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Approximate Approximations and their Applications

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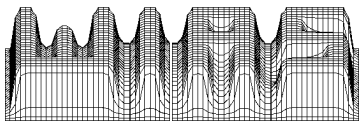

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ABSTRACT. This paper gives a survey of an approximation method which was proposed by V. Maz'ya as underlying procedure for numerical algorithms to solve initial and boundary value problems of mathematical physics. Due to a greater flexibility in the choice of approximating functions it allows efficient approximations of multi–dimensional integral operators often occuring in applied problems. Its application especially in connection with integral equation methods is very promising, which has been proved already for different classes of evolution equations. The survey describes some basic results concerning error estimates for quasi–interpolation and cubature of integral operators with singular kernels as well as a multiscale and wavelet approach to approximate those operators over bounded domains. Finally a general numerical method for solving nonlocal nonlinear evolution equations is presented.

1. Introduction

V. Maz'ya proposed the method of "Approximate Approximations" in the late 80s when dealing with applied problems as chair of the division "Mathematical models in mechanics" of the Leningrad Research Engineering Institute of the USSR Academy of Sciences (see some of its applications in [8], [20], [21], and the first announcement in English [9]). The main idea of this method is based on the fact that the numerical solution of applied problems is always required only within some prescribed accuracy. Therefore it is possible to use numerical algorithms which provide good approximations only up to some prescribed error level, but do not converge in rigorous mathematical sense. By this way one can

- enlarge the set of approximating functions significantly
- obtain numerically cheap and accurate approximation formulas
- deduce effective formulas for the approximation of various integral and other pseudodifferential operators of mathematical physics

For the first time I heard about this concept at a lecture of V. Maz'ya in 1991 and I became interested in the rather unusual idea to use non–converging approximation methods in numerical computations together with the wide area of possible applications. At the end of 1992 we began our joint work on different aspects of approximate approximations which is still in progress. The present report gives an outline of some of the obtained results.

The starting point is the error analysis of approximate quasi–interpolation on uniform lattices. The quasi–interpolants are linear combinations of scaled translates of a sufficiently smooth and rapidly decaying basis function η and depend on two parameters, the "small" mesh width h and the "large" parameter $\mathcal{D}$ to scale the Fourier image. Under quite general assumptions on η the approximants converge if $\mathcal{D} \to \infty$ and the product $\sqrt{\mathcal{D}}h \to 0$. However, in practical computations it is advantageous to keep $\mathcal{D}$ fixed, which results in an approximation of some order N up to a saturation error which can be made arbitrarily small if $\mathcal{D}$ is chosen large enough. The order N is determined by the moments of the function η and can be increased to any integer value by some analytic or algebraic transformations of the given basis function. Thus one derives a new class of simple multi–variate formulas which behave in numerical computations like high order approximations. This quasi–interpolation procedure was recently extended to the approximation of functions on domains and manifolds with nonuniformly distributed nodes.

The great flexibility in the choice of generating functions η makes it easier to find approximations for which the action of a given operator can be effectively determined. For example, suppose one has to evaluate the convolution with a singular radial kernel as in the case of many potentials in mathematical physics. If the density is replaced by a quasi–interpolant with radial η then after passing to spherical coordinates the convolution is approximated by one–dimensional integrals. For many important integral operators K one can choose η even such that $K\eta$ is analytically known, which results in semi–analytic cubature formulas for these operators. The special structure of

the quasi–interpolation error gives rise to an interesting effect. Since the saturation error is a fast oscillating function and converges weakly to zero, the cubature formulas for potentials, for example, converge even in rigorous sense.

The presence of the two scaling parameters h and $\mathcal{D}$ is also useful to develop a multiscale approach within the concept of approximate approximations. This approach enables the construction of high order approximations to discontinuous functions and of accurate cubature formulas for integral operators over bounded domains. Here a two scale relation for generating functions from the Schwartz class is used which is valid within any prescribed tolerance for sufficiently large $\mathcal{D}$. This relation allows also to perform an approximate multiresolution analysis of spaces generated by those functions. Therefore a wavelet basis can be constructed in which elements of fine scale spaces are representable within a given tolerance. These approximate wavelets provide most of the properties utilized in wavelet based numerical methods and possess additionally simple analytic representations. Therefore the sparse approximation of important integral operators in the new basis can be computed using special functions or simple quadrature.

