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The shock location for a class of sensitive boundary value problems

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Abstract

The sensitive internal layer behaviour in an autonomous nonlinear singularly perturbed boundary value problem is investigated. For this problem we show that the internal layers solutions exhibit either an exponential or an algebraic sensitivity in reponse to small changes in the boundary conditions as well as in the coefficients of the equation and we derive a geometric method to determine the shock location as a function of the perturbations.

The results are then applied to study the behavior of both the viscous shock location for the two-point problem for the stationary Burgers equation and the supersonic-subsonic shock that arises in modelling compressible flows as a result of perturbations of the boundary conditions of order $O(e^{-1/\varepsilon})$. For the corresponding time-dependent partial differential equation we also show how the exponentially small perturbations in the ordinary differential equation are associated with the metastable viscous shock layer motion. Some associated boundary layer resonance problems with turning points are also considered.

1 Introduction

We consider the nonlinear singularly perturbed boundary value problem $P_\delta(A, B)$:

$$\varepsilon x'' = (g(x) + \delta)\ f(x'), \qquad c < t < d, \tag{1}$$

$$x(c) = A, \qquad x(d) = B, \tag{2}$$

where A and B are real values, δ is an infinitesimal and ε is a fixed positive infinitesimal. For our study we use the language of E. Nelson 's Internal Set Theory (IST) [16].

We assume that f and g are standard functions that satisfy the following hypotheses H:

1. $g \in C(\mathbb{R})$ and g locally Lipschitz ,
2. $f \in C^1(\mathbb{R})$; $f(0) = 0$; $f(v) > 0$ if $v > 0$ and $f(v) < 0$ if $v < 0$,
3. $\displaystyle\int_{v_0}^{+\infty} \frac{v}{f(v)} dv = +\infty$ where v_0 is a limited positive constant,

   f gives the *the type of growth* of (1) for unlimited v,
4. The function $G : \mathbb{R} \to \mathbb{R}$, $G(x) = \displaystyle\int_{A^*}^{x} g(u)du$ satisfies:

   $G(B^*) = 0$, $G'(A^*) \underset{\not\simeq}{>} 0$, $G'(B^*) \underset{\not\simeq}{<} 0$, $G(x) \underset{\not\simeq}{>} 0$ for all $A^* \underset{\not\simeq}{<} x \underset{\not\simeq}{<} B^*$ (or $G(x) \underset{\not\simeq}{<} 0$ for all $B^* \underset{\not\simeq}{<} x \underset{\not\simeq}{<} A^*$

This paper concerns the sensitivity of the boundary and the internal layer solutions to small changes in the boundary data as well as in the coefficient of the differential equation for (1), (2) and the location t_0 of the internal layers. We are especially interested in the behavior of the solution of $P_\delta(A,B)$ for $A \simeq A^*$, $B \simeq B^*$ and $\delta \simeq 0$. In a previous work [1] we have shown that the solution of $P_0(A,B)$ exhibits the phenomenon of sensitivity: slight variations of the boundary values significantly change the position of the shock.

This phenomenon is connected with the fact that for the boundary data (A^*, B^*) the classical Rankine-Hugoniot condition fails to determine the location ${t_0}^*$ of the shock layer as we have shown recently in [2] where we extended the study of sensitivity to a more general class of boundary value problems including certain non autonomous equations.

Because of the sensitivity exhibited by the solution of $P_\delta(A,B)$ the usual method of matched asymptotic expansions and a phase plane analysis do not determine the shock location uniquely. Some methods have been successfully applied to determine t_0 in some special cases. When $G(x)$ is even and $\delta = 0$, symmetry arguments show that $t_0 = (c+d)/2$ for any f we have considered. When $f(x')$ is linear, an extension of the matched asymptotic expansions that includes exponentially small terms [11] or the projection method [17] may be also used.

For arbitrary functions g and f satisfying our hypotheses, a method to determine t_0 was given in [1]. It uses the fact that the solution is strictly monotone and it is based on studying the inverse solution $t(x)$ in an appropriate observability phase space [7] to show how the two boundary layers of $t(x)$ are connected by the unknown constant t_0.

The estimate obtained in this way shows that the shock location for the unperturbed boundary value problem $P_0(A^*, B^*)$ is the same for any function f since this location only depends on the values of $g(x)$ at the endvalues. On the contrary, the thickness of the jump depends on the type of growth of the equation with x' for $x' \to +\infty$.

It also shows that this location changes by order $O(1)$ with small variations of the boundary data and that the order of the perturbation depends on f.

In this paper, we study the sensitive dependence of the shock layer on small changes in both the boundary values and the coefficient of the differential equation and we quantify the behavior of the internal layer position . We give estimates of the shock location as a function of the perturbations for different functions f.

Depending on f, this class of equations may exhibit either an exponential or an algebraic sensitivity . The transition point is sensitive to perturbations of order $O(\varepsilon^{1/(s-1)})$ if $f(x') = x'^s$ with $1 < s \leq 2$ but when $s = 1$ the phenomenon is extremely sensitive since the shock location moves by $O(1)$ with exponentially small changes in the boundary values.

Our main results can be summarized as follows.

In the case of exponential sensitivity, we consider small perturbations of the form $\pm e^{-b/\varepsilon}$ and determine the shock location t_0 as a function of the parameter b. We prove the existence of a critical value b^* such that $t_0(b) \simeq t_0{}^*$ for $b \geq b^*$. We show that any other internal layer location is obtained for appropriated small positive or negative perturbations with $0 \underset{\not\simeq}{<} b < b^*$.

The results are then applied to study the behavior of both the viscous shock location for the two-point problem for the stationary Burgers equation and the supersonic-subsonic shock that arises in modelling compressible flows as a result of perturbations of the boundary values of order $O(e^{-1/\varepsilon})$. Our results extend those of J. Laforgue and R. O'Malley [11] who have found the shock location for the Burgers problem by using exponential asymptotics.

Exponential sensitivity also arises in some nonlinear singularly perturbed partial differential equations including Burgers equation . See, for example, [12, 13, 17, 18, 4] where the sensitivity of the solutions to small perturbations was studied as well as the phenomenon known as dynamic metastability.

