A UNIFORM DIMENSION RESULT FOR THE BROWNIAN SNAKE

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April 7, 1999

AMS Subject Classification (1991): 60G57, 60G17, 28A80.
Key words: super-Brownian motion, Brownian snake, Hausdorff dimension.

*On leave from: Fachbereich Mathematik, Universität Kaiserslautern, 67663 Kaiserslautern, Germany. This project was supported by the DFG Graduiertenkolleg "Stochastische Prozesse" of HU, TU and WIAS Berlin.

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1 Introduction

A very useful result in the study of the Hausdorff dimension of exceptional subsets of a Brownian path is the following uniform dimension theorem of Kaufman [RK69].

Theorem 1 *Suppose B is a Brownian motion in dimension at least 2. Then, almost surely, for every Borel set $A \subset [0,1]$, we have $\dim B(A) = 2\dim(A)$, where* dim *denotes the Hausdorff dimension.*

The strength of the statement lies in the "uniformity", the fact that the exceptional nullset of paths does not depend on A. It is due to this fact that the theorem can be used to reduce the study of random subsets of a Brownian path to the study of random subsets $A \subset [0,1]$.

It is the aim of the present note to prove an analogous theorem that enables us to reduce the study of the Hausdorff dimension of exceptional subsets in the support of a super-Brownian motion X_t to the study of random subsets of the real line. The necessary time dynamics is provided by Le Gall's Brownian snake, which we briefly recall now.

The Brownian snake is a continuous strong Markov process $\{W_s = (w_s, \zeta_s) : s \geq 0\}$ with values in the set

$$\mathcal{W} := \Big\{(w,\zeta) \,:\, w : [0,\infty) \to \mathbb{R}^d \text{ continuous with } w(s) = w(\zeta) \text{ for all } s \geq \zeta \geq 0\Big\}$$

of stopped paths equipped with the metric

$$d\Big((w_1,\zeta_1),(w_2,\zeta_2)\Big) = \|w_1 - w_2\|_\infty + |\zeta_1 - \zeta_2| \,.$$

Denote by $\Omega = \mathcal{C}([0,\infty),\mathcal{W})$ the space of continuous mappings from $[0,\infty)$ to $\mathcal{W}$ equipped with the Borel-σ-algebra $\mathcal{A}$ coming from the compact-open topology and by $W = \{W_s\}_{s\geq 0}$ the coordinate process on Ω. Let $\hat{W}_s = w_s(\zeta_s)$ be the endpoint of W_s. By $\mathbb{P}_x$ denote the law on $(\Omega,\mathcal{A})$ of the path-valued process associated with d-dimensional Brownian motion, or *Brownian snake*, starting in the constant path at x of length 0. This law was introduced in [LG91] and [LG93] and can be described as follows. First, the *lifetime process* $\{\zeta_s\}$ is a reflected Brownian motion. Second, the distribution of $\{w_s\}$ given $\{\zeta_s\}$ is that of an inhomogenuous Markov process whose transition kernel from $s > 0$ to $r > s$ is described by

(i) $w_r(u) = w_s(u)$ for all $u \leq m := \inf_{s \leq u \leq r} \zeta_u$.

(ii) conditionally given m the process $\{w_r(m+t) \,:\, 0 \leq t \leq \zeta_r - m\}$ is independent of w_s and distributed as a Brownian path started in $w_s(m)$ and stopped at time $\zeta_r - m$.

The intuitive picture is that $\{w_s\}$ grows like a Brownian motion in $\mathbb{R}^d$ when $\{\zeta_s\}$ is increasing and is erased, when $\{\zeta_s\}$ is decreasing (this is merely heuristical, as $\{\zeta_s\}$, of course, has neither points of increase, nor of decrease). We denote by $\{L^t_s : s \geq 0\}$ the continuous local time of $\{\zeta_s\}$ at level t, normalized to be a density of the occupation measure of $\{\zeta_s\}$, and by $\tau = \tau(\zeta) = \inf\{s \geq 0 : L^0_s = 1\}$ the inverse local time at 1. Then

$$X_t(A) = \int_0^\tau 1_A(\hat{W}_s)\, L^t_\cdot(ds) \quad \text{, for } A \subseteq \mathbb{R}^d \text{ Borel,} \tag{1}$$

defines a Markov process with values in the space $\mathcal{M}_F(\mathbb{R}^d)$ of finite measures on $\mathbb{R}^d$ with the vague topology. This process is a *super-Brownian motion started in* δ_x. Here is the main theorem of this paper.

