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# FINITE ELEMENT APPROXIMATION OF TRANSPORT OF REACTIVE SOLUTES IN POROUS MEDIA*

## Part 1 Error Estimates for Non-Equilibrium Adsorption Processes

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ABSTRACT

In this paper we analyse a fully practical piecewise linear finite element approximation; involving regularization, numerical integration and backward Euler time discretisation; of the following degenerate parabolic system arising in a model of reactive solute transport in porous media: Find $\{u(x,t),v(x,t)\}$ such that

$$\partial_t u + \partial_t v - \Delta u = f \quad \text{in} \quad \Omega \times (0,T] \qquad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T]$$

$$\partial_t v = k(\varphi(u)-v) \quad \text{in} \quad \Omega \times (0,T]$$

$$u(\cdot,0) = g_1(\cdot) \qquad v(\cdot,0) = g_2(\cdot) \qquad \text{in } \Omega \subset \mathbb{R}^d,\ 1 \le d \le 3$$

for given data $k \in \mathbb{R}^+$, $f$, $g_1$, $g_2$ and a monotonically increasing $\varphi \in C^0(\mathbb{R}) \cap C^1(-\infty,0] \cup (0,\infty)$ satisfying $\varphi(0) = 0$; which is only locally Hölder continuous, with exponent $p \in (0,1)$, at the origin; e.g. $\varphi(s) \equiv [s]_+^p$. This lack of Lipschitz continuity at the origin limits the regularity of the unique solution $\{u,v\}$ and leads to difficulties in the finite element error analysis. Nevertheless we arrive at error bounds which in some cases exhibit the full approximation power of the trial space.

## 1. INTRODUCTION

In these papers we study finite element approximations of degenerate parabolic systems and equations, as they arise in the modelling of reactive solute transport in porous media, as soils or aquifers. The reaction, that we are going to take into account, is adsorption; that is, a retention/release reaction of the solute, e.g. a contaminant, with the porous skeleton. Adsorption is a major concern in soil science and hydrology as it is often the primary factor determining the mobility of a solute.

We consider the process on a macroscopic level, i.e. averaged/homogenized scale, where single grains and pores do not appear anymore. A macroscopic model has the form (c.f. Knabner (1991a) and van Duijn & Knabner (1992) for a derivation)

$$\partial_t(\Theta u) + \rho\partial_t[\lambda_1\psi(u)+\lambda_2 v] - \nabla\cdot(\Theta\underline{\underline{D}}\nabla u-\underline{q}u) = f \quad \text{in } Q_T \tag{1.1a}$$

$$\partial_t v = r(u,v) \quad \text{in } Q_T\ , \tag{1.1b}$$

supplemented by initial conditions for u and v and appropriate boundary conditions for u. Here u and v are the unknowns of the system, the dissolved concentration (with reference to the water-filled part of the pore space) and the adsorbed concentration in non-equilibrium (with reference to the mass of the porous skeleton). The process takes place in a bounded domain $\Omega$ in $\mathbb{R}^d$, $1 \le d \le 3$. Let $[0,T]$ be the fixed time interval and $Q_t \equiv \Omega \times (0,t]$, for $t \in (0,T]$. The other quantities, all assumed to be known, either describe the underlying water flow regime and geology, as the water content $\Theta$, the volumetric water flux $\underline{q}$, the sum of diffusion and dispersion matrix $\underline{\underline{D}}$ and the bulk density $\rho$, or the adsorption process: Here it is assumed that two classes of adsorption sites may be distinguished (with relative specific grain surfaces $\lambda_i \in [0,1]$). The sites in class 2 are in (chemical) non-equilibrium and the kinetics are described by (1.1b), which applies to adsorption reaction at a time scale comparable to transport. Whereas for sites where the reaction is considerably faster, a quasistationary approach is feasible, assuming the reaction to be equilibrium. This approach is used for sites in class 1, leading to an algebraic expression for the adsorbed concentration in terms of the dissolved concentration, - the adsorption isotherm $\psi$.

A common heuristic approach for the rate function r consists of taking it proportional to the deviation from equilibrium, i.e.

$$r(u,v) \equiv k(\varphi(u)-v), \qquad (1.2)$$

where $\varphi$ is the adsorption isotherm for sites of class 2 and $k > 0$ is a rate parameter. We will restrict ourselves to this form. The quasilinear, respectively semilinear (for $\lambda_1 = 0$), system (1.1) may be degenerate because there are typical examples for the isotherms $\varphi$ or $\psi$, which are not Lipschitz continuous at $u = 0$ such as is the Freundlich isotherm

$$\varphi(u) \equiv \alpha u^p \quad \text{for } u \geq 0, \text{ where } \alpha \in \mathbb{R}^+ \text{ and } p \in (0,1). \qquad (1.3)$$

On the other hand isotherms are monotone increasing , such that in the following we will consider monotone nonlinearities allowing for degenerate behaviour like (1.3) at the origin.

In the first part of this paper we consider only non-equilibrium adsorption, such that we assume $\lambda_1 = 0$ from now on. The underlying water flow regime in general leads to time and space dependent coefficients, but with a linear uniformly parabolic operator on u, due to

$$\partial_t \Theta + \nabla.\underline{q} = 0 , \quad \Theta(x,t) \geq \Theta_0 > 0 \quad \text{in } Q_T. \qquad (1.4)$$

The degenerate semilinear system (1.1), supplemented by in and outflow conditions has been extensively studied by Knabner (1991a) and van Duijn & Knabner (1990). The boundary conditions read as

$$(\Theta D\nabla u-\underline{q}u).\underline{n} = F \quad \text{on } S_1 \times (0,T] \quad \text{and} \quad D\nabla u.\underline{n} = 0 \quad \text{on } S_2 \times (0,T], \qquad (1.5)$$

where $\underline{n}$ is the outward normal to $\partial\Omega \equiv S_1 \cup S_2$, $S_1$ is defined by $\underline{q}.\underline{n} \leq 0$ (the inflow boundary) and $S_2$ by $\underline{q}.\underline{n} \geq 0$ (the outflow/noflow boundary). A specific sequence of testing leads to a uniqueness result (see II Th.2.2 in Knabner

(1991a)), which can be extended to the usual energy norm stability estimate for the u-components of the solutions, but only under certain structural conditions on the coefficients (II Th.2.6). These conditions are fulfilled for time-independent coefficients, i.e. for stationary water flow.

Our aim is to prove order of convergence estimates in energy norms for the corresponding finite element approximation, therefore we consider this stability estimate to be important. In fact it turns out that the same approach enables us to reduce the error estimation (for the continuous in time conformal Galerkin approximation) to problems, which have already been studied by Barrett & Shanahan (1991), see Knabner (1991b) for a preliminary account. In fact the problem considered in Barrett & Shanahan (1991) can be viewed as a stationary version of the present problem by neglecting the desorption term -kv. Therefore we restrict ourselves to situations where this reasoning for the stability estimate is possible, by considering only stationary water flow. We substantially extend and refine the aforementioned preliminary analysis by improving on the error bounds there and considering a fully practical scheme involving numerical integration on the nonlinear term and time discretisation using the backward Euler method. The analysis is centred on introducing a regularized system $(P_\varepsilon)$ obtained by substituting $\varphi$ by a Lipschitz continuous $\varphi_\varepsilon$, differing only near u = 0. In fact if the solution u satisfies a non-degeneracy condition, see below, by adapting the regularization parameter $\varepsilon$ to the discretization parameters one can prove better rates of convergence for the approximation of $(P_\varepsilon)$ to (P) than for the approximation of (P) directly. This situation is not uncommon for the finite element approximation of degenerate problems (e.g. see Nochetto & Verdi (1988)).

The non-Lipschitzian behaviour of $\varphi$ at u = 0 can only play an important role if fronts, given by the boundary of the support of u (or v) in $\Omega$, do not vanish instantaneously, as for the heat equation, but are preserved; i.e. if the problem exhibits a finite speed of propagation property. This property is analysed by Knabner (1991a) for the one-dimensional case and found to be characterised by

$$\Phi^{-1/2} \in L^1(0,\delta) \text{ for some } \delta > 0, \tag{1.6}$$

where $\Phi(s) \equiv \int_0^s \varphi(\sigma)\, d\sigma$. This is fulfilled by the example (1.3) and may be considered as the typical case in the following. The non-degeneracy condition describes the minimal growth of u away from the front. This local behaviour of the profile has only been analysed for travelling wave solutions ( see van Duijn & Knabner (1991)). We will assume later on, that $\varphi$ is Hölder continuous near u = 0 with exponent $p \in (0,1]$. If in addition the exponent is sharp, i.e.

$$\varphi(u) \geq \alpha u^p \quad \text{for } u \in [0,\delta_0] \text{ and for some } \alpha,\ \delta_0 > 0 \tag{1.7}$$

then:

**(N.D.)**
$$A_\varepsilon(t) \leq C\varepsilon^{1/2}\ , \tag{1.8a}$$

where

$$A_\varepsilon(t) \equiv \int_0^t \underline{m}(\Omega_\varepsilon(s))\, ds, \tag{1.8b}$$

$$\Omega_\varepsilon(t) \equiv \{\ x \in \Omega : u(x,t) \in (0,\varepsilon^{1/(1-p)})\ \}, \tag{1.8c}$$

and $\underline{m}$ is the Lebesgue measure.

Our analysis applies to the case of general time-independent coefficients (assuming they are sufficiently regular). However, the fact that we analyse the Galerkin procedure implies the requirement that the process is not convection-dominated, where we would encounter the well-known difficulties. There are alternative procedures for this situation like the streamline diffusion method or the modified method of characteristics. We expect that the techniques that we are going to develop here will enable us to analyse also variants of these methods. We refer to Dawson, van Duijn & Wheeler (1992) for a first account with respect to the modified method of characteristics.

For ease of exposition we will develop our results for the following model problem, which keeps the specific difficulty of the non-Lipschitz nonlinearity, but reduces the handling of standard terms:

**(P)** Find $\{u(x,t),v(x,t)\}$ such that

$$\partial_t u + \partial_t v - \Delta u = f \quad \text{in } Q_T \qquad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T]$$

$$\partial_t v = k(\varphi(u)-v) \quad \text{in } Q_T$$

$$u(\cdot,0) = g_1(\cdot) \qquad v(\cdot,0) = g_2(\cdot) \qquad \text{in } \Omega,$$

where we make the following assumptions on the given data:

Assumptions **(D1)**: $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, with either $\Omega$ convex polyhedral or $\partial\Omega \in C^{1,1}$, $k \in \mathbb{R}^+$, $f \in L^\infty(Q_T)$, $g_1 \in L^\infty(\Omega)\cap H^1_0(\Omega)$, $g_2 \in L^\infty(\Omega)$ and $\varphi \in C^0(\mathbb{R})$ is such that

(i) $\varphi(0) = 0$, $\varphi(s) > 0 \;\forall\; s > 0$ and $\varphi$ is monotonically increasing (1.9a)

(ii) $\varphi \in C^1(-\infty,0]\cup(0,\infty)$ (1.9b)

(iii) there exist $L \in \mathbb{R}^+$ and $\varepsilon_0$, $p \in (0,1]$ such that

$$|\varphi(a)-\varphi(b)| \le L|a-b|^p \qquad \text{for all } a,b \in [0,\varepsilon_0]. \tag{1.9c}$$

The layout of this paper is as follows. In the next section we establish the existence and uniqueness of a solution to (P) by firstly establishling these results for a regularized version $(P_\varepsilon)$. In section 3 we consider a continuous in time continuous piecewise linear finite element approximation in space. In section 4 we consider a more practical approximation employing numerical integration on the nonlinear term. Finally in section 5 we consider a fully practical approximation involving discretisation in time using the backward Euler method.

Throughout the paper we adopt the standard notation for Sobolev spaces. We note that the seminorm $|\cdot|_{H^1(\Omega)}$ and norm $\|\cdot\|_{H^1(\Omega)}$ are equivalent on $H^1_0(\Omega)$. The standard $L^2$ inner product over $\Omega$ is denoted by $(\cdot,\cdot)$. Throughout $C$ or $C_i$ denote generic positive constants independent of $\varepsilon$ the regularization parameter, h the mesh spacing and k the reaction rate parameter. If a constant does depend on k say, this will be written as $C(k)$. We track the constant k in the analysis as we use nearly all the results in this paper to study the case of k infinite, equilibrium adsorption, in part II. This often makes the present analysis more complicated than it need be if we were just interested in the case k finite.

## 2. THE CONTINUOUS PROBLEM

In this section we establish existence and uniqueness of a solution to (P). These results have been proved by Knabner(1991a) for (1.1) with boundary conditions (1.5). However the model problem (P) allows for a more direct account and furthermore in doing so, we will develop various bounds that will be useful in analysing the error in the finite element approximation of (P). Firstly we introduce a regularized version of (P), for $\varepsilon \in (0,\varepsilon_0]$ ($\varepsilon_0$ as in (1.9c)):

**($P_\varepsilon$)** Find $\{u_\varepsilon(x,t),v_\varepsilon(x,t)\}$ such that

$$\partial_t u_\varepsilon + \partial_t v_\varepsilon - \Delta u_\varepsilon = f \quad \text{in } Q_T \qquad u_\varepsilon = 0 \quad \text{on } \partial\Omega \times (0,T]$$

$$\partial_t v_\varepsilon = k(\varphi_\varepsilon(u_\varepsilon)-v_\varepsilon) \quad \text{in } Q_T$$

$$u_\varepsilon(\cdot,0) = g_1(\cdot) \qquad v_\varepsilon(\cdot,0) = g_2(\cdot) \quad \text{in } \Omega,$$

where $\varphi_\varepsilon \in C^0(\mathbb{R})$ is such that

(i) $\varphi_\varepsilon(s) \equiv \varphi(s)$ for $s \notin (0,\varepsilon^{1/(1-p)})$ (2.1a)

(ii) $\varphi_\varepsilon(s)$ is strictly monotonically increasing on $[0,\varepsilon^{1/(1-p)}]$ (2.1b)

(iii) for $m \in \mathbb{N}$ there exists a $M(m) \in \mathbb{R}^+$:

$$\varphi_\varepsilon'(s) \le M(m)\varepsilon^{-1} \text{ for almost all } |s| \le m. \tag{2.1c}$$

Note that M can be chosen independently of m, if $\varphi'$ is bounded in $\mathbb{R}\backslash(0,\delta)$ for some $\delta > 0$. In addition we set

$$\Phi_\varepsilon(s) \equiv \int_0^s \varphi_\varepsilon(\sigma)\, d\sigma . \tag{2.2}$$

It is a simple matter to deduce from the conditions (2.1) that for all $|a|$, $|b| \le m$

$$[M(m)]^{-1}\varepsilon|\varphi_\varepsilon(a)-\varphi_\varepsilon(b)|^2 \le [\varphi_\varepsilon(a)-\varphi_\varepsilon(b)](a-b) \le M(m)\varepsilon^{-1}|a-b|^2 \tag{2.3a}$$

and

$$\varphi_\varepsilon(\varepsilon^{1/(1-p)}) \equiv \varphi(\varepsilon^{1/(1-p)}) \le L\varepsilon^{p/(1-p)} \tag{2.3b}$$

with L as in (1.9c). The simplest choice for $\varphi_\varepsilon$ is the linear regularization

$$\varphi_\varepsilon(s) \equiv \varepsilon^{-1/(1-p)}\varphi(\varepsilon^{1/(1-p)})s \quad \text{for } s \in (0,\varepsilon^{1/(1-p)}). \tag{2.4}$$

Definition: $\{u_\varepsilon, v_\varepsilon\}$ is a <u>weak</u> <u>upper</u> <u>(lower)</u> <u>solution</u> to $(P_\varepsilon)$ if $u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)) \equiv W_2^{1,1}(Q_T)$, $\varphi_\varepsilon(u_\varepsilon) \in L^2(Q_T)$ and $v_\varepsilon \in H^1(0,T;L^2(\Omega))$ are such that for all test functions $\eta \in L^2(0,T;H_0^1(\Omega))$ with $\eta \geq 0$ in $Q_T$

$$\int_{Q_T} [\partial_t u_\varepsilon \eta + \partial_t v_\varepsilon \eta + \nabla u_\varepsilon . \nabla \eta - f\eta]\, dxdt \geq (\leq)\ 0 \qquad u_\varepsilon \geq (\leq)\ 0 \text{ on } \partial\Omega \times (0,T]$$

$$\partial_t v_\varepsilon \geq (\leq)\ k(\varphi_\varepsilon(u_\varepsilon) - v_\varepsilon) \quad \text{in } Q_T$$

$$u_\varepsilon(\cdot,0) \geq (\leq)\ g_1(\cdot) \qquad v_\varepsilon(\cdot,0) \geq (\leq)\ g_2(\cdot) \qquad \text{in } \Omega.$$

$\{u_\varepsilon, v_\varepsilon\}$ is a <u>weak</u> <u>solution</u> to $(P_\varepsilon)$ if it is both a weak lower solution and a weak upper solution to $(P_\varepsilon)$. Similar definitions hold for (P) with $\varphi_\varepsilon$ in the above replaced by $\varphi$.

**Theorem 2.1**

Let the Assumptions (D1) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ there exists a unique weak solution $\{u_\varepsilon, v_\varepsilon\}$ to $(P_\varepsilon)$ such that

$$\underline{u} \leq u_\varepsilon \leq \bar{u} \quad \text{and} \quad \underline{v} \leq v_\varepsilon \leq \bar{v} \quad \text{in } Q_T \tag{2.5a}$$

$$|\nabla u_\varepsilon|^2_{L^2(Q_T)} + |\partial_t u_\varepsilon|^2_{L^2(Q_T)} + |\partial_t v_\varepsilon|^2_{L^\infty(Q_T)} \leq C(k), \tag{2.5b}$$

where $\underline{u}$, $\bar{u}$, $\underline{v}$, $\bar{v} \in C^0(\bar{\Omega})$ are all independent of $\varepsilon$ and $k$. Furthermore, if $g_1$, $g_2$ and $f \geq 0$ one can take $\underline{u} = \underline{v} = 0$.

Proof: Firstly, we prove the existence of weak lower and upper solutions to $(P_\varepsilon)$. Let $w \in H^2(\Omega)$ $(\subset C^0(\bar{\Omega}))$ be such that $-\Delta w = 1$ in $\Omega$ and $w = 1$ on $\partial\Omega$. It follows that $w \geq 1$ in $\Omega$. Let $\gamma \equiv \max\,\{\|f\|_{L^\infty(Q_T)}, \|g_1\|_{L^\infty(\Omega)}, 1\}$. Then for all $\varepsilon \in (0,\varepsilon_0]$ $\{\bar{u},\bar{v}\}$ $(\{\underline{u},\underline{v}\})$ is an upper (lower) solution of $(P_\varepsilon)$, where $\bar{u} \equiv \gamma w$, $\underline{u} \equiv -\bar{u}$, $\bar{v} \equiv \max\,\{\|g_2\|_{L^\infty(\Omega)}, \|\varphi(\bar{u})\|_{L^\infty(\Omega)}\}$ and $\underline{v} \equiv -\max\,\{\|g_2\|_{L^\infty(\Omega)}, \|\varphi(\underline{u})\|_{L^\infty(\Omega)}\}$. Note that $\varphi_\varepsilon(\bar{u}) \equiv \varphi(\bar{u})$ and $\varphi_\varepsilon(\underline{u}) \equiv \varphi(\underline{u})$ as $\bar{u} \geq 1$ and $\underline{u} \leq 0$. If $g_1$, $g_2$ and $f \geq 0$ we note that one can alternatively choose $\{\underline{u},\underline{v}\} \equiv \{0,0\}$.