As a second application of our results, we show how the exponentially small perturbations for the ordinary differential equation are associated with the metastable viscous shock layer motion for the corresponding time-dependent partial differential equation.

In case of algebraic sensitivity we consider perturbations of the form $b\varepsilon^{1/(s-1)}$ with $1 < s \leq 2$ and determine the shock location $t_0(b)$ as a function of b. Then we study some associated boundary layer resonance problems. We show how the estimate of the shock location we have found determines the solution of certain linear turning point problems exhibiting the phenomenon of boundary layer resonance . The sensitivity of the internal layer location to perturbations in the boundary data also allows us to explain another phenomenon observed by Matkowsky [15] in the study of resonance in a quasilinear boundary value problem: a solution with two boundary layers changes into a solution with only one boundary layer in reponse to perturbations at one endpoint of the interval of order $O(\varepsilon^\gamma)$ for any γ. Our results also precise the order of the perturbation that makes one boundary layer disappear.

Notation: We shall denote $x \simeq y$ for "x infinitely close to y", $x \underset{\sim}{<} y$ for"$x < y$ or $x \simeq y$", $x \underset{\not\simeq}{<} y$ for "$x < y$ and not $x \simeq y$" and similarly $x \underset{\sim}{>} y$ and $x \underset{\not\simeq}{>} y$. When x is limited, we shall denote by ${}^0x$ the standard part of x. The symbol $\pounds$ is used for a limited value.

2 Sensitivity of $P_\delta(A,B)$

The existence of a solution $x(t)$ of $P_\delta(A,B)$ such that $A \le x(t) \le B$ is ensured by [10]. The monotonicity of $x(t)$ follows as a consequence of the uniqueness of the solution of the initial value problem associated with (1). The solution $x(t)$ increases monotonically if $A < B$ while we have $x'(t) < 0$ if $B < A$. We shall only consider the case $A < B$ since the other follows analogously. Then the inverse $t(x)$ of $x(t)$ exists, is also strictly monotonic in $[A,B]$ and satisfies the following boundary value problem $\hat{P}_\delta(A,B)$:

$$\begin{aligned} \varepsilon \ddot{t} &= -(g(x)+\delta)\ f(1/\dot{t})\ \dot{t}^3, \qquad (3)\\ t(A) &= c, \qquad t(B) = d. \qquad (4) \end{aligned}$$

The uniqueness of $t(x)$ follows from using the maximum principle and this ensures the uniqueness of $x(t)$.

Finally the assumptions on f and g imply that the slow portions of the solution are almost constant. Then, $x(t)$ must necessarily jump in order to satisy both boundary conditions when $A \neq B$.

Now, in order to study these rapid motions and to determine how many jumps the solution of (1), (2) has we use the observability plane method [7]. It consists on introducing a rescaling of the fast variable $v = x'(t)$ by considering the transformation $v = h(V/\varepsilon)$ where $h(\sigma)$ is defined by:

$$h\frac{dh}{d\sigma} = f(h) \quad ; h(0) = v_0 \qquad \text{with } v_0 \text{ a limited value.} \qquad (5)$$

After performing the change of variable, the jumps of $x(t)$ at some t_0 are contained near the vertical plane $t = t_0$ (called the observability plane) and satisfy

$$\frac{dV}{dx} \simeq g(x) \qquad \text{for } \delta \simeq 0. \qquad (6)$$

They are then described, up to an infinitesimal, by the curves

$$V(x) \simeq G(x) + K \quad \text{with } K \text{ a constant.} \qquad (7)$$

The increasing jumps are contained in the positive half-plane $V > 0$ while any slow motion of the trajectory associated with the solution $x(t)$ appears in this plane near $V = 0$.

Then, taking into account this qualitative behavior of the trajectories, the constant K is used as a shooting parameter for the initial value problem associated with (1) in the phase space (t,x,V) with initial data $t = c$, $x = A$ and $V \simeq K$. The value of K is selected in such a way that the corresponding solution also attains the other boundary value B at time $t = d$.

Under the assumption on the function $G(x)$ any jump of the solution of $P_\delta(A, B)$ with $A \simeq A^*$, $B \simeq B^*$ and $\delta \simeq 0$ satisfies $V(x) \simeq G(x)$ in order to connect A and B in $[c, d]$.

In addition, we note that a slow motion of our increasing solution of the form $x(t) \simeq e$, $x'(t) \simeq 0$ with $A \underset{\not\simeq}{<} e \underset{\not\simeq}{<} B$ is not possible since in this case we must have $V(e) \simeq G(e) \simeq 0$ but $G(x) \underset{\not\simeq}{>} 0$ for $A \underset{\not\simeq}{<} x \underset{\not\simeq}{<} B$.

Therefore the solution of $P_\delta(A, B)$ has only one jump at some t_0 and satisfies

$$x(t) \simeq \begin{cases} A & \text{for} \;\; c \leq t < t_0 - \eta, \\ B & \text{for} \;\; t_0 + \eta < t \leq d \quad \text{and} \;\; \eta \simeq 0^+ \end{cases}$$

We remark that the above results give us the **same** qualitative behavior of the solution for this entire family of boundary value problems. The equation that describes the jumps is determined by $g(x)$ and is the same for all f. Only the rescaling h depends on f.

For example, the internal layer of $\varepsilon x'' = -2x \; x'^s$ with $1 \leq s \leq 2$ and $A = -1$ and $B = 1$ is close to the parabola

$$V(x) = 1 - x^2$$

but the rescaling is given by $x' = ((2-s)V/\varepsilon)^{1/(2-s)}$ for $1 \leq s < 2$ and $x' = \exp(V/\varepsilon)$ in the quadratic case. In the first case h is defined by selecting $v_0 = 0$ while $v_0 = 1$ in the other case. The choice of v_0 is rather arbitrary and it does not change the behavior of the jumps. See [7].

3 The equation for the shock location

In this section we will now derive an equation for the shock location t_0 as a function of the A, B and δ. There are two steps in our approach.