The capability of approximate approximations to treat multi–dimensional integral operators very efficiently is very promising for new integral equation based numerical methods. In the last section I present some examples of these methods which were developed by V. Maz'ya together with V. Karlin to solve linear and nonlinear problems. In particular, numerical results for evolution equations with nonlocal operators are given which cannot be solved by standard finite–difference or finite–element methods.

2. Quasi–interpolation

2.1. Example

To illustrate the idea of approximate approximations let us consider the following example of a simple quasi–interpolation formula in $\mathbf{R}^1$

$$\mathcal{M}_h u(x) = \frac{1}{\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} u(mh)\,\mathrm{sech}\,\frac{x-mh}{\sqrt{\mathcal{D}}h} \tag{2.1}$$

If u is 2–times continuously differentiable then the Taylor expansion

$$u(mh) = u(x) + u'(x)(mh-x) + u''(x_m)\frac{(mh-x)^2}{2}$$

with some x_m between x and mh leads to

$$\mathcal{M}_h u(x) = \frac{u(x)}{\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} \mathrm{sech}\,\frac{x-mh}{\sqrt{\mathcal{D}}h} + \frac{u'(x)}{\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} (mh-x)\,\mathrm{sech}\,\frac{x-mh}{\sqrt{\mathcal{D}}h}$$
$$+ \frac{1}{2\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} u''(x_m)(mh-x)^2\,\mathrm{sech}\,\frac{x-mh}{\sqrt{\mathcal{D}}h}\,.$$

The Fourier transform of $\mathrm{sech}(x)$ is given as

$$(\mathcal{F}\mathrm{sech})(\lambda) := \int_{\mathbf{R}^1} \mathrm{sech}\,x\, e^{-2\pi i x\lambda}\,dx = \pi\,\mathrm{sech}\,\pi^2\lambda\,,$$

therefore Poisson's summation formula results in the relations

$$\frac{1}{\pi\sqrt{\mathcal{D}}} \sum_{m=-\infty}^{\infty} \mathrm{sech}\,\frac{x-m}{\sqrt{\mathcal{D}}} = 1 + 2\sum_{\nu=1}^{\infty} \mathrm{sech}(\pi^2\sqrt{\mathcal{D}}\nu)\,\cos(2\pi\nu x)\,,$$

$$\frac{1}{\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty}\frac{x-m}{\sqrt{\mathcal{D}}}\operatorname{sech}\frac{x-m}{\sqrt{\mathcal{D}}}=\pi\sum_{\nu=1}^{\infty}\operatorname{sech}(\pi^2\sqrt{\mathcal{D}}\nu)\tanh(\pi^2\sqrt{\mathcal{D}}\nu)\,\sin(2\pi\nu x)\,.$$

It is clear that $\mathcal{M}_h u(x)$ does not converge to $u(x)$. However, one has

$$\left|\frac{1}{\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty}\operatorname{sech}\frac{x-m}{\sqrt{\mathcal{D}}}-1\right|\le 2\,\varepsilon(\sqrt{\mathcal{D}})\,,$$
$$\left|\frac{1}{\pi\sqrt{\mathcal{D}}}\sum_{m=-\infty}^{\infty}\frac{x-m}{\sqrt{\mathcal{D}}}\operatorname{sech}\frac{x-m}{\sqrt{\mathcal{D}}}\right|\le \pi\,\varepsilon(\sqrt{\mathcal{D}})\,,$$

where the number

$$\varepsilon(\sqrt{\mathcal{D}}):=\sum_{\nu=1}^{\infty}\operatorname{sech}(\pi^2\sqrt{\mathcal{D}}\nu)$$

can be made arbitrarily small by choosing $\mathcal{D}$ large enough. For example, if $\mathcal{D}=4$ then $\varepsilon(\sqrt{\mathcal{D}})=0.000000005351$. Since

$$\frac{1}{2\pi\sqrt{\mathcal{D}}}\left|\sum_{m=-\infty}^{\infty}u''(x_m)\frac{(mh-x)^2}{\mathcal{D}h^2}\operatorname{sech}\frac{x-mh}{\sqrt{\mathcal{D}}h}\right|\le\frac{5}{4}\sup_{t\in\mathbf{R}}|u''(t)|$$

the difference between u and $\mathcal{M}_h u$ can be estimated by

$$|\mathcal{M}_h u(x)-u(x)|\le\frac{5}{4}\mathcal{D}h^2\sup_{\mathbf{R}}|u''|+\varepsilon(\sqrt{\mathcal{D}})\big(2\ \ |u(x)|+\pi\sqrt{\mathcal{D}}h\,|u'(x)|\big)\,.$$