Theorem 2 *Suppose W is a Brownian snake in dimension at least 2. Then, $\mathbb{P}_x$-almost surely, for every $t > 0$ and every Borel set $A \subset \{s \in [0, \tau] : \zeta_s = t\}$, we have* $\dim \hat{W}(A) = 4 \dim(A)$.

Remarks:
- Note that the $\mathbb{P}_x$-nullset in the theorem is independent of t and A.
- Recall that $A \subset \{s \in [0, \tau] : \zeta_s = t\}$ implies that $\hat{W}(A) \subset \mathrm{supp} X_t$ except for an at most countable set of points.
- An analogous result can be proved for *packing* instead of Hausdorff dimension.

Theorem 2 is new only in dimensions $d = 2, 3$. In dimensions $d \geq 4$ Serlet [LS95] proved the following stronger result.

Theorem 3 *Suppose W is a Brownian snake in dimension at least 4. Then, $\mathbb{P}_x$-almost surely, for every Borel set $A \subset [0, \tau]$ we have* $\dim \hat{W}(A) = 4 \dim(A)$.

It is not hard to convince oneself, that this stronger result cannot hold in dimension $d = 2, 3$ even if we restrict attention to sets A of small dimension. For the study of exceptional sets in the support of X_t, however, the statement of Theorem 2 is generally sufficient. We hope to give full evidence of the applicability of the theorem in future research. For the moment we just mention one immediate consequence: We obtain the dimension of X_t uniformly for all $t > 0$, a result first established by Perkins in [EP88].

Corollary 4 *For super-Brownian motion in dimension at least 2, almost surely, for every $t > 0$, the carrying dimension of X_t is 2. In fact,* $\dim(\mathrm{supp}\ X_t) = 2$ *and for every Borel set $A \subset \mathbb{R}^d$ with* $\dim(A) < 2$ *we have* $X_t(A) = 0$.

Proof. It is known, see for example [LG91], that $\mathrm{supp} X_t$ and $\{\hat{W}_s : \zeta_s = t\}$ coincide except for an at most countable set. Hence, by Theorem 2, $\dim(\mathrm{supp} X_t) = 2$ follows from the fact that, almost surely, for all $t > 0$, $\dim\{s : \zeta_s = t\} = 1/2$. On the other hand, if $A \subset \mathrm{supp} X_t$ and $\dim(A) < 2$, then $\dim \hat{W}^{-1}(A) < 1/2$ and hence $\hat{W}^{-1}(A)$ is not charged by Brownian local time. Therefore, by (1), $X_t(A) = 0$. ■

In order to prove Theorem 2 we follow a modification of the general plan of Serlet and we make direct use of some of the steps of his proof. The crucial new part is Lemma 9 below, which may also be used as an alternative to the corresponding statement [LS95, Lemma 8] (proved in [LS93]) in Serlet's proof of Theorem 3.

2 Proof of the theorem

We begin by recalling from [LG93] that, for every $\delta > 0$, the mapping $s \mapsto \hat{W}_s$ is locally Hölder continuous with exponent $(4+\delta)^{-1}$. Hence we get the upper bound $\dim \hat{W}(A) \leq 4 \dim(A)$ without any effort. It remains to prove the lower bound. For this purpose we have to show that, on a set of full $\mathbb{P}_x$-measure, for every subset $H \subset \mathbb{R}^d$ and every $t > 0$, we have

$$\dim\{s \in (0,\tau) \,:\, \zeta_s = t\,,\, \hat{W}_s \in H\} \leq \frac{\dim H}{4}\,. \tag{2}$$

The lower bound then follows by choosing $H = \{\hat{W}_s \,:\, s \in A\}$. Without loss of generality we may assume that our Brownian snake is started in the origin, that H is contained in a unit cube $C = a + [0,1)^d \subset \mathbb{R}^d$ at positive distance from the origin and t is contained in a unit interval $I = b + [0,1]$, for some $b > 0$.