First we integrate once the equation (1) by setting $x'(t) = p(x)$ and use the boundary conditions to get

$$\int_{x'(c)}^{x'(d)} \frac{p}{f(p)} dp = \frac{G(B) - G(A) + \delta(B - A)}{\varepsilon} \tag{8}$$

Note that the integral in (8) can be written in terms of the inverse function $\sigma(h) := h^{-1}(h)$ of the transformation h defined in (5).

Since $hh'/f(h) = 1$, $h(0) = v_0$ we have $\sigma(h) = \int_{v_0}^{h(\sigma)} \frac{p}{f(p)} dp$.

Thus the solution $x(t)$ of $P_\delta(A, B)$ satisfies

$$h^{-1}(x'(d)) - h^{-1}(x'(c)) = \frac{G(B) - G(A) + \delta(A - B)}{\varepsilon} \tag{9}$$

The latter step will be used to obtain estimates of the first derivative x' at both endpoints of the interval $[c, d]$. These estimates which depend on t_0 will be obtained by studying the inverse problem $\hat{P}_\delta(A, B)$ as follows.

If we seek a solution $x(t)$ with an internal layer at t_0, the inverse solution $t(x)$ will have two boundary layers at $x = A$ and $x = B$ connected by the limiting solution $t(x) \simeq t_0$. We shall study the boundary layers of $t(x)$ by again applying the observability plane method.

The fast-slow system for (3), in the $(x, t, w = \dot{t})$ phase space, is

$$\left\{ \begin{array}{rcl} \dot{x} & = & 1 \\ \dot{t} & = & w \\ \varepsilon \dot{w} & = & -(g(x) + \delta) f(1/w) w^3 \end{array} \right. \tag{10}$$

The jumps of $t(x)$ for w unlimited may be observed in the (t, W) observability plane by rescaling the fast variable w by introducing $w = \hat{h}(W/\varepsilon)$ where the appropriate rescaling $\hat{h}(\theta)$ for the system (10) is defined by:

$$\hat{h}\frac{d\hat{h}}{d\theta} = f(1/\hat{h})\hat{h}^3 \quad ; \hat{h}(0) = w_0 \qquad \text{with } w_0 \text{ a limited value.} \tag{11}$$

Then, the rapid trajectories of (10), related to an increasing jump of $t(x)$ near $x = x_0$ are, in the (x, t, W) observability space contained near the vertical plane $x = x_0$, and they satisfy:

$$\frac{dW}{dt} \simeq -g(x_0) \qquad \text{up to an infinitesimal.} \tag{12}$$

Thus, each boundary layer for the solution $t(x)$ is, in the (t, W) plane, infinitely close to straight lines of slope $-g(A^*)$ or $-g(B^*)$ when $A \simeq A^*$ and $B \simeq B^*$.

The boundary layer at $x = A$ is close to

$$W_A(t) \simeq -g(A^*)(t - c) + W_A(c) \tag{13}$$

while for the boundary layer at $x = B$

$$W_B(t) \simeq -g(B^*)(t - d) + W_B(d) \tag{14}$$

where

$$\dot{t}(A) = 1/x'(c) = \hat{h}(W_A(c)/\varepsilon). \tag{15}$$

$$\dot{t}(B) = 1/x'(d) = \hat{h}(W_B(d)/\varepsilon), \tag{16}$$

The constants $W_A(c)$ and $W_B(d)$ correspond, in the new variable W, to the values of $\dot{t}(A)$ and $\dot{t}(B)$ respectively.

The two boundary layers are connected by the almost constant solution $t(x) \simeq t_0$. In the (t, W) plane, the slow motions of (10) lie near $W = 0$, so we must have both $W_A(t_0) \simeq 0$ and $W_B(t_0) \simeq 0$ and from (13) and (14) it follows that

$$W_A(c) \simeq g(A^*)(t_0 - c). \tag{17}$$

$$W_B(d) \simeq g(B^*)(t_0 - d). \tag{18}$$

But $c \lesssim_{\not\simeq} t_0 \lesssim_{\not\simeq} d$ and $g(A^*)$ and $g(B^*)$ are two appreciable values, so there exist $\eta \simeq 0$ and $\xi \simeq 0$ such that

$$W_A(c) = g(A^*)(t_0 - c)(1 + \eta)$$
$$W_B(d) = g(B^*)(t_0 - d)(1 + \xi).$$

As a consequence of (15) and (16)

$$x'(c) = 1/\hat{h}[g(A^*)(t_0 - c)(1 + \eta)/\varepsilon]$$

and

$$x'(d) = 1/\hat{h}[g(B^*)(t_0 - d)(1 + \xi)/\varepsilon].$$

Finally, since $x'(c)$ and $x'(d)$ satisfy (9), we obtain

Proposition 1 *If problem $P_\delta(A, B)$ has an internal layer at t_0, there exist two infinitesimals $\eta \simeq 0$ and $\xi \simeq 0$ such that*

$$\begin{aligned} h^{-1}(1/\hat{h}[\frac{g(B^*)(t_0 - d)}{\varepsilon}(1 + \xi)]) \quad &- \quad h^{-1}(1/\hat{h}[\frac{g(A^*)(t_0 - c)}{\varepsilon}(1 + \eta)]) \\ &= \quad \frac{G(B) - G(A) + \delta(B - A)}{\varepsilon} \end{aligned} \tag{19}$$

Remark : This equation gives a relationship between t_0, the boundary values and the parameter δ through the diffeomorphisms h and $\hat{h}$ which are strictly increasing due to the sign of f (in the case of increasing solutions.

To illustrate equation (19) let us reconsider the family of problems $\varepsilon x'' = -2x\ x'^s$ with $1 \leq s \leq 2$ with boundary values $A = -1$ and $B = 1$. Since h satisfies $hh' = h^s$, the inverse function $\sigma(h) = h^{2-s}/(2-s)$ for $1 \leq s < 2$ and $\sigma(h) = \ln h$ for $s = 2$.