This means, above the tolerance $\varepsilon(\sqrt{\mathcal{D}})\big(2\,|u(x)|+\pi\sqrt{\mathcal{D}}h\,|u'(x)|\big)$ the quasi–interpolant $\mathcal{M}_h u$ approximates any 2-times continuously differentiable function like usual second order approximations and any prescribed accuracy can be reached if $\mathcal{D}$ is chosen appropriately.

2.2. Quasi–interpolation with general basis functions

A similar approximation behavior as in the previous example can be obtained for quite arbitrary basis functions in arbitrary space dimension n and approximation order N. Consider the quasi–interpolation formula

$$\mathcal{M}_h u(\mathbf{m})=\mathcal{D}^{-\mathbf{n/2}}\sum_{\mathbf{m}\in\mathbf{Z^n}}\mathbf{u(hm)}\,\eta\Big(\frac{\mathbf{x-hm}}{\sqrt{\mathcal{D}}\mathbf{h}}\Big)\tag{2.2}$$

with a continuous generating function η satisfying

A.1 Decay condition

$$|\eta(\mathbf{x})|\le A_K\,(1+|\mathbf{x}|^2)^{-K/2}\ ,\ \mathbf{x}\in\mathbf{R^n}\,,\tag{2.3}$$

for some constant A_K, $K>N+n$ and $N\ge 1$ is a given positive integer.

Denoting by

$$\sigma_\alpha(\mathbf{x},\mathcal{D}):=\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\Big(\frac{\mathbf{x-m}}{\sqrt{\mathcal{D}}}\Big)^\alpha\eta\Big(\frac{\mathbf{x-m}}{\sqrt{\mathcal{D}}}\Big)\,,$$
$$\rho_\alpha(\mathbf{x},\mathcal{D}):=\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\left|\Big(\frac{\mathbf{x-m}}{\sqrt{\mathcal{D}}}\Big)^\alpha\eta\Big(\frac{\mathbf{x-m}}{\sqrt{\mathcal{D}}}\Big)\right|\,,$$

one gets after substituting Taylors expansion of u into (2.2)

$$\mathcal{M}_h u(\mathbf{x})=\sum_{|\alpha|=0}^{N-1}\frac{\partial^\alpha u(\mathbf{x})}{\alpha!}(-\sqrt{\mathcal{D}}h)^{|\alpha|}\sigma_\alpha(\mathbf{x}/h,\mathcal{D})+R_N(\mathbf{x},\sqrt{\mathcal{D}}h)\,,\tag{2.4}$$

where

$$|R_N(\mathbf{x},\sqrt{\mathcal{D}}h)|\le(\sqrt{\mathcal{D}}h)^N\sum_{|\alpha|=N}\frac{\|\partial^\alpha u\|_{L_\infty(\mathbf{R^n})}}{\alpha!}\,\rho_\alpha(\mathbf{x}/h,\mathcal{D})$$

The application of Poisson's summation formula to σ_α yields the equality

$$\sigma_\alpha(\mathbf{x},\mathcal{D}) = \Big(\frac{i}{2\pi}\Big)^{|\alpha|} \sum_{\nu\in\mathbf{Z}^n} \partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)\, e^{2\pi i(\nu,\mathbf{x})}\,,$$

which holds if the sequence $\{\partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\,\cdot)\} \in l_1(\mathbf{Z}^n)$ (see e.g. [18]).