We first define a set of full $\mathbb{P}_0$-measure such that our dimension property (2) holds for all W in the set. For this purpose let $\{x_m \,:\, m \geq 1\}$ be the set of dyadic points in the cube C indexed such that, for every n,

$$\Big\{x_m \,:\, 1 \leq m \leq 2^{nd}\Big\} = \Big\{a + \Big(\frac{k_1}{2^n}, \ldots, \frac{k_d}{2^n}\Big) \,:\, 0 \leq k_i \leq 2^n - 1\Big\}\,.$$

Next, let $\{a_m \,:\, m \geq 1\}$ be the set of dyadic points in the interval I indexed such that, for every n, $\{a_m \,:\, 0 \leq m \leq 2^n\} = \{b + \frac{k}{2^n} \,:\, 0 \leq k \leq 2^n\}$. Define $I_m(\varepsilon) = [a_m - \varepsilon, a_m + \varepsilon]$. The *occupation measure* Z for the Brownian snake is defined by

$$Z(J,A) = \int_0^\infty 1_J(\zeta_s) 1_A(\hat{W}_s)\, ds \text{ for all } A \subset \mathbb{R}^d \text{ and } J \subset (0,\infty) \text{ Borel.}$$

Also, we have to define the succesive entrance times in a ball $B(x,\varepsilon)$ and exit times from the ball $B(x, 2\varepsilon)$ by

$$T_1(x,\varepsilon) = \inf\{s \geq 0 \,:\, \hat{W}_s \in B(x,\varepsilon)\}\,,$$

and, for $k \geq 1$,

$$\begin{aligned} S_k(x,\varepsilon) &= \inf\{s > T_k(x,\varepsilon) \,:\, \hat{W}_s \notin B(x,2\varepsilon)\}\,,\\ T_{k+1}(x,\varepsilon) &= \inf\{s > S_k(x,\varepsilon) \,:\, \hat{W}_s \in B(x,\varepsilon)\}\,. \end{aligned}$$

By passing to a subsequence, if necessary, we may assume that $\zeta([S_k(x,\varepsilon), T_k(x,\varepsilon)]) \cap I \neq \emptyset$ for all k. Finally, denote $L(x) = 1 + |\log(x)|$. This completes the notation needed to formulate our lemma.

Lemma 5 *For every $\delta > 0$ there is a set $\Omega(\delta) \subset \Omega$ of full $\mathbb{P}_0$-measure, a constant $c > 1$ and a random variable Q, which is finite on $\Omega(\delta)$, such that, for every path $W = (w, \zeta) \in \Omega(\delta)$, every $\varepsilon \in \{2^{-q} \,:\, q \geq Q\}$ and all integers $m, n \geq 1$,*

(i) $Z\Big(I_m(\varepsilon^2), B(x_n, 2\varepsilon)\Big) \leq cL(m)^2 L(n)^2 L(\varepsilon)^2\, \varepsilon^4$,

(ii) $S_k(x_n,\varepsilon) - T_k(x_n,\varepsilon) \leq cL(n)^3 L(k)^3 L(\varepsilon)^3 \varepsilon^4$, *for every k such that $T_k(x_n,\varepsilon) < \tau$,*

(iii) $|\hat{W}_s - \hat{W}_t| \le |s-t|^{1/(4+\delta)}$, *for all* $0 \le s, t \le \tau$ *such that* $|s-t| \le 2^{-Q}$,

(iv) $|\zeta_s - \zeta_t| \le |s-t|^{1/(2+\delta)}$, *for all* $0 \le s, t \le \tau$ *such that* $|s-t| \le 2^{-Q}$.

Proof. It is clear from the fact that $s \mapsto \hat{W}_s$ is locally $1/(4+\delta)$-Hölder continuous and that $s \mapsto \zeta_s$ is locally $1/(2+\delta)$-Hölder continuous that properties (iii) and (iv) can be realized on a set of full measure. For property (ii) we use the following lemma from Serlet [LS95, Lemma 9].

Lemma 6 *There exist positive constants* c_1 *and* c_2 *such that, for every* $x \in C$, $0 < \varepsilon < 1/4$, $0 < \lambda < \infty$ *and integers* $k \ge 1$,

$$\mathbb{P}_0\Big(S_k(x,\varepsilon) - T_k(x,\varepsilon) > \lambda\varepsilon^4 \,\Big|\, T_k(x,\varepsilon) < \tau\Big) \le c_1 \exp(-c_2 \sqrt[3]{\lambda}) .$$

We can add up the probabilities above and obtain a constant $c > 0$ such that

$$\sum_{q=1}^{\infty} \mathbb{P}_0\Big\{ S_k(x_n, 2^{-q}) - T_k(x_n, 2^{-q}) > cL(n)^3 L(k)^3 L(2^q)^3 2^{-4q} \\ \text{for some } k, n \text{ with } T_k(x_n, 2^{-q}) < \tau \Big\} < \infty .$$

Using the Borel-Cantelli-Lemma we infer that property (ii) can be realized on a set of full measure. The new part is the proof of property (i). We formulate two lemmas to prepare the proof.