For the inverse problem $\varepsilon \ddot{t} = 2x\dot{t}^{3-s}$ with $t(-1) = c$ and $t(1) = d$, the corresponding $\hat{h}$ is the solution of $\hat{h}\hat{h}' = \hat{h}^{3-s}$. Thus, $\hat{h}(\theta) = \exp\theta$ for $s = 1$ and $\hat{h}(\theta) = ((s-1)\theta)^{1/(s-1)}$ for $1 < s \leq 2$. Moreover,

the left-hand side of (19) takesf the form

$$h^{-1}(1/\hat{h}(\theta)) = \begin{cases} \exp(-\theta) & s = 1 \\ ((s-1)\theta)^{(2-s)/(1-s)}/(2-s) & 1 < s < 2 \\ -\ln\theta & s = 2 \end{cases} \tag{20}$$

Remark : For the unperturbed boundary value problem $P_0(A^*, B^*)$ with $\delta = 0$ the boundary conditions imply that the Rankine-Hugoniot condition $G(B^*) = 0$ is satisfied and the two terms on the left-hand side of (19) are equal. Then there exist $\eta^* \simeq 0$ and $\xi^* \simeq 0$ such that

$$g(B^*)(t_0^* - d)(1 + \xi^*) = g(A^*)(t_0^* - c)(1 + \eta^*) \tag{21}$$

which finally gives the shock location

$$t_0^* = \frac{g(A^*)c - g(B^*)d}{g(A^*) - g(B^*)} + \rho^* \quad with\ \rho^* \simeq 0. \tag{22}$$

This formula gives the zeroth order term in the asymptotic expansion for t_0^* and shows that this term does not depend on the type of growth of f as $x' \to +\infty$.

For the unperturbed boundary values $A^* = -1$, $B^* = 1$, the solution of $\varepsilon x'' = -2x\ x'^s$ with $1 \leq s \leq 2$ or of $\varepsilon x'' = -2x\ x' \tanh x'$ has a jump at $t_0^* \simeq 1/2$ in $[0, 1]$.

For the boundary values $A^* > 0$ and $B^* = 1/A^*$ the solutions of $\varepsilon x'' = (1 - 1/x^2)x'^s$ or $\varepsilon x'' = (1 - 1/x^2)x'^s \tanh x'$ have a jump at $t_0^* \simeq A^{*2}/(A^* + 1)$ with $[0, 1]$.

Our method does not allow us to obtain more terms in the exapnsion for t_0^*, but it has the main advantage of displaying the independence of the standard part of t_0^* on f. Of course, the thickness of the shock depends on f.

A similar result holds in the case of problems like $P_\delta(A, B)$ with turning points. In [3] we determined the position of the internal layers when $g(x)$ is zero at the equilibrium states . We showed that the location of the shock depends on the number of turning points $e_i \in [A,B]$, where $g(e_i)=0$, the position of $e_i \in [A,B]$ and the order r_i of each turning point e_i, but not on f.

For the perturbed boundary value problem $P_\delta(A,B)$, the position of t_0 changes significantly by slight variations of A^* and B^* or taking $\delta \simeq 0$. The order of magnitude of the perturbation for which there is an internal layer characterizes the sensitivity of $P_\delta(A,B)$ and it is given by the thickness of the jumps of the inverse problem $\hat{P}_\delta(A,B)$. Since the thickness of the jumps of $\hat{P}_\delta(A,B)$ only depends on the type of growth of the equation as $\dot{t} \to +\infty$, the sensitivity of the shock location depends on $f(1/\dot{t})\dot{t}^3$.

4 The perturbed boundary value problem

We now consider $P_\delta(A,B)$ with $A \simeq A^*$ and $B \simeq B^*$ and $\delta \simeq 0$ and determine the location of the transition as a function of the perturbations by solving the equation (19) for different functions f. To include the effect of the small perturbations in the boundary data we rewrite the right-hand side of (19) taking into account that for $A \simeq A^*$ and $B \simeq B^*$ we have

$$G(B) = g(B^*)(B-B^*)(1+\rho)$$
$$G(A) = g(A^*)(A-A^*)(1+\rho); \quad \rho \simeq 0$$

4.1 The case $f(x') = x'$

The quasilinear problem $P_\delta(A,B)$ exhibits an extreme sensitivity since the shock location is sensitive to perturbations that are exponentially small of order $O(\exp(-1/\varepsilon))$. For the perturbed problem:

$$\left\{ \begin{array}{rcl} \varepsilon x'' &=& (g(x)+\delta)\ x' \\ x(c) &=& A^* + \alpha\ e^{-a/\varepsilon} \\ x(d) &=& B^* + \beta\ e^{-b/\varepsilon} \end{array} \right. \tag{23}$$

where $\delta = \pm e^{-k/\varepsilon}$, a, b and k are positive and not infinitesimal values and $\alpha, \beta = \pm 1$, equation (19) which gives the shock location as a function of the perturbations becomes

$$e^{-(\frac{g(B^*)(t_0-d)}{\varepsilon}(1+\xi))} - e^{-(\frac{g(A^*)(t_0-c)}{\varepsilon}(1+\eta))} = \frac{g(B^*)\beta e^{-b/\varepsilon}(1+\eta) - g(A^*)\alpha e^{-a/\varepsilon}(1+\eta) + \delta(B-A)}{\varepsilon} \tag{24}$$

with $\xi \simeq \eta \simeq 0$.

For simplicity we first consider problem (23) where only the boundary value B^* is perturbed. The shock location for the more general problem where both boundary conditions are perturbed will be given afterwards.

From the value of t_0^* we deduce the existence of a critical value

$$b^* = g(B^*)(t_0^* - d)(1+\xi^*) = g(A^*)(t_0^* - c)(1+\eta^*)$$

such that, by multiplying the equation (24) by $e^{b^*/\varepsilon}$ with $\alpha = \delta = 0$ we obtain

$$e^{\frac{-g(B^*)(t_0-t_0^*)+\rho_1)}{\varepsilon}} - e^{\frac{-g(A^*)(t_0-t_0^*)+\rho_2}{\varepsilon}} = \frac{g(B^*)(1+\eta)\beta}{\varepsilon} e^{\frac{-(b-b^*)}{\varepsilon}} \tag{25}$$

for some $\rho_1 \simeq \rho_2 \simeq 0$.

- For a perturbation of B^* with $\beta < 0$, $\;g(B^*)\beta > 0$ implies that we must necessarily have either $t_0 \underset{\not\simeq}{>} t_0^*$ or $t_0 \simeq t_0^*$.