Let $B \equiv \{\ u \in L^2(Q_T) \ :\ \underline{u} \leq u \leq \bar{u}\ \}$. We now define an operator $T : B \to L^2(Q_T)$. Firstly, given $u \in B$, we define

$$v(x,t) \equiv e^{-kt}g_2(x) + k\int_0^t e^{-k(t-s)}\varphi_\varepsilon(u(x,s))\,ds. \tag{2.6}$$

Clearly, $v$ is such that $\partial_t v = k(\varphi_\varepsilon(u)-v)$ in $Q_T$ and $v(\cdot,0) = g_2(\cdot)$ in $\Omega$ and $u \in B \Rightarrow u \in K \equiv [\inf \underline{u}\ ,\ \sup \bar{u}] \Rightarrow u \in L^\infty(Q_T) \Rightarrow \varphi_\varepsilon(u) \in L^\infty(Q_T) \Rightarrow v$, $\partial_t v \in L^\infty(Q_T)$ with norms bounded uniformly for all $u \in B$. Then $\tilde{u} \equiv Tu$ is defined to be the unique weak solution of

$$\partial_t\tilde{u} - \Delta\tilde{u} + M_\varepsilon\tilde{u} = k(v-\varphi_\varepsilon(u)) + M_\varepsilon u + f \quad \text{in } Q_T \tag{2.7a}$$

$$\tilde{u} = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \tilde{u}(\cdot,0) = g_1(\cdot) \quad \text{in } \Omega, \tag{2.7b}$$

where $M_\varepsilon \equiv kM(m)\varepsilon^{-1}$ is the Lipschitz constant of $k\varphi_\varepsilon(\cdot)$ and $m$ is such that $K \subseteq [-m,m]$, see (2.1c). We now show that $T : B \to L^2(Q_T)$ is (i) a compact operator, (ii) a continuous operator and (iii) $T[B] \subset B$.

(i) Using $\tilde{u}$ $(\partial_t\tilde{u})$ as a test function for (2.7) $\Rightarrow$

$$|\nabla\tilde{u}|^2_{L^2(Q_T)} + |\partial_t\tilde{u}|^2_{L^2(Q_T)} \le C\,[M_\varepsilon^2|u|^2_{L^2(Q_T)} + k^2|\varphi_\varepsilon(u)|^2_{L^2(Q_T)} + k^2|v|^2_{L^2(Q_T)} +$$
$$+ |f|^2_{L^2(Q_T)} + M_\varepsilon|g_1|^2_{H^1(\Omega)}] \le C(k,\varepsilon), \tag{2.8}$$

where we have noted the bounds on $u$ and $v$ above and the assumptions (D1). The testing with $\partial_t\tilde{u}$ is justified for sufficiently smooth solutions of the linear equation (2.7). The desired estimate also holds true in general, as the smooth solutions are dense in the space of weak solutions (see e.g. III §1 in Ladyzhenskaya (1985)). Therefore $T[B] \subset W_2^{1,1}(Q_T)$ and hence $T : B \to L^2(Q_T)$ is a compact operator.

(ii) Let $\{u_n\} \in B$ be such that $u_n \to u$ in $L^2(Q_T)$ as $n \to \infty$. We obtain, in a similar way to (2.8), with $\tilde{u}_n \equiv Tu_n$ that

$$|\tilde{u}-\tilde{u}_n|^2_{L^2(Q_T)} \le C(k,\varepsilon)\ [\,|u-u_n|^2_{L^2(Q_T)} + |\varphi_\varepsilon(u)-\varphi_\varepsilon(u_n)|^2_{L^2(Q_T)} + |v-v_n|^2_{L^2(Q_T)}]. \tag{2.9}$$

From (2.1c), (2.6) and (2.9) it follows that $\varphi_\varepsilon(u_n) \to \varphi_\varepsilon(u)$, $v_n \to v$ and hence $\tilde{u} \to \tilde{u}_n$ in $L^2(Q_T)$ as $n \to \infty$. Therefore $T : B \to L^2(Q_T)$ is a continuous operator.

(iii) We have that $v$ given by (2.6) is such that

$$\int_{Q_T} (\partial_t+k)(\underline{v}-v)\ (\underline{v}-v)_+\,dxdt \le \int_{Q_T} k[(\varphi_\varepsilon(\underline{u})-\varphi_\varepsilon(u)]\ (\underline{v}-v)_+\,dxdt \le 0,$$

since $u \geq \underline{u}$. Hence it follows that $v \geq \underline{v}$ in $Q_T$. In an analogous way we have that $v \leq \bar{v}$ in $Q_T$. From (2.7) it then follows that $\tilde{u}-\underline{u}$ satisfies weakly

$$\partial_t(\tilde{u}-\underline{u}) - \Delta(\tilde{u}-\underline{u}) + M_\varepsilon(\tilde{u}-\underline{u}) \geq k(v-\underline{v}) + [M_\varepsilon(u-\underline{u})-k(\varphi_\varepsilon(u)-\varphi_\varepsilon(\underline{u}))] \geq 0 \quad \text{in } Q_T$$

$$\tilde{u}-\underline{u} \geq 0 \quad \text{on} \quad \partial\Omega \times (0,T] \quad \text{and} \quad (\tilde{u}-\underline{u})(\cdot,0) \geq 0 \quad \text{in } \Omega,$$

since $v \geq \underline{v}$, $u \geq \underline{u}$, $M_\varepsilon$ is the Lipschitz constant of $k\varphi_\varepsilon(\cdot)$ and $\varphi_\varepsilon(\cdot)$ is monotonically increasing. From the weak maximum principle it follows that $\tilde{u} \geq \underline{u}$ in $Q_T$. In an analogous way it follows that $\tilde{u} \leq \bar{u}$ in $Q_T$. Therefore we have that $T[B] \subset B$.

As $T$ satisfies the above properties it follows from the Schauder fixed point theorem that $T$ has a fixed point $u_\varepsilon$, i.e. $u_\varepsilon = Tu_\varepsilon$. Moreover, it follows that $\{u_\varepsilon, v_\varepsilon\}$, where $v_\varepsilon$ is defined by (2.6) with $u \equiv u_\varepsilon$, is a weak solution of $(P_\varepsilon)$ satisfying (2.5a), where we have noted from the above that $u_\varepsilon \in W_2^{1,1}(Q_T)$ and $\varphi_\varepsilon(u)$, $v$, $\partial_t v \in L^\infty(Q_T)$. In addition for the fixed point $u_\varepsilon$ the term $M_\varepsilon u_\varepsilon$ cancels on both sides of (2.7a) and therefore the bound (2.8) holds for $u_\varepsilon$ with a constant $C(k)$ independent of $\varepsilon$. Hence the desired result (2.5b). To prove uniqueness we can argue as we will do for the non-regularized problem in the proof of Theorem 2.2 leading to (2.17) and then exploit the monotonicity of $\varphi_\varepsilon$. □

For $k \in \mathbb{R}^+$ and for sufficiently smooth $w$ we set

$$\|w\|^2_{E_1(k,t)} \equiv |w|^2_{L^2(Q_t)} + \tfrac{1}{2}\, k^{-1} |w(\cdot,t)|^2_{L^2(\Omega)}$$

and

$$\|w\|^2_{E_2(k,t)} \equiv \|w\|^2_{E_1(k,t)} + \tfrac{1}{2}\Big|\nabla\int_0^t w(\cdot,s)\,ds\Big|^2_{L^2(\Omega)} + k^{-1}|\nabla w|^2_{L^2(Q_t)}.$$

**Lemma 2.1**

Let the Assumptions (D1) hold and for $0 < \varepsilon_1 \le \varepsilon_2 \le \varepsilon_0$ let $\{u_{\varepsilon_i}, v_{\varepsilon_i}\}$ be a weak solution to $(P_{\varepsilon_i})$, $i = 1, 2$. Then for all $t \in (0,T]$ we have that

$$\|u_{\varepsilon_1}-u_{\varepsilon_2}\|^2_{E_2(k,t)} + \varepsilon_2|\varphi_{\varepsilon_1}(u_{\varepsilon_1})-\varphi_{\varepsilon_2}(u_{\varepsilon_2})|^2_{L^2(Q_t)} + \varepsilon_2\|v_{\varepsilon_1}-v_{\varepsilon_2}\|^2_{E_1(k,t)}$$
$$\le C\varepsilon_2^{-1}|\zeta-u_{\varepsilon_1}|^2_{L^2(Q_t)} \le C\varepsilon_2^{(1+p)/(1-p)}, \tag{2.10}$$

where $\zeta \equiv \varphi^{-1}_{\varepsilon_2}(\varphi_{\varepsilon_1}(u_{\varepsilon_1}))$ if $\varphi_{\varepsilon_1}(u_{\varepsilon_1}) \in (0, \varphi(\varepsilon_2^{1/(1-p)}))$ and $\zeta \equiv u_{\varepsilon_1}$ otherwise.

Proof: Let $e^u \equiv u_{\varepsilon_1}-u_{\varepsilon_2}$, $e^v \equiv v_{\varepsilon_1}-v_{\varepsilon_2}$ and $t \in (0,T]$. Subtracting the first equation in $(P_{\varepsilon_2})$ from that in $(P_{\varepsilon_1})$, using the test function $\eta(\cdot,s) \equiv \int_s^t e^u(\cdot,\sigma)d\sigma$ for $s \in [0,t]$, $\eta(\cdot,s) \equiv 0$ for $s \in (t,T]$ and performing integration by parts yields that

$$|e^u|^2_{L^2(Q_t)} + \tfrac{1}{2}|\nabla\int_0^t e^u(\cdot,s)ds|^2_{L^2(\Omega)} = -\int_0^t (e^v(\cdot,s),e^u(\cdot,s))ds. \tag{2.11}$$

Using the test function $\eta(\cdot,s) \equiv e^u(\cdot,s)$ for $s \in [0,t]$, $\eta(\cdot,s) \equiv 0$ for $s \in (t,T]$ yields that

$$\tfrac{1}{2}|e^u(\cdot,t)|^2_{L^2(\Omega)} + |\nabla e^u|^2_{L^2(Q_t)} = -\int_0^t (\partial_s e^v(\cdot,s),e^u(\cdot,s))ds. \tag{2.12}$$

Therefore combining (2.11) and (2.12) we have that

$$\|e^u\|^2_{E_2(k,t)} = -\int_0^t (k^{-1}\partial_s e^v(\cdot,s)+e^v(\cdot,s),e^u(\cdot,s))ds$$
$$= -\int_0^t (\varphi_{\varepsilon_1}(u_{\varepsilon_1}(\cdot,s))-\varphi_{\varepsilon_2}(u_{\varepsilon_2}(\cdot,s)),e^u(\cdot,s))ds. \tag{2.13}$$

Noting that $\varphi_{\varepsilon_1}(u_{\varepsilon_1}) \equiv \varphi_{\varepsilon_2}(\zeta)$, it follows from (2.13) and (2.3) that

$$\|e^u\|^2_{E_2(k,t)} + [M(m)]^{-1}\varepsilon_2|\varphi_{\varepsilon_1}(u_{\varepsilon_1})-\varphi_{\varepsilon_2}(u_{\varepsilon_2})|^2_{L^2(Q_t)}$$
$$\le \int_0^t (\varphi_{\varepsilon_1}(u_{\varepsilon_1}(\cdot,s))-\varphi_{\varepsilon_2}(u_{\varepsilon_2}(\cdot,s)),(\zeta-u_{\varepsilon_1})(\cdot,s))ds$$
$$\le \tfrac{1}{2}[M(m)]^{-1}\varepsilon_2|\varphi_{\varepsilon_1}(u_{\varepsilon_1})-\varphi_{\varepsilon_2}(u_{\varepsilon_2})|^2_{L^2(Q_t)} + \tfrac{1}{2}M(m)\varepsilon_2^{-1}|\zeta-u_{\varepsilon_1}|^2_{L^2(Q_t)}$$
$$\le M(m)\varepsilon_2^{-1}|\zeta-u_{\varepsilon_1}|^2_{L^2(Q_t)} \le C\varepsilon_2^{-1}\varepsilon_2^{2/(1-p)}; \tag{2.14}$$

where $[\inf \underline{u}, \sup \bar{u}] \subseteq [-m,m]$, see (2.1c) and Theorem 2.1.

Finally subtracting the second equation in $(P_{\varepsilon_2})$ from that in $(P_{\varepsilon_1})$, multiplying by $e^v$ and integrating over $Q_t$ yields

$$\|v_{\varepsilon_1}-v_{\varepsilon_2}\|^2_{E_1(k,t)} = \int_0^t(\varphi_{\varepsilon_1}(u_{\varepsilon_1}(\cdot,s))-\varphi_{\varepsilon_2}(u_{\varepsilon_2}(\cdot,s)),e^v(\cdot,s))ds$$

$$\le C|\varphi_{\varepsilon_1}(u_{\varepsilon_1})-\varphi_{\varepsilon_2}(u_{\varepsilon_2})|^2_{L^2(Q_t)}. \tag{2.15}$$

Combining (2.14) and (2.15) yields the desired result (2.10). □

**Theorem 2.2**

Let the Assumptions (D1) hold. Then there exists a unique weak solution $\{u,v\}$ to (P) and for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in (0,T]$

$$\|u-u_\varepsilon\|^2_{E_2(k,t)} + k^{-2}\varepsilon|\nabla(u-u_\varepsilon)(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\varphi(u)-\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} +$$

$$+ \varepsilon\|v-v_\varepsilon\|^2_{E_1(k,t)} \le C\ A_\varepsilon(t)\ \varepsilon^{(1+p)/(1-p)}. \tag{2.16}$$

In addition the bounds (2.5a&b) hold true for $\{u,v\}$ and in particular if $g_1$, $g_2$ and $f \ge 0$ then $u,\ v \ge 0$ in $Q_T$.

Proof: We first establish existence of a solution to (P). Let $\varepsilon_n \to 0$ as $n \to \infty$ and let $\{u_{\varepsilon_n},v_{\varepsilon_n}\}$ be the unique weak solution to $(P_{\varepsilon_n})$. It follows from (2.10) that $\{u_{\varepsilon_n},v_{\varepsilon_n}\}$ is Cauchy in $L^2(0,T;H^1(\Omega)) \times L^\infty(0,T;L^2(\Omega))$ and therefore $\{u_{\varepsilon_n},v_{\varepsilon_n}\} \to \{u,v\}$ in $L^2(0,T;H^1(\Omega)) \times L^\infty(0,T;L^2(\Omega))$ as $n \to \infty$. The inclusions (2.5a) also hold true for $\{u,v\}$. In addition from (2.5b) we have that there exists a subsequence of $\{\partial_t u_{\varepsilon_n},\partial_t v_{\varepsilon_n}\}$ converging weakly to $\{\partial_t u,\partial_t v\}$ in $L^2(Q_T) \times L^2(Q_T)$. Finally we have from (1.9), (2.1) and (2.3b) that

$$|\varphi(u)-\varphi_{\varepsilon_n}(u_{\varepsilon_n})|_{L^2(Q_t)} \le |\varphi(u)-\varphi(u_{\varepsilon_n})|_{L^2(Q_t)} + |\varphi(u_{\varepsilon_n})-\varphi_{\varepsilon_n}(u_{\varepsilon_n})|_{L^2(Q_t)}$$

$$\le C|u-u_{\varepsilon_n}|^p_{L^2(Q_t)} + C\varepsilon_n^{p/(1-p)}.$$

Hence $\varphi_{\varepsilon_n}(u_{\varepsilon_n}) \to \varphi(u)$ as $n \to \infty$. Therefore $\{u,v\}$ is a weak solution of (P). The bounds (2.5b) also hold true for $\{u,v\}$.

We now prove uniqueness. Assume there exist two weak solutions $\{u_i, v_i\}$, $i = 1, 2$ to (P). Setting $e^u \equiv u_1-u_2$ and $e^v \equiv v_1-v_2$, the analogue of (2.13) with $\varepsilon_1 = \varepsilon_2 = 0$, yields

$$\|e^u\|^2_{E_2(k,t)} + \int_0^t (\varphi(u_1(\cdot,s))-\varphi(u_2(\cdot,s)), e^u(\cdot,s))ds = 0. \tag{2.17}$$

From (1.9a) and (2.17) it follows that $u_1 = u_2$ and hence $v_1 = v_2$.

Finally setting $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon$ in the proof of (2.10), noting (1.8) and that, with a justification analogous to the proof of (2.8)

$$\begin{aligned}
&\int_0^t |\partial_s(u-u_\varepsilon)(\cdot,s)|^2_{L^2(\Omega)}ds + \tfrac{1}{2}|\nabla(u-u_\varepsilon)(\cdot,t)|^2_{L^2(\Omega)} \\
&\qquad = -\int_0^t (\partial_s(v-v_\varepsilon)(\cdot,s), \partial_s(u-u_\varepsilon)(\cdot,s)ds \\
&\qquad \le Ck^2[\,|\varphi(u)-\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)} + |v-v_\varepsilon|^2_{L^2(Q_t)}]
\end{aligned}$$

yields the desired result (2.16). □

Because of the bounds in (2.5a) we now can fix M in (2.1c) when dealing with u or $u_\varepsilon$. We end this section by proving some useful bounds on the unique weak solution $\{u_\varepsilon, v_\varepsilon\}$ of $(P_\varepsilon)$, $\varepsilon \in (0,\varepsilon_0]$.

**Lemma 2.2**

Under Assumptions (D1) we have for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in (0,T]$ that

$$\begin{aligned}
&\varepsilon|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_T)} + (\Phi_\varepsilon(u_\varepsilon(\cdot,t)),1) + k|\varphi_\varepsilon(u_\varepsilon)-v_\varepsilon|^2_{L^2(Q_T)} + \\
&\qquad + |v_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t v_\varepsilon|^2_{L^2(Q_T)} \le C.
\end{aligned} \tag{2.18}$$

Proof: From $(P_\varepsilon)$ we have that

$$\begin{aligned}
&\int_0^t [(\nabla u_\varepsilon(\cdot,s), \nabla\varphi_\varepsilon(u_\varepsilon(\cdot,s))) + (\partial_t u_\varepsilon(\cdot,s), \varphi_\varepsilon(u_\varepsilon(\cdot,s))) + (\partial_t v_\varepsilon(\cdot,s), v_\varepsilon(\cdot,s)) + \\
&\qquad + k|\varphi_\varepsilon(u_\varepsilon(\cdot,s))-v_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}]ds = \int_0^t (f(\cdot,s), \varphi_\varepsilon(u_\varepsilon(\cdot,s)))ds.
\end{aligned} \tag{2.19}$$

From (1.9) and (2.1) it follows that for all $w \in H^1_0(\Omega)$ with $|w(x)| \le m$ for a.e. $x \in \Omega$ that

$$[M(m)]^{-1}\varepsilon|\nabla\varphi_\varepsilon(w)|^2_{L^2(\Omega)} \le (\nabla w, \nabla\varphi_\varepsilon(w)). \tag{2.20}$$

Noting (2.20) and (2.5a) yields that

$$\varepsilon|\nabla\varphi_\varepsilon(u_\varepsilon)]|^2_{L^2(Q_t)} + (\Phi_\varepsilon(u_\varepsilon(\cdot,t)),1) + |v_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + k|\varphi_\varepsilon(u_\varepsilon)-v_\varepsilon|^2_{L^2(Q_t)}$$
$$\le C\left[\int_0^t(f(\cdot,s),\varphi_\varepsilon(u_\varepsilon(\cdot,s)))ds + (\Phi_\varepsilon(u_\varepsilon(\cdot,0)),1) + |v_\varepsilon(\cdot,0)|^2_{L^2(\Omega)}\right]$$
$$\le C_1 + C_2|\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_t)}$$
$$\le C_1 + C_2|\varphi_\varepsilon(u_\varepsilon)-v_\varepsilon|^2_{L^2(Q_t)} + C_2|v_\varepsilon|^2_{L^2(Q_t)} \le C_3, \tag{2.21}$$

where we can choose $C_2$ sufficiently small. Hence the desired result (2.18) then follows from (2.21) and the second equation in $(P_\varepsilon)$. □

For the final result we need further assumptions on the data.