We first need the following statement about Brownian motion stopped at its inverse local time. Recall that the local time of $\{|B_s|\}$ at level 0 is twice the local time of $\{B_s\}$ at level 0.

Lemma 7 *Let* $\{B_s\}$ *be a linear Brownian motion,* L_s^0 *its local time at level* 0 *and* $\tau = \tau(B) = \inf\{s : L_s^0 = 1/2\}$. *Then, for all* $0 < \lambda < (\log 2)/2$, *we have*

$$\mathbb{E}\Big\{ \exp\Big(\lambda \int_0^\tau 1_{\{|B_s| \le 1\}}\, ds\Big)\Big\} \le \frac{1}{2 - e^{2\lambda}} .$$

Proof. Recall that τ itself does not even have finite first moment, so that the statement is not trivial. We first check that, for the process $\{X_t\}$ defined as

$$X_t = \int_0^{t\wedge\tau} 1_{\{|B_s| \le 1\}}\, ds ,$$

we have, for all stopping times T and associated σ-algebras $\mathcal{F}_T$, that $\mathbb{E}\{|X_\infty - X_T|\,|\mathcal{F}_T\} \le 1$ almost surely. Indeed,

$$\mathbb{E}\{X_\infty - X_T | \mathcal{F}_T\} \le \sup_{|a|\le 1} P_a\Big\{ \int_0^{\tau(B)} 1_{\{|B_s|\le 1\}}\, ds \Big\} = \sup_{|a|\le 1} \int_0^\infty P_a\{|B_s| \le 1, L_s^0 < 1/2\}\, ds,$$

where P_a refers to a Brownian motion B starting in $B_0 = a$. Recall the explicit density of the joint distribution of (B_s, L_s^0), see for example [BS96, (1.3.8)], which yields

$$\begin{aligned}
&\sup_{|a|\le 1} \int_0^\infty P_a\{|B_s| \le 1, L_s^0 < 1/2\}\, ds \\
&= \sup_{|a|\le 1} \int_0^\infty ds \int_{-1}^1 dx \int_0^{1/2} dy \Big\{ \frac{1}{s\sqrt{2\pi s}} (|x| + y + a) \exp(-(|x| + y + a)^2/2s) \Big\} \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{e^{-t/2}}{\sqrt{t}}\, dt = 1\,.
\end{aligned}$$

With this property of $\{X_t\}$ at hand, we can see that, for every $a > 0$ and $T = \inf\{t : X_t = a\}$,

$$\begin{aligned}
P_0\{X_\infty > a+2\} &= \mathbb{E}\Big\{ P_0\{X_\infty - X_T > 2 \,|\, \mathcal{F}_T\} \Big| T < \infty \Big\} \cdot P\{T < \infty\} \\
&\le \frac{1}{2} \mathbb{E}\Big\{ \mathbb{E}\{X_\infty - X_T \,|\, \mathcal{F}_T\} \Big| T < \infty \Big\} \cdot P\{T < \infty\} \\
&\le P_0\{X_\infty \ge a\}/2\,.
\end{aligned}$$

From this it is easy to see that $\mathbb{E}\{\exp(\lambda X_\infty)\} < \infty$ for all $0 < \lambda < (\log 2)/2$ and, similarly, we see that

$$\begin{aligned}
\mathbb{E}\{\exp(\lambda X_\infty)\} &= \int_1^\infty P_0\Big\{ X_\infty > \log a/\lambda \Big\}\, da + 1 \\
&= e^{2\lambda} \cdot \int_1^\infty P\{X_\infty > 2 + \log a/\lambda\}\, da + 1 \\
&\le \frac{e^{2\lambda}}{2} \mathbb{E}\{\exp(\lambda X_\infty)\} + 1/2\,,
\end{aligned}$$

from which we infer that $\mathbb{E}\{\exp(\lambda X_\infty)\} \le (2 - e^{2\lambda})^{-1}$, as required. ■

We also need the following fact, see [LP95, Lemma 3.1] for an easy proof. Denote by $p(y, s) = (1/\sqrt{2\pi s})^d \exp(-\|x\|^2/2s)$ the standard heat kernel.

