Extinction properties of super-Brownian motions with additional spatially dependent mass production

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Abstract

Consider the finite measure-valued continuous super-Brownian motion X on $\mathbb{R}^d$ corresponding to the evolution equation $u_t = \frac{1}{2}\Delta u + \beta u - u^2$, where $\beta \in C^\gamma(\mathbb{R}^d)$ with $\gamma \in (0, 1]$ is bounded from above. We prove criteria for (finite time) extinction and local extinction of X in terms of β. It turns out that for $d \leq 2$, local extinction is equivalent with extinction. For general d, we show that if β has a suitable decay rate at infinity then it can be changed on a compact set in order to guarantee local extinction. On the other hand, if β is above this decay rate, the process does not exhibit local extinction. If $d \leq 6$, then extinction has the same threshold rate as local extinction, while for $d > 6$ one observes a phase transition. Last, we show that in dimension 1, if β is no longer bounded from above and, in fact, degenerates to a single point source, then X does not exhibit local extinction, and the expectation of the rescaled process $t \mapsto \mathrm{e}^{-t/2} X_t$ has a limit as $t \to \infty$. In the proofs pde techniques and Laplace transforms are used together with h-transforms for measure-valued processes.

1 Introduction and statement of results

1.1 Motivation

In [Pin96, Theorem 6] an abstract (spectral theoretical) criterion has been presented for the local extinction of supercritical superdiffusions with spatially constant branching mechanism. In [EP99] this criterion has been generalized for a spatially dependent branching mechanism resulting into so-called $(L, \beta, \alpha; D)$-superdiffusions, and also abstract conditions have been derived for extinction and for the compact support property. Here L is a diffusion operator on a domain $D \subset \mathbb{R}^d$, and, loosely speaking, $\beta(x)v - \alpha(x)v^2$ refers to the branching mechanism. These abstract theorems however do not give a straightforward way to decide whether a given superdiffusion becomes (locally) extinct or possesses the compact support property. (Note nevertheless that a sufficient condition has already been given for having the compact support property by Theorem 3.5 in [EP99]; see also Theorem 3.6 there.) Recently ([Eng99]) this gap has been partially filled by giving concrete criteria for the compact support property in a simple setting, namely, when the underlying migration process is a time-changed Brownian motion (that is $L = \varrho(x)\Delta$ with $\varrho > 0$) and the spatially constant branching mechanism is critical (that is $\beta(x) \equiv 0$).

In this paper we are going to derive similar concrete criteria for (finite time) extinction and local extinction, again in a relatively simple setup. In fact, we consider a continuous super-Brownian motion ($L = \frac{1}{2}\Delta$) in $D = \mathbb{R}^d$ with constant α, but with additional spatially dependent mass "production" β. See Theorems 1 and 2 below.

A second purpose is to begin studying what happens if this mass production coefficient β varies in space in an irregular way. Here we restrict our attention to the simplest case, namely, if it degenerates to a single point source δ_0 (Theorems 3 and 4). Our inspiration comes from the so-called catalytic branching models (see [Fle94], [DFL95], or [Kle99] for surveys).

1.2 Preparation

Let $\mathcal{M}_f = \mathcal{M}_f(\mathbb{R}^d)$ denote the set of finite measures μ on $\mathbb{R}^d$, and $\mathcal{M}_c = \mathcal{M}_c(\mathbb{R}^d)$ the subset of all compactly supported μ. Write $C^\gamma = C^\gamma(\mathbb{R}^d)$ and $C^{k,\gamma} = C^{k,\gamma}(\mathbb{R}^d)$, $\gamma \in (0,1]$, $k = 1,2$, for the usual Hölder spaces.

Let L be an elliptic operator on $\mathbb{R}^d$ of the form

$$L = \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \quad \text{on } \mathbb{R}^d, \tag{1}$$

where $a_{i,j},\ b_i \in C^{1,\gamma}$, $i,j = 1,...,d$, for some $\gamma \in (0,1]$ and the symmetric matrix $a = \{a_{i,j}\}$ satisfies $\sum_{i,j=1}^d a_{ij}(x)v_iv_j > 0$, for all $v \in \mathbb{R}^d \backslash \{0\}$ and all $x \in \mathbb{R}^d$. In addition, let $\alpha, \beta \in C^\gamma$ denote functions satisfying $\alpha > 0$ and $\sup_{x\in\mathbb{R}^d} \beta(x) < \infty$.

Notation 1 (superdiffusion) Let $(X, P_\mu\,,\ \mu \in \mathcal{M}_f)$ denote the $(L, \beta, \alpha; \mathbb{R}^d)$-*superdiffusion*. That is, X is the unique $\mathcal{M}_f$-valued (time-homogeneous) continuous Markov process which satisfies, for any bounded continuous $g : \mathbb{R}^d \mapsto \mathbb{R}_+$,

$$E_\mu \exp \langle X_t\,, -g\rangle = \exp\left(-\int_{\mathbb{R}^d} \mu(\mathrm{d}x)\ u(x,t)\right), \tag{2}$$

where u is the minimal non-negative solution to

$$\left.\begin{aligned} u_t = Lu + \beta u - \alpha u^2 \quad \text{on } \mathbb{R}^d \times (0,\infty), \\ \lim_{t\to 0+} u(\cdot, t) = g(\cdot) \end{aligned}\right\} \tag{3}$$

(see [EP99]). Here $\langle \nu, f\rangle$ denotes the integral $\int_{\mathbb{R}^d} \nu(\mathrm{d}x)\, f(x)$. ◇

Definition 2 (extinction) A measure-valued path X *becomes extinct* (in finite time) if $X_t = 0$ for all sufficiently large t. It *exhibits local extinction* if $X_t(B) = 0$ for all sufficiently large t, for each ball $B \subset \mathbb{R}^d$. The measure-valued process X corresponding to P_μ is said to possess any one of these properties if that property is true with P_μ-probability one. ◇

Remark 3 (process properties) In [EP99] it is shown that, for fixed L, β and α, if any one of the properties in Definition 2 holds for some $P_\mu\,,\ \mu \in \mathcal{M}_c$ with $\mu \neq 0$, then it in fact holds for every $P_\mu\,,\ \mu \in \mathcal{M}_c\,.$ ◇

1.3 Criteria for extinction

Local extinction can be characterized in terms of L and β (see [Pin96] and [EP99]):

Lemma 4 (local extinction) *The $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X exhibits local extinction if and only if there exists a (strictly) positive solution u to the equation $(L + \beta)u = 0$ on $\mathbb{R}^d$.*

The following sufficient condition for extinction will be proved in Subsection 4.2:

Proposition 5 (extinction via local extinction) *Assume the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X exhibits local extinction. If there exists a function $h \in C^{2,\gamma}$ and an (non-empty) open ball $B \subset \mathbb{R}^d$ such that $\inf_{\mathbb{R}^d} h > 0$ and $(L+\beta)h \leq 0$ on $\mathbb{R}^d \backslash \overline{B}$, then X becomes extinct.* [1)]

In the remaining part of this section, we specialize to $L = \frac{1}{2}\Delta$ and to $\alpha(x) \equiv 1$; that is, X is the superdiffusion (super-Brownian motion) corresponding to the quadruple $\left(\frac{1}{2}\Delta, \beta, 1; \mathbb{R}^d\right)$.