**Assumptions (D2):** In addition to the Assumptions (D1) we assume that $f \in H^1(0,T;L^2(\Omega))$, $g_1 \in H^2(\Omega)$ and to simplify the analysis that $k \ge k_0$.

By the last assumption we do not neglect any important features, as for $k \to 0$ we expect convergence to the case of no reaction, i.e. to the linear diffusion equation.

**Lemma 2.3**

Under Assumptions (D2) we have for all $\varepsilon \in (0,\varepsilon_0]$ and $t \in (0,T]$ that

$$|\nabla u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t u_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t v_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t[\varphi_\varepsilon(u_\varepsilon)]|^2_{L^2(Q_T)} +$$
$$+ k^{-1}\left[|\partial_t u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t v_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t u_\varepsilon)|^2_{L^2(Q_T)}\right]$$
$$\le C[1+k|\varphi_\varepsilon(g_1)-g_2|^2_{L^2(\Omega)}] \le Ck. \tag{2.22}$$

Proof: Differentiating the first equation in $(P_\varepsilon)$ with respect to $t$ yields that

$$k^{-1}\partial_{tt}u_\varepsilon + (\varphi'_\varepsilon(u_\varepsilon)\partial_t u_\varepsilon - \partial_t v_\varepsilon) - k^{-1}\Delta(\partial_t u_\varepsilon) = k^{-1}\partial_t f \quad \text{in } Q_T \tag{2.23a}$$

and hence that

$$k^{-1}\partial_{tt}u_\varepsilon + (1+\varphi'_\varepsilon(u_\varepsilon))\partial_t u_\varepsilon - \Delta(k^{-1}\partial_t u_\varepsilon + u_\varepsilon) = k^{-1}\partial_t f + f \quad \text{in } Q_T. \tag{2.23b}$$

This formal procedure can be justified as follows: Consider an auxiliary linear initial-boundary value problem for $\partial_t u_\varepsilon$, i.e. an equation analogous to (2.23a) and initial condition $\Delta g_1+f(\cdot,0)-k[\varphi_\varepsilon(g_1)-g_2]$. Due to (D2) and (2.5a) a weak solution $w_\varepsilon$ exists. We have that $u_\varepsilon(\cdot,t) = g_1(\cdot)+\int_0^t w_\varepsilon(\cdot,s)ds$ as both satisfy the same linear initial-boundary value problem. Thus $w_\varepsilon=\partial_t u_\varepsilon$. Multiplying (2.23b) by $\partial_s u_\varepsilon(\cdot,s)$, integrating over $Q_t$, where s is the integration variable in time, and performing integration by parts yields that

$$\begin{aligned} k^{-1}\int_0^t|\nabla\partial_s u_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds + \int_0^t([1+\varphi_\varepsilon'(u_\varepsilon(\cdot,s))]\partial_s u_\varepsilon(\cdot,s),\partial_s u_\varepsilon(\cdot,s))ds + \\ + ½\left[k^{-1}|\partial_t u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)}\right] \\ = \int_0^t(k^{-1}\partial_s f(\cdot,s)+f(\cdot,s),\partial_s u_\varepsilon(\cdot,s))ds + \\ + ½\left[k^{-1}|\partial_t u_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot,0)|^2_{L^2(\Omega)}\right] \end{aligned}$$

and hence

$$\begin{aligned} k^{-1}\int_0^t|\nabla\partial_s u_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds + \int_0^t([1+\varphi_\varepsilon'(u_\varepsilon(\cdot,s))]\partial_s u_\varepsilon(\cdot,s),\partial_s u_\varepsilon(\cdot,s))ds + \\ + k^{-1}|\partial_t u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} \\ \le \int_0^t|k^{-1}\partial_s f(\cdot,s)+f(\cdot,s)|^2_{L^2(\Omega)}ds + k^{-1}|\partial_t u_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} + |\nabla u_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} \\ \le \int_0^t|k^{-1}\partial_s f(\cdot,s)+f(\cdot,s)|^2_{L^2(\Omega)}ds + |\nabla g_1|^2_{L^2(\Omega)} + 2k^{-1}|\Delta g_1(\cdot)+f(\cdot,0)|^2_{L^2(\Omega)} + \\ + 2k|\varphi_\varepsilon(g_1)-g_2|^2_{L^2(\Omega)} . \end{aligned} \tag{2.24}$$

Noting (2.1c) we have that

$$M^{-1}\varepsilon\int_0^t|\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot,s))]|^2_{L^2(\Omega)}ds \le \int_0^t(\varphi_\varepsilon'(u_\varepsilon(\cdot,s))\partial_s u_\varepsilon(\cdot,s),\partial_s u_\varepsilon(\cdot,s))ds. \tag{2.25}$$

In addition we have that

$$\begin{aligned} ½k^{-1}|\partial_t v_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \int_0^t|\partial_s v_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds \\ = ½k^{-1}|\partial_t v_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} + \int_0^t(\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot,s))],\partial_s v_\varepsilon(\cdot,s))ds \end{aligned}$$

and hence that

$$\begin{aligned} k^{-1}|\partial_t v_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \int_0^t|\partial_s v_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds \\ \le k^{-1}|\partial_t v_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} + \int_0^t|\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot,s))]|^2_{L^2(\Omega)}ds \\ \le k|\varphi_\varepsilon(u_\varepsilon(\cdot,0))-v_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} + \int_0^t|\partial_s[\varphi_\varepsilon(u_\varepsilon(\cdot,s))]|^2_{L^2(\Omega)}ds. \end{aligned} \tag{2.26}$$

Combining (2.24) - (2.26) yields the desired result (2.22). □

## 3. A CONTINUOUS IN TIME FINITE ELEMENT APPROXIMATION

We now consider the continuous piecewise linear finite element approximation to $(P_\varepsilon)$. We make the following assumptions on the data and triangulation:

Assumptions **(D3)**: In addition to the assumptions (D2) we assume that $g_2 \in H^1(\Omega)$ and the constant $M$ in (2.1c) can be chosen uniformly for all $s \in \mathbb{R}$. Let $\Omega^h$ be a polyhedral approximation to $\Omega$ defined by $\bar{\Omega}^h \equiv \cup_{\kappa \in T^h} \bar{\kappa}$, where $T^h$ is a quasi-uniform partition consisting of simplices $\kappa$ with maximum diameter not exceeding $h$ and with $\text{dist}(\partial\Omega, \partial\Omega^h) \leq Ch^2$. For ease of exposition we assume that $\Omega^h \subseteq \Omega$.

We introduce

$$S^h \equiv \{ \chi \in C(\bar{\Omega}^h) : \chi|_\kappa \text{ is linear for all } \kappa \in T^h \}$$

and

$$S_0^h \equiv \{ \chi \in C(\bar{\Omega}) : \chi|_{\bar{\Omega}^h} \in S^h \text{ and } \chi|_{\Omega\setminus\bar{\Omega}^h} = 0 \}.$$

Let $\pi_h : C^0(\bar{\Omega}) \to S^h$ denote the interpolation operator such that for any $w \in C^0(\bar{\Omega})$, $\pi_h w \in S^h$ satisfies

$$(\pi_h w)(x_i) = w(x_i) \quad \text{for all nodes } x_i \text{ of the partition } T^h.$$

Let $P_h^0 : L^2(\Omega) \to S^h$ denote the $L^2$ projection such that for any $w \in L^2(\Omega)$, $P_h^0 w \in S^h$ satisfies

$$(w-P_h^0 w, \chi) = 0 \quad \forall\ \chi \in S^h.$$

Let $P_h^1 : H_0^1(\Omega) \to S_0^h$ denote the $H^1$ semi-norm projection such that for any $w \in H_0^1(\Omega)$, $P_h^1 w \in S_0^h$ satisfies

$$(\nabla(w-P_h^1 w), \nabla\chi) = 0 \quad \forall\ \chi \in S_0^h.$$

We recall the standard approximation results, for all $\kappa \in T^h$

$$|w-\pi_h w|_{W^{m,q}(\kappa)} \leq Ch^{2-m}|w|_{W^{2,q}(\kappa)} \quad \text{for } m = 0 \text{ and } 1 \text{ and}$$
$$\forall\ q \in [1,\infty] \text{ if } d \leq 2 \text{ and } \forall\ q \in (3/2,\infty] \text{ if } d = 3 \tag{3.1a}$$

$$|w-P_h^0 w|_{L^2(\Omega)} \leq Ch^m|w|_{H^m(\Omega)} \quad \text{for } m = 0,\ 1 \text{ and } 2 \tag{3.1b}$$

and

$$|w-P_h^1w|_{L^2(\Omega)} + h|w-P_h^1w|_{H^1(\Omega)} \le Ch^m|w|_{H^m(\Omega)} \quad \text{for } m = 1 \text{ and } 2; \tag{3.1c}$$

where in (3.1a) we note the imbedding $W^{2,1}(\kappa) \subset C^0(\bar{\kappa})$ in the case $d = 2$, see for example p300 in Kufner et al. (1977).

As the partition is quasi-uniform we have the inverse inequalities

$$|\chi|_{L^\infty(\Omega)} \le Ch^{-d/2}|\chi|_{L^2(\Omega)} \quad \text{and} \quad |\chi|_{H^1(\Omega)} \le Ch^{-1}|\chi|_{L^2(\Omega)} \ \forall\ \chi \in S^h, \tag{3.2a}$$

and for $d \le 2$ the discrete Sobolev imbedding result

$$|\chi|_{L^\infty(\Omega)} \le C[\ln(1/h)]^r|\chi|_{H^1(\Omega)} \le C[\ln(1/h)]^r|\nabla\chi|_{L^2(\Omega)} \ \forall\ \chi \in S_0^h, \tag{3.2b}$$

where $r = 0$ if $d = 1$ and $r = ½$ if $d = 2$; see for example p67 in Thomée (1984). It follows from (3.1b&c) and (3.2a) that for any $w \in H_0^1(\Omega)$

$$|P_h^0w|_{H^1(\Omega)} \le Ch^{-1}|(P_h^0-P_h^1)w|_{L^2(\Omega)} + |P_h^1w|_{H^1(\Omega)} \le C|w|_{H^1(\Omega)}. \tag{3.3}$$

Another result that will be useful later is that

$$|(I-\pi_h)\varphi_\varepsilon(\chi)|_{L^2(\Omega)} \le h|\nabla\pi_h[\varphi_\varepsilon(\chi)]|_{L^2(\Omega)} \quad \forall\ \chi \in S_0^h\ . \tag{3.4}$$

This result is proved in Elliott (1987), p68, with h replaced by Ch on the righthand side of (3.4). However, it is easy to see from this proof that C can be taken as 1.

The approximation to $(P_\varepsilon)$ we wish to consider first is :

$(P_\varepsilon^h)$ Find $u_\varepsilon^h \in H^1(0,T;S_0^h)$ and $v_\varepsilon^h \in H^1(0,T;S^h)$ such that

$$(\partial_t u_\varepsilon^h + \partial_t v_\varepsilon^h,\chi) + (\nabla u_\varepsilon^h,\nabla\chi) = (f,\chi) \quad \forall\ \chi \in S_0^h$$

$$(\partial_t v_\varepsilon^h,\chi) = k(\varphi_\varepsilon(u_\varepsilon^h)-v_\varepsilon^h,\chi) \quad \forall\ \chi \in S^h$$

$$u_\varepsilon^h(\cdot,0) = P_h^1g_1(\cdot) \qquad v_\varepsilon^h(\cdot,0) = P_h^0g_2(\cdot).$$

**Theorem 3.1**

Let the Assumptions (D3) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ there exists a unique solution $\{u_\varepsilon^h,v_\varepsilon^h\}$ to $(P_\varepsilon^h)$.

Proof: Existence and uniqueness of a solution follows from standard ordinary differential equation theory and the bounds

$$\|u_\varepsilon^h\|^2_{E_2(k,t)} \le C[\,|g_1|^2_{L^2(\Omega)} + |g_2|^2_{L^2(\Omega)} + |f|^2_{L^2(Q_T)}]$$

and $$\|v_\varepsilon^h\|^2_{E_1(k,t)} \le C|\varphi_\varepsilon(u_\varepsilon^h)|^2_{L^2(Q_T)} \qquad \text{for } t \in (0,T].$$

These are shown along the lines of the proof of Lemma 2.1, taking into account that $\varphi_\varepsilon(u^h_\varepsilon)u^h_\varepsilon \ge 0$ due to (1.9a) and (2.1a&b). The first estimate and (3.2a) implies that $\|u^h_\varepsilon\|_{L^\infty(Q_T)} \le C(k,h)$, which in turn yields that $\|v^h_\varepsilon\|_{L^\infty(Q_T)} \le C(k,h)$. Therefore the unique local in time solution $\{u^h_\varepsilon, v^h_\varepsilon\}$, assured by the Picard-Lindelof theorem, has to exist globally in time. □

Firstly, we have the following analogue of Lemma 2.3.

**Lemma 3.1**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$
\begin{aligned}
&|\nabla u^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t u^h_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t v^h_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t[\varphi_\varepsilon(u^h_\varepsilon)]|^2_{L^2(Q_T)} + \\
&\quad + k^{-1}\Big[|\partial_t u^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t v^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t u^h_\varepsilon)|^2_{L^2(Q_T)}\Big] \\
&\quad \le C[1+k|\varphi_\varepsilon(g_1)-g_2|^2_{L^2(\Omega)}+k\varepsilon^{-2}h^4] \le Ck(1+\varepsilon^{-2}h^4). \qquad (3.5)
\end{aligned}
$$

Proof: A direct analogue of (2.23) and (2.24) yields that

$$
\begin{aligned}
&k^{-1}\int_0^t|\nabla\partial_s u^h_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds + \int_0^t([1+\varphi'_\varepsilon(u^h_\varepsilon(\cdot,s))]\partial_s u^h_\varepsilon(\cdot,s),\partial_s u^h_\varepsilon(\cdot,s))ds + \\
&\quad + k^{-1}|\partial_t u^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\nabla u^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} \\
&\le \int_0^t|k^{-1}\partial_s f(\cdot,s)+f(\cdot,s)|^2_{L^2(\Omega)}ds + |\nabla g_1|^2_{L^2(\Omega)} + 2k^{-1}|\Delta g_1(\cdot)+f(\cdot,0)|^2_{L^2(\Omega)} + \\
&\quad + 2k|\varphi_\varepsilon(P^1_h g_1)-g_2|^2_{L^2(\Omega)}. \qquad (3.6)
\end{aligned}
$$

In addition the analogue of (2.26) yields that

$$
\begin{aligned}
&k^{-1}|\partial_t v^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \int_0^t|\partial_s v^h_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds \\
&\quad \le k|\varphi_\varepsilon(P^1_h g_1)-g_2|^2_{L^2(\Omega)} + \int_0^t|\partial_s[\varphi_\varepsilon(u^h_\varepsilon(\cdot,s))]|^2_{L^2(\Omega)}ds. \qquad (3.7)
\end{aligned}
$$

Combining (3.6) and (3.7) with the analogue of (2.25) yields that the lefthand side of (3.5) is bounded by $C[1+k|\varphi_\varepsilon(P^1_h g_1)-g_2|^2_{L^2(\Omega)}]$. Finally we note from (2.1c) and (3.1c) that

$$
|\varphi_\varepsilon(P^1_h g_1)-g_2|_{L^2(\Omega)} \le |\varphi_\varepsilon(g_1)-g_2|_{L^2(\Omega)} + C\varepsilon^{-1}h^2
$$

and hence the desired result (3.5). $\square$

In order to analyse the approximation $(\mathrm{P}_\varepsilon^h)$ it is convenient to introduce the associated linear problem :

$(\mathrm{P}_\varepsilon^{h,*})$ Find $u_\varepsilon^{h,*} \in H^1(0,T;S_0^h)$ and $v_\varepsilon^{h,*} \in H^1(0,T;S^h)$ such that

$$(\partial_t u_\varepsilon^{h,*} + \partial_t v_\varepsilon^{h,*},\chi) + (\nabla u_\varepsilon^{h,*},\nabla\chi) = (f,\chi) \quad \forall\ \chi \in S_0^h$$

$$(\partial_t v_\varepsilon^{h,*},\chi) = k(\varphi_\varepsilon(u_\varepsilon)-v_\varepsilon^{h,*},\chi) \quad \forall\ \chi \in S^h$$

$$u_\varepsilon^{h,*}(\cdot,0) = P_h^1 g_1(\cdot) \qquad v_\varepsilon^{h,*}(\cdot,0) = P_h^0 g_2(\cdot).$$

The existence and uniqueness of $\{u_\varepsilon^{h,*},v_\varepsilon^{h,*}\}$ solving $(\mathrm{P}_\varepsilon^{h,*})$ for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ is easily established and we have the following result.