  Then the shock location behaves as follows:

  i) If $\frac{b-b^*}{\varepsilon} \simeq +\infty$ the term in the right handside of (25) is exponentially small, so $t_0 \simeq t_0^*$. If instead, if $t_0 \underset{\not\simeq}{>} t_0^*$, the first term would be exponentially large since $g(B^*) < 0$ while the second would be exponentially small because $g(A^*) < 0$ which is absurd.

  ii) If $\frac{b-b^*}{\varepsilon} = \pounds$, with $\pounds$ limited (i.e. of $O(1)$), the right hand side of the (25) is large of order $O(1/\varepsilon)$ and we still have $t_0 \simeq t_0^*$.

  iii) Finally if $\frac{b-b^*}{\varepsilon} \simeq -\infty$ (and $b > 0$) the right member in (25) is exponentially large. Then $t_0 \underset{\not\simeq}{>} t_0^*$ and the second term satisfies

  $$L = e^{-(\frac{g(A^*)(t_0-t_0^*)+\rho_2}{\varepsilon})} < 1$$

  (so has a limited value) and the first term on the left dominates so

  $$e^{-(\frac{g(B^*)(t_0-t_0^*)+\rho_1}{\varepsilon})} = g(B^*)\beta(1+\eta)e^{-\frac{b-b^*}{\varepsilon}}/\varepsilon + L$$

  which finally gives

  $$t_0 = t_0^* + \left(\frac{b-b^*}{g(B^*)} + \frac{\varepsilon}{g(B^*)}\ln\left(\frac{\varepsilon}{g(B^*)\beta(1+\eta)}\right) - \frac{\varepsilon^2}{g(B^*)}Le^{\frac{b-b^*}{\varepsilon}}\right)(1+\rho) \tag{26}$$

  where $\eta \simeq \rho \simeq 0$.

- For a perturbation of B^* with $\beta > 0$ we show in a similar way that $t_0 \simeq t_0^*$ for $\frac{b-b^*}{\varepsilon} \simeq +\infty$ or $= \pounds$. For $\frac{b-b^*}{\varepsilon} \simeq -\infty$ with $b > 0$, we instead have

  $$t_0 = t_0^* + \left(\frac{b-b^*}{g(A^*)} + \frac{\varepsilon}{g(A^*)}\ln\left(\frac{\varepsilon}{g(B^*)\beta(1+\eta)}\right) - \frac{\varepsilon^2}{g(A^*)}Le^{\frac{b-b^*}{\varepsilon}}\right)(1+\rho) \tag{27}$$

for some $\eta \simeq \rho \simeq 0$.

Note that according to (26) the shock location t_0 moves toward the right endpoint d as b decreases for $\beta < 0$ (since $t_0^* - \frac{b^*}{g(B^*)} \simeq d$) and toward t_0^* as b tends toward b^*. When $\beta > 0$ it follows from (28) that t_0 moves toward the left endpoint c (where $t_0^* - \frac{b^*}{g(A^*)} \simeq c$) as b decreases and again toward t_0^* as b tends towards b^* .

Arguing in a similar way the shock location in the more general perturbed boundary value problem (23) where both boundary conditions are perturbed is given as follows:

- If $b < a$ the right member in (24) may be written as $e^{-b/\varepsilon}\mu/\varepsilon$ with $\mu \simeq g(B^*)\beta - g(A^*)\alpha e^{(b-a)/\varepsilon}$ and only the perturbation of B^* influences the movement of the internal layer. Depending on the sign of μ the shock position for $0 < b$ such that $\frac{b-b^*}{\varepsilon} \simeq -\infty$ is given as follows:

  For $\mu > 0$

$$t_0 = t_0^* + (\frac{b-b^*}{g(B^*)} + \frac{\varepsilon}{g(B^*)} \ln(\frac{\varepsilon}{g(B^*)\mu(1+\eta)}) - \frac{\varepsilon^2}{g(B^*)} L e^{\frac{b-b^*}{\varepsilon}})(1+\rho) \tag{28}$$

  For $\mu < 0$

$$t_0 = t_0^* + (\frac{b-b^*}{g(A^*)} + \frac{\varepsilon}{g(A^*)} \ln(\frac{\varepsilon}{g(B^*)\mu(1+\eta)}) - \frac{\varepsilon^2}{g(A^*)} L e^{\frac{b-b^*}{\varepsilon}})(1+\rho) \tag{29}$$

- When $b = a$ the influence of both perturbations is such that the shock layer moves monotonically from the left or from the right of t_0^*, depending on the sign of $\mu \simeq g(B^*)\beta - g(A^*)\alpha$, to the endpoints d or c respectively as b tends to 0. In this case (28) or (29) gives the shock position if $\mu > 0$ or $\mu < 0$.

- Finally if $b > a$ the opposite situation occurs and only the perturbation of A^* affects the behavior of the shock layer. Depending on the sign of $\mu \simeq g(B^*)\beta e^{-(b-a)/\varepsilon} - g(A^*)\alpha$ the shock location may be easily obtained replacing b by a in either (28) or (29).

Remark : For

$$\left\{ \begin{array}{rcl} \varepsilon x'' & = & (g(x) + \delta)\ x' \\ x(c) & = & A^* + \alpha\ e^{-b/\varepsilon} \\ x(d) & = & B^* + \beta\ e^{-b/\varepsilon} \end{array} \right. \tag{30}$$

the perturbation $\delta = \pm e^{-b/\varepsilon}$ in the coefficient as well as in the boundary values also change the shock location. Actually, these perturbations are not independent.

The sign of $\mu' \simeq g(B^*)\beta - g(A^*)\alpha \pm (B^* - A^*)$ determines if the shock layer moves to the left or to the right of t_0^* while the exponentially small term $e^{-b/\varepsilon}$ and the values of $g(x)$ at the unperturbed boundary values A^* and B^* determine the standard part of the shock location .