It is well-known that if β is constant, this super-Brownian motion X becomes extinct if and only if $\beta \leq 0$. Using Lemma 4 one can show that for constant $\beta > 0$ there is even no local extinction. If however β is spatially dependent, then the local branching mechanism may be supercritical (that is $\beta(x) > 0$) in certain regions and critical or subcritical ($\beta(x) \leq 0$) in others. We are interested in obtaining more specific criteria for extinction and local extinction of X in terms of $\beta \in C^\gamma$. In the following subsection we will consider a non-regular β as well.

First, we will show that for our $\beta \in C^\gamma$ there exists a *threshold decay rate* $K_d/|x|^2$ concerning local extinction. We will use the notation $r >> 1$ for the phrase "r large enough", and $r << -1$ is defined similarly.

Theorem 1 (threshold decay rate for local extinction) *Consider the $(\frac{1}{2}\Delta, \beta, 1; \mathbb{R}^d)$-superdiffusion X.*

(a) *If*

$$\beta(x) \leq \frac{K_d}{|x|^2} \quad \textit{for } |x| >> 1, \textit{ where } K_d := \frac{(d-2)^2}{8}, \tag{4}$$

then there exists a $\beta^* \in C^\gamma$ satisfying $\beta^* = \beta$ outside some compact set such that X exhibits local extinction.

(b) *On the other hand, if*

$$\beta(x) \geq \frac{K}{|x|^2} \quad \textit{for } |x| >> 1 \textit{ and some } K > K_d\,, \tag{5}$$

then X does not exhibit local extinction.

Remark 6 (one-dimensional case) In one dimension, Theorem 1 (b) can be replaced by a stronger statement: If

$$\beta(x) \geq \frac{K}{x^2} \quad \text{for } x >> 1 \text{ or } x << -1, \text{ and some } K > K_1 = \frac{1}{8}, \tag{6}$$

then X does not exhibit local extinction. See Subsection 4.2 for a proof. $\diamond$

[1)] $\overline{B}$ denotes the closure of B.

It is well-known that for any given ball $B \subset \mathbb{R}^d$ (with positive radius), β can be chosen large enough on B in order to guarantee non-existence of positive solutions to the equation $(L+\beta)u = 0$ on B (or, equivalently, the positivity of the principal eigenvalue for $L+\beta$ on B (see [Pin95, Chapter 4] for more elaboration). Then, a fortiori, there is no positive solution u to the equation $(L+\beta)u = 0$ on $\mathbb{R}^d$. By Lemma 4 then, X does not exhibit local extinction. This shows that a small 'tail' for β alone will never guarantee local extinction.

Since, by Lemma 4, local extinction is completely determined by a property of the *linear* operator $L+\beta$, it is relatively easy to get conditions on local extinction using techniques from linear pde. Characterizing extinction of the superdiffusion however is a subtler question. We will show that if $d \le 2$ or if β is below a threshold decay rate $k_d/|x|^2$ then local extinction implies extinction, while, on the other hand, extinction does not hold for any β above this threshold. If $d \le 6$, then $k_d = K_d$ where K_d is defined in (5). However, if $d > 6$, a phase transition occurs: $k_d < K_d$. In fact, our first *main result* reads as follows.

Theorem 2 (extinction versus local extinction) *The $(\frac{1}{2}\Delta, \beta, 1; \mathbb{R}^d)$-superprocess X has the following properties:*

(a) *Let $d \le 2$. Then local extinction implies extinction.*

(b) *If*

$$\beta(x) \le \frac{k_d}{|x|^2} \quad \textit{for } |x| >> 1, \ \textit{where } k_d := \begin{cases} K_d & \textit{if} \quad d \le 6, \\ d-4 & \textit{if} \quad d > 6, \end{cases} \tag{7}$$

then local extinction implies extinction.

(c) *However, if*

$$\beta(x) \ge \frac{k}{|x|^2} \quad \textit{for } |x| >> 1 \textit{ and some } k > k_d\,, \tag{8}$$

then extinction does not hold.

Remark 7 (generalization) The claim in Theorem 2 (a) remains true for any $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion whenever L corresponds to a *recurrent* diffusion on $\mathbb{R}^d$, and α is bounded away from zero. This can easily be seen from the proof in Subsection 4.3. $\diamond$

Remark 8 (non-negative β) In the case $\beta \ge 0$ but $\beta(x) \not\equiv 0$, one can show using Lemma 4, that X does not exhibit local extinction (and consequently extinction does not hold for X) if $d \le 2$, while extinction will hold for $d \ge 3$ in some cases. See the end of Subsection 4.3 for a proof. In particular, if $d \le 2$ and β has the maximal tail in Theorem 1 (a), then β^* must change the sign. $\diamond$

1.4 A single point source

In the light of the previous remark, it seems to be interesting to ask what happens in the one-dimensional case when β degenerates to a *single point source*, that is, when the additional mass production is zero everywhere except at a single point (the origin, say) where the mass production is infinite (in a δ-function sense). In other words, we drop now our requirement that β is bounded from above and even consider the superdiffusion X corresponding to the quadruple $\left(\frac{1}{2}\Delta, \delta_0, 1; \mathbb{R}\right)$, where δ_0 denotes the Dirac δ-function at zero. More precisely, from the partial differential equation (3) we pass to the *integral equation*

$$\begin{aligned} u(\cdot\,,t) = \int_{-\infty}^{\infty} \mathrm{d}y\ p(t,\cdot\,,y)g(y) + \int_0^t \mathrm{d}s\ p(t-s,\cdot\,,0)u(s,0) \\ - \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ p(t-s,\cdot\,,y)u^2(s,y), \qquad t>0, \end{aligned} \tag{9}$$

where $\{p(t,x,y) = p(t,x-y);\ t>0,\ x,y \in \mathbb{R}\}$ denote the Brownian transition densities. The construction of this continuous $\mathcal{M}_f$-valued process X having again the Laplace transition functionals (2) [but with the new u from (9)] goes along standard lines via regularization of δ_0; in particular, the limiting log-Laplace equation (9) makes sense and enjoys the needed continuity properties. (See e.g. [DF97] and references therein.) The corresponding probabilities will be denoted by $\{P_\mu^{\mathrm{sin}},\ \mu \in \mathcal{M}_f\}$.

It turns out that the (additional) mass production at this single point is enough to guarantee that the process does not exhibit local extinction (and consequently extinction does not hold):

Theorem 3 (single point source) *For any $\mu \in \mathcal{M}_f \backslash \{0\}$, the superdiffusion X corresponding to P_μ^{sin} does not exhibit local extinction.*

We mention that for the case when $\beta = 0$ and $\alpha = \delta_0$ instead, it is known, that

$$P_\mu\left(\|X_t\| > 0,\ \forall t > 0,\ \text{ but } \|X_t\| \to 0 \text{ as } t \to \infty \right) = 1 \tag{10}$$

for all $\mu \in \mathcal{M}_f \backslash \{0\}$; see [FL95] or [DFL95, Corollary 5]. (Here $\|\nu\|$ denotes the total mass of a measure ν.) Furthermore, $X_t(B) \to 0$ in probability for any ball $B \subset \mathbb{R}$, even if the starting measure μ is Lebesgue (see [DF94]).

