\psi\in\bar{\mathcal{B}}_{\rho+\bar\zeta}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}\\ \forall t\in[0,T],\,\mathrm{dist}(\psi(t),\Lambda)\leq\bar\zeta}} \int_0^T dt \mathcal{L}_\epsilon^{(r)*}\left(t,\psi(t),\dot\psi(t)\right) - 3\varepsilon_0 T \tag{5.119}$$

To conclude the proof of the upper bound what is left to do is to pass to the limits $\epsilon\downarrow 0$, $\varepsilon_0\downarrow 0$, and $r\downarrow 0$ in (5.119). Note that by Lemma 4.12, for all $r>0$, the function $\mathcal{L}_\epsilon^{(r)*}(t,u,v^*)$ is uniformly bounded for all $t\in\mathbb{R}^+, v^*\in D$, and u such that $\mathrm{dist}(u,\Lambda)\leq r/2$. Moreover, on the same set it converges uniformly to $\mathcal{L}^{(r)*}(t,u,v^*)$. Thus we can use that

$$\begin{aligned}
&\inf_{\substack{\psi\in\bar{\mathcal{B}}_{\rho+\bar\zeta}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}\\ \forall t\in[0,T],\,\mathrm{dist}(\psi(t),\Lambda)\leq\bar\zeta}} \int_0^T dt \mathcal{L}_\epsilon^{(r)*}\left(t,\psi(t),\dot\psi(t)\right) \geq \inf_{\substack{\psi\in\bar{\mathcal{B}}_{\rho+\bar\zeta}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}\\ \forall t\in[0,T],\,\mathrm{dist}(\psi(t),\Lambda)\leq\bar\zeta}} \int_0^T dt \mathcal{L}^{(r)*}\left(t,\psi(t),\dot\psi(t)\right)\\
&- \sup_{\substack{\psi\in\bar{\mathcal{B}}_{\rho+\bar\zeta}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}\\ \forall t\in[0,T],\,\mathrm{dist}(\psi(t),\Lambda)\leq\bar\zeta}} \int_0^T dt \left[\mathcal{L}_\epsilon^{(r)*}\left(t,\psi(t),\dot\psi(t)\right) - \mathcal{L}^{(r)*}\left(t,\psi(t),\dot\psi(t)\right)\right]
\end{aligned} \tag{5.120}$$

But

$$\begin{aligned}
&\sup_{\substack{\psi\in\bar{\mathcal{B}}_{\rho+\bar\zeta}(\phi):\\ |\psi(t_0)-\phi_0|\leq\bar\zeta+\epsilon\sqrt{d}\\ \forall t\in[0,T],\,\mathrm{dist}(\psi(t),\Lambda)\leq\bar\zeta}} \int_0^T dt \left[\mathcal{L}_\epsilon^{(r)*}\left(t,\psi(t),\dot\psi(t)\right) - \mathcal{L}^{(r)*}\left(t,\psi(t),\dot\psi(t)\right)\right]\\
&\leq \sup_{\substack{\psi\in\bar{\mathcal{B}}_{\rho+r/2}(\phi):\\ |\psi(t_0)-\phi_0|\leq r/2\\ \forall t\in[0,T],\,\mathrm{dist}(\psi(t),\Lambda)\leq r/2}} \int_0^T dt \left[\mathcal{L}_\epsilon^{(r)*}\left(t,\psi(t),\dot\psi(t)\right) - \mathcal{L}^{(r)*}\left(t,\psi(t),\dot\psi(t)\right)\right]
\end{aligned} \tag{5.121}$$

By Lemma 4.13 and dominated convergence, the last integral in (5.121) converges to zero as $\epsilon\downarrow 0$ uniformly for any $\psi\in\bar{\mathcal{B}}_{\rho+r/2}(\phi)$, and so (5.121) converges to zero. Recall from the proof of the lower bound that η and τ were chosen such that both $\epsilon\tau\downarrow 0$ and $\eta\downarrow 0$ as $\epsilon\downarrow 0$. Hence $\bar\zeta\downarrow 0$ as $\epsilon\downarrow 0$. Taking the limit $\epsilon\downarrow 0$ first and $\varepsilon_0\downarrow 0$ in (5.119) yields that, for any $r>0$,

$$\limsup_{\epsilon\downarrow 0} \epsilon \log \widetilde{\mathcal{P}}_{\epsilon,\phi_0}\left(\bar{\mathcal{B}}_\rho(\phi)\right) \leq - \inf_{\substack{\psi\in\bar{\mathcal{B}}_\rho(\phi):\\ \psi(t_0)=\phi_0\\ \forall t\in[0,T],\psi(t)\in\Lambda}} \int_0^T dt \mathcal{L}^{(r)*}\left(t,\psi(t),\dot\psi(t)\right) \tag{5.122}$$

Finally we must pass to the limit as $r\downarrow 0$. Here the argument is identical to the one given in [DEW]. It basically relies on Theorem 3.3 in [WF] which states that if $\mathcal{I}$ is a rate function with compact level sets $K(s) \equiv \{\psi : \mathcal{I}(\psi) \leq s\}$, than an upper bound of the form (5.122) with rate function $\mathcal{I}$ is equivalent to the statement that for any $c, c' > 0$, there is $\epsilon_0 > 0$ such that for all $\epsilon \leq \epsilon_0$,

$$\mathcal{P}_{\epsilon,\phi_0}\left(\operatorname{dist}(\psi, K(s)\right) \leq e^{-\frac{1}{\epsilon}(s-c')} \tag{5.123}$$

Therefore, it is enough to show that with $K^{(r)}(\psi) \equiv \int_0^T dt \mathcal{L}^{(r)*}\left(t, \psi(t), \dot{\psi}(t)\right)$, and $\bar{K}(\psi) \equiv \int_0^T dt \bar{\mathcal{L}}^*\left(t, \psi(t), \dot{\psi}(t)\right)$, for any $s, c, c' > 0$, there exists $r > 0$ such that

$$K^{(r)}(s-c) \subset \{\psi : \operatorname{dist}(\psi, K(s)) \leq c'\} \tag{5.124}$$

which is established in Proposition 2.10 of [DEW]. This gives the upper bound of Proposition 3.2.$\diamondsuit$

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