Lemma 8 *Denote $G(r, t) = \int_0^t [\int_{\|y\|\le r} p(y, s)\, dy]\, ds$. Then, for all $t > 0, r > 0$ and $x \in C$,*

$$\mathbb{P}_0\Big\{ \exp\Big(\lambda X_t(B(x, r))\Big) \Big\} \le \exp\Big(2\lambda \int_{\|y\|\le r} p(y, t)\, dy\Big)\,,$$

for all $0 \le \lambda \le (4G(r, t))^{-1}$.

The following lemma serves as a replacement for [LS95, Lemma 8]. For its formulation define

$$\Delta(W) = \sup\Big\{ \delta \ge 0 : |w_s(t_1) - w_s(t_2)| \le 3\sqrt{|t_1 - t_2| L(|t_1 - t_2|)} \text{ for all } |t_1 - t_2| \le \delta, 0 \le s \le \tau \Big\}\,.$$

The modulus of continuity for the historical process, established in [DP91, Theorem 8.7], makes sure that $\Delta(W) > 0$ for $\mathbb{P}_0$-almost all W. This will be used in the next lemma. The basic idea is the following: If a particle is outside $B(x, 7\varepsilon L(\varepsilon))$ at time t and $\Delta(W) > \varepsilon^2$, the particle and its offspring cannot move into $B(x, \varepsilon)$ in time less than ε^2. Hence, if $\int_t^{t+\varepsilon^2} X_s B(x, \varepsilon)\, ds$ is large and $\Delta(W) > \varepsilon^2$, then either $X_t(B(x, 7\varepsilon L(\varepsilon)))$ is large or the particles which are in $B(x, 7\varepsilon L(\varepsilon))$ at time t are unusually fertile. Both these events are shown to be unlikely.

Lemma 9 *For every bounded interval $I \subset (0,\infty)$ there exist positive constants c_3 and c_4 such that, for all $t \in I$, $x \in C$, all $\lambda > 0$ and all $\varepsilon = 2^{-q}$, $q \geq 1$.*

$$\mathbb{P}_0\Big\{\Delta(W) > \varepsilon^2\,,\, \int_t^{t+\varepsilon^2} X_s B(x,\varepsilon)\,ds \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\} \leq c_3 \exp(-c_4\sqrt{\lambda})\,.$$

Proof. Denote by

$$\mathcal{M} = \Big\{\mu \in \mathcal{M}_F(\mathbb{R}^d) \,:\, \mu(\mathbb{R}^d) \leq \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\Big\}\,.$$

We have, by the Markov property of super-Brownian motion and the definition of Δ,

$$\begin{aligned}
&\mathbb{P}_0\Big\{\Delta(W) > \varepsilon^2, \int_t^{t+\varepsilon^2} X_s B(x,\varepsilon)\,ds \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\} \\
&\leq\ \mathbb{P}_0\Big\{X_t B(x, 7\varepsilon L(\varepsilon)) \geq \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\Big\} + \sup_{\mu\in\mathcal{M}} \mathbb{P}_\mu\Big\{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\}\,,
\end{aligned}$$

where $\{X_t\}$ under $\mathbb{P}_\mu$ is a super-Brownian motion started in μ. We give seperate estimates of these two probabilities. To investigate the first probability we use Lemma 8. We have

$$G(r,t) = \int_{\|y\|\leq r} \Big[\int_0^t p(y,s)\,ds\Big]\,dy = r^2 \int_{\|y\|\leq 1} \Big[\int_0^{t/r^2} p(y,s)\,ds\Big]\,dy\,,$$

which, for all $t \in I$ and $0 < r < 1/2$, is bounded from above by a constant multiple of $r^2\log(1/r)$ in dimension $d=2$ and by a constant multiple of r^2 in dimension $d \geq 3$. In all cases we find a positive constant k_1 such that

$$t \int_{\|y\|\leq 7\varepsilon L(\varepsilon)} p(y,t)\,dy \leq G(7\varepsilon L(\varepsilon), t) \leq k_1 \varepsilon^2 L(\varepsilon)^4\,.$$