Formulas (28) and (29) give an estimate for the shock location in reponse to small perturbations tothe boundary value problem. From these estimates we show that the zeroth order term in the asymptotic expansion for the new shock location (i.e. the standard part of t_0) is

$$^0(t_0) = \begin{cases} ^0(t_0^* + (\frac{b-b^*}{g(B^*)}) = d + \frac{b}{g(B^*)} & \text{if } \mu' > 0 \\ ^0(t_0^* + (\frac{b-b^*}{g(A^*)}) = c + \frac{b}{g(A^*)} & \text{if } \mu' < 0 \end{cases} \tag{31}$$

Our method does not allow us to determine the infinitesimal ρ^* as a function of ε in the expresion for t_0^* therefore further terms in powers of ε in (28) and (29) might change due to ρ^*. For symmetric functions $G(x)$ the corresponding location t_0^* is exactly $(c+d)/2$ and $\rho^* = 0$. In this case formulas (28) and (29) provide further terms in the asymptotic expansion for the shock location as, for example, for Burgers' equation.

4.1.1 Example

A classical example of a supersensitive boundary value problem with a linear function f is the two-point problem for the steady state Burgers'equation:

$$\begin{cases} \varepsilon x'' &= -x\, x' \\ x(-1) &= -1 \\ x(1) &= 1 \end{cases} \tag{32}$$

For these boundary values (actually for any symmetric values $B^* = -A^* > 0$) the shock layer which joins the limiting solutions -1 and 1 is located at the midpoint $t_0^* = 0$. The inclusion of exponentially small perturbations in

$$\begin{cases} \varepsilon x'' &= -x\, x' \\ x(-1) &= -1 + \alpha e^{-b/\varepsilon} \\ x(1) &= 1 + \beta e^{-b/\varepsilon} \end{cases} \tag{33}$$

moves the shock location away from $t_0^* = 0$. We consider (33) with small perturbations only in the boundary values . For an exponentially small variation $\pm e^{-b/\varepsilon}$ in $g(x) = -x$ the simple change of variable $z = x \pm e^{-b/\varepsilon}$ converts the full problem into (33).

Here $\mu = -(\alpha+\beta)(1+(\beta-\alpha)e^{-b/\varepsilon})$ and $b^* = 1$. For $0 < b \lesssim_{\neq} 1$, according to (28) and (29) we have:

If $\alpha + \beta < 0$

$$t_0 = 1 - (b + \varepsilon \ln(\varepsilon / -(\alpha+\beta)) - \varepsilon(\beta-\alpha)e^{-b/\varepsilon} - \varepsilon^2 \pounds^+ e^{(b-1)/\varepsilon})(1+\rho) \tag{34}$$

If $\alpha + \beta > 0$

$$t_0 = -1 + (b + \varepsilon \ln(\varepsilon/(\alpha+\beta)) - \varepsilon(\beta-\alpha)e^{-b/\varepsilon} + \varepsilon^2 \pounds^+ e^{(b-1)/\varepsilon})(1+\rho) \tag{35}$$

For $b \gtrsim 1$, $t_0 \simeq 0$. When $\alpha + \beta = 0$ the shock occurs near 0. To illustrate the behavior of t_0 in the case of different small perturbations let $A = -1 + \alpha e^{-\varepsilon/2}$ and $B = 1 + \beta e^{-\varepsilon/3}$, then $\mu \simeq -\beta$ and it follows from (25) or (26) that $t_0 \simeq 2/3$ if $\beta < 0$ or $t_0 \simeq -2/3$ if $\beta > 0$.

Laforgue and O'Malley [11] have found the same location for this problem by using asymptotic expansions which include exponentially small terms.

4.1.2 Example

Another example of exponential sensitivity is given by a model for compressible fluid flow in nozzles:

$$\left\{ \begin{array}{rcl} \varepsilon x'' &=& ((\gamma+1)/2 - x^{-2})\, x' \\ x(0) &=& A \\ x(1) &=& B \end{array} \right. \tag{36}$$

This problem arises when a gas is injected at a supersonic velocity A in a duct of uniform cross-sectional area and a back pressure is applied. Here x is the dimensionless velocity of the gas relative to the velocity of sound, t is the dimensionless distance with $t = 0$ at the entrance of the duct and γ is the adiabatic index $(1 \leq \gamma \leq 5/3)$; cf.[5, 6].

For coupled boundary conditions which satisfy the well-known Prandtl relation $A^*B^* = 2/(\gamma+1)$, the solution of (36) is a steady wave with a transition at $t_0^* \simeq \frac{A^*}{A^*+B^*}$ from a supersonic velocity A^* to a subsonic velocity B^* as we have shown in [1].

If we now allow the boundary values and the coefficient of x' in (36) to change by exponentially small amount as follows:

$$\left\{ \begin{array}{rcl} \varepsilon x'' &=& ((\gamma+1)/2 - x^{-2} + \lambda e^{-b/\varepsilon})\, x' \\ x(0) &=& A^* + \alpha e^{-b/\varepsilon} \\ x(1) &=& B^* + \beta e^{-b/\varepsilon} \end{array} \right. \tag{37}$$

the location of the supersonic-subsonic transition at t_0 is given as follows:

For $0 < b \lesssim_{\neq} b^* \simeq \frac{A^*-B^*}{A^*+B^*}\frac{1}{A^*B^*}$ and if $\mu' \simeq \frac{A^*-B^*}{A^{*2}B^{*2}}(-A^*\beta - B^*\alpha + \lambda) > 0$,

$$t_0 = 1 - (\frac{A^*B^{*2}}{A^* - B^*})(b + \varepsilon \ln(\frac{\varepsilon}{\mu'}) - \varepsilon^2 \pounds^+ e^{(b-b^*)/\varepsilon})(1+\rho) \tag{38}$$

while if $\mu' < 0$

$$t_0 = \frac{A^{*2}B^*}{A^* - B^*}(b + \varepsilon \ln(\frac{\varepsilon}{-\mu'}) + \varepsilon^2 \pounds^+ e^{(b-b^*)/\varepsilon})(1+\rho) \tag{39}$$

while $t_0 \simeq t_0^* \simeq \frac{A^*}{A^*+B^*}$ if $b \underset{\simeq}{>} b^*$.

If the effect of all the perturbations makes t_0 moves to the right of t_0^*, it moves to the end of the duct as b decreases, i.e. as the perturbation increases.