Next, we will show that the total mass of the superdiffusion corresponding to $P_{\delta_0}^{\mathrm{sin}}$ grows exponentially in expectation. For this aim, for simplicity we assume that the process starts with a unit mass situated at the origin.

Theorem 4 (exponential growth)

(a) *For all $t \geq 0$,*

$$E_{\delta_0}^{\mathrm{sin}} \|X_t\| = \frac{2}{\sqrt{\pi}}\, \mathrm{e}^{t/2} \int_{-\sqrt{t/2}}^{\infty} \mathrm{d}x\ \mathrm{e}^{-x^2}. \tag{11}$$

Thus,

$$E_{\delta_0}^{\sin} \|X_t\| \sim 2\mathrm{e}^{t/2} \quad \text{as } t \to \infty. \tag{12}$$

(b) *For all bounded continuous* $g : \mathbb{R} \mapsto \mathbb{R}_+$,

$$\lim_{t\to\infty} \mathrm{e}^{-t/2} E_{\delta_0}^{\sin} \langle X_t\,, g\rangle = \int_{\mathbb{R}} \mathrm{d}x \; g(x)\, \mathrm{e}^{-|x|}. \tag{13}$$

In particular,

$$\lim_{t\to\infty} \frac{1}{t} \log E_{\delta_0}^{\sin} \langle X_t\,, g\rangle = -\frac{1}{2}\,, \tag{14}$$

provided that $g \neq 0$.

Remark 9 (generalizations) Our results on the model with a single point source suggest to deal with the following further questions (we will address in a forthcoming paper):
(i) Extend the model to more general *non-regular* coefficients β.
(ii) Verify that the rescaled process $\mathrm{e}^{-t/2} X_t$ *itself has a limit* in law as $t \to \infty$ (instead of considering only its expectation). $\diamond$

1.5 Outline

The remainder of this paper is organized as follows. In Section 2 we present some auxiliary material. Section 3 gives a pde interpretation of some of the results stated in Subsection 1.3. Finally, the last section is devoted to the proofs.

For standard facts on superprocesses in general, we refer to [Daw93] and [Dyn93].

2 Auxiliary definitions and tools

First we give a short review of some definitions and results for $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusions which we will need and which can be found in [EP99].

Definition 10 Consider the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X corresponding to P_μ with $\mu \in \mathcal{M}_c \backslash \{0\}$.

(a) (compact support property) X possesses the *compact support property* if

$$P_\mu \left(\bigcup_{0 \le s \le t} \operatorname{supp}(X_s) \text{ is bounded} \right) = 1 \quad \text{for all } t \ge 0. \tag{15}$$

(b) (recurrence) X is said to be *recurrent* if

$$P_\mu\left(X_t(B) > 0 \text{ for some } t \geq 0 \,\middle|\, E^{\mathrm{c}}\right) = 1 \tag{16}$$

for every (non-empty) open ball $B \subset \mathbb{R}^d$. Here E^{c} denotes the complement of the event that X becomes extinct. (Roughly speaking, each ball is charged given survival.)

(c) (transience) X is called *transient* if

$$P_\mu\left(X_t(B) > 0 \text{ for some } t \geq 0 \,\middle|\, E^{\mathrm{c}}\right) < 1 \tag{17}$$

(if $\mathbf{d \geq 2}$) for all open balls $B \subset \mathbb{R}^d$ such that $\overline{B} \cap \operatorname{supp}(\mu) = \emptyset$;

(if $\mathbf{d = 1}$) for all finite intervals $B \subset \mathbb{R}$ satisfying $\sup B < \inf \operatorname{supp}(\mu)$, or for all finite intervals $B \subset \mathbb{R}$ satisfying $\inf B > \sup \operatorname{supp}(\mu)$. $\diamond$

In [EP99] it is shown that X is either recurrent or transient, and that if any one of the properties in Definition 10 holds for some P_μ, $\mu \in \mathcal{M}_c \backslash \{0\}$, then it in fact holds for every P_μ, $\mu \in \mathcal{M}_c \backslash \{0\}$.

We mention that recurrence and transience for superdiffusions were first defined and studied in [Pin96] in the case when α and β are positive constants. (In [Pin96], [EP99], and [Eng99] the terminology is actually slightly different: Instead of calling X recurrent/transient, the *support* of X is called recurrent/transient respectively.)

Definition 11 (h-transformed superdiffusion X^h) Let $0 < h \in C^{2,\gamma}$ and consider the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X. Define

$$X_t^h := hX_t \quad \left(\text{that is } \frac{\mathrm{d}X_t^h}{\mathrm{d}X_t} = h\right), \qquad t \geq 0. \tag{18}$$

Then X^h is the $(L_0^h, \beta^h, \alpha^h; \mathbb{R}^d)$-superdiffusion, where

$$L_0^h := L + a\,\frac{\nabla h}{h} \cdot \nabla, \quad \beta^h := \frac{(L+\beta)h}{h}, \quad \text{and} \quad \alpha^h := \alpha h. \tag{19}$$

X^h makes sense even if β^h is unbounded from above (see [EP99, Section 2] for more elaboration). X^h is called the *h-transformed superdiffusion.* $\diamond$

Remark 12 (h-transforms) **(i)** L_0^h is just the diffusion part of the usual *linear* h-transformed operator L^h (see [Pin95, Chapter 4]).
(ii) The operators $\mathcal{A}(u) := Lu + \beta u - \alpha u^2$ and $\mathcal{A}^h(u) := L_0^h u + \beta^h u - \alpha^h u^2$ are related by $\mathcal{A}^h(u) = \frac{1}{h}\mathcal{A}(hu)$. $\diamond$

An obvious but important property of the h-transform is that it leaves the support process $t \mapsto \operatorname{supp}(X_t)$ invariant. It is also important to point out that extinction, local extinction, recurrence/transience, as well as the compact support property are in fact properties of the *support process,* and that these properties are therefore *invariant under h-transforms.*

Remark 13 In the particular case when h satisfies the equation $(L+\beta)h=0$ on $\mathbb{R}^d$, the superdiffusion X^h coincides with Overbeck's [Ove94] additive h-transform in a time-independent case. $\diamond$

The following lemma collects some more detailed facts taken from [EP99].

Lemma 14 (details) *Consider the $(L,\beta,\alpha;\mathbb{R}^d)$-superdiffusion X.*

(a) (w-function and extinction) *There exists a function $w:\mathbb{R}^d \mapsto \mathbb{R}_+$ which solves the 'stationary' equation*

$$Lu+\beta u-\alpha u^2=0 \quad \text{on } \mathbb{R}^d, \tag{20}$$

and for which

$$P_\mu(X \text{ becomes extinct}) = \mathrm{e}^{-\langle \mu,w\rangle}, \qquad \mu\in\mathcal{M}_c\,. \tag{21}$$

If $\inf_{\mathbb{R}^d}\alpha>0$ and $\beta\le 0$ then $w=0$. On the other hand, if $w\neq 0$, then w is actually positive. Also, if L corresponds to a conservative diffusion on $\mathbb{R}^d$ and α and β are constants then $w=(\beta\vee 0)/\alpha$.