**Lemma 3.2**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$\|u_\varepsilon^{h,*}-u_\varepsilon^h\|^2_{E_2(k,t)} + k^{-2}\varepsilon|\nabla(u_\varepsilon^{h,*}-u_\varepsilon^h)(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\varphi_\varepsilon(u_\varepsilon)-\varphi_\varepsilon(u_\varepsilon^h)|^2_{L^2(Q_t)} +$$

$$+ \varepsilon\|v_\varepsilon^{h,*}-v_\varepsilon^h\|^2_{E_1(k,t)} \le C\varepsilon^{-1}|u_\varepsilon-u_\varepsilon^{h,*}|^2_{L^2(Q_t)}. \tag{3.8}$$

Proof: The proof is very similar to Lemma 2.1. Let $e_\varepsilon^{u,h} \equiv u_\varepsilon^{h,*}-u_\varepsilon^h$ and $e_\varepsilon^{v,h} \equiv v_\varepsilon^{h,*}-v_\varepsilon^h$. Subtracting the first equation in $(\mathrm{P}_\varepsilon^h)$ from that in $(\mathrm{P}_\varepsilon^{h,*})$, choosing $\chi \equiv \int_s^t e_\varepsilon^{u,h}(\cdot,\sigma)d\sigma$, integrating over $(0,t)$ in time, where $s$ is the integration variable in time, and performing integration by parts yields that

$$\int_0^t|e_\varepsilon^{u,h}(\cdot,s)|^2_{L^2(\Omega)}ds + \tfrac{1}{2}\Big|\nabla\int_0^t e_\varepsilon^{u,h}(\cdot,s)ds\Big|^2_{L^2(\Omega)} = -\int_0^t(e_\varepsilon^{v,h}(\cdot,s),e_\varepsilon^{u,h}(\cdot,s))ds. \tag{3.9}$$

Similarly choosing $\chi \equiv e_\varepsilon^{u,h}$ and $\chi \equiv \partial_s e_\varepsilon^{u,h}$ yields respectively that

$$\tfrac{1}{2}|e_\varepsilon^{u,h}(\cdot,t)|^2_{L^2(\Omega)} + \int_0^t|\nabla e_\varepsilon^{u,h}(\cdot,s)|^2_{L^2(\Omega)}ds = -\int_0^t(\partial_s e_\varepsilon^{v,h}(\cdot,s),e_\varepsilon^{u,h}(\cdot,s))ds \tag{3.10a}$$

and

$$\int_0^t |\partial_s e_\varepsilon^{u,h}(\cdot,s)|^2_{L^2(\Omega)} ds + \tfrac{1}{2}|\nabla e_\varepsilon^{u,h}(\cdot,t)|^2_{L^2(\Omega)} = -\int_0^t (\partial_s e_\varepsilon^{v,h}(\cdot,s), \partial_s e_\varepsilon^{u,h}(\cdot,s)) ds$$

$$= k\int_0^t (e_\varepsilon^{v,h}(\cdot,s) - [\varphi_\varepsilon(u_\varepsilon(\cdot,s)) - \varphi_\varepsilon(u_\varepsilon^h(\cdot,s))], \partial_s e_\varepsilon^{u,h}(\cdot,s)) ds. \tag{3.10b}$$

Therefore from (3.9), (3.10a) and (2.3a) it follows that

$$\begin{aligned} \|e_\varepsilon^{u,h}\|^2_{E_2(k,t)} &+ M^{-1}\varepsilon|\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(u_\varepsilon^h)|^2_{L^2(Q_t)} \\ &\le \|e_\varepsilon^{u,h}\|^2_{E_2(k,t)} + \int_0^t (\varphi_\varepsilon(u_\varepsilon(\cdot,s)) - \varphi_\varepsilon(u_\varepsilon^h(\cdot,s)), u_\varepsilon(\cdot,s) - u_\varepsilon^h(\cdot,s)) ds \\ &= \int_0^t (\varphi_\varepsilon(u_\varepsilon(\cdot,s)) - \varphi_\varepsilon(u_\varepsilon^h(\cdot,s)), u_\varepsilon(\cdot,s) - u_\varepsilon^{h,*}(\cdot,s)) ds \\ &\le M\varepsilon^{-1}|u_\varepsilon - u_\varepsilon^{h,*}|^2_{L^2(Q_t)}. \end{aligned} \tag{3.11}$$

Subtracting the second equation in $(P_\varepsilon^h)$ from that in $(P_\varepsilon^{h,*})$, choosing $\chi \equiv e_\varepsilon^{v,h}$ and integrating over $(0,t)$ yields

$$\begin{aligned} \|e_\varepsilon^{v,h}\|^2_{E_1(k,t)} &= \int_0^t (\varphi_\varepsilon(u_\varepsilon(\cdot,s)) - \varphi_\varepsilon(u_\varepsilon^h(\cdot,s)), e_\varepsilon^{v,h}) ds \\ &\le |\varphi_\varepsilon(u_\varepsilon) - \varphi_\varepsilon(u_\varepsilon^h)|^2_{L^2(Q_t)} . \end{aligned} \tag{3.12}$$

Combinining (3.11), (3.12) and (3.10b) yields the desired result (3.8). □

**Lemma 3.3**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$\begin{aligned} |u_\varepsilon - u_\varepsilon^{h,*}|^2_{L^2(Q_t)} &+ h^2\left|\nabla\int_0^t (u_\varepsilon - u_\varepsilon^{h,*})(\cdot,s) ds\right|^2_{L^2(\Omega)} \\ &\le Ch^4\left[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |g_1|^2_{H^2(\Omega)}\right] \le Ckh^4, \end{aligned} \tag{3.13a}$$

$$\begin{aligned} |(u_\varepsilon - u_\varepsilon^{h,*})(\cdot,t)|^2_{L^2(\Omega)} & \\ &\le Ch^2\left[|u_\varepsilon|^2_{H^1(0,t;H^1(\Omega))} + |\nabla u_\varepsilon(\cdot,t)|^2_{L^2(\Omega)}\right] \le Ck^2h^2, \end{aligned} \tag{3.13b}$$

$$\begin{aligned} |\nabla(u_\varepsilon - u_\varepsilon^{h,*})|^2_{L^2(Q_t)} & \\ &\le Ch^2\left[|u_\varepsilon|^2_{H^1(0,t;H^1(\Omega))} + |u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))}\right] \le Ck^2h^2 \end{aligned} \tag{3.13c}$$

$$\begin{aligned} |\nabla(u_\varepsilon - u_\varepsilon^{h,*})(\cdot,t)|^2_{L^2(\Omega)} & \\ &\le Ch^2\left[|u_\varepsilon|^2_{H^1(0,t;H^1(\Omega))} + |u_\varepsilon|^2_{L^\infty(0,t;H^2(\Omega))}\right] \le C\varepsilon^{-1}k^2h^2 \end{aligned} \tag{3.13d}$$

and

$$\|v_\varepsilon - v_\varepsilon^{h,*}\|^2_{E_1(k,t)} \le Ch^2[\,|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_T)} + k^{-1}|g_2|^2_{H^1(\Omega)}] \le C\varepsilon^{-1}h^2. \tag{3.13e}$$

Proof: The problem $(P_\varepsilon)$ can be restated as: Find $u_\varepsilon(x,t)$ such that

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = f + F_\varepsilon(t,u_\varepsilon) \quad \text{in } Q_T \tag{3.14a}$$

$$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \qquad u_\varepsilon(\cdot,0) = g_1(\cdot), \tag{3.14b}$$

where

$$F_\varepsilon(t,w(\cdot,t)) \equiv k\left[e^{-kt}g_2 - \varphi_\varepsilon(w(\cdot,t)) + k\int_0^t e^{-k(t-s)}\varphi_\varepsilon(w(\cdot,s))ds\right]. \tag{3.14c}$$

Similarly, $(P_\varepsilon^{h,*})$ can be restated as: Find $u_\varepsilon^{h,*} \in H^1(0,T;S_0^h)$ such that

$$(\partial_t u_\varepsilon^{h,*},\chi) + (\nabla u_\varepsilon^{h,*},\nabla\chi) = (f + F_\varepsilon(t,u_\varepsilon),\chi) \quad \forall\ \chi \in S_0^h$$

$$u_\varepsilon^{h,*}(\cdot,0) = P_h^1 g_1(\cdot).$$

Let $e_{u,\varepsilon}^{h,*} \equiv u_\varepsilon - u_\varepsilon^{h,*}$ and so we have that

$$(\partial_t e_{u,\varepsilon}^{h,*},\chi) + (\nabla e_{u,\varepsilon}^{h,*},\nabla\chi) = 0 \quad \forall\ \chi \in S_0^h \tag{3.15a}$$

$$e_{u,\varepsilon}^{h,*}(\cdot,0) = g_1(\cdot) - P_h^1 g_1(\cdot). \tag{3.15b}$$

With $e_{u,\varepsilon}^{h,*} \equiv (u_\varepsilon - P_h^1 u_\varepsilon) + (P_h^1 u_\varepsilon - u_\varepsilon^{h,*}) \equiv \rho + \vartheta$, it follows by choosing $\chi \equiv \int_s^t \vartheta(\cdot,\sigma)d\sigma$, integrating over $(0,t)$ in time, where $s$ is the integration variable in time, and performing integration by parts yields that

$$\int_0^t |\vartheta(\cdot,s)|^2_{L^2(\Omega)}ds + \tfrac{1}{2}\left|\nabla\int_0^t\vartheta(\cdot,s)ds\right|^2_{L^2(\Omega)}$$
$$= -\int_0^t (\rho(\cdot,s)),\vartheta(\cdot,s))ds + (g_1(\cdot) - P_h^1 g_1(\cdot),\int_0^t\vartheta(\cdot,s)ds). \tag{3.16}$$

Under the stated assumptions on $\Omega$ we have from $(P_\varepsilon)$ that $u_\varepsilon \in L^r(0,T;H^2(\Omega))$ for all $r \in [1,\infty]$ and

$$|u_\varepsilon|_{L^r(0,T;H^2(\Omega))}$$
$$\leq C\left[|\partial_t u_\varepsilon|_{L^r(0,T;L^2(\Omega))} + |\partial_t v_\varepsilon|_{L^r(0,T;L^2(\Omega))} + |f|_{L^r(0,T;L^2(\Omega))}\right].$$

Hence from (2.18) and (2.22) we have that

$$|u_\varepsilon|^2_{L^2(0,T;H^2(\Omega))} \leq Ck \quad \text{and} \quad |u_\varepsilon|^2_{L^\infty(0,T;H^2(\Omega))} \leq C\varepsilon^{-1}k^2. \tag{3.17}$$

From (3.16), (3.1c) and (3.17) we have that

$$|e_{u,\varepsilon}^{h,*}|^2_{L^2(Q_t)} \leq C\left[|\vartheta|^2_{L^2(Q_t)} + |\rho|^2_{L^2(Q_t)}\right] \leq C|\rho|^2_{L^2(Q_t)} + Ch^4|g_1|^2_{H^2(\Omega)}$$
$$\leq Ch^4[\,|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |g_1|^2_{H^2(\Omega)}] \leq Ckh^4$$

and

$$|\nabla\int_0^t e_{u,\varepsilon}^{h,*}(\cdot,s)ds|^2_{L^2(\Omega)} \le C\left[|\nabla\int_0^t \rho(\cdot,s)ds|^2_{L^2(\Omega)} + |\rho|^2_{L^2(Q_t)}\right] + Ch^2|g_1|^2_{H^1(\Omega)}$$
$$\le Ch^2[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |g_1|^2_{H^1(\Omega)}] \le Ckh^2.$$

Hence we obtain (3.13a).

In addition choosing $\chi \equiv \vartheta$ and $\chi \equiv \partial_s\vartheta$ in (3.15a) yields that

$$|\vartheta(\cdot,t)|^2_{H^1(\Omega)} \le C\int_0^t |\partial_s\rho(\cdot,s)|^2_{L^2(\Omega)}ds.$$

Hence from (3.1c), (2.22) and (3.17) we obtain the results (3.13b-d).

Finally setting $e_{v,\varepsilon}^{h,*} \equiv v_\varepsilon - v_\varepsilon^{h,*}$ we have from (3.1b) and (2.18) that

$$\|e_{v,\varepsilon}^{h,*}\|^2_{E_1(k,t)} = \int_0^t((I-P_h^0)\varphi_\varepsilon(u_\varepsilon(\cdot,s)),e_{v,\varepsilon}^{h,*}(\cdot,s))ds + ½\, k^{-1}|(I-P_h^0)g_2|^2_{L^2(\Omega)}$$
$$\le Ch^2[|\nabla\varphi_\varepsilon(u_\varepsilon)|^2_{L^2(Q_T)} + k^{-1}|g_2|^2_{H^1(\Omega)}] \le C\varepsilon^{-1}h^2. \qquad (3.18)$$

Hence the desired result (3.13e). □

**Theorem 3.2**

Under Assumptions (D3) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$|u-u_\varepsilon^h|^2_{L^2(Q_t)} + \varepsilon|\varphi(u)-\varphi_\varepsilon(u_\varepsilon^h)|^2_{L^2(Q_t)} \le C[A_\varepsilon(t)\varepsilon^{(1+p)/(1-p)} + \varepsilon^{-1}kh^4] \qquad (3.19a)$$

$$\varepsilon\|v-v_\varepsilon^h\|^2_{E_1(k,t)} \le C[A_\varepsilon(t)\varepsilon^{(1+p)/(1-p)} + h^2 + \varepsilon^{-1}kh^4]. \qquad (3.19b)$$

Proof: The results (3.19a&b) follow directly from (2.16), (3.8) and (3.13). □

**Corollary 3.2**

Let Assumptions (D3) hold, then for all $h > 0$ and $t \in (0,T]$:

(i) Under no assumptions on non-degeneracy, we have on choosing $\varepsilon = Ch^{2(1-p)} \le \varepsilon_0$ that

$$|u-u_\varepsilon^h|_{L^2(Q_T)} \le C(k)h^{1+p}, \qquad (3.20a)$$

$$|(u-u_\varepsilon^h)(\cdot,t)|_{L^2(\Omega)} + |\int_0^t(u-u_\varepsilon^h)(\cdot,s)ds|_{H^1(\Omega)} + |\nabla(u-u_\varepsilon^h)|_{L^2(Q_T)} \le C(k)h \qquad (3.20b)$$

$$|\nabla(u-u_\varepsilon^h)(\cdot,t)|_{L^2(\Omega)} \le C(k)h^p \qquad (3.20c)$$

and

$$|\varphi(u)-\varphi_\varepsilon(u_\varepsilon^h)|_{L^2(Q_T)} + |(v-v_\varepsilon^h)(\cdot,t)|^2_{L^2(\Omega)} \le C(k)h^{2p}. \tag{3.20d}$$

(ii) Assuming (N.D.) and choosing $\varepsilon = Ch^{8(1-p)/(5-p)} \le \varepsilon_0$ we have that

$$|u-u_\varepsilon^h|_{L^2(Q_T)} \le C(k)h^{2(3+p)/(5-p)}, \tag{3.21a}$$

$$|(u-u_\varepsilon^h)(\cdot,t)|_{L^2(\Omega)} + \left|\int_0^t(u-u_\varepsilon^h)(\cdot,s)ds\right|_{H^1(\Omega)} + |\nabla(u-u_\varepsilon^h)|_{L^2(Q_T)} \le C(k)h, \tag{3.21b}$$

$$|\nabla(u-u_\varepsilon^h)(\cdot,t)|_{L^2(\Omega)} \le C(k)h^{(1+3p)/(5-p)} \tag{3.21c}$$

and

$$|\varphi(u)-\varphi_\varepsilon(u_\varepsilon^h)|_{L^2(Q_T)} + |(v-v_\varepsilon^h)(\cdot,t)|^2_{L^2(\Omega)} \le C(k)h^{2(1+3p)/(5-p)}. \tag{3.21d}$$

Proof: Noting the non-degeneracy condition (N.D.)≡(1.8a) in the case of (3.21); (3.20a&d) and (3.21a&d) follow directly from (3.19a&b). (3.20b&c) and (3.21b&c) follow from (2.16), (3.8) and (3.13). □

**Remark 3.1**

We note that one can improve on the error bound for $v_\varepsilon^h$ in (3.20d) and (3.21d) by choosing $\varepsilon$ to maximize the rate of convergence of $v_\varepsilon^h$ to $v$ in $L^2(Q_T)$ as opposed to the present choice which maximizes the rate of convergence of $u_\varepsilon^h$ to $u$ in $L^2(Q_T)$. For example under no assumptions on non-degeneracy, choosing $\varepsilon = Ch^{2(1-p)/(1+p)} \le \varepsilon_0$ one obtains $O(h^{2p/(1+p)})$ convergence for $v_\varepsilon^h$ to $v$ in $L^2(Q_T)$, but only $O(h)$ convergence for $u_\varepsilon^h$ to $u$ in $L^2(Q_T)$. □

One could approximate directly the problem (P) without regularising by introducing problem $(P^h)$, the same as $(P_\varepsilon^h)$ with $\varphi_\varepsilon$ replaced by $\varphi$.

**Theorem 3.3**

Let the Assumptions (D3) hold. Then there exists a unique solution $\{u^h, v^h\}$ to $(P^h)$ for all $h > 0$ and for all $\varepsilon \in (0, \varepsilon_0]$ and $t \in (0,T]$

$$\|u^h - u^h_\varepsilon\|^2_{E_2(k,t)} + k^{-2}\varepsilon|\nabla(u^h - u^h_\varepsilon)(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\varphi(u^h) - \varphi_\varepsilon(u^h_\varepsilon)|^2_{L^2(Q_t)} +$$
$$+ \varepsilon\|v^h - v^h_\varepsilon\|^2_{E_1(k,t)} \le C\varepsilon^{(1+p)/(1-p)}. \tag{3.22}$$

Moreover, we have that the error bounds (3.20a-d) hold with $\{u^h_\varepsilon, v^h_\varepsilon, \varphi_\varepsilon(u^h_\varepsilon)\}$ replaced by $\{u^h, v^h, \varphi(u^h)\}$ for all $h > 0$ and $t \in (0,T]$.

Proof: Existence and uniqueness of a solution and (3.22) follow from a discrete analogue of the proof of Theorem 2.2. Combining this with (3.20a-d) yields the desired error bounds. □

**Remark 3.2**

In proving (3.22) we have made no assumptions on the non-degeneracy of $u^h$, as such assumptions would be difficult to verify in practice. If we know that $u$ satisfies the non-degeneracy condition (N.D.), then from the error estimates above it is better to approximate (P) by $(P^h_\varepsilon)$, with the appropriate choice of $\varepsilon$, rather than $(P^h)$. □

**Remark 3.3**

One could of course analyse the error between $u$ and $u^h$ without using the regularization procedure by introducing problem $(P^{h,*})$, the same as $(P^{h,*}_\varepsilon)$ with $\varphi_\varepsilon$ replaced by $\varphi$. If we assume that (1.9c) holds for all $a, b \in \mathbb{R}$, as it does for $\varphi(s) \equiv [s]^p_+$, then we have in place of (2.3a) that

$$L^{-1/p}|\varphi(a) - \varphi(b)|^{(1+p)/p} \le [\varphi(a) - \varphi(b)](a-b) \le L|a-b|^{1+p}.$$

Let $u^{h,*}$ be the solution of $(P^{h,*})$. It is then a simple matter to adapt the proof of Lemma 3.2; to prove for all $t \in (0,T]$ that

$$\|u^{h,*}-u^h\|^2_{E_2(k,t)} + L^{-1/p}\int_0^t|\varphi(u(\cdot,s))-\varphi(u^h(\cdot,s))|^{(1+p)/p}_{L^{(1+p)/p}(\Omega)}\,ds$$

$$\le \|u^{h,*}-u^h\|^2_{E_2(k,t)} + \int_0^t(\varphi(u(\cdot,s))-\varphi(u^h(\cdot,s)),u(\cdot,s)-u^h(\cdot,s))ds$$

$$\le \int_0^t(\varphi(u(\cdot,s))-\varphi(u^h(\cdot,s)),u(\cdot,s)-u^{h,*}(\cdot,s))ds$$

$$\le C\int_0^t|(u-u^{h,*})(\cdot,s))|^{1+p}_{L^{1+p}(\Omega)}\,ds. \tag{3.22}$$

Letting $\varepsilon \to 0$ in (3.13a-c) and combining this with (3.22) yields the results (3.20a&b) with $\varepsilon = 0$ and in place of (3.20c&d) with $\varepsilon = 0$ we have that

$$\left[\int_0^t|\varphi(u(\cdot,s))-\varphi(u^h(\cdot,s))|^{(1+p)/p}_{L^{(1+p)/p}(\Omega)}\,ds\right]^{p/(1+p)} \le C(k)h^{2p}. \tag{3.23}$$

Therefore, bypassing the regularization procedure yields no error bound for v. □

## 4. A MORE PRACTICAL CONTINUOUS IN TIME FINITE ELEMENT APPROXIMATION

The standard Galerkin approximation analysed above is not practical as it requires the term $(\varphi_\varepsilon(u^h_\varepsilon),\chi)$ to be integrated exactly. This is obviously difficult in practice and it is computationally more convenient to consider a scheme where numerical integration is applied to all the terms and the initial data is interpolated as opposed to being projected. Below we introduce and analyse such a scheme.