Thus $\mathcal{M}_h u$ would approximate a sufficiently smooth functions u with the order $\mathcal{O}((\sqrt{\mathcal{D}}h)^N)$ if $\sigma_\alpha(\mathbf{x},\mathcal{D}) = \delta_{|\alpha|0}$ for all $\mathbf{x}\in\mathbf{R^n}$, δ_{ik} is the Kronecker symbol. These equalities imply the well–known Strang–Fix condition

$$\partial^\alpha \mathcal{F}\eta(\mathbf{0}) = \delta_{|\alpha|0}\,, \quad \partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu) = 0\,, \quad \nu\in\mathbf{Z}^n\setminus\{\mathbf{0}\}\,,\ 0\le|\alpha|\le N-1\,.$$

The idea of approximate quasi–interpolation is to use generating functions η for which $\sigma_\alpha(\mathbf{x},\mathcal{D})$ can be made arbitrarily close to $\delta_{|\alpha|0}$ by choosing appropriate values of $\mathcal{D}$. Therefore η and its Fourier transform $\mathcal{F}\eta$ have to satisfy

A.2 Moment condition

$$\int_{\mathbf{R}^n} \eta(\mathbf{y})\,d\mathbf{y} = 1\,,\ \int_{\mathbf{R}^n} \mathbf{y}^\alpha \eta(\mathbf{y})\,d\mathbf{y} = 0, \quad \forall\,\alpha\,,\ 1\le|\alpha|<N\,, \tag{2.5}$$

A.3 For any $\varepsilon>0$ there exists $\mathcal{D}>0$ so that

$$\sum_{\nu\in\mathbf{Z}^n\setminus\{0\}} |\partial^\alpha \mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)| < \varepsilon\,, \quad \forall\,\alpha\,,\ 0\le|\alpha|\le N\,. \tag{2.6}$$

In [17] it is shown that for M–times differentiable functions η (M the smallest integer greater than $n/2$), which satisfy together with all derivatives $\partial^\beta\eta$, $|\beta|\le M$, the decay condition A.1, assumption A.2 is true.

Consequently, if the generating function η satisfies the assumptions A.1–3 one obtains from (2.4) the error estimate

Theorem 2.1. ([14]) *If $u\in C^N(\mathbf{R}^n)\cap W_\infty^N(\mathbf{R}^n)$ then for any $\varepsilon>0$ there exists $\mathcal{D}>0$ such that*

$$|\mathcal{M}_h u(\mathbf{x}) - u(\mathbf{x})| \le c\,(\sqrt{\mathcal{D}}h)^N \sum_{|\alpha|=N} \frac{\|\partial^\alpha u\|_{L_\infty(\mathbf{R^n})}}{\alpha!} + \varepsilon \sum_{|\alpha|=0}^{N-1} (\sqrt{\mathcal{D}}h)^{|\alpha|}\,\frac{|\partial^\alpha u(\mathbf{x})|}{\alpha!}\,,$$

where the constant c depends only on η.

Similar error estimates for quasi–interpolation in the L_p–norm are proved in [15]. Note that for u from the Hölder class $C^{L,\mu}$, $0\le L<N$, $0<\mu<1$, the error $|\mathcal{M}_h u(\mathbf{x}) - u(\mathbf{x})|$ is of the order $\mathcal{O}((\sqrt{\mathcal{D}}h)^{L+\mu})$ plus small remainder terms.

The local nature of the quasi–interpolant $\mathcal{M}_h$ should be emphasised, its behaviour at a given point $\mathbf{x}$ depends within some error level ε only on the values of u at grid points in some neighborhood of this point. Since in practice one has to truncate the summation in (2.2) it is important that depending on $\varepsilon>0$ there exist $N_s>0$ ensuring that the truncated quasi–interpolant

$$\mathcal{M}_h^s u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{|\mathbf{x}-h\mathbf{m}|\le N_s h} u(h\mathbf{m})\,\eta\Big(\frac{\mathbf{x}-\mathbf{hm}}{\sqrt{\mathcal{D}}\mathbf{h}}\Big)\,, \tag{2.7}$$

shows the same approximation behaviour as $\mathcal{M}_h u$. Thus, the number of summands in (2.7) does not depend on the mesh size h.

One interesting consequence of Theorem 2.1 is that the quasi–interpolant $\mathcal{M}_h u$ converges to u if the parameter $\mathcal{D}$ is chosen depending on h with $\mathcal{D}(h)\to\infty$ as $h\to 0$. For any generating functions η satisfying the assumptions A.1–3 one can specify this dependence and determine the optimal

convergence order, which is of course less than N. For special one–dimensional or tensor–product generating functions this problem has been addressed in the literature ([19],[2],[1]).