(b) ($w_{\max}$ and the compact support property) *There exists a* maximal non-negative solution $w_{\max}$ *to* (20)*. Furthermore, $w_{\max}=w$ with w from* (a) *if X has the compact support property. If $w=0$, then $w_{\max}=0$ if and only if X has the compact support property.*

(c) ($\varphi_{\min}$ and recurrence/transience) *Take an open ball $B\subset\mathbb{R}^d$. There exists a* minimal positive solution $\varphi_{\min}$ *to*

$$\left.\begin{aligned} Lu+\beta u-\alpha u^2=0 \quad \text{on } \mathbb{R}^d\backslash\overline{B},\\ \lim_{x\to\partial B} u(x)=\infty. \end{aligned}\right\} \tag{22}$$

Moreover, exactly one of the following two possibilities occurs:

(c1) *$\varphi_{\min}>w$ on $\mathbb{R}^d\backslash\overline{B}$ for any open ball B, and X is* recurrent.

(c2) *$\liminf_{|x|\to\infty}\frac{\varphi_{\min}}{w}(x)=\inf_{x\in\mathbb{R}^d\backslash\overline{B}}\frac{\varphi_{\min}}{w}(x)=0$ for any open ball B, and X is* transient.

Remark 15 (construction of $\varphi_{\min}$) Take balls $B_n\supset\overline{B}$ centered at the origin and with (sufficiently large) radius n, where B is from (c). Moreover, let φ_n be the unique solution to

$$\left.\begin{aligned} Lu+\beta u-\alpha u^2 &= 0 \quad \text{on } B_n\backslash\overline{B}\\ u &= n \quad \text{on } \partial B,\\ u &= 0 \quad \text{on } \partial B_n\,. \end{aligned}\right\} \tag{23}$$

Then $\varphi_{\min}=\lim_{n\to\infty}\varphi_n$ (see [Pin95, p.250]). $\diamond$

For relations between extinction and the compactness of the range of super-Brownian motions with constant β but otherwise general branching mechanism, see [She97].

3 A pde interpretation of some of our results

Recall that $\beta \in C^{\gamma}$ is assumed to be bounded from above. Consider the following two possibilities.

(I) There is no positive solution to $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$.

(II) There exists a positive solution to $\frac{1}{2}\Delta u + \beta u - u^2 = 0$ on $\mathbb{R}^d$.

By Lemma 4, case (I) is equivalent to exhibiting no local extinction for the $(\frac{1}{2}\Delta, \beta, 1, \mathbb{R}^d)$-superdiffusion X. In the light of this correspondence we point out that conditions for (I) like the ones appearing in Theorem 1 and Remark 6 are, of course, well-known from standard pde literature. By [EP99, Theorem 3.5], the compact support property holds for X, and thus, by Lemma 14 (b), $w = w_{\max}$, where w and $w_{\max}$ are defined in (a) and (b) of Lemma 14 respectively. Putting this together with the first sentence in Lemma 14 (a), it follows that (II) is satisfied if and only if extinction does not hold for X. Using this together with Theorem 2, we immediately obtain the following relations between (I) and (II), and condition on (II); we omit the trivial proof.

Corollary 16 (relations between (I) and (II))

(a) (I) *implies* (II).

(b) (I) *and* (II) *are equivalent if* $d \leq 2$, *or if* $\beta(x) \leq k_d/|x|^2$ *for* $|x| >> 1$ [*with* k_d *from* (7)].

(c) (II) *holds, if* $\beta(x) \geq k/|x|^2$ *for* $|x| >> 1$ *and some* $k > k_d$.

4 Proofs

4.1 Preparation

We will utilize the following two lemmata.

Lemma 17 (condition for extinction) *X becomes extinct if all of the following conditions are true:*
(i) *the $(L, \beta, \alpha; \mathbb{R}^d)$-superdiffusion X exhibits local extinction,*
(ii) *$\beta \leq 0$ outside a compact set, and*
(iii) $\inf_{\mathbb{R}^d} \alpha > 0$.

Lemma 18 (condition for non-extinction) *Let X^i be the $(L_i, \beta_i, \alpha_i; \mathbb{R}^d)$-superdiffusions, $i = 1, 2$, and assume that, outside a compact set, α_1, β_1, and the coefficients of L_1 coincide with α_2, β_2, and the coefficients of L_2 respectively. Furthermore, assume that*
(i) *X^1 exhibits local extinction,*
(ii) *X^2 does not become extinct, and*
(iii) *X^2 is transient.*
Then X^1 does not become extinct either.

For the proofs of the Lemmas 17 and 18, we refer to [Eng99, Theorem 1.1], more precisely, to the proof of part a) and to the end of the proof of part b) there respectively.

4.2 Proof of Proposition 5 and Theorem 1

Proof of Proposition 5 Take h and B as in the proposition, and consider the h-transformed superdiffusion X^h according to Definition 11. Then, by assumption, $\beta^h \leq 0$ on $\mathbb{R}^d \backslash \overline{B}$. Note that $\alpha^h = h$, and thus α^h is bounded away from 0, also by assumption. Since X exhibits local extinction, also X^h does, and from Lemma 17 it follows that X^h becomes extinct. Then the same is true for X. ∎

Remark 19 (monotonicity) We will use the following comparison, for simplicity we refer to this as *"monotonicity"*: If $\beta_1 \leq \beta_2$ and there is no positive solution for the equation $(\frac{1}{2}\Delta + \beta_1)v = 0$ on $\mathbb{R}^d$, then there is no positive solution to $(\frac{1}{2}\Delta + \beta_2)v = 0$ on $\mathbb{R}^d$ either. In fact, similarly to the discussion following Remark 6, the non-existence of positive solutions for $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$ is equivalent to $\lambda_c^{(\beta)} > 0$, where $\lambda_c^{(\beta)}$ denotes the so-called generalized principal eigenvalue of $\frac{1}{2}\Delta + \beta$ on $\mathbb{R}^d$. Using the well-known probabilistic characterization of $\lambda_c^{(\beta)}$ ([Pin95, Theorem 6.4.4]) it is immediate that $\lambda_c^{(\beta)}$ is monotone non-decreasing in β. This implies the mentioned monotonicity. ◇

Proof of Remark 6 Let $d = 1$. By Lemma 4 it is sufficient to show that there is no positive solution to the equation $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}$. We may assume, that $\beta(x) \geq K/x^2,\ x >> 1$, where $K > \frac{1}{8}$. By monotonicity (Remark 19), it is enough to verify the statement for $\beta(x) = K/x^2,\ x >> 1$. Suppose on the contrary that there exists a function $f > 0$ satisfying $\frac{1}{2}f'' + \beta f = 0$. Then $\frac{1}{2}f'' + \frac{K}{x^2} f = 0$ for $x >> 1$. But the two-dimensional space of complex solutions to this equation is spanned by the power functions x^{ϱ_+} and x^{ϱ_-}, where $\varrho_\pm = \frac{1}{2}(1 \pm \sqrt{1-8K})$. Since $\mathrm{Im}(\varrho_\pm) \neq 0$, there is no positive solution, getting a contradiction. This already finishes the proof. ∎