Choose $k_2 = 1/(4k_1)$ and k_3 such that $\exp(1/(2t)) < k_3$ for all $t \in I$. Then, using Chebyshev,

$$\begin{aligned}
&\mathbb{P}_0\Big\{X_t(B(x,7\varepsilon L(\varepsilon))) \geq \sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\Big\} \\
&\leq\ \exp\Big(-\frac{\sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4}{4G(7\varepsilon L(\varepsilon),t)}\Big)\,\mathbb{P}_0\Big\{\exp\Big(\frac{X_t(B(x,7\varepsilon L(\varepsilon)))}{4G(7\varepsilon L(\varepsilon),t)}\Big)\Big\} \\
&\leq\ \exp\Big(-\sqrt{\lambda}/(4k_1)\Big)\cdot\exp\Big(2\frac{\int_{\|y\|\leq 7\varepsilon L(\varepsilon)} p(y,t)\,dy}{4G(7\varepsilon L(\varepsilon),t)}\Big) \\
&\leq\ k_3\exp(-k_2\sqrt{\lambda})\,.
\end{aligned} \tag{3}$$

This is the required estimate for the first term. To investigate the second probability we use our Lemma 7. We choose a positive constant k_4 smaller than $L(\varepsilon)^4(\log 2)/2$ for all $\varepsilon = 2^{-q}$, $q \geq 1$. Then, by Chebyshev,

$$\mathbb{P}_\mu\Big\{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt \geq \lambda\varepsilon^4 L(\varepsilon)^4\Big\} \leq e^{-k_4\lambda}\,\mathbb{P}_\mu\Big\{\exp\Big(k_4\frac{\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt}{\varepsilon^4 L(\varepsilon)^4}\Big)\Big\}\,.$$

Now let $a = k_4/L(\varepsilon)^4$ and observe $0 < a < (\log 2)/2$. We have, by the branching property of super-Brownian motion,

$$\begin{aligned}
\mathbb{P}_\mu\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\} &= \exp\Big(\int \log \mathbb{P}_x\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^2} X_t(\mathbb{R}^d)\,dt\Big)\Big\}\,d\mu(x)\Big) \\
&= \exp\Big(\int \log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\tau} 1_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\}\,d\mu(x)\Big) \\
&\le \exp\Big(\sqrt{\lambda}\varepsilon^2 L(\varepsilon)^4\cdot \log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\tau} 1_{\{|B_s|\le\varepsilon^2\}}\,ds\Big)\Big\}\Big)\,.
\end{aligned}$$

We now use a scaling argument. Define $\tau_0 = 0$ and $\{B^0_s\} = \{B_s\}$. Proceed inductively by putting, for $j \ge 0$, $\tau_{j+1} = \inf\{s : L^0_s(B^j_s) = 1/2\}$ and $\{B^{j+1}_s\} = \{B_{\tau_{j+1}+s}\}$. Finally, let $\tilde\tau = \inf\{s : L^0_s = 1/(2\varepsilon^2)\}$. Then, using Lemma 7 in the final step,

$$\begin{aligned}
&\log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\tau} 1_{\{|B_s|\le\varepsilon^2\}} ds\Big)\Big\} = \log P_0\Big\{ \exp\Big(\frac{a}{\varepsilon^4}\int_0^{\varepsilon^4\tilde\tau} 1_{\{|B_{s/\varepsilon^4}|\le 1\}}\,ds\Big)\Big\} \\
&= \log P_0\Big\{ \exp\Big(a\sum_{j=1}^{1/\varepsilon^2}\int_0^{\tau_j} 1_{\{|B^{j-1}_s|\le 1\}} ds\Big)\Big\} = \sum_{j=1}^{1/\varepsilon^2}\log P_0\Big\{ \exp\Big(a\int_0^{\tau_j} 1_{\{|B^{j-1}_s|\le 1\}} ds\Big)\Big\} \\
&\le \frac{1}{\varepsilon^2}\cdot\log\Big(\frac{1}{2-e^{2a}}\Big)\,.
\end{aligned}$$

Therefore, as, for some constant $k_5 > 0$, $\log(1/(2-e^{2a})) \le k_5 a$,

$$\begin{aligned}
\mathbb{P}_\mu\Big\{\int_0^{\varepsilon^2} X_t(B(x,\varepsilon))\,dt \ge \sqrt{\lambda}\varepsilon^4 L(\varepsilon)^4\Big\} &\le e^{-k_4\lambda}\cdot \exp\Big(\sqrt{\lambda}L(\varepsilon)^4\log\Big(\frac{1}{2-e^{2a}}\Big)\Big) \\
&\le \exp\Big(-\lambda k_4(1-k_5/\sqrt{\lambda})\Big)\,. \qquad (4)
\end{aligned}$$