For all $w_1$, $w_2 \in C^0(\bar{\Omega}^h)$ we set

$$(w_1,w_2)^h \equiv \int_{\Omega^h} \pi_h(w_1w_2)$$

as an approximation to $(w_1,w_2)$. On setting

$$|w|_h \equiv [(w,w)^h]^{½} \quad \text{for } w \in C^0(\bar{\Omega}^h),$$

we recall the well-known results

$$|\chi|_{L^2(\Omega^h)} \le |\chi|_h \le C_1|\chi|_{L^2(\Omega^h)} \quad \forall\ \chi \in S^h, \tag{4.1a}$$

$$\Big|\int_{\Omega^h} \chi_1\chi_2 - (\chi_1,\chi_2)^h\Big| \le C_2h^2\|\chi_1\|_{H^1(\Omega^h)}\|\chi_2\|_{H^1(\Omega^h)} \quad \forall\ \chi_1,\ \chi_2 \in S^h. \tag{4.1b}$$

We make the following assumptions on the data.

Assumptions **(D4)**: In addition to the Assumptions (D3) we assume that

$$f \in H^1(0,T;C^0(\bar{\Omega}))\cap L^2(0,T;H^2(\Omega)) \text{ and } g_2 \in H^2(\Omega).$$

A more practical approximation to $(P_\varepsilon)$ than $(P^h_\varepsilon)$ is then :

$(\hat{P}^h_\varepsilon)$ Find $\hat{u}^h_\varepsilon \in H^1(0,T;S^h_0)$ and $\hat{v}^h_\varepsilon \in H^1(0,T;S^h)$ such that

$$(\partial_t\hat{u}^h_\varepsilon + \partial_t\hat{v}^h_\varepsilon,\chi)^h + (\nabla\hat{u}^h_\varepsilon,\nabla\chi) = (f,\chi)^h \ \forall\ \chi \in S^h_0$$

$$(\partial_t\hat{v}^h_\varepsilon,\chi)^h = k(\varphi_\varepsilon(\hat{u}^h_\varepsilon)-\hat{v}^h_\varepsilon,\chi)^h \ \forall\ \chi \in S^h$$

$$\hat{u}^h_\varepsilon(\cdot,0) = \pi_hg_1(\cdot) \qquad \hat{v}^h_\varepsilon(\cdot,0) = \pi_hg_2(\cdot).$$

We have the following analogues of Theorem 3.1 and Lemmas 3.1.

**Theorem 4.1**

Let the Assumptions (D4) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$ and $h > 0$ there exists a unique solution $\{\hat{u}^h_\varepsilon,\hat{v}^h_\varepsilon\}$ to $(\hat{P}^h_\varepsilon)$.

Proof: A simple adaption of the proof of Theorem 3.1. □

**Lemma 4.1**

Under Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$|\nabla \hat{u}^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\partial_t \hat{u}^h_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t \hat{v}^h_\varepsilon|^2_{L^2(Q_T)} + \varepsilon|\partial_t \pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_T)} +$$
$$+ k^{-1}\Big[|\partial_t \hat{u}^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\partial_t \hat{v}^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + |\nabla(\partial_t \hat{u}^h_\varepsilon)|^2_{L^2(Q_T)}\Big]$$
$$\le C[1+k|\varphi_\varepsilon(g_1)-g_2|^2_h] \le Ck. \tag{4.2}$$

Proof: A direct analogue of (2.24) and (3.6) yields

$$k^{-1}\int_0^t|\nabla\partial_s \hat{u}^h_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds + \int_0^t([1+\varphi'_\varepsilon(\hat{u}^h_\varepsilon(\cdot,s))]\partial_s \hat{u}^h_\varepsilon(\cdot,s),\partial_s \hat{u}^h_\varepsilon(\cdot,s))^h ds +$$
$$+ k^{-1}|\partial_t \hat{u}^h_\varepsilon(\cdot,t)|^2_h + |\nabla \hat{u}^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)}$$
$$\le \int_0^t|k^{-1}\partial_s f(\cdot,s)+f(\cdot,s)|^2_h ds + k^{-1}|\partial_t \hat{u}^h_\varepsilon(\cdot,0)|^2_h + |\nabla \hat{u}^h_\varepsilon(\cdot,0)|^2_{L^2(\Omega)}$$
$$\le \int_0^t|k^{-1}\partial_s f(\cdot,s)+f(\cdot,s)|^2_h ds + C|g_1|^2_{H^2(\Omega)} + 2k|\varphi_\varepsilon(g_1)-g_2|^2_h +$$
$$+ 2k^{-1}\Big[|g_1|^2_{H^2(\Omega)}+|f(\cdot,0)|^2_{L^\infty(\Omega)}\Big], \tag{4.3}$$

where we have noted from (3.1a) and (3.2a) that for all $\chi \in S^h_0$

$$|(\nabla\pi_h g_1,\nabla\chi)| = |(\nabla(\pi_h g_1-g_1),\nabla\chi) - (\Delta g_1,\chi)| \le C|g_1|_{H^2(\Omega)}|\chi|_{L^2(\Omega)}.$$

A direct analogue of (3.7) yields

$$k^{-1}|\partial_t \hat{v}^h_\varepsilon(\cdot,t)|^2_h + \int_0^t|\partial_s \hat{v}^h_\varepsilon(\cdot,s)|^2_h ds$$
$$\le k^{-1}|\partial_t \hat{v}^h_\varepsilon(\cdot,0)|^2_h + \int_0^t|\partial_s[\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,s))]|^2_h ds$$
$$\le \int_0^t|\partial_s[\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,s))]|^2_h ds + k|\varphi_\varepsilon(g_1)-g_2|^2_h.$$

Combining the above with (4.3) and noting (4.1a) and the analogue of (2.25) yields the desired result (4.2). □

Assumptions (D5): In addition to the Assumptions (D4) we assume that

The triangulation $T^h$ is such that (i) for d = 2 it is weakly acute; that is, for any pair of adjacent triangles the sum of opposite angles relative to the common side does not exceed $\pi$; and (ii) for d = 3 the angle between the vectors normal to any two faces of the same tetrahedron must not exceed $\pi/2$, see Kerkhoven & Jerome (1990).

Let $B \equiv \{b_{ij}\}_{i,j=1}^{I} \equiv \{(\nabla\chi_i,\nabla\chi_j)\}_{i,j=1}^{I}$ and $A \equiv \{(\chi_i,\chi_j)^h\}_{i,j=1}^{I}$; where $\{x_i\}_{i=1}^{I}$ are the internal nodes of the partitioning and $\chi_j \in S_0^h$ is such that $\chi_j(x_i) = \delta_{ij}$, $i,j = 1 \to I$. It follows that A is diagonal matrix with positive entries and that B and $\tilde{B} \equiv A^{-1}B$ are positive definite. Under Assumption (D5) it follows that $b_{ij} \le 0$ for $i \ne j$ and hence B, and $\tilde{B}$, are M-matrices. From this property one can deduce the discrete analogue of (2.20)

$$M^{-1}\varepsilon|\nabla\pi_h[\varphi_\varepsilon(\chi)]|^2_{L^2(\Omega)} \le (\nabla\chi,\nabla\pi_h[\varphi_\varepsilon(\chi)]) \quad \forall\ \chi \in S_0^h, \tag{4.4}$$

see §2.4.2 of Nochetto (1991).

**Corollary 4.1**

Let the Assumptions (D5) hold. If $g_1$, $g_2$ and $f \ge 0$ then the unique solution $\{\hat{u}_\varepsilon^h,\hat{v}_\varepsilon^h\}$ to $(\hat{P}_\varepsilon^h)$, $\varepsilon \in (0,\varepsilon_0]$, is such that $\hat{u}_\varepsilon^h$, $\hat{v}_\varepsilon^h \ge 0$ in $Q_T$.

Proof: Adopting the notation above, it follows as $\varphi_\varepsilon(0)=0$ that $\hat{u}_\varepsilon^h(x,t) \equiv \sum_{j=1}^{I}\alpha_j(t)\chi_j(x)$ and $\hat{v}_\varepsilon^h(x,t) \equiv \sum_{j=1}^{I}\beta_j(t)\chi_j(x) + e^{-kt}\sum_{j=I+1}^{J} g_2(x_j)\chi_j(x)$, where $\{x_j\}_{j=I+1}^{J}$ are the boundary nodes and $\{\chi_j\}_{j=I+1}^{J}$ the corresponding basis functions $\in S^h$. As $\chi_j \ge 0$, $j = 1 \to J$; to prove the assertion we need to show that $\underline{\alpha}(t) \equiv \{\alpha_j(t)\}_{j=1}^{I}$, $\underline{\beta}(t) \equiv \{\beta_j(t)\}_{j=1}^{I} \ge \underline{0}$ for all $t \in [0,T]$.

Problem $(\hat{P}_\varepsilon^h)$ and the assumptions on the data yield that for all $t \in [0,T]$

$$\underline{\alpha}'(t) + k\underline{\varphi}_\varepsilon(\underline{\alpha}(t)) + \tilde{B}\underline{\alpha}(t) \ge k\underline{\beta}(t) \qquad \underline{\alpha}(0) \ge \underline{0} \tag{4.5a}$$

$$\underline{\beta}'(t) = k(\underline{\varphi}_\varepsilon(\underline{\alpha}(t))-\underline{\beta}(t)) \qquad \underline{\beta}(0) \ge \underline{0}\ , \tag{4.5b}$$

where $\underline{\varphi}_\varepsilon(\underline{\alpha}) \equiv \{\varphi_\varepsilon(\alpha_j)\}_{j=1}^{I}$. Setting $\underline{\alpha} \equiv \underline{\alpha}^+ + \underline{\alpha}^-$, where $\underline{\alpha}^+ \equiv \{[\alpha_j]_+\}_{j=1}^{I}$ and $[\alpha]_+ \equiv \max\ \{\alpha,0\}$, and noting that $[\underline{\alpha}^-]^T\underline{\alpha}^+ = [\underline{\alpha}^-]^T\underline{\alpha}^{+\prime} = 0$, $[\underline{\alpha}^-]^T\tilde{B}\underline{\alpha} \ge [\underline{\alpha}^-]^T\tilde{B}\underline{\alpha}^-$

$\geq 0$ and $[\underline{\alpha}^-]^T\underline{\varphi}_\varepsilon(\underline{\alpha}) \equiv [\underline{\alpha}^-]^T\underline{\varphi}_\varepsilon(\underline{\alpha}^-) \geq 0$; we have that $[\underline{\alpha}^-(t)]^T$(4.5a) and $[\underline{\beta}^-(t)]^T$(4.5b) yield for all $t \in [0,T]$

$$\tfrac{1}{2}\frac{d}{dt}|\underline{\alpha}^-(t)|^2 \leq k[\underline{\alpha}^-(t)]^T[\underline{\beta}(t)] \leq k[\underline{\alpha}^-(t)]^T[\underline{\beta}^-(t)] \qquad \underline{\alpha}^-(0) = \underline{0}$$

$$\tfrac{1}{2}\frac{d}{dt}|\underline{\beta}^-(t)|^2 \leq k[\underline{\beta}^-(t)]^T(\underline{\varphi}_\varepsilon(\underline{\alpha}(t)) \leq kM\varepsilon^{-1}[\underline{\beta}^-(t)]^T[\underline{\alpha}^-(t)] \qquad \underline{\beta}^-(0) = \underline{0}.$$

Adding the above, applying a Cauchy Schwartz and a Gronwall inequality yields that $\underline{\alpha}^-(t) \equiv \underline{\beta}^-(t) \equiv \underline{0}$ for all $t \in [0,T]$ and hence the desired result. □

We now have the analogue of Lemma 2.2.

**Lemma 4.2**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$\varepsilon|\nabla\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_T)} + (\Phi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,t)),1)^h + k|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]-\hat{v}^h_\varepsilon|^2_{L^2(Q_T)} + $$
$$+ |\hat{v}^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t\hat{v}^h_\varepsilon|^2_{L^2(Q_T)} \leq C. \tag{4.6}$$

Proof: We have on choosing $\chi \equiv \pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]$ in $(\hat{P}^h_\varepsilon)$ that

$$(\nabla\hat{u}^h_\varepsilon,\nabla\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]) + (\partial_t\hat{u}^h_\varepsilon,\varphi_\varepsilon(\hat{u}^h_\varepsilon))^h + (\partial_t\hat{v}^h_\varepsilon,\hat{v}^h_\varepsilon)^h + k|\varphi_\varepsilon(\hat{u}^h_\varepsilon)-\hat{v}^h_\varepsilon|^2_h$$
$$= (f,\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)])^h . \tag{4.7}$$

Integrating (4.7) in time over $(0,t)$, noting (4.4), (4.1a) and a Gronwall inequality yields the analogue of (2.21)

$$\varepsilon|\nabla\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_t)} + (\Phi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,t)),1)^h + |\hat{v}^h_\varepsilon(\cdot,t)|^2_h + k|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]-\hat{v}^h_\varepsilon|^2_{L^2(Q_t)}$$
$$\leq C\left[\int_0^t(f(\cdot,s),\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,s))])^h ds + (\Phi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,0)),1)^h + |\hat{v}^h_\varepsilon(\cdot,0)|^2_h\right]$$
$$\leq C + \tfrac{1}{6}k_0|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_t)}$$
$$\leq C + \tfrac{1}{3}k_0|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]-\hat{v}^h_\varepsilon|^2_{L^2(Q_t)} + \tfrac{1}{3}k_0|\hat{v}^h_\varepsilon|^2_{L^2(Q_t)} \leq C. \tag{4.8}$$

Hence the desired result (4.6) then follows from (4.8) and the second equation in $(\hat{P}^h_\varepsilon)$ and noting (4.1a). □

We now prove the analogue of Lemma 4.2 for the solution $\{u^h_\varepsilon, v^h_\varepsilon\}$ of $(P^h_\varepsilon)$.

**Lemma 4.3**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$ and for all h, provided $M\varepsilon^{-1}kh^2 \le 1$, and $t \in (0,T]$ that

$$\varepsilon|\nabla[\pi_h\varphi_\varepsilon(u^h_\varepsilon)]|^2_{L^2(Q_T)} + (\Phi_\varepsilon(u^h_\varepsilon(\cdot,t)),1) + k|\varphi_\varepsilon(u^h_\varepsilon)-v^h_\varepsilon|^2_{L^2(Q_T)} + \\ + |v^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + k^{-1}|\partial_t v^h_\varepsilon|^2_{L^2(Q_T)} \le C. \tag{4.9}$$

Proof: We have on choosing $\chi \equiv \pi_h[\varphi_\varepsilon(u^h_\varepsilon)]$ in $(P^h_\varepsilon)$ that

$$(\nabla u^h_\varepsilon,\nabla\pi_h[\varphi_\varepsilon(u^h_\varepsilon)]) + (\partial_t u^h_\varepsilon,\varphi_\varepsilon(u^h_\varepsilon)) + (\partial_t v^h_\varepsilon, v^h_\varepsilon) + k|\varphi_\varepsilon(u^h_\varepsilon)-v^h_\varepsilon|^2_{L^2(\Omega)} \\ = (f,\varphi_\varepsilon(u^h_\varepsilon)) + (\partial_t u^h_\varepsilon + k[\varphi_\varepsilon(u^h_\varepsilon)-v^h_\varepsilon] - f,(I-\pi_h)\varphi_\varepsilon(u^h_\varepsilon)). \tag{4.10}$$

Integrating (4.10) in time over $(0,t)$, noting (4.4) and the bounds (3.4) and (3.1b) yields the analogue of (4.8) that

$$M^{-1}\varepsilon|\nabla\pi_h[\varphi_\varepsilon(u^h_\varepsilon)]|^2_{L^2(Q_t)} + (\Phi_\varepsilon(u^h_\varepsilon(\cdot,t)),1) + \tfrac{1}{2}|v^h_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \\ + k|\varphi_\varepsilon(u^h_\varepsilon)-v^h_\varepsilon|^2_{L^2(Q_t)} \\ \le C + \tfrac{1}{3}k_0|\varphi_\varepsilon(u^h_\varepsilon)-v^h_\varepsilon|^2_{L^2(Q_t)} + \tfrac{1}{3}k_0|v^h_\varepsilon|^2_{L^2(Q_t)} + (\Phi_\varepsilon(P^1_h g_1),1) + \\ + \tfrac{1}{2}kh^2|\nabla\pi_h[\varphi_\varepsilon(u^h_\varepsilon)]|^2_{L^2(Q_t)} + \tfrac{1}{2}k^{-1}[|\partial_t u^h_\varepsilon|^2_{L^2(Q_t)} + k^2|\varphi_\varepsilon(u^h_\varepsilon)-v^h_\varepsilon|^2_{L^2(Q_t)}]. \tag{4.11}$$

Hence the desired result (4.9) then follows from (4.11), the second equation in $(P^h_\varepsilon)$, the bound (3.5) for $|\partial_t u^h_\varepsilon|^2_{L^2(Q_t)}$, (3.1c) and a Gronwall inequality provided $M\varepsilon^{-1}kh^2 \le 1$. □

**Lemma 4.4**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$ that

$$\|u^h_\varepsilon-\hat{u}^h_\varepsilon\|^2_{E_2(k,t)} + \varepsilon|\varphi_\varepsilon(u^h_\varepsilon)-\varphi_\varepsilon(\hat{u}^h_\varepsilon)|^2_{L^2(Q_t)} + \varepsilon\|v^h_\varepsilon-\hat{v}^h_\varepsilon\|^2_{E_1(k,t)}$$
$$\le C[\varepsilon^{-1} + k|\varphi_\varepsilon(g_1)-g_2|^2_h + \|\pi_h g_2\|^2_{H^1(\Omega)}]h^2 + C[h^4|g_1|^2_{H^2(\Omega)} + |(I-\pi_h)g_2|^2_{L^2(\Omega)}]$$
$$\le C[\varepsilon^{-1}+k]h^2. \tag{4.12}$$

Proof: The proof is very similar to Lemma 2.1. Let $\hat{e}^{u,h}_\varepsilon \equiv u^h_\varepsilon-\hat{u}^h_\varepsilon$ and $\hat{e}^{v,h}_\varepsilon \equiv v^h_\varepsilon-\hat{v}^h_\varepsilon$. Subtracting the first equation in $(\hat{P}^h_\varepsilon)$ from that in $(P^h_\varepsilon)$, choosing $\chi \equiv \int_s^t\hat{e}^{u,h}_\varepsilon(\cdot,\sigma)d\sigma$, integrating over $(0,t)$ in time, where s is the integration variable in time, and performing integration by parts yields that

$$\int_0^t|\hat{e}^{u,h}_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds + ½|\nabla\int_0^t\hat{e}^{u,h}_\varepsilon(\cdot,s)ds|^2_{L^2(\Omega)}$$
$$= -\int_0^t(\hat{e}^{v,h}_\varepsilon(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))ds + ((\hat{e}^{u,h}_\varepsilon+\hat{e}^{v,h}_\varepsilon)(\cdot,0),\int_0^t\hat{e}^{u,h}_\varepsilon(\cdot,s)ds) +$$
$$+ \int_0^t\Big[(\xi(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s)) - (\xi(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))^h\Big]ds, \tag{4.13a}$$

where

$$\xi(\cdot,t) \equiv \int_0^t(f-\partial_s\hat{u}^h_\varepsilon-\partial_s\hat{v}^h_\varepsilon)(\cdot,s)ds$$
$$= (\hat{u}^h_\varepsilon+\hat{v}^h_\varepsilon)(\cdot,0)-(\hat{u}^h_\varepsilon+\hat{v}^h_\varepsilon)(\cdot,t) + \int_0^t(f(\cdot,s)ds. \tag{4.13b}$$