Proof of Theorem 1 (b) Because of the previous proof, we could assume that $d \geq 2$. Recall that it suffices to show that there is no positive solution to the equation $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$. Again, by monotonicity, it is enough to verify the statement for $\beta(x) = K/|x|^2,\ x >> 1$. Suppose that there exists a function $f > 0$ satisfying $\frac{1}{2}\Delta f + \beta f = 0$ in $\mathbb{R}^d$. Then $\frac{1}{2}\Delta f + \frac{K}{|x|^2} f = 0$ on some annulus of the form $\{x \in \mathbb{R}^d : |x| > c\},\ c > 0$. Using a scaling argument, it then follows that there exists a positive solution to $\frac{1}{2}\Delta f + \frac{K}{|x|^2} f = 0$ on any annulus of the above form. Then, by a compactness argument, there exists a positive solution on $\mathbb{R}^d \backslash \{0\}$ as well. (For compactness arguments see [Pin95, Chapter 4].) But this is known to be false (see [Pin95, Example 3.12 on p.153]). Consequently, part (b) of Theorem 1 is proved.

(a) Assume that $\beta(x)|x|^2 \le K^d$ for $|x| >> 1$, and let h be a positive $C^{2,\gamma}$-function satisfying $h(x) = |x|^{-(d-2)/2}$ for $|x| >> 1$. Note that

$$\frac{\frac{1}{2}\Delta h}{h} = -K^d \frac{1}{|x|^2} \quad \text{for } |x| >> 1. \tag{24}$$

Moreover, let $\widehat{\beta} \le 0$ be a C^γ-function satisfying

$$\widehat{\beta}(x) = \beta(x) - K^d/|x|^2, \qquad |x| >> 1. \tag{25}$$

(The existence of such a $\widehat{\beta}$ is guaranteed by the growth rate assumption on β.) Define $\beta^* := \widehat{\beta} - \frac{1}{2}\frac{\Delta h}{h}$. It is easy to see that β^* belongs to C^γ, and moreover, using (24) and (25) we have $\beta^*(x) = \beta(x)$ for $|x| >> 1$. Taking the linear h-transform (see [Pin95, Chapter 4]) of the operator

$$\frac{1}{2}\Delta + \beta^*, \tag{26}$$

we get

$$\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla + \widehat{\beta}. \tag{27}$$

Since $\widehat{\beta} \le 0$, it is well-known (see e.g. [Pin95, Theorem 4.3.3 (iii)]) that there exists a positive solution for

$$\left(\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla + \widehat{\beta}\right) u = 0 \quad \text{on } \mathbb{R}^d. \tag{28}$$

Therefore,

$$\left(\frac{1}{2}\Delta + \beta^*\right)(hu) = 0 \tag{29}$$

[recall Remark 12 (ii)], and thus, by Lemma 4, the $(\frac{1}{2}\Delta, \beta^*, 1; \mathbb{R}^d)$-superdiffusion exhibits local extinction, finishing the proof. ■

4.3 Proof of Theorem 2

(a) Let $d \le 2$, and suppose to the contrary that X does not become extinct but exhibits local extinction. Since β is bounded from above, using the recurrence of the Brownian motion and Theorem 4.5 (a) of [EP99], it follows that X is recurrent. But this contradicts the local extinction (see the remark after Theorem 4.2 in [EP99]), giving the claim (a).

(b) If $d \le 2$, then the statement follows from (a).

Assume now that $3 \le d \le 6$ and that X exhibits local extinction. Similarly to the argument in part (a) of the proof of Theorem 1, X can be h-transformed into the $\left(\frac{1}{2}\Delta + \frac{\nabla h}{h} \cdot \nabla, \beta^h, h; \mathbb{R}^d\right)$-superdiffusion X^h, where $\beta^h = \widehat{\beta}$ for $|x| >> 1$

with $\widehat{\beta}$ as in (25). Recall that $\beta^h \leq 0$ for $|x| >> 1$. According to [EP99, Theorem 3.5], the compact support property holds for X, thus the same is true for X^h. Therefore, using Lemma 14 (b), it follows that the extinction of X^h is equivalent to the non-existence of positive solutions for the corresponding semilinear elliptic equation. Dividing through by h, we see that X^h (and also X) becomes extinct if and only if there is no positive solution to

$$\frac{1}{2h}\Delta u + \frac{\nabla h}{h^2} \cdot \nabla u + \frac{\beta^h}{h} u - u^2 = 0 \quad \text{on } \mathbb{R}^d, \tag{30}$$

that is, if and only if the corresponding maximal solution $w_{\max}$ is zero. In order to prove that $w_{\max} = 0$, let X^* denote the superdiffusion corresponding to the quadruple

$$\left(\frac{1}{2h}\Delta + \frac{\nabla h}{h^2} \cdot \nabla, \frac{\beta^h}{h}, 1, \mathbb{R}^d\right). \tag{31}$$

We will show that X^* becomes extinct (the w-function of Lemma 14 (a) is zero), and that $w = w_{\max}$. For the first statement, note that by the local extinction assumption on X and Lemma 4, $(\frac{1}{2}\Delta + \beta)u = 0$ with some $u > 0$. By Remark 12 (ii) then

$$\left(\frac{1}{2h}\Delta u + \frac{\nabla h}{h^2} \cdot \nabla + \frac{\beta^h}{h}\right) \frac{u}{h} = 0, \tag{32}$$

and therefore by Lemma 4, also X^* exhibits local extinction. Since $\beta^h \leq 0$ for $|x| >> 1$, and $\alpha = 1$, Lemma 17 yields that X^* becomes extinct.

For the present $3 \leq d \leq 6$ part, it remains to show that $w = w_{\max}$. By Lemma 14 (b), it is enough to verify that the compact support property holds for X^*. Since in particular $d \leq 6$, for the diffusion coefficient in (31) we have

$$\frac{1}{2h(x)} = O(|x|^2) \quad \text{as } |x| \to \infty. \tag{33}$$

Using this, the fact that the drift term $\frac{\nabla h}{h^2}(x)$ is negative for $|x| >> 1$, and that β^h/h is bounded from above (non-positive outside a compact set), the compact support property is implied by [EP99, Theorem 3.5].

Assume now that $d > 6$. Take an $h \in C^{2,\gamma}$ satisfying $h(x) = |x|^{-2}$ for $|x| >> 1$. Resolving the Laplacian in radial form, an elementary computation shows that if $\beta(x)|x|^2 \leq d - 4$ is satisfied for $|x| >> 1$, then

(i) $(\frac{1}{2}\Delta + \beta)h\,(x) \leq 0$ and

(ii) $\nabla h(x) \leq 0$

for $|x| >> 1$. Then the rest of the proof works similarly as in the case $3 \leq d \leq 6$. In fact, reading carefully the proof, one can see that it relies only on the fact that the h chosen there satisfies (i) and (ii) of the present case as well as (33). Indeed, we replaced the previous h by the present one in order to guarantee (33) for $d > 6$. This completes the proof of (b).