4.1.3 Supersensitivity and Metastability

The ordinary differential equation $\varepsilon u'' = g(u)u'$ where $u' = du/dx$ provides travelling wave solutions for the corresponding time-dependent partial differential equation $u_t = \varepsilon u_{xx} - g(u)u_x$.

Let us consider the initial boundary value problem

$$(P) \qquad \begin{cases} u_t & = \varepsilon u_{xx} - g(u)u_x \;, \qquad c < x < d, \; t > 0 \\ u(c,t) & = A^*, \quad u(d,t) = B^*, \qquad t > 0 \\ u(x,0) & = u_0(x) \end{cases}$$

where $u_0(x)$ is a solution of the perturbed boundary value problem

$$\begin{cases} \varepsilon u'' & = & (g(u) + \delta) \; u' \\ u(c) & = & A^* \\ u(d) & = & B^* \end{cases} \tag{40}$$

with $\delta = \pm e^{-b/\varepsilon}$ and $0 \lesssim_{\neq} b \lesssim_{\neq} b^*$.

This δ-parameter family of problems provides a suitable set of initial data $u_0(x)$ that leads to metastable behavior of the time-dependent solution $u(x,t)$. By using comparison principles, we can show that $u(x,t)$ is bounded by the travelling wave solution $u_0(x - \delta t)$ and the inital condition $u_0(x)$. Then the initial configuration of $u(x,t)$ will reamin almost the same over the exponentially long time interval $0 \le t \le \tau = e^{-k_0/\varepsilon}$ where $k_0 \simeq b$, giving rise to a metastable behavior.

4.2 The case $f(x') = x'^2$

For the quadratically nonlinear problem the shock location is sensitive to perturbations of order $O(\varepsilon)$. For the perturbed problem

$$\left\{\begin{array}{rcl} \varepsilon x'' & = & (g(x)+s\varepsilon)x'^2 \\ x(c) & = & A^* + a\varepsilon \\ x(d) & = & B^* + b\varepsilon \end{array}\right. \tag{41}$$

equation (19) becomes:

$$\ln\Big(\frac{g(A^*)(t_0-c)}{g(B^*)(t_0-d)}(1+\eta)\Big) = g(B^*)b - g(A^*)a + s(B^*-A^*) + s(b-a)\varepsilon \tag{42}$$

and the shock location is given by:

$$t_0 = \frac{g(A^*)c - (1+\eta)g(B^*)de^{g(B^*)b-g(A^*)a+s(B^*-A^*)+s(b-a)\varepsilon}}{g(A^*) - (1+\eta)g(B^*)e^{g(B^*)b-g(A^*)a+s(B^*-A^*)+s(b-a)\varepsilon}} \tag{43}$$

Note that the shock layer moves toward c as $g(B^*)b - g(A^*)a + s(B^*-A^*) \to -\infty$ toward d as $g(B^*)b-g(A^*)a+s(B^*-A^*) \to +\infty$ and near t_0^* when $g(B^*)b-g(A^*)a+s(B^*-A^*) \to 0$.

Sensitivity and Resonance

For singularly perturbed problems like $P(A,B)$ a straightforward application of the method of matched asymptotic expansions fails to determine the internal layer locations uniquely. A similar difficulty with this approach arises in a class of problems exhibiting the phenomena of boundary layer resonance. In this case the matched asymptotic expansion for the solution is given in terms of an undetermined constant K_0. The method given in this paper allows us not only to find the shock location but also to study its behavior in reponse to small changes in the boundary conditions as well as in the coefficient $g(x)$ in the equation. In addition, for problems like (41), the shock location determines the constant K_0 uniquely, that is, it determines the outer solution $t(x) \simeq t_0$ for the associated boundary layer resonance problem:

$$\left\{\begin{array}{rcl} \varepsilon\ddot{t} & = & -g(x)\dot{t} \\ t(A) & = & c \\ t(B) & = & d \end{array}\right. \tag{44}$$

4.2.1 Example

The sensitivity of the solutions of (41) explains a phenomena observed by Matkowsky [15] in the study of resonance for a quasilinear boundary value problem. Matkowsky remarked that the solution changes significantly when one of the endpoints of the interval is slightly varied by $O(\varepsilon^\gamma)$ for any γ. More precisely, he has noted that for the problem

$$\left\{\begin{array}{rcl} \varepsilon\ddot{t} & = & x\dot{t} \\ t(A) & = & c \\ t(B) & = & d, \qquad\quad A<0<B \end{array}\right. \tag{45}$$

algebraically small changes at one of the endpoints of the interval $[A, B]$ change a solution $t(x)$ with two boundary layers into a solution with only one boundary layer. Obviously, the corresponding inverse problem

$$\left\{\begin{array}{rcl} \varepsilon x'' & = & -x\ x'^2 \\ x(c) & = & A \\ x(d) & = & B, \end{array}\right. \tag{46}$$

is a supersensitive problem with respect to the symmetric boundary values $B^* = -A^*$. Since, for these boundary values, the solution $x(t)$ has an internal layer located at the middle of the interval $t_0^* \simeq \frac{c+d}{2}$, $t(x)$ has two boundary layers connected by the "outer solution" t_0^*. As the shock location is sensitive to small changes of the boundary data of order $O(\varepsilon)$, these slight variations of "the endpoint" B^* move the position of t_0^* by $O(1)$. Then, from (43), the new location of t_0 for $B = B^* + b\varepsilon$ is

$$t_0 = \frac{c + (1+\eta)\ de^{-B^*b}}{1 + (1+\eta)\ e^{-B^*b}} \tag{47}$$

From (47) we can easily describe the initial formation of the boundary layers near the endpoints. Note that the two boundary layers of $t(x)$ exist for $B - B^* = O(\varepsilon)$ and that one of the them disappears when the slight variations of B^* are of order ε^γ with $\gamma < 1$. The lefthand boundary layer for $t(x)$ disappears as b increases since the shock location t_0 tends to the left endpoint c according to (47). The righthand boundary layer likewise disappears as b decreases.