In addition subtracting the first equation in $(\hat{P}^h_\varepsilon)$ from that in $(P^h_\varepsilon)$, choosing $\chi \equiv \hat{e}^{u,h}_\varepsilon$, integrating over $(0,t)$ and performing integration by parts yields that

$$½|\hat{e}^{u,h}_\varepsilon(\cdot,t)|^2_{L^2(\Omega)} + \int_0^t|\nabla\hat{e}^{u,h}_\varepsilon(\cdot,s)|^2_{L^2(\Omega)}ds$$
$$= -\int_0^t(\partial_s\hat{e}^{v,h}_\varepsilon(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))ds + ½|\hat{e}^{u,h}_\varepsilon(\cdot,0)|^2_{L^2(\Omega)} +$$
$$+ \int_0^t\Big[(\partial_s\xi(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s)) - (\partial_s\xi(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))^h\Big]ds. \tag{4.14}$$

Therefore from (4.13), (4.14) and the second equations in $(P^h_\varepsilon)$ and $(\hat{P}^h_\varepsilon)$, it follows that

$$\|\hat{e}^{u,h}_\varepsilon\|^2_{E_2(k,t)} + \int_0^t(\varphi_\varepsilon(u^h_\varepsilon(\cdot,s))-\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,s)),\hat{e}^{u,h}_\varepsilon(\cdot,s))ds$$

$$= \int_0^t\Big[((1+k^{-1}\partial_s)\xi(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s)) - ((1+k^{-1}\partial_s)\xi(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))^h\Big]ds +$$

$$+ \int_0^t((1+k^{-1}\partial_s)\hat{v}^h_\varepsilon(\cdot,s)-\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,s)),\hat{e}^{u,h}_\varepsilon(\cdot,s))ds +$$

$$+ ((\hat{e}^{u,h}_\varepsilon+\hat{e}^{v,h}_\varepsilon)(\cdot,0),\int_0^t\hat{e}^{u,h}_\varepsilon(\cdot,s)ds) + \tfrac{1}{2}k^{-1}|\hat{e}^{u,h}_\varepsilon(\cdot,0)|^2_{L^2(\Omega)}$$

$$= \Big[((\hat{e}^{u,h}_\varepsilon+\hat{e}^{v,h}_\varepsilon)(\cdot,0),\int_0^t\hat{e}^{u,h}_\varepsilon(\cdot,s)ds) + \tfrac{1}{2}k^{-1}|\hat{e}^{u,h}_\varepsilon(\cdot,0)|^2_{L^2(\Omega)}\Big] +$$

$$+ \int_0^t\Big[(\eta(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s)) - (\eta(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))^h\Big]ds$$

$$\equiv T_1 + T_2, \tag{4.15a}$$

where

$$\eta(\cdot,t) \equiv (1+k^{-1}\partial_t)(\xi+\hat{v}^h_\varepsilon)(\cdot,t) - \varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,t))$$

$$\equiv \eta_1(\cdot,t) + \eta_2(\cdot,t) + \eta_3(\cdot,t), \tag{4.15b}$$

$$\eta_1(\cdot,t) \equiv (\hat{u}^h_\varepsilon+\hat{v}^h_\varepsilon)(\cdot,0) - (1+k^{-1}\partial_t)\hat{u}^h_\varepsilon(\cdot,t) \in S^h \tag{4.15c}$$

$$\eta_2(\cdot,t) \equiv (1+k^{-1}\partial_t)\int_0^t(f(\cdot,s)ds \tag{4.15d}$$

and

$$\eta_3(\cdot,t) \equiv -\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,t)). \tag{4.15e}$$

From (3.1a&c) and as $\int_0^t\hat{e}^{u,h}_\varepsilon(\cdot,s)ds \in S^h$ it follows that

$$T_1 \le ((P^1_h-\pi_h)g_1+(I-\pi_h)g_2,\int_0^t\hat{e}^{u,h}_\varepsilon(\cdot,s)ds) + C|(P^1_h-\pi_h)g_1|^2_{L^2(\Omega)}$$

$$\le C[h^4|g_1|^2_{H^2(\Omega)} + |(I-\pi_h)g_2|^2_{L^2(\Omega)}] + \tfrac{1}{2}\|\hat{e}^{u,h}_\varepsilon\|^2_{E_2(k,t)}. \tag{4.16}$$

Next we note that

$$T_2 \equiv \int_0^t\Big[(\pi_h\eta(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s)) - (\pi_h\eta(\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))^h\Big]ds +$$

$$+ \int_0^t([(I-\pi_h)(\eta_2+\eta_3)](\cdot,s),\hat{e}^{u,h}_\varepsilon(\cdot,s))ds \equiv T_{2,1} + T_{2,2}. \tag{4.17}$$

We have from (4.1b), (3.2a), (3.1a), (4.2), (4.6) and (3.4) that

$$T_{2,1} \le Ch \int_0^t\|\pi_h\eta(\cdot,s)\|_{H^1(\Omega)}\|\hat{e}^{u,h}_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds$$

$$\le Ch[\int_0^t\|\pi_h\eta(\cdot,s)\|^2_{H^1(\Omega)}ds]^{1/2}\ \|\hat{e}^{u,h}_\varepsilon\|_{E_2(k,t)}$$

$$\le Ch[\varepsilon^{-1/2} + k^{1/2}|\varphi_\varepsilon(g_1)-g_2|_h + \|\pi_h g_2\|_{H^1(\Omega)}]\|\hat{e}^{u,h}_\varepsilon\|_{E_2(k,t)} \tag{4.18a}$$

and

$$T_{2,2} \le \int_0^t |(I-\pi_h)(\eta_2+\eta_3)(\cdot,s)|_{L^2(\Omega)} |\hat{e}_\varepsilon^{u,h}(\cdot,s)|_{L^2(\Omega)} ds$$
$$\le C\varepsilon^{-\frac{1}{2}} h \| \hat{e}_\varepsilon^{u,h} \|_{E_2(k,t)}. \tag{4.18b}$$

Combining (4.15a), (4.16), (4.17) and (4.18) and noting (2.3a) and (3.1a) yields the desired result (4.12) for $u_\varepsilon^h - \hat{u}_\varepsilon^h$ and $\varphi_\varepsilon(u_\varepsilon^h) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)$.

Finally, we have from (3.4) and (4.6) that

$$\| \hat{e}_\varepsilon^{v,h} \|^2_{E_1(k,t)}$$
$$= \int_0^t (\varphi_\varepsilon(u_\varepsilon^h(\cdot,s)) - \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,s))], \hat{e}_\varepsilon^{v,h}(\cdot,s)) ds + \tfrac{1}{2} k^{-1} |(P_h^0 - \pi_h) g_2|^2_{L^2(\Omega)}$$
$$\le C[ |\varphi_\varepsilon(u_\varepsilon^h) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)|^2_{L^2(Q_t)} + |(I-\pi_h)[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)} + k^{-1} |(I-\pi_h) g_2|^2_{L^2(\Omega)} ]$$
$$\le C[ |\varphi_\varepsilon(u_\varepsilon^h) - \varphi_\varepsilon(\hat{u}_\varepsilon^h)|^2_{L^2(Q_t)} + \varepsilon^{-1} h^2 + k^{-1} |(I-\pi_h) g_2|^2_{L^2(\Omega)} ] \tag{4.19}$$

and hence the desired result (4.12) for $v_\varepsilon^h - \hat{v}_\varepsilon^h$. □

We now improve on the bound (4.12) in the physically interesting case of given data $g_1$, $g_2$ and $f \ge 0$ yielding $u$, $\hat{u}_\varepsilon^h \ge 0$ in $Q_T$.

Assumptions (D6): In addition to the Assumptions (D5) we assume that (i) $\Omega \subset \mathbb{R}^d$, $d = 1$ or $2$, (ii) $g_1$, $g_2$ and $f \ge 0$ and (iii) $\varphi \in C^2(0,\infty)$ such that $\varphi''(s) \le 0$ for all $s > 0$ and there exist an $s_0$ such that $\varphi(s) \ge s\varphi'(s)$ for all $s \in (0,s_0)$. We set $\varphi_\varepsilon$ to be the following quadratic regularization of $\varphi$

$$\varphi_\varepsilon(s) \equiv \begin{cases} \varphi(s) & \text{for } s \ge \delta \\ as^2 + bs & \text{for } s \in [0,\delta] \ ; \\ bs & \text{for } s \le 0 \end{cases} \tag{4.20}$$

where $a \equiv \delta^{-1}\varphi'(\delta) - \delta^{-2}\varphi(\delta)$, $b \equiv -\varphi'(\delta) + 2\delta^{-1}\varphi(\delta)$ and $\delta \equiv \varepsilon^{1/(1-p)}$ so that $\varphi_\varepsilon \in C^1(\mathbb{R})$.

As (ii) $\Rightarrow u \ge 0$ in $Q_T$, see Theorem 2.1, we can choose $\varphi(s)$ for $s < 0$ as we please. As (iii) holds it follows for $\varepsilon$ sufficiently small that $0 < b \le C_1\varepsilon^{-1}$ and $-C_2\varepsilon^{(p-2)/(1-p)} \le a \le 0$, see (2.3b), and hence $\varphi_\varepsilon$ satisfies the conditions (2.1b&c). Extending $\varphi$ so that $\varphi(s) \equiv \varphi_\varepsilon(s)$ for $s \le 0$, we have that (1.9) holds and $\varphi_\varepsilon$ satisfies (2.1a). Therefore all the results proved so far in this paper hold under the Assumptions (D6). We note for example that

$\varphi(s) \equiv s^p$ for $s \geq 0$ with $p \in (0,1)$ satisfies (1.9) and (iii) above.

**Lemma 4.5**

Under Assumptions (D6) there exists an $\tilde{\varepsilon}_0 \leq \varepsilon_0$ such that we have for all $\varepsilon \in (0,\tilde{\varepsilon}_0]$ and for all h, provided $M\varepsilon^{-1}kh^2 \leq 1$, and $t \in (0,T]$ that

$$\|u_\varepsilon^h-\hat{u}_\varepsilon^h\|^2_{E_2(k,t)} + \varepsilon|\pi_h[\varphi_\varepsilon(u_\varepsilon^h)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)}$$

$$\leq Ck[\ln(1/h)]^{2r}\varepsilon^{-2}h^4[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + \varepsilon^{-2/(1-p)}|\nabla(u_\varepsilon-u_\varepsilon^h)|^4_{L^4(0,t;L^2(\Omega))}] +$$

$$+ Ck[\varepsilon^{-1} + k|\varphi_\varepsilon(g_1)-g_2|^2_h + \|\pi_h g_2\|^2_{H^1(\Omega)}]h^4 + C[\ln(1/h)]^{2r}|(I-\pi_h)g_2|^2_{L^1(\Omega)}$$

$$\leq C(k)\varepsilon^{-2}h^4[\ln(1/h)]^{2r}[1 + \varepsilon^{-(3-p)/(1-p)}h^4] \tag{4.21a}$$

and

$$\varepsilon\|v_\varepsilon^h-\hat{v}_\varepsilon^h\|^2_{E_1(k,t)} \leq C[h^2 + \varepsilon|\pi_h[\varphi_\varepsilon(u_\varepsilon^h)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)} + \varepsilon k^{-1}|(I-\pi_h)g_2|^2_{L^2(\Omega)}]$$

$$\leq C[h^2 + \varepsilon|\pi_h[\varphi_\varepsilon(u_\varepsilon^h)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)}], \tag{4.21b}$$

where $r = 0$ if $d = 1$ and $r = ½$ if $d = 2$.

Proof: Adopting the notation of Lemma 4.4 we have from (4.15a) that

$$\|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + M^{-1}\varepsilon|\pi_h[\varphi_\varepsilon(u_\varepsilon^h)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)}$$

$$\leq \|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)} + \int_0^t(\varphi_\varepsilon(u_\varepsilon^h(\cdot,s))-\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,s)),\hat{e}_\varepsilon^{u,h}(\cdot,s))^h ds$$

$$\equiv T_1 + \hat{T}_2, \tag{4.22a}$$

where $T_1$ is given by (4.15a),

$$\hat{T}_2 \equiv \int_0^t\Big[(\hat{\eta}(\cdot,s),\hat{e}_\varepsilon^{u,h}(\cdot,s)) - (\hat{\eta}(\cdot,s),\hat{e}_\varepsilon^{u,h}(\cdot,s))^h\Big]ds, \tag{4.22b}$$

$\hat{\eta} \equiv \eta_1 + \eta_2 + \hat{\eta}_3$, with $\eta_1$, $\eta_2$ as given by (4.15c&d) and $\hat{\eta}_3 \equiv -\varphi_\varepsilon(u_\varepsilon^h)$. Next we write $\hat{T}_2 \equiv \hat{T}_{2,1} + \hat{T}_{2,2}$, where $\hat{T}_{2,1}$ and $\hat{T}_{2,2}$ are the same as $T_{2,1}$ and $T_{2,2}$, see (4.17), with $\eta$ and $\eta_3$ replaced by $\hat{\eta}$ and $\hat{\eta}_3$, respectively. We then have from (4.15a), (4.16), (3.2b), (4.1b), (3.1a), (4.2), (3.4) and (4.9) that

$$T_1 \leq C[h^4|g_1|^2_{H^2(\Omega)} + [\ln(1/h)]^{2r}|(I-\pi_h)g_2|^2_{L^1(\Omega)}] + ½\|\hat{e}_\varepsilon^{u,h}\|^2_{E_2(k,t)}, \tag{4.23a}$$

$$\hat{T}_{2,1} \le Ch^2 \int_0^t \|\pi_h\eta(\cdot,s)\|_{H^1(\Omega)} \|\hat{e}_\varepsilon^{u,h}(\cdot,s)\|_{H^1(\Omega)} ds$$

$$\le Ck^{½}h^2[\varepsilon^{-½} + k^{½}|\varphi_\varepsilon(g_1)-g_2|_h + \|\pi_h g_2\|_{H^1(\Omega)}]\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)}, \tag{4.23b}$$

$$\hat{T}_{2,2} \le Ch^2\|\hat{e}_\varepsilon^{u,h}\|_{E_2(k,t)} + \hat{T}_3 \tag{4.23c}$$

and

$$\hat{T}_3 \equiv \left|\int_0^t ((I-\pi_h)\varphi_\varepsilon(u_\varepsilon^h(\cdot,s)),\hat{e}_\varepsilon^{u,h}(\cdot,s))ds\right|$$

$$\le C\int_0^t |(I-\pi_h)\varphi_\varepsilon(u_\varepsilon^h(\cdot,s))|_{L^1(\Omega)} |\hat{e}_\varepsilon^{u,h}(\cdot,s)|_{L^\infty(\Omega)} ds$$

$$\le Ck^{½}[\ln(1/h)]^r\|\hat{e}_\varepsilon^{u,h}\|_{E(k,t)} \left[\int_0^t |(I-\pi_h)\varphi_\varepsilon(u_\varepsilon^h(\cdot,s))|^2_{L^1(\Omega)} ds\right]^{½}. \tag{4.23d}$$

We have from (3.1a) that

$$\int_0^t |(I-\pi_h)\varphi_\varepsilon(u_\varepsilon^h(\cdot,s))|^2_{L^1(\Omega)} ds \le Ch^4 \int_0^t \left[\sum_{K\in T^h} |\varphi_\varepsilon(u_\varepsilon^h(\cdot,s))|_{W^{2,1}(K)}\right]^2 ds$$

$$\le Ch^4\, \hat{T}_4, \tag{4.24a}$$

where by the fact that $\varphi_\varepsilon''(s) \le 0$ for almost all s

$$\hat{T}_4 \equiv \int_0^t |(\varphi_\varepsilon''(u_\varepsilon^h(\cdot,s))\nabla u_\varepsilon^h(\cdot,s),\nabla u_\varepsilon^h(\cdot,s))|^2 ds = \int_0^t |(\nabla[\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s))],\nabla u_\varepsilon^h(\cdot,s))|^2 ds$$

$$\le \int_0^t \left[|(\nabla[\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s))],\nabla u_\varepsilon(\cdot,s))| + |(\nabla[\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s))],\nabla(u_\varepsilon-u_\varepsilon^h)(\cdot,s))|\right]^2 ds$$

$$\le C\int_0^t \left[|(\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s)),\Delta u_\varepsilon(\cdot,s))| + \left|\int_{\partial\Omega}\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s))\nabla u_\varepsilon(\cdot,s).\underline{n}\right|\right]^2 ds +$$

$$+ C\int_0^t \left[|\nabla[\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s))]|^2_{L^2(\Omega)} |\nabla(u_\varepsilon-u_\varepsilon^h)(\cdot,s)|^2_{L^2(\Omega)}\right] ds$$

$$\le C\varepsilon^{-2}|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + |\nabla[\varphi_\varepsilon'(u_\varepsilon^h)]|^2_{L^4(0,t;L^2(\Omega))} |\nabla(u_\varepsilon-u_\varepsilon^h)|^2_{L^4(0,t;L^2(\Omega))}. \tag{4.24b}$$

From (4.20) it follows that $-\varphi_\varepsilon''(u_\varepsilon^h(\cdot,s)) \le -2a \le C\varepsilon^{(p-2)/(1-p)}$ and hence

$$|\nabla[\varphi_\varepsilon'(u_\varepsilon^h)]|^4_{L^4(0,t;L^2(\Omega))} \equiv \int_0^t |(\varphi_\varepsilon''(u_\varepsilon^h(\cdot,s))\nabla[\varphi_\varepsilon'(u_\varepsilon^h(\cdot,s))],\nabla u_\varepsilon^h(\cdot,s))|^2 ds$$

$$\le C\varepsilon^{2(p-2)/(1-p)}\hat{T}_4. \tag{4.25}$$

Combining the bounds (4.24b) and (4.25) yields that

$$\hat{T}_4 \le C\varepsilon^{-2}[|u_\varepsilon|^2_{L^2(0,t;H^2(\Omega))} + \varepsilon^{-2/(1-p)}|\nabla(u_\varepsilon-u_\varepsilon^h)|^4_{L^4(0,t;L^2(\Omega))}]. \tag{4.26}$$

Combining (4.22), (4.23), (4.24a) and (4.26) yields the first inequality in (4.21a). From (3.13) and (3.8) it follows that

$$|\nabla(u_\varepsilon-u_\varepsilon^h)|^2_{L^2(Q_T)} \le C(k)[1+\varepsilon^{-1}h^2]h^2 \le C(k)h^2,$$

$$|\nabla(u_\varepsilon-u_\varepsilon^h)|^2_{L^\infty(0,t;L^2(\Omega))} \le C(k)\varepsilon^{-1}[1+\varepsilon^{-1}h^2]h^2 \le C(k)\varepsilon^{-1}h^2$$

and hence we have that

$$|\nabla(u_\varepsilon-u_\varepsilon^h)|^4_{L^4(0,t;L^2(\Omega))} \le C(k)\varepsilon^{-1}h^4.$$

Noting this with (3.17) and (3.1a) yields the second inequality in (4.21a).