(c) Obviously, we can assume that $d > 6$, otherwise the assertion follows from Theorem 1 (b). Also, by comparison, we can set $\beta(x)|x|^2 = d - 4 + \varepsilon_0$ for $|x| >> 1$, with some $0 < \varepsilon_0 \leq 1$. In fact, for the comparison one has to check that for larger β we have a larger w-function, that is, less chance for extinction. This can easily be seen from the construction of the w-function and the parabolic maximum principle (see [EP99], Theorem 3.1 and Proposition 7.2 respectively). Last, we will assume that the process exhibits local extinction (otherwise the assertion is trivial).

Let h be a radially symmetric positive $C^{2,\gamma}$-function satisfying

$$h(x) = |x|^{-2} \quad \text{for } |x| >> 1. \tag{34}$$

Making the h-transform and dividing by h in the quadruple corresponding to X, we obtain the quadruple (31) [but now with h as in (34)]. Let X^1 denote the corresponding superdiffusion. Note, that by a simple computation, $\beta^h/h = \varepsilon_0$ outside a large closed ball $B \subset \mathbb{R}^d$. The same argument as in part (b) shows that X^1 exhibits local extinction.

Similarly to the argument preceding (30), the extinction of X is equivalent to the non-existence of a positive solution to (30) [but now with h as in (34)]. Our goal is to prove that extinction does not hold for X^1. In fact, then by Lemma 14 (a), the corresponding w-function is a positive solution to (30).

Using (34) and Feller's test for explosion (see e.g. [Pin95, Theorem 5.1.5]), we conclude that the operator $\frac{1}{2h}\Delta + \frac{\nabla h}{h^2} \cdot \nabla$ corresponds to a conservative diffusion on $\mathbb{R}^d$. Thus, by the last part of Lemma 14 (a) applied to X^2, which denotes the superdiffusion corresponding to the quadruple

$$\left(\frac{1}{2h}\Delta + \frac{\nabla h}{h^2} \cdot \nabla, \varepsilon_0, 1, \mathbb{R}^d \right), \tag{35}$$

we obtain $w(x) \equiv \varepsilon_0$. In particular, X^2 does not become extinct.

Applying Lemma 18 to X^1 and X^2 it will suffice to show that the latter process is transient. Then non-extinction of X^1 will follow.

Consider the $\varphi_{\min}$-function according to Lemma 14 (c) applied to X and with B, the ball introduced above. Resolving the Laplacian in radial form, and using $\varepsilon_0 \leq 1$, a simple computation reveals that if

$$0 < \varepsilon \leq \frac{1}{2} \left(d - 6 + \sqrt{(d-6)^2 + 8(1 - \varepsilon_0)} \right), \tag{36}$$

then $u(x) = |x|^{-2-\varepsilon}$ satisfies $\frac{1}{2}\Delta u + \beta u - u^2 \leq 0$ for $|x| >> 1$. Thus, by the elliptic maximum principle ([EP99, Proposition 7.1]) and Remark 15, there exists a constant $c > 0$ such that

$$\varphi_{\min}(x) \leq c\, u(x), \qquad |x| >> 1. \tag{37}$$

(Cf. the end of the proof of Theorem 4.2 in [EP99].) Since $\varphi^h_{\min} = \varphi_{\min}/h$ by Remark 12 (ii), the $\varphi_{\min}$-function for X^1 (and also for X^2) on $\mathbb{R}^d \backslash \overline{B}$ is $\varphi_{\min}/h$. Putting this together with (34) and (37), the $\varphi_{\min}$-function for X^2 tends to zero as $|x| \to \infty$. Therefore $\lim_{|x|\to\infty} \frac{\varphi_{\min}(x)}{w(x)} = 0$ for X^2. Thus, X^2 is transient, by Lemma 14 (c2).

This completes the proof of (c) and of Theorem 2 altogether. ■

Proof of Remark 8 First, let $d \leq 2$. By [Pin95, Theorem 6.3 (i)], there is no positive solution to the equation $(\frac{1}{2}\Delta + \beta)u = 0$ on $\mathbb{R}^d$. Thus, the statement is true by Lemma 4. On the other hand, if $d \geq 3$, $\beta \geq 0$, $\beta \neq 0$, and β is compactly supported, then by [Pin95, Theorem 4.6.2], there exists an $\varepsilon > 0$ and a function $u > 0$ such that $(\frac{1}{2}\Delta + \varepsilon\beta)u = 0$ on $\mathbb{R}^d$. Then, by Lemma 4, the $(\frac{1}{2}\Delta, \varepsilon\beta, 1, \mathbb{R}^d)$-superdiffusion X exhibits local extinction, hence by Lemma 17 it even becomes extinct. ■

4.4 Proof of Theorem 3

We need a lemma. Define the δ_0-regularization

$$\beta_\varepsilon(x) := \frac{1}{\varepsilon}\beta\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0, \quad x \in \mathbb{R}, \tag{38}$$

where $\beta \geq 0$ is a compactly supported non-vanishing smooth symmetric function with $\beta'(x) \leq 0$ for $x \geq 0$.

Lemma 20 (subsolutions for approximating equations) *There is a number $\ell > 0$ and there are functions $v_\varepsilon^- = v_{\varepsilon,\ell}^-$, $\varepsilon > 0$, defined on the interval $D_\ell := (-\ell, \ell)$, such that, for ε sufficiently small,*

(i) $v_\varepsilon^- \geq 0$, *and* $v_\varepsilon^- = 0$ *on* $\partial D_\ell := \{\pm\ell\}$,

(ii) $\frac{1}{2}(v_\varepsilon^-)'' + \beta_\varepsilon v_\varepsilon^- - (v_\varepsilon^-)^2 \geq 0$ *on* D_ℓ,

(iii) $\sup_{D_\ell} v_\varepsilon^- = v_\varepsilon^-(0)$,

and that $v_\varepsilon^-(0)$ is bounded away from zero as $\varepsilon \downarrow 0$.

Proof Denote by λ_ε^ℓ the leading eigenvalue for $\frac{1}{2}\Delta + \beta_\varepsilon$ on D_ℓ with zero boundary condition and with corresponding eigenfunction $\psi_\varepsilon^\ell > 0$. Furthermore, denote by λ^ℓ the leading eigenvalue for $\frac{1}{2}\Delta$ on D_ℓ with zero boundary condition and with corresponding eigenfunction $\psi_\ell > 0$, where ψ_ℓ has been normalized by $\int_{D_\ell} \mathrm{d}x\, \psi_\ell^2(x) = 1$. In other words,

$$\psi_\ell(x) = \frac{1}{\sqrt{\ell}}\cos\left(\frac{\pi x}{2\ell}\right) \quad \text{and} \quad \lambda^\ell = -\frac{\pi^2}{8\ell^2}. \tag{39}$$

Define

$$v_{\varepsilon,\ell}^- := \frac{\lambda_\varepsilon^\ell}{\sup_{D_\ell}\psi_\varepsilon^\ell}\,\psi_\varepsilon^\ell \quad \text{on } D_\ell. \tag{40}$$