The required result follows from (3) and (4). ■

We can now add the probabilities and obtain, for some $c > 0$,

$$\sum_{q=1}^\infty \mathbb{P}_0\Big\{\Delta(W) > 2^{-2q}, \int_{I_m(2^{-2q})} X_t B(x_n, 2^{-q})\,dt \ge c2^{-4q}L(n)^2L(m)^2L(2^q)^2 \text{ for some } n,m\Big\} < \infty,$$

and hence Borel-Cantelli gives a random integer Q_1 such that, whenever $\Delta(W) > 2^{-2Q_1}$, we have

$$\int_{I_m(2^{-2q})} X_t B(x_n, 2^{-q})\,dt \le c2^{-4q}L(n)^2L(m)^2L(2^q)^2\,,$$

for all integers n, m and $q \ge Q_1$. By the modulus of continuity result of [DP91, Theorem 8.7] there is a random integer Q_2 such that $\Delta(W) > 2^{-2Q_2}$ almost surely and this implies that (i) can be realized on a set of full measure and hence we obtain the statement of the lemma. ■

We finish the proof of Theorem 2 by showing that our dimension property holds for all W which are in the intersection of the sets $\Omega(\delta)$. We fix the set $H \subset C$, which we may assume to have

$\dim H < 4$, and the time $t \in I$ and let $4 > \gamma > \dim H$ and $0 < \delta < \gamma - \dim H$. We further fix an integer K with $2^K \geq \sqrt{d} + 1$.

We choose a covering of H by a sequence $\{B(y_j, r_j) : j \geq 1\}$ of balls centred in C whose radii satisfy $r_j \leq 2^{-(Q+K+1)}$, such that

$$\sum_{j=1}^{\infty} r_j^{\gamma} L(2^K r_j)^{24} \leq 2^{-(K+1)\gamma} . \tag{5}$$

We can replace this covering by a covering consisting of balls $B(x_{m_j}, \varepsilon_j)$ such that $\varepsilon_j \in \{2^{-q} : q \geq Q\}$, $2^{-(K+1)}\varepsilon_j \leq r_j \leq 2^{-K}\varepsilon_j$ and

$$|y_j - x_{m_j}| \leq \sqrt{d}\frac{\varepsilon_j}{2^K} \text{ for some } m_j \leq \left(\frac{2^K}{\varepsilon_j}\right)^d .$$

We write T_k^j, S_k^j for $T_k(x_{m_j}, \varepsilon_j)$, $S_k(x_{m_j}, \varepsilon_j)$, and denote by

$$\mathcal{N}_j = \left\{ k \geq 1 \,:\, T_k^j < \tau \text{ and } t \in \zeta[T_k^j, S_k^j] \right\} ,$$

and

$$\mathcal{M}_j = \left\{ k \geq 1 \,:\, T_k^j < \tau \text{ and } \zeta[T_k^j, S_k^j] \subset \bigcup_{m=0}^{\varepsilon_j^{-2}} I_m(\varepsilon_j^2) \right\} .$$

Let $N_j = \#\mathcal{N}_j$ and $M_j = \#\mathcal{M}_j$. Because $B(y_j, r_j) \subset B(x_{m_j}, \varepsilon_j)$ we have that

$$\bigcup_{j=1}^{\infty} \bigcup_{k \in \mathcal{N}_j} [T_k^j, S_k^j] \supset \left\{ s \in [0, \sigma] \,:\, \hat{W}_s \in H, \zeta_s = t \right\} .$$

This is the covering which will give a good upper bound of the Hausdorff measure. Property (ii) ensures that we have control over the length of the individual covering intervals $[T_k^j, S_k^j]$. We now argue that also the cardinality of the sets $\mathcal{N}_j$ is not too large. We begin by observing that $|\hat{W}_{S_k^j} - \hat{W}_{T_k^j}| \geq \varepsilon_j$. As $\varepsilon_j \leq 2^{-Q}$ we infer from property (iii) that

$$S_k^j - T_k^j \geq \varepsilon_j^{4+\delta} . \tag{6}$$

From (i) applied to the family of intervals $I_m(\varepsilon_j^2)$, $m = 0, \ldots, \varepsilon_j^{-2}$, we infer that