Finally, in similar manner to (4.19), (3.4) and (4.9) yield that

$$\begin{aligned}\|\hat{e}_\varepsilon^{v,h}\|^2_{E_1(k,t)} &\le C[\,|(I-\pi_h)[\varphi_\varepsilon(u_\varepsilon^h)]|^2_{L^2(Q_t)} + |\pi_h[\varphi_\varepsilon(u_\varepsilon^h)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)} + \\ &\qquad + k^{-1}|(P_h^0-\pi_h)g_2|^2_{L^2(\Omega)}] \\ &\le C[\varepsilon^{-1}h^2 + |\pi_h[\varphi_\varepsilon(u_\varepsilon^h)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)} + k^{-1}|(I-\pi_h)g_2|^2_{L^2(\Omega)}].\end{aligned}$$

Hence the desired result (4.21b). □

**Theorem 4.2**

Under Assumptions (D5) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h > 0$ and $t \in (0,T]$

$$\begin{aligned}|u-\hat{u}_\varepsilon^h|^2_{L^2(Q_t)} + \varepsilon|\varphi(u)-\varphi_\varepsilon(\hat{u}_\varepsilon^h)|^2_{L^2(Q_t)} + \varepsilon\|v-\hat{v}_\varepsilon^h\|^2_{E_1(k,t)} \\ \le CA_\varepsilon(t)\,\varepsilon^{(1+p)/(1-p)} + C(k)\varepsilon^{-1}h^2.\end{aligned} \tag{4.27}$$

Under Assumptions (D6) there exists an $\tilde{\varepsilon}_0 \le \varepsilon_0$ such that we have for all $\varepsilon \in (0,\tilde{\varepsilon}_0]$ and for all $h$, provided $M\varepsilon^{-1}kh^2 \le 1$, and $t \in (0,T]$ that

$$\begin{aligned}|u-\hat{u}_\varepsilon^h|^2_{L^2(Q_t)} &\le CA_\varepsilon(t)\varepsilon^{(1+p)/(1-p)} + \\ &\quad + C(k)\varepsilon^{-2}[1+\varepsilon^{-(3-p)/(1-p)}h^4]h^4[\ln(1/h)]^{2r},\end{aligned} \tag{4.28a}$$

$$\begin{aligned}|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_t)} + \|v-\hat{v}_\varepsilon^h\|^2_{E_1(k,t)} \\ \le C\varepsilon^{-1}[A_\varepsilon(t)\varepsilon^{(1+p)/(1-p)}+h^2] + \\ + C(k)\varepsilon^{-3}[1+\varepsilon^{-(3-p)/(1-p)}h^4]h^4[\ln(1/h)]^{2r};\end{aligned} \tag{4.28b}$$

where $r = 0$ if $d = 1$ and $r = ½$ if $d = 2$.

Proof: The result (4.27) follows immediately from (3.19) and (4.12). (4.28) follows similarly with (4.12) replaced by (4.21) and noting (3.1a), (3.4) with $\chi \equiv u_\varepsilon^h$ and (4.9). □

**Corollary 4.2**

Let Assumptions (D5) hold, then for all $h > 0$ and $t \in (0,T]$:

(i) Under no assumptions on non-degeneracy on choosing $\varepsilon = Ch^{1-p} \le \varepsilon_0$, we have that

$$|(u-\hat{u}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)} + |\int_0^t (u-\hat{u}^h_\varepsilon)(\cdot,s)ds|_{H^1(\Omega)} + |\nabla(u-\hat{u}^h_\varepsilon)|_{L^2(Q_T)} \le C(k)h^{(1+p)/2}$$

and (4.29a)

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|_{L^2(Q_T)} + |(v-\hat{v}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)} \le C(k)h^p. \qquad (4.29b)$$

(ii) On assuming (N.D.) and choosing $\varepsilon = Ch^{4(1-p)/(5-p)} \le \varepsilon_0$ we have that

$$|(u-\hat{u}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)} + |\int_0^t (u-\hat{u}^h_\varepsilon)(\cdot,s)ds|_{H^1(\Omega)} + |\nabla(u-\hat{u}^h_\varepsilon)|_{L^2(Q_T)}$$
$$\le C(k)h^{(3+p)/(5-p)} \qquad (4.30a)$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|_{L^2(Q_T)} + |(v-\hat{v}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)} \le C(k)h^{(1+3p)/(5-p)}. \qquad (4.30b)$$

Proof: The results follow directly from (4.27), (4.12), (2.16), (3.8), (3.13), (1.8), (3.4) and (4.6). □

**Corollary 4.3**

Let Assumptions (D6) then for all $t \in (0,T]$

(i) Under no assumptions on non-degeneracy and on choosing $\varepsilon = C\{h^2[\ln(1/h)]^r\}^{2(1-p)/(3-p)} \le \tilde{\varepsilon}_0$, we have for all $h \le h_0(k)$

$$|u-\hat{u}^h_\varepsilon|_{L^2(Q_T)} \le C(k)\{h^2[\ln(1/h)]^r\}^{(1+p)/(3-p)}, \qquad (4.31a)$$

$$|(u-\hat{u}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)} + |\int_0^t (u-\hat{u}^h_\varepsilon)(\cdot,s)ds|_{H^1(\Omega)} + |\nabla(u-\hat{u}^h_\varepsilon)|_{L^2(Q_T)}$$
$$\le C(k) \min\left[ h\ ,\ \{h^2[\ln(1/h)]^r\}^{(1+p)/(3-p)} \right] \qquad (4.31b)$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|_{L^2(Q_T)} + |(v-\hat{v}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)}$$
$$\le C(k)\{h^2[\ln(1/h)]^r\}^{2p/(3-p)}. \qquad (4.31c)$$

(ii) On assuming (N.D.) and choosing $\varepsilon = C\{h^2[\ln(1/h)]^r\}^{4(1-p)/(7-3p)} \leq \tilde{\varepsilon}_0$, we have for all $h \leq h_0(k)$

$$|(u-\hat{u}^h_\varepsilon)|_{L^2(Q_T)} \leq C(k)\{h^2[\ln(1/h)]^r\}^{(3+p)/(7-3p)}, \tag{4.32a}$$

$$|(u-\hat{u}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)} + \left|\int_0^t (u-\hat{u}^h_\varepsilon)(\cdot,s)ds\right|_{H^1(\Omega)} + |\nabla(u-\hat{u}^h_\varepsilon)|_{L^2(Q_T)}$$
$$\leq C(k)\min\left[h\ ,\ \{h^2[\ln(1/h)]^r\}^{(3+p)/(7-3p)}\right] \tag{4.32b}$$

and

$$|\varphi(u)-\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|_{L^2(Q_T)} + |(v-\hat{v}^h_\varepsilon)(\cdot,t)|_{L^2(\Omega)}$$
$$\leq C(k)\{h^2[\ln(1/h)]^r\}^{(1+3p)/(7-3p)}. \tag{4.32c}$$

Proof: The results follow directly from (4.28), (4.21), (2.16), (3.8), (3.13) and (1.8). □

Let problem $(\hat{P}^h)$ be the same as $(\hat{P}^h_\varepsilon)$ with $\varphi_\varepsilon$ replaced by $\varphi$.

**Theorem 4.3**

Under the Assumptions (D4) there exists a unique solution $\{\hat{u}^h, \hat{v}^h\}$ to $(\hat{P}^h)$ for all $h > 0$ and for all $\varepsilon \in (0, \varepsilon_0]$ and $t \in (0,T]$ we have that

$$\|\hat{u}^h-\hat{u}^h_\varepsilon\|^2_{E_2(k,t)} + k^{-2}\varepsilon|\nabla(\hat{u}^h-\hat{u}^h_\varepsilon)(\cdot,t)|^2_{L^2(\Omega)} + \varepsilon|\pi_h[\varphi(\hat{u}^h)-\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_t)} +$$
$$+ \varepsilon\|\hat{v}^h-\hat{v}^h_\varepsilon\|^2_{E_1(k,t)} \leq C\varepsilon^{(1+p)/(1-p)}. \tag{4.33}$$

Under the Assumptions (D5) if $g_1$, $g_2$ and $f \geq 0$ then $\hat{u}^h$, $\hat{v}^h \geq 0$ in $Q_T$. Moreover, under the Asumptions (D5) and (D6) the error bounds (4.29) and (4.31), respectively, hold with $\{\hat{u}^h_\varepsilon, \hat{v}^h_\varepsilon, \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ replaced by $\{\hat{u}^h, \hat{v}^h, \varphi(\hat{u}^h)\}$.

Proof: Existence and uniqueness of a solution and (4.33) follow from a discrete analogue of the proof of Theorem 2.2. Combining (4.33) with (4.29a&b) and (4.31a-c) yields the desired error bounds. □

## 5. A FULLY DISCRETE AND PRACTICAL FINITE ELEMENT APPROXIMATION

In this section we analyse the following fully discrete practical approximation to $(P_\varepsilon)$ with timestep $\tau = T/N$ :

$(\hat{P}^{h,\tau}_\varepsilon)$ For $n = 1 \to N$ find $\hat{u}^{h,n}_\varepsilon \in S^h_0$ and $\hat{v}^{h,n}_\varepsilon \in S^h$ such that

$$\tau^{-1}((\hat{u}^{h,n}_\varepsilon - \hat{u}^{h,n-1}_\varepsilon) + (\hat{v}^{h,n}_\varepsilon - \hat{v}^{h,n-1}_\varepsilon),\chi)^h + (\nabla\hat{u}^{h,n}_\varepsilon,\nabla\chi) = (f^n,\chi)^h \quad \forall\, \chi \in S^h_0$$

$$\tau^{-1}(\hat{v}^{h,n}_\varepsilon - \hat{v}^{h,n-1}_\varepsilon,\chi)^h = k\,(\varphi_\varepsilon(\hat{u}^{h,n}_\varepsilon) - \hat{v}^{h,n}_\varepsilon,\chi)^h \quad \forall\, \chi \in S^h$$

$$\hat{u}^{h,0}_\varepsilon(\cdot) = \pi_h g_1(\cdot) \qquad \hat{v}^{h,0}_\varepsilon(\cdot) = \pi_h g_2(\cdot),$$

where $f^n(\cdot) \equiv f(\cdot,n\tau)$.

Let $\hat{U}_\varepsilon \in L^\infty(0,T;S^h_0)$ and $\hat{V}_\varepsilon \in L^\infty(0,T;S^h)$ be such that for $n = 1 \to N$

$$\hat{U}_\varepsilon(\cdot,t) \equiv \hat{u}^{h,n}_\varepsilon(\cdot) \quad\text{and}\quad \hat{V}_\varepsilon(\cdot,t) \equiv \hat{v}^{h,n}_\varepsilon(\cdot) \quad \text{if } t \in ((n-1)\tau,n\tau].$$

**Theorem 5.1**

Let the Assumptions (D4) hold. Then for all $\varepsilon \in (0,\varepsilon_0]$, $h$, $\tau > 0$ there exists a unique solution $\{\hat{U}_\varepsilon,\hat{V}_\varepsilon\}$ to $(\hat{P}^{h,\tau}_\varepsilon)$. Moreover, if the Assumptions (D5) hold and $g_1$, $g_2$ and $f \geq 0$ then $\hat{U}_\varepsilon$, $\hat{V}_\varepsilon \geq 0$.

Proof: Adopting the notation of the proof of Corollary 4.1, $(\hat{P}^{h,\tau}_\varepsilon)$ can be restated as find $\underline{\alpha}^n \equiv \{\alpha^n_j\}^I_{j=1}$, $\underline{\beta}^n \equiv \{\beta^n_j\}^I_{j=1}$, where $\hat{u}^{h,n}_\varepsilon(x) \equiv \sum_{j=1}^{I} \alpha^n_j\chi_j(x)$, $\hat{v}^{h,n}_\varepsilon(x) \equiv \sum_{j=1}^{I}\beta^n_j\chi_j(x) + (1+k\tau)^{-n}\sum_{j=I+1}^{J} g_2(x_j)\chi_j(x)$, such that $\alpha^0_j \equiv g_1(x_j)$, $\beta^0_j \equiv g_2(x_j)$ $j = 1 \to I$ and for $n = 1 \to N$

$$F(\underline{\alpha}^n) \equiv (I+\tau\tilde{B})\underline{\alpha}^n + (1+k\tau)^{-1}k\tau\underline{\varphi}_\varepsilon(\underline{\alpha}^n) = \underline{\alpha}^{n-1} + (1+k\tau)^{-1}k\tau\underline{\beta}^{n-1} + \tau\underline{f}^n \tag{5.1a}$$

$$\underline{\beta}^n = (1+k\tau)^{-1}[\underline{\beta}^{n-1} + k\tau\underline{\varphi}_\varepsilon(\underline{\alpha}^n)]. \tag{5.1b}$$

As $I+\tau\tilde{B}$ is positive definite and $\underline{\varphi}_\varepsilon$ is a continuous diagonal isotone mapping, the existence of a unique solution to (5.1a), and hence (5.1b), is easily established. Furthermore, under Assumptions (D5) $I+\tau\tilde{B}$ is a M-matrix and hence the mapping $F(\cdot)$ is inverse isotone and a homeomorphism of $\mathbb{R}^I$ into itself, see §13.5 of Ortega & Rheinboldt (1970). Therefore as $F(\underline{0}) = \underline{0}$, it follows that $\underline{\alpha}^0$, $\underline{\beta}^0$, $\underline{f}^n \geq \underline{0} \Rightarrow \underline{\alpha}^n$, $\underline{\beta}^n \geq \underline{0}$ for $n = 1 \to N$. □

**Lemma 5.1**

Under the Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h$, $\tau > 0$ and $m = 0 \to N$ that

$$\|\hat{u}^h_\varepsilon - \hat{U}_\varepsilon\|^2_{E_2(k,m\tau)} + \varepsilon|\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon) - \varphi_\varepsilon(\hat{U}_\varepsilon)]|^2_{L^2(Q_{m\tau})} + \varepsilon\|\hat{v}^h_\varepsilon - \hat{V}_\varepsilon\|^2_{E_1(k,m\tau)}$$
$$\leq C\tau^2\Big\{ |\partial_t\hat{u}^h_\varepsilon|^2_{L^2(Q_T)} + k|\nabla(\partial_t\hat{u}^h_\varepsilon)|^2_{L^2(Q_T)} + |\partial_t\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)]|^2_{L^2(Q_T)} +$$
$$+ |\partial_t\hat{v}^h_\varepsilon|^2_{L^2(Q_T)} + |\partial_t[\pi_h f]|^2_{L^2(Q_T)} \Big\}. \qquad (5.2)$$

Proof: Let $E^n_u \equiv (\hat{u}^h_\varepsilon - \hat{U}_\varepsilon)(\cdot,n\tau)$, $E^n_v \equiv (\hat{v}^h_\varepsilon - \hat{V}_\varepsilon)(\cdot,n\tau)$. Defining $I^n(w)(\cdot) \equiv w(\cdot,n\tau) - \tau^{-1}\int_{(n-1)\tau}^{n\tau} w(\cdot,s)ds$, we then set $\eta^n \equiv I^n(\hat{u}^h_\varepsilon)$, $\xi^n \equiv I^n(\hat{v}^h_\varepsilon)$, $\mu^n \equiv I^n(\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon)])$ and $\sigma^n \equiv I^n(f)$.. It then follows from $(\hat{P}^h_\varepsilon)$ and $(\hat{P}^{h,\tau}_\varepsilon)$ that $E^0_u = E^0_v = 0$ and for $n = 1 \to N$

$$\tau^{-1}((E^n_u - E^{n-1}_u) + (E^n_v - E^{n-1}_v),\chi)^h + (\nabla E^n_u,\nabla\chi) = (\nabla\eta^n,\nabla\chi) - (\sigma^n,\chi)^h$$
$$\forall\, \chi \in S^h_0 \qquad (5.3a)$$

$$\tau^{-1}(E^n_v - E^{n-1}_v,\chi)^h = k\,([\varphi_\varepsilon(\hat{u}^h_\varepsilon(\cdot,n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau))] - E^n_v,\chi)^h +$$
$$+ k\,(\xi^n - \mu^n,\chi)^h \qquad \forall\, \chi \in S^h. \qquad (5.3b)$$

We note the following identities, assuming $a^0 = 0$,

$$\sum_{n=1}^m [(a^n - a^{n-1})\sum_{i=n}^m b^i] \equiv \sum_{n=1}^m a^n b^n \qquad (5.4a)$$

$$\sum_{n=1}^m [a^n \sum_{i=n}^m b^i] + \sum_{n=1}^m [\sum_{i=n}^m a^i] b^n \equiv (\sum_{n=1}^m a^n \sum_{n=1}^m b^n) + \sum_{n=1}^m a^n b^n \qquad (5.4b)$$

$$\sum_{n=1}^m [(a^n - a^{n-1})a^n] \equiv \tfrac{1}{2}\,[(a^m)^2 + \sum_{n=1}^m (a^n - a^{n-1})^2]\,. \qquad (5.4c)$$

Choosing $\chi \equiv \sum_{i=n}^m E^i_u$ in (5.3a), then summing the equations from $n = 1 \to m$ and noting (5.4a&b) yields

$$\tau\sum_{n=1}^m (E^n_u + E^n_v, E^n_u)^h + \tfrac{1}{2}\,[|\nabla(\tau\sum_{n=1}^m E^n_u)|^2_{L^2(\Omega)} + \tau^2\sum_{n=1}^m |\nabla E^n_u|^2_{L^2(\Omega)}]$$
$$= \{ (\nabla(\tau\sum_{n=1}^m \eta^n),\nabla(\tau\sum_{n=1}^m E^n_u)) + \tau^2\sum_{n=1}^m (\nabla\eta^n,\nabla E^n_u) - \tau\sum_{n=1}^m (\nabla(\tau\sum_{i=n}^m \eta^i),\nabla E^n_u) \} -$$
$$- \{ (\tau\sum_{n=1}^m \sigma^n,\tau\sum_{n=1}^m E^n_u)^h + \tau^2\sum_{n=1}^m (\sigma^n,E^n_u)^h - \tau\sum_{n=1}^m (\tau\sum_{i=n}^m \sigma^i,E^n_u)^h \}. \qquad (5.5)$$

Choosing $\chi \equiv E_u^n$ in (5.3a), then summing the equations from $n = 1 \to m$ and noting (5.4c) yields

$$\tfrac{1}{2} [\, |E_u^m|_h^2 + \sum_{n=1}^{m} |E_u^n - E_u^{n-1}|_h^2 ] + \sum_{n=1}^{m} (E_v^n - E_v^{n-1}, E_u^n)^h + \tau \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)}$$
$$= \tau \sum_{n=1}^{m} [(\nabla\eta^n, \nabla E_u^n) - (\sigma^n, E_u^n)^h]. \tag{5.6}$$