Then $v_{\varepsilon,\ell}^-$ satisfies the boundary condition in (i), and a simple computation shows that (ii) also holds. We are going to show that there exists an $\ell > 0$ such that $\liminf_{\varepsilon\downarrow 0}\lambda_\varepsilon^\ell > 0$. This will prove that $v_{\varepsilon,\ell}^- \geq 0$ for ε sufficiently small and that $\sup_{D_\ell} v^-_{\varepsilon,\ell}$ is bounded away from zero as $\varepsilon \downarrow 0$. In order to do this, we invoke the following minimax representation of λ^ℓ_ε (see [Pin95, Theorem 3.7.1]):

$$\lambda^\ell_\varepsilon \;=\; \sup_\mu \inf_{\substack{u>0 \text{ on } D_\ell \\ u\in C^2(D_\ell)}} \int_{D_\ell} \mu(\mathrm{d}x)\, \left(\frac{1}{2}\frac{u''}{u} + \beta_\varepsilon\right)(x), \tag{41}$$

where the supremum is taken over all probability measures μ on D_ℓ with densities f satisfying $\sqrt{f} \in C^1(\overline{D_\ell})$ and $f(\pm\ell) \equiv 0$. (Of course, C^m, $m \geq 1$, refers to the set of all m-times continuously differentiable functions.) Take $\mu(\mathrm{d}x) = \psi^2_\ell(x)\,\mathrm{d}x$ in (41). Then,

$$\lambda^\ell_\varepsilon \geq \inf_{0<u\in C^2(D_\ell)} \int_{D_\ell} \mathrm{d}x\; \frac{1}{2}\frac{u''}{u}\psi^2_\ell + \int_{D_\ell} \mathrm{d}x\; \beta_\varepsilon \psi^2_\ell =: I + II \tag{42}$$

(with the obvious correspondence). Using [Pin95, Theorem 3.7.1] again, we get $I = \lambda^\ell$. Thus

$$\lambda^\ell_\varepsilon \geq \lambda^\ell + \int_{D_\ell} \mathrm{d}x\; \beta_\varepsilon \psi^2_\ell = -\frac{\pi^2}{8\ell^2} + \int_{D_\ell} \mathrm{d}x\; \frac{1}{\ell}\cos^2\left(\frac{\pi x}{2\ell}\right)\, \beta_\varepsilon(x). \tag{43}$$

Since $\beta_\varepsilon(x)\mathrm{d}x \to \delta_0(\mathrm{d}x)$ weakly as $\varepsilon \downarrow 0$, the latter inequality yields $\liminf_{\varepsilon\downarrow 0} \lambda^\ell_\varepsilon > 0$, provided that ℓ is sufficiently large.

It remains to show that $\sup_{D_\ell} \psi^\ell_\varepsilon = \psi^\ell_\varepsilon(0)$ and consequently $\sup_{D_\ell} v^-_{\varepsilon,\ell} = v^-_{\varepsilon,\ell}(0)$. For this purpose, we consider the equation

$$\frac{1}{2}(\psi^\ell_\varepsilon)'' = (\lambda^\ell_\varepsilon - \beta_\varepsilon)\psi^\ell_\varepsilon\,. \tag{44}$$

Clearly, $(\psi^\ell_\varepsilon)''(x) \geq 0$ if and only if $\beta_\varepsilon(x) \leq \lambda^\ell_\varepsilon$, and consequently $\lambda^\ell_\varepsilon \leq \sup_{D_\ell} \beta_\varepsilon = \beta_\varepsilon(0)$. Putting this together with the positivity, symmetry and compact support of ψ^ℓ_ε, we conclude that $\sup_{D_\ell} \psi^\ell_\varepsilon = \psi^\ell_\varepsilon(0)$. This completes the proof of the lemma. ■

Proof of Theorem 3 *Step* 1° Let $\ell > 0$ and let $v^-_\varepsilon = v^-_{\varepsilon,\ell}$ be as in Lemma 20. By that lemma, one can pick a constant $c > 0$ such that

$$\sup_{D_\ell} v^-_\varepsilon = v^-_\varepsilon(0) > c \quad \text{for all small } \varepsilon > 0. \tag{45}$$

Fix a non-negative continuous function g satisfying

$$g = c \text{ on } D_\ell \quad \text{and} \quad g = 0 \text{ on } \mathbb{R}\backslash D_{2\ell}\,. \tag{46}$$

Put

$$u^-_\varepsilon := \frac{c \cdot v^-_\varepsilon}{\sup_{D_\ell} v^-_\varepsilon}\,. \tag{47}$$

Note, that $u_\varepsilon^-(0) = c$ by Lemma 20 (iii). Using (i)-(ii) of the same lemma and the statement (45), an easy computation shows that, for $\varepsilon > 0$ sufficiently small, u_ε^- satisfies

$$\left.\begin{aligned} \frac{1}{2}(u_\varepsilon^-)'' + \beta_\varepsilon u_\varepsilon^- - (u_\varepsilon^-)^2 \geq 0 \quad \text{on } D_\ell\,, \\ u_\varepsilon^-(x) \leq g(x) \quad \text{on } D_\ell\,, \\ u_\varepsilon^- = 0 \quad \text{on } \partial D_\ell\,. \end{aligned}\right\} \tag{48}$$

Then, by the parabolic maximum principle ([EP99, Proposition 7.2]), for all $\varepsilon > 0$ small enough,

$$u_\varepsilon^-(\cdot) \leq u_\varepsilon^g(\,\cdot\,, t), \qquad t \geq 0, \tag{49}$$

where u_ε^g denotes the minimal non-negative solution to the evolution equation (3) with $d = 1$, $L = \frac{1}{2}\Delta$, β replaced by β_ε, $\alpha = 1$, and g from (46).

Step 2° First we verify the claim in the special case $\mu = r\delta_0$ with $r > 0$. Let E^ε denote the expectations corresponding to the $(\frac{1}{2}\Delta, \beta_\varepsilon, 1; \mathbb{R})$-superdiffusion. By (2) specialized to the present case, (49), and using

$$u_\varepsilon^-(0) \equiv c > 0, \tag{50}$$

we obtain for all $\varepsilon > 0$ small enough and $t > 0$,

$$E_{r\delta_0}^\varepsilon \exp\langle X_t\,, -g\rangle = \exp\big[-ru_\varepsilon^g(0,t)\big] \leq \exp\big[-ru_\varepsilon^-(0)\big] = \mathrm{e}^{-rc}. \tag{51}$$

Since this holds for all $\varepsilon > 0$ small and $t > 0$, letting $\varepsilon \downarrow 0$, we get

$$E_{r\delta_0}^{\sin} \exp\langle X_t\,, -g\rangle \leq \mathrm{e}^{-rc} < 1, \qquad t > 0. \tag{52}$$

Assume for the moment that

$$P_{r\delta_0}^{\sin}\Big(X_t(D_{2\ell}) = 0 \text{ for all large } t\Big) = 1, \tag{53}$$

then the left hand side of (52) tends to one as $t \to \infty$, and this is a contradiction. Consequently, the superdiffusion X with law $P_{r\delta_0}^{\sin}$ does not exhibit local extinction.