We remark that a boundary layer like (44) is also algebraically sensitive to a small change of order $O(\varepsilon)$ in the coefficient $g(x)$ of the first derivative $\dot{t}$. For example, a small perturbation $s\varepsilon$ of $g(x) = x$ in (45) with $B^* = -A^*$ changes the outer solution t_0^* to $t_0 = \frac{c+(1+\eta)\ de^{2bB^*}}{1+(1+\eta)\ e^{2bB^*}}$ according to (43). We conclude that if the turning point is perturbed from $x = 0$ by an algebraically small amount $s\varepsilon$, the value of t_0 varies from $t_0^* = (c+d)/2$ by $O(1)$ and t_0 moves toward d or c if the turning point $x = -b\varepsilon$ moves to the left or to the right of $x = 0$.

4.2.2 Example

The next problem was also considered by Matkowsky to illustrate the resonance phenomenon for functions $g(x)$ having more than a single simple zero in the interval $[A, B]$:

$$\left\{ \begin{array}{rcl} \varepsilon \ddot{t} & = & x^3(x^2-1)(x-2)^2 \ \dot{t} \\ t(A) & = & c \\ t(B) & = & d, \qquad A=-2, \ B>1 \text{ and } B \neq 2 \end{array} \right. \tag{48}$$

For this turning point problem the associated inverse problem is

$$\left\{ \begin{array}{rcl} \varepsilon x'' & = & -x^3(x^2-1)(x-2)^2 \ x'^2 \\ x(c) & = & -2 \\ x(d) & = & B \end{array} \right. \tag{49}$$

For this choice of $A^* = -2$ the corresponding B^* satisfies

$$\int_{-2}^{B^*} x^3(x^2-1)(x-2)^2 dx = 0$$

and $B^* > 2$. Depending on the values of B we can deduce as in [1] the following behavior for the solution of (49).

If $-2 < B < B^*$ ($B > B^*$), there is a boundary layer at $t = d$ (at $t = c$) and there is an internal layer at $t_0^* = \frac{g(-2)c-(1+\eta)g(B^*)d}{g(-2)-(1+\eta)g(B^*)}$ when $B = B^*$. Then the solution $t(x)$ has a single boundary layer at $x = -2$ (at $x = B$) or there are two boundary layers at both endpoints -2 and B^*. In addition, this is a supersensitive problem with respect to (A^*, B^*). Then, as before, we can explain how the boundary layers of $t(x)$ may disappear due to small changes of order ε in the endpoints of the interval $[A^*, B^*]$ and in the function $g(x)$.

4.3 Case $f(x') = x'^s, \ 1 < s < 2$

Another case of algebraic sensitivity is given by the following family of problems $P(A, B)$:

$$\left\{ \begin{array}{rcl} \varepsilon x'' & = & g(x) \ x'^s \\ x(c) & = & A \\ x(d) & = & B \end{array} \right. \tag{50}$$

with $1 < s < 2$.

From (19) the shock position satisfies:

$$
\begin{aligned}
&(g(B^*)(1+\xi)(t_0-d))^{s-2/s-1}-(g(A^*)(1+\eta)(t_0-c))^{s-2/s-1} \\
&\quad = K\varepsilon^{1/1-s}(g(B^*)(B-B^*)(1+\eta)-g(A^*)(A-A^*)(1+\eta))
\end{aligned} \tag{51}
$$

with $K=((2-s)(s-1))^{s-2/s-1}$.

Since the associated inverse problem is $\varepsilon\ddot{t} = g(x)\ \dot{t}^{(3-s)}$, $t(A)=c$, $t(B)=d$, the thickness of the jumps of $\hat{P}$ at A^* and B^* are of $O(\varepsilon^{1/s-1})$. Then the small perturbations of the boundary conditions that make the internal layer move away from t_0^* are of order $O(\varepsilon^{1/s-1})$ as we can also deduce from (51) (and not of order $O(\varepsilon^{3-s})$ as was unfortunately written in [1]).

In the special case $s=3/2$, equation (51) becomes:

$$
\begin{aligned}
&(g(B^*)(1+\xi)(t_0-d))^{-1}-(g(A^*)(1+\eta)(t_0-c))^{-1} \\
&\qquad = 3(g(B^*)b(1+\eta)-g(A^*)a(1+\eta))
\end{aligned} \tag{52}
$$

where $A=A^*+a\varepsilon^2,\ B=B^*+b\varepsilon^2$, and the internal layer can be found explicitly as the unique root in $[c,d]$ of:

$$
\mu\ t_0^2-(\mu(c+d)+\beta-\alpha)\ t_0+(\mu dc+\beta c-\alpha d)=0 \tag{53}
$$

where $\mu=3(g(B^*)b(1+\eta)-g(A^*)a(1+\eta))$, $\alpha=(g(A^*)(1+\eta))^{-1}$ and $\beta=(g(B^*)(1+\eta))^{-1}$. If the small perturbations almost cancel each other to give $\mu\simeq 0$ then $t_0\simeq t_0^*$ as we can easily deduce from (53).

However, if $\mid\mu\mid \gtrsim 0$, then

$$
t_0=\frac{c+d}{2}+\frac{\beta-\alpha}{2\mu}+\frac{d-c}{2}\frac{\mid\mu\mid}{\mu}\left(1+2\frac{(\alpha+\beta)}{(d-c)\mu}+\frac{(\beta-\alpha)^2}{(d-c)^2\mu^2}\right)^{1/2}. \tag{54}
$$

As expected t_0 moves toward c or d as $\mu\to-\infty$ or $\mu\to+\infty$.

4.3.1 Example

For the boundary value problem:

$$
\left\{\begin{array}{rcl}
\varepsilon x'' & = & -x\ x'^{3/2} \\
x(-1) & = & -1+a\varepsilon^2 \\
x(1) & = & 1+b\varepsilon^2
\end{array}\right. \tag{55}
$$

the value $\mu=-3(a+b)(1+\eta)$ determines the shock location.

If $a+b \lneq 0$ then $t_0 \simeq \frac{1}{3(a+b)} + (1+\frac{1}{9(a+b)^2)})^{1/2}$, if $a+b \gneq 0$ then $t_0 \simeq \frac{1}{3(a+b)} - (1+\frac{1}{9(a+b)^2)})^{1/2}$, and if $a+b \simeq 0$ the shock is located near 0.

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