From (5.5) and (5.6) it follows that

$$\tau \sum_{n=1}^{m} |E_u^n|_h^2 + \tfrac{1}{2} [\, |\nabla(\tau \sum_{n=1}^{m} E_u^n)|^2_{L^2(\Omega)} + \tau^2 \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} ] +$$
$$+ k^{-1}\{ \tfrac{1}{2} [\, |E_u^m|_h^2 + \sum_{n=1}^{m} |E_u^n - E_u^{n-1}|_h^2 ] + \tau \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} \} +$$
$$+ \tau \sum_{n=1}^{m} ([\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot, n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot, n\tau))], E_u^n)^h$$
$$= - k^{-1}\tau \sum_{n=1}^{m} (\tau^{-1}(E_v^n - E_v^{n-1}) + kE_v^n - k[\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot, n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot, n\tau))], E_u^n)^h +$$
$$+ \{ (\nabla(\tau \sum_{n=1}^{m} \eta^n), \nabla(\tau \sum_{n=1}^{m} E_u^n)) + \tau^2 \sum_{n=1}^{m} (\nabla\eta^n, \nabla E_u^n) - \tau \sum_{n=1}^{m} (\nabla(\tau \sum_{i=n}^{m} \eta^i), \nabla E_u^n) \} -$$
$$- \{ (\tau \sum_{n=1}^{m} \sigma^n, \tau \sum_{n=1}^{m} E_u^n)^h + \tau^2 \sum_{n=1}^{m} (\sigma^n, E_u^n)^h - \tau \sum_{n=1}^{m} (\tau \sum_{i=n}^{m} \sigma^i, E_u^n)^h \} +$$
$$+ k^{-1}\tau \sum_{n=1}^{m} [(\nabla\eta^n, \nabla E_u^n) - (\sigma^n, E_u^n)^h]. \tag{5.7}$$

From (5.7) and (5.3b) it follows that

$$\tau \sum_{n=1}^{m} |E_u^n|_h^2 + \tfrac{1}{2} [\, |\nabla(\tau \sum_{n=1}^{m} E_u^n)|^2_{L^2(\Omega)} + \tau^2 \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} ] +$$
$$+ k^{-1}\{ \tfrac{1}{2} [\, |E_u^m|_h^2 + \sum_{n=1}^{m} |E_u^n - E_u^{n-1}|_h^2 ] + \tau \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} \} +$$
$$+ \tau \sum_{n=1}^{m} ([\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot, n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot, n\tau))], E_u^n)^h$$
$$\le C\Big\{ |\nabla(\tau \sum_{n=1}^{m} \eta^n)|^2_{L^2(\Omega)} + \tau(\tau + k^{-1}) \sum_{n=1}^{m} |\nabla\eta^n|^2_{L^2(\Omega)} + k\tau \sum_{n=1}^{m} |\nabla(\tau \sum_{i=n}^{m} \eta^i)|^2_{L^2(\Omega)} +$$
$$+ |\tau \sum_{n=1}^{m} \sigma^n|_h^2 + \tau(\tau^2 + k^{-2}) \sum_{n=1}^{m} |\sigma^n|_h^2 + \tau \sum_{n=1}^{m} |\tau \sum_{i=n}^{m} \sigma^i|_h^2 +$$
$$+ \tau \sum_{n=1}^{m} |\mu^n - \xi^n|_h^2 \Big\}. \tag{5.8}$$

Next we note that

$$|\tau \sum_{i=n}^{m} I^i(w)(\cdot)|^2 \le (m-n)\tau^2 \sum_{i=n}^{m} |I^i(w)(\cdot)|^2 \tag{5.9a}$$

and

$$\tau \sum_{i=n}^{m} |I^i(w)(\cdot)|^2 \equiv \tau^{-1} \sum_{i=n}^{m} \Big| \int_{(i-1)\tau}^{i\tau} [s - (i-1)\tau]\, w_s(\cdot, s)\,ds \Big|^2$$
$$\le C\tau^2 \int_{(n-1)\tau}^{m\tau} |w_s(\cdot, s)|^2 ds. \tag{5.9b}$$

Therefore it follows from (5.8), (5.9) and (2.3a) that

$$\tau \sum_{n=1}^{m} |E_u^n|_h^2 + \tfrac{1}{2} \left[ |\nabla(\tau \sum_{n=1}^{m} E_u^n)|^2_{L^2(\Omega)} + \tau^2 \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} \right] +$$

$$+ k^{-1}\{ \tfrac{1}{2} [ |E_u^m|_h^2 + \sum_{n=1}^{m} |E_u^n - E_u^{n-1}|_h^2 ] + \tau \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} \} +$$

$$+ M^{-1}\varepsilon\, \tau \sum_{n=1}^{m} |\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau))|_h^2$$

$$\le C\tau^2 \Big\{ |\partial_t\{\hat{v}_\varepsilon^h - \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\}|^2_{L^2(Q_T)} + k|\nabla(\partial_t \hat{u}_\varepsilon^h)|^2_{L^2(Q_T)} +$$

$$+ \int_0^T |\partial_t f(\cdot,t)|_h^2 \, dt \Big\}. \qquad (5.10)$$

Noting bounds like

$$\int_0^{m\tau} |(\hat{u}_\varepsilon^h - \hat{U}_\varepsilon)(\cdot,t)|_h^2 \, dt \le 2\tau \sum_{n=1}^{m} |E_u^n|_h^2 + 2 \sum_{n=1}^{m} \int_{(n-1)\tau}^{n\tau} |\hat{u}_\varepsilon^h(\cdot,t) - \hat{u}_\varepsilon^h(\cdot,n\tau)|_h^2 \, dt, \qquad (5.11a)$$

$$\sum_{n=1}^{m} \int_{(n-1)\tau}^{n\tau} |\hat{u}_\varepsilon^h(\cdot,t) - \hat{u}_\varepsilon^h(\cdot,n\tau)|_h^2 \, dt \le \sum_{n=1}^{m} \int_{(n-1)\tau}^{n\tau} \Big| \int_{n\tau}^{t} \partial_s \hat{u}_\varepsilon^h(\cdot,s) \, ds \Big|_h^2 \, dt$$

$$\le \tau^2 \int_0^{m\tau} |\partial_t \hat{u}_\varepsilon^h(\cdot,t)|_h^2 \, dt, \qquad (5.11b)$$

(5.9) and (4.1a) it follows that

$$\| \hat{u}_\varepsilon^h - \hat{U}_\varepsilon \|^2_{E_2(k,m\tau)} \le$$

$$C_1 \Big\{ \tau \sum_{n=1}^{m} |E_u^n|_h^2 + \tfrac{1}{2} |\nabla(\tau \sum_{n=1}^{m} E_u^n)|^2_{L^2(\Omega)} + k^{-1}[ \tfrac{1}{2} |E_u^m|_h^2 + \tau \sum_{n=1}^{m} |\nabla E_u^n|^2_{L^2(\Omega)} ] \Big\} +$$

$$+ C_2 \tau^2 \Big\{ |\partial_t \hat{u}_\varepsilon^h|^2_{L^2(Q_T)} + |\nabla(\partial_t \hat{u}_\varepsilon^h)|^2_{L^2(Q_T)} \Big\} \qquad (5.12a)$$

and

$$|\pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h) - \varphi_\varepsilon(\hat{U}_\varepsilon)]|^2_{L^2(Q_{m\tau})} \le C_3 \tau \sum_{n=1}^{m} |\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau))|_h^2 +$$

$$+ C_4 \tau^2 |\partial_t \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]|^2_{L^2(Q_T)}. \qquad (5.12b)$$

Finally, choosing $\chi \equiv E_v^n$ in (5.3b), then summing the equations from $n = 1 \to m$ and noting (5.4c), (5.10) and (5.9) yields

$$\tfrac{1}{2} k^{-1} [ (E_v^m)^2 + \sum_{n=1}^{m} (E_v^n - E_v^{n-1})^2 ] + \tau \sum_{n=1}^{m} |E_v^n|_h^2$$

$$= \tau \sum_{n=1}^{m} [ ([\varphi_\varepsilon(\hat{u}_\varepsilon^h(\cdot,n\tau)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,n\tau))], E_v^n)^h + (\xi^n - \mu^n, E_v^n)^h ]$$

$$\le C\varepsilon^{-1}\tau^2 \Big\{ |\partial_t\{\hat{v}_\varepsilon^h - \pi_h[\varphi_\varepsilon(\hat{u}_\varepsilon^h)]\}|^2_{L^2(Q_T)} + |\nabla(\partial_t \hat{u}_\varepsilon^h)|^2_{L^2(Q_T)} +$$

$$+ \int_0^T |\partial_t f(\cdot,t)|_h^2 \, dt \Big\}. \qquad (5.13)$$

Similarly to (5.12) we have that

$$\|\hat{v}^h_\varepsilon - \hat{V}_\varepsilon\|^2_{E_1(k,m\tau)} \le C_1 \; [\tau \sum_{n=1}^{m} |E^n_v|^2_h + \tfrac{1}{2} k^{-1} |E^m_v|^2] + C_2 \tau^2 |\partial_t \hat{v}^h_\varepsilon|^2_{L^2(Q_T)}. \tag{5.14}$$

Combining (5.10), (5.12), (5.13) and (5.14) yields the desired result (5.2).□

**Corollary 5.1**

Under the Assumptions (D4) we have for all $\varepsilon \in (0,\varepsilon_0]$, $h$, $\tau > 0$ and $m = 0 \to N$ that

$$\|\hat{u}^h_\varepsilon - \hat{U}_\varepsilon\|^2_{E_2(k,m\tau)} + \varepsilon |\pi_h[\varphi_\varepsilon(\hat{u}^h_\varepsilon) - \varphi_\varepsilon(\hat{U}_\varepsilon)]|^2_{L^2(Q_{m\tau})} + \varepsilon \|\hat{v}^h_\varepsilon - \hat{V}_\varepsilon\|^2_{E_1(k,m\tau)}$$
$$\le C(k)\varepsilon^{-1}\tau^2. \tag{5.15}$$

Proof: The result (5.15) follows immediately from (5.2) and the bounds (4.2).

□

**Theorem 5.2**

(a) Let Assumptions (D5) hold. Then for the stated choice of $\varepsilon$, on choosing $\tau \le Ch$ we have that the error bounds (4.29) and (4.30) hold for $t = m\tau$, $m = 0 \to N$, with $\{\hat{u}^h_\varepsilon, \hat{v}^h_\varepsilon\}$ replaced by $\{\hat{U}_\varepsilon, \hat{V}_\varepsilon\}$.

(b) Let Assumptions (D6) hold. Then for the stated choice of $\varepsilon$, on choosing $\tau \le C\varepsilon^{-1/2} h^2[\ln(1/h)]^r$ we have that the error bounds (4.31) and (4.32) hold for $t = m\tau$, $m = 0 \to N$, with $\{\hat{u}^h_\varepsilon, \hat{v}^h_\varepsilon\}$ replaced by $\{\hat{U}_\varepsilon, \hat{V}_\varepsilon\}$.

Proof: These results follow from balancing the terms (5.15), (4.27), (4.28) (4.21), (4.12) , (2.16), (3.8), (3.13) and (1.8). In case (a) it follows that we require $\varepsilon^{-1}\tau^2 \le C\varepsilon^{-1}h^2$ and in case (b) $\varepsilon^{-1}\tau^2 \le C\varepsilon^{-2}\{h^2[\ln(1/h)]^r\}^2$. □

Let $(\hat{P}^{h,\tau})$ be the same as problem $(\hat{P}^{h,\tau}_\varepsilon)$ with $\varphi_\varepsilon$ replaced by $\varphi$.

**Theorem 5.3**

Under the Assumptions (D4) hold there exists a unique solution $\{\hat{U},\hat{V}\}$ to $(\hat{P}^{h,\tau})$ for all h and $\tau > 0$ and for all $\varepsilon \in (0,\varepsilon_0]$ and $m = 0 \to N$ we have that

$$\|\hat{U}-\hat{U}_\varepsilon\|^2_{E_2(k,m\tau)} + \varepsilon|\pi_h[\varphi(\hat{U})-\varphi_\varepsilon(\hat{U}_\varepsilon)]|^2_{L^2(Q_{m\tau})} + \varepsilon\|\hat{V}-\hat{V}_\varepsilon\|^2_{E_1(k,m\tau)}$$
$$\leq C\varepsilon^{(1+p)/(1-p)}. \qquad (5.16)$$

Moreover, under the Assumptions (D5) we have (i) if $g_1$, $g_2$ and $f \geq 0$ then $\hat{U}$, $\hat{V} \geq 0$ in $Q_T$, (ii) on choosing $\tau \leq Ch$ we have that the error bounds (4.29) hold for $t = m\tau$, $m = 0 \to N$, with $\{\hat{u}^h_\varepsilon, \hat{v}^h_\varepsilon, \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ replaced by $\{\hat{U},\hat{V},\varphi(\hat{U})\}$. Under the Assumptions (D6) on choosing $\tau \leq C\{h^2[\ln(1/h)]^r\}^{2/(3-p)}$ the error bounds (4.31) hold for $t = m\tau$, $m = 0 \to N$, with $\{\hat{u}^h_\varepsilon, \hat{v}^h_\varepsilon, \varphi_\varepsilon(\hat{u}^h_\varepsilon)\}$ replaced by $\{\hat{U},\hat{V},\varphi(\hat{U})\}$.

Proof: Existence and uniqueness of a solution follows as for $(\hat{P}^{h,\tau}_\varepsilon)$, see the proof of Theorem 5.1. Let $E^n_{u,\varepsilon}(\cdot) \equiv (\hat{u}^{h,n}-\hat{u}^{h,n}_\varepsilon)(\cdot) \equiv (\hat{U}-\hat{U}_\varepsilon)(\cdot,n\tau)$ and $E^n_{v,\varepsilon}(\cdot) \equiv (\hat{v}^{h,n}-\hat{v}^{h,n}_\varepsilon)(\cdot) \equiv (\hat{V}-\hat{V}_\varepsilon)(\cdot,n\tau)$. It follows from $(\hat{P}^{h,\tau})$ and $(\hat{P}^{h,\tau}_\varepsilon)$ that $E^n_{u,\varepsilon} = 0$ and $E^n_{v,\varepsilon} = 0$ and for $n = 1 \to N$

$$\tau^{-1}((E^n_{u,\varepsilon} - E^{n-1}_{u,\varepsilon}) + (E^n_{v,\varepsilon} - E^{n-1}_{v,\varepsilon}),\chi)^h + (\nabla E^n_{u,\varepsilon},\nabla\chi) = 0 \quad \forall\ \chi \in S^h_0 \qquad (5.17a)$$

$$\tau^{-1}(E^n_{v,\varepsilon} - E^{n-1}_{v,\varepsilon},\chi)^h = k\ ([\varphi(\hat{u}^{h,n})-\varphi_\varepsilon(\hat{u}^{h,n}_\varepsilon)] - E^n_{v,\varepsilon},\chi)^h \quad \forall\ \chi \in S^h. \qquad (5.17b)$$

Choosing $\chi \equiv \sum_{i=n}^{m} E^i_{u,\varepsilon}$ and $\chi \equiv E^n_{u,\varepsilon}$ in (5.17a), summing from $n = 1 \to m$ and noting (5.4) yields, in a similar manner to (5.5) and (5.6), that

$$\tau\sum_{n=1}^{m}(E^n_{u,\varepsilon} + E^n_{v,\varepsilon},E^n_{u,\varepsilon})^h + \tfrac{1}{2}\ [\ |\nabla(\tau\sum_{n=1}^{m}E^n_{u,\varepsilon})|^2_{L^2(\Omega)} + \tau^2\sum_{n=1}^{m}|\nabla E^n_{u,\varepsilon}|^2_{L^2(\Omega)}] = 0$$
$$(5.18a)$$

and

$$\tfrac{1}{2}\ [\ |E^m_{u,\varepsilon}|^2_h + \sum_{n=1}^{m}|E^n_{u,\varepsilon}-E^{n-1}_{u,\varepsilon}|^2_h] + \sum_{n=1}^{m}(E^n_{v,\varepsilon} - E^{n-1}_{v,\varepsilon},E^n_{u,\varepsilon})^h + \tau\sum_{n=1}^{m}|\nabla E^n_{u,\varepsilon}|^2_{L^2(\Omega)} = 0,$$
$$(5.18b)$$

respectively. From (5.18) and (5.17b) we have the discrete analogue of (2.13):

$$\tau \sum_{n=1}^{m} |E^n_{u,\varepsilon}|^2_h + \tfrac{1}{2} [\, |\nabla(\tau \sum_{n=1}^{m} E^n_{u,\varepsilon})|^2_{L^2(\Omega)} + \tau^2 \sum_{n=1}^{m} |\nabla E^n_{u,\varepsilon}|^2_{L^2(\Omega)} ] +$$
$$+ k^{-1}\{ \tfrac{1}{2} [\, |E^m_{u,\varepsilon}|^2_h + \sum_{n=1}^{m} |E^n_{u,\varepsilon} - E^{n-1}_{u,\varepsilon}|^2_h ] + \tau \sum_{n=1}^{m} |\nabla E^n_{u,\varepsilon}|^2_{L^2(\Omega)} \}$$
$$= -\tau \sum_{n=1}^{m} (\varphi(\hat{u}^{h,n}) - \varphi_\varepsilon(\hat{u}^{h,n}_\varepsilon), E^n_{u,\varepsilon})^h. \tag{5.19}$$

From (5.19), (4.1a) and (2.3) we have that for all m = 0 → N that

$$\|\hat{U} - \hat{U}_\varepsilon\|^2_{E_2(k,m\tau)} + M^{-1}\varepsilon |\pi_h[\varphi(\hat{U}) - \varphi_\varepsilon(\hat{U}_\varepsilon)]|^2_{L^2(Q_{m\tau})}$$
$$\le \int_0^{m\tau} (\varphi(\hat{U}(\cdot,s)) - \varphi_\varepsilon(\hat{U}_\varepsilon(\cdot,s)), (\zeta - \hat{U})(\cdot,s))^h ds \le C\varepsilon^{(1+p)/(1-p)}, \tag{5.20}$$

where $\zeta \equiv \varphi_\varepsilon^{-1}(\varphi(\hat{U}))$ if $\varphi(\hat{U}) \in (0, \varphi(\varepsilon^{1/(1-p)}))$ and $\zeta \equiv \hat{U}$ otherwise. Choosing $\chi \equiv E^n_v$ in (5.17b), summing from n = 1 → m and noting (4.1a) and (5.4c), yields for all m = 0 → N that

$$\|\hat{V} - \hat{V}_\varepsilon\|^2_{E_1(k,m\tau)} \le \tfrac{1}{2} k^{-1} [\, |E^m_{v,\varepsilon}|^2_h + \sum_{n=1}^{m} |E^n_{v,\varepsilon} - E^{n-1}_{v,\varepsilon}|^2_h ] + \tau \sum_{n=1}^{m} |E^n_{v,\varepsilon}|^2_h$$
$$= \tau \sum_{n=1}^{m} (\varphi(\hat{u}^{h,n}) - \varphi_\varepsilon(\hat{u}^{h,n}_\varepsilon), E^n_{v,\varepsilon})^h. \tag{5.21}$$

Combining (5.20), (5.21) and (4.1a) yields the desired result (5.16). The desired error bounds then follow from combining (5.16) with (4.29) and (4.31). □

**Remark 5.1**

It is a simple matter to adapt the results in this section to analyse a time discretisation of $(P^h_\varepsilon)$. However this is not so interesting as it is not a practical scheme, requiring the nonlinear term to be integrated exactly.

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