Step 3° Before turning to general starting measures, we need a slight generalization of (52). To this end, we modify the superdiffusion X with law $P_{r\delta_0}^{\sin}$ a bit: Instead of starting at time 0 with the measure $r\delta_0$, we choose a starting time s according to a non-vanishing finite measure $\eta(\mathrm{d}s)$ on $\mathbb{R}_+$. Then, by definition,

$$E_\eta^{\sin} \exp\langle X_t\,, -g\rangle = \exp\left[-\int_{[0,t]} \eta(\mathrm{d}s)\, u(0, t-s)\right], \qquad t \geq 0, \tag{54}$$

with u satisfying the integral equation (9) with g from (46). Moreover, by (49) and (50), instead of (52) we then get

$$E_\eta^{\sin} \exp \langle X_t\,, -g \rangle \le \exp \Big[-\eta\left([0,t]\right) c \Big] < 1, \qquad t >> 1. \tag{55}$$

Step 4° Finally, for our original superdiffusion X with general starting measure $\mu \in \mathcal{M}_f \backslash \{0\}$ (at time 0), we use Dynkin's *stopped (or exit) measures* X_τ and their so-called *special Markov property* (see [Dyn91a]). In our case, τ is the Brownian (first) hitting time of 0, where the additional mass source is sitting. Having in mind a historical setting of the superdiffusion X (see, for instance [DP91] or [Dyn91b]), then intuitively the present $X_\tau\,(\mathrm{d}s)$ is a measure on $\mathbb{R}_+$ which describes the mass distribution of all superdiffusion's particles which hit 0 the first time in the moment $\tau = s$. Of course, the formal description of stopped measures as X_τ along the historical setting and their special Markov property requires some technicalities, but we skip such details here and in the sequel.

Now,

$$E_\mu^{\sin} \exp \langle X_t\,, -g \rangle = E_\mu^{\sin} E_\mu^{\sin} \left\{ \exp \langle X_t\,, -g \rangle \;\middle|\; \mathcal{G}_{\tau \wedge t} \right\} \tag{56}$$

where $\mathcal{G}_{\tau\wedge t}$ denotes the pre-$(\tau \wedge t)$ σ-field (concerning the stopped historical superdiffusion and the Brownian stopping time $\tau \wedge t$). By the special Markov property and (55) we may continue with

$$= E_\mu^{\sin} E_{X_{\tau\wedge t}}^{\sin} \exp \langle X_t\,, -g \rangle \le E_\mu^{\sin} \exp \Big[-X_{\tau\wedge t}\left([0,t]\right) c \Big]\,. \tag{57}$$

But, as $t \to \infty$, the right hand side converges to

$$E_\mu^{\sin} \exp \big[-\|X_\tau\|\, c \big] \le E_\mu \exp \big[-\|X_\tau\|\, c \big]\,, \tag{58}$$

where E_μ refers to the $(\frac{1}{2}\Delta, 0, 1, \mathbb{R})$-superdiffusion. (Indeed, dropping the additional mass source δ_0, we may loose some population mass.) However,

$$P_\mu \left(\|X_\tau\| \neq 0 \right) > 0 \tag{59}$$

since by the expectation formula for X_τ-measures (see [Dyn91a, (1.50a)]),

$$E_\mu \,\|X_\tau\| = \|\mu\| > 0. \tag{60}$$

Hence, $E_\mu \exp \big[-\|X_\tau\|\, c \big] < 1$, and therefore altogether

$$\limsup_{t\to\infty} E_\mu^{\sin} \exp \langle X_t\,, -g \rangle < 1. \tag{61}$$

Again arguments as in the end of step 2° will finish the proof. ■

4.5 Proof of Theorem 4

(a) Set $u(x,t) := E^{\sin}_{\delta_x}\|X_t\|$. Then using the equation (9), it is standard to verify the following integral equation for the expectations:

$$u(x,t) = 1 + \int_0^t \mathrm{d}s\, p(t-s,x)u(0,s), \qquad x \in \mathbb{R}, \quad t \geq 0. \tag{62}$$

(Symbolically, $u_t = \frac{1}{2}\Delta u + \delta_0 u$ with $u(x,0) \equiv 1$.) Setting $x = 0$ and exploiting the notations $f(t) := u(0,t)$ and $p_x(t) := p(t,x)$, we realize that f satisfies

$$f(t) = 1 + \int_0^t \mathrm{d}s\, p_0(t-s)f(s), \qquad t \geq 0. \tag{63}$$

Taking Laplace transforms on both sides (where the Laplace transform of a function g is denoted by $\widehat{g}$), the convolution on the right hand side transforms into a product. Thus,

$$\widehat{f}(\lambda) = \frac{\widehat{1}}{1-\widehat{p_0}(\lambda)} = \frac{1}{\lambda\bigl(1-\frac{1}{\sqrt{2\lambda}}\bigr)}, \qquad \lambda > 0. \tag{64}$$

Statement (a) follows by an inverse Laplace transform.

(b) Fix a bounded continuous g. Set

$$u(x,t) := E^{\sin}_{\delta_x}\langle X_t\,, g\rangle \quad \text{and} \quad f(t) := u(0,t). \tag{65}$$

Put $F(t) := \mathrm{e}^{-t/2}f(t)$. Finally, let $C(g) := \int_{\mathbb{R}} \mathrm{d}x\; g(x)\,\mathrm{e}^{-|x|}$. Our goal is to verify that $F(t) \to C(g)$ as $t \to \infty$. By a well-known Tauberian theorem ([Fel71, formula (13.5.22)]), it is enough to show that

$$\widehat{F}(\lambda) \sim C(g)\,\frac{1}{\lambda}, \quad \text{as } \lambda \downarrow 0. \tag{66}$$

Set $k(t) := \int_{\mathbb{R}} \mathrm{d}x\; p(t,x)g(x)$. By a similar computation as in (a), for the Laplace transforms one obtains,

$$\widehat{F}(\lambda) = \widehat{f}\left(\lambda + \frac{1}{2}\right) = \widehat{k}\left(\lambda + \frac{1}{2}\right)\,\frac{1}{1-\dfrac{1}{\sqrt{2\lambda+1}}}, \qquad \lambda > 0. \tag{67}$$

Using Fubini's Theorem,

$$\lim_{\lambda\to 0}\widehat{k}\left(\lambda + \frac{1}{2}\right) = \widehat{k}\left(\frac{1}{2}\right) = \int_{\mathbb{R}} \mathrm{d}x\;\widehat{p_x}\left(\frac{1}{2}\right)g(x). \tag{68}$$

Since $\widehat{p_x}(1/2) = \mathrm{e}^{-|x|}$, we get

$$\widehat{k}\left(\frac{1}{2}\right) = C(g). \tag{69}$$

Furthermore, an elementary computation shows that

$$\frac{1}{1-\frac{1}{\sqrt{2\lambda+1}}} \sim \frac{1}{\lambda} \quad \text{as } \lambda \downarrow 0. \tag{70}$$

This completes the proof of (b), hence of Theorem 4 altogether. ■

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