$$M_j \varepsilon_j^{4+\delta} \leq \sum_{k \in \mathcal{M}_j} S_k^j - T_k^j \leq Z\Big(\bigcup_{m=0}^{\varepsilon_j^{-2}} I_m(\varepsilon_j^2), B(x_{m_j}, 2\varepsilon_j) \Big) \leq cL(\varepsilon_j^2)^2 L(m_j)^2 L(\varepsilon_j)^2 \varepsilon_j^4 (\varepsilon_j^{-2} + 1) ,$$

and hence, for some positive constant c_5,

$$M_j \leq c_5 L(\varepsilon_j)^6 \varepsilon_j^{-2-\delta} .$$

Properties (iv) and (ii) imply that, for some positive constant c_6, for every k,

$$\operatorname{diam} \zeta[T_k^j, S_k^j] \leq (S_k^j - T_k^j)^{1/(2+\delta)} \leq cL(m_j)^3 L(M_j)^3 L(\varepsilon_j)^3 \varepsilon_j^{4/(2+\delta)} \leq c_6 L(\varepsilon_j)^9 \varepsilon_j^{4/(2+\delta)} . \tag{7}$$

We infer from this and our numbering of the points a_m that

$$\zeta[T_k^j, S_k^j] \subset [a_m - 2^{-n}, a_m + 2^{-n}) ,$$

for some $m \leq 2^n$ and n satisfying

$$2^{-(n+1)} \leq c_6 L(\varepsilon_j)^9 \varepsilon_j^{4/(2+\delta)} < 2^{-n} .$$

We are interested in $m = m(t) \leq 2^n$ with the property that $t \in [a_m - 2^{-n}, a_m + 2^{-n})$. On the one hand we then have

$$Z\Big([a_m - 2^{-n}, a_m + 2^{-n}], B(x_{m_j}, 2\varepsilon_j)\Big) \geq \sum_{k \in \mathcal{N}_j} (S_k^j - T_k^j) \geq N_j \cdot \varepsilon_j^{4+\delta} ,$$

and on the other hand, $[a_m - 2^{-n}, a_m + 2^{-n}]$ can be covered by at most $2c_6 L(\varepsilon_j)^9 \varepsilon_j^{-\delta}$ intervals from the collection $\{I_k(\varepsilon_j^2) : 0 \leq k \leq 1/\varepsilon_j^2\}$. From (i), we hence know that

$$Z\Big([a_m - 2^n, a_m + 2^n], B(x_{m_j}, 2\varepsilon_j)\Big) \leq 2cc_6 L(m)^2 L(m_j)^2 L(\varepsilon_j)^{11} \varepsilon_j^{4-\delta} ,$$

which together gives, for some positive constant c_7,

$$N_j \leq c_7 L(\varepsilon_j)^{15} \varepsilon_j^{-2\delta} . \tag{8}$$

Using again (ii) and (8), we get, for every $\alpha \in [0,1]$, and a suitable positive constant c_8,

$$\sum_{j=1}^{\infty} \sum_{k \in \mathcal{N}_j} (S_k^j - T_k^j)^\alpha \leq c \sum_{j=1}^{\infty} N_j \Big(L(m_j)^3 L(M_j)^3 L(\varepsilon_j)^3 \varepsilon_j^4\Big)^\alpha \leq c_8 \sum_{j=1}^{\infty} \varepsilon_j^{4\alpha - 2\delta} L(\varepsilon_j)^{24} .$$

We choose $\alpha = (\gamma + 2\delta)/4$, and get, upon recalling (5) and $2^{-K}\varepsilon_j \geq r_j \geq 2^{-(K+1)}\varepsilon_j$,

$$\sum_{j=1}^{\infty} \sum_{k \in N_j} (S_k^j - T_k^j)^{(\gamma+2\delta)/4} \leq c_8 \sum_{j=1}^{\infty} \varepsilon_j^\gamma L(\varepsilon_j)^{24} \leq c_8 \sum_{j=1}^{\infty} (2^{K+1} r_j)^\gamma L(2^K r_j)^{24} \leq c_8 < \infty .$$

This implies that $\dim\{s : \hat{W}_s \in H, \zeta_s = a\} \leq (\gamma + 2\delta)/4$ and, as this holds for all $\delta > 0$, the required result follows and Theorem 2 is proved.

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