We prefer to have $\mathcal{D}$ as additional parameter because this may be advantageous in some numerical applications, where the quasi–interpolation is used, as well as for further developments of approximate approximations mentioned in Sections 7 and 8.

2.3. Quasi–interpolation with variable nodes

The quasi–interpolation formula (2.2) is not restricted to the case of uniform spacing of the nodes or grid points. In [17] an extension of this formula to the form

$$\mathcal{D}^{-n/2} \sum_{\mathbf{x_m} \in \Omega} u(\mathbf{x_m})\, \eta\Big(\frac{\mathbf{x}-\mathbf{x_m}}{\sqrt{\mathcal{D}}\, V_{\mathbf{m}}}\Big)\,, \quad \mathbf{x} \in \Omega\,, \tag{2.8}$$

is discussed, where Ω is some domain in $\mathbf{R^n}$ or an n–dimensional manifold in $\mathbf{R^s}$ and the nodes $\mathbf{x_m} \in \Omega$ are the images of a lattice of width h under smooth parametrization of Ω. To retain the local character of the quasi–interpolant the scaling $V_{\mathbf{m}}$ should be proportional to h.

After applying a partition of unity it can be supposed that $\Omega = \phi(\omega)$, where $\omega \subset \mathbf{R}^n$ is a bounded domain and $\phi = (\varphi_1, \ldots, \varphi_s) : \mathbf{R}^n \to \mathbf{R}^s$, $n \le s$ is a sufficiently smooth and non-singular one-to-one mapping. In particular,

$$|\phi'(\mathbf{y})| = \Big(\sum_{(i)} (\kappa_{(i)}(\mathbf{y}))^2\Big)^{1/2} > 0\,, \quad \mathbf{y} \in \omega\,,$$

where $\kappa^{(i)}$ denotes the minor of order n of the matrix $\phi'(\mathbf{y}) = (\partial\varphi_j/\partial y_k)_{j,k=1}^{s,n}$, corresponding to indices $i_1 < \ldots < i_n$. The sum is extended over all distinct tuples $(i) = (i_1, \ldots, i_n)$, $1 \le i_p \le s$, of this kind.

It turns out that if in (2.8) the nodes are given by $\mathbf{x_m} = \phi(h\mathbf{m})$, $h\mathbf{m} \in \omega$, then this quasi–interpolant has similar approximation properties as in the case of uniformly distributed nodes. For the sake of simplicity we assume here that η belongs to the Schwartz class, $\eta \in \mathcal{S}(\mathbf{R}^s)$. Further one has suppose that in the case $s > n$ the generating function η is radial.

Theorem 2.2. ([17]) *Assume that η satisfies the assumption A.2, $\phi : \omega \to \Omega$ is in the class C^{N+1} with $|\phi'(\mathbf{y}| > 0$, $\mathbf{y} \in \overline{\omega}$, and the numbers $V_{\mathbf{m}}$ are chosen such that*

$$\big|(h^{-1}V_{\mathbf{m}})^n - |\phi'(h\mathbf{m})|\big| \le c\, h^N\,.$$

If $u \in C_0^N(\Omega)$, then for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that at any point $\mathbf{x} \in \Omega$

$$|u_h(\mathbf{x}) - u(\mathbf{x})| \le c_\eta\, (\sqrt{\mathcal{D}}h)^N\, \|u\|_{C^N(\overline{\Omega})} + \varepsilon \sum_{k=0}^{N-1} c_k\, (\sqrt{\mathcal{D}}h)^k\,, \tag{2.9}$$

where c_η does not depend on u, h and $\mathcal{D}$ and the constants c_k can be obtained from the values $\partial^\alpha u(\mathbf{x})$, $|\alpha| \le k$.

Consider as simple example a fourth order quasi–interpolation formula on a surface Γ in $\mathbf{R}^3$. To define the numbers $V_{\mathbf{m}}$ denote by $Q^j_{\mathbf{m}} = h\mathbf{m} + hQ^j$, $j = 1, 2$, where $Q^1 = [-1, 1]^2$ and Q^2 is the sqaure with corners at the points $(\pm 1, 0)$ and $(0, \pm 1)$. Then formula (2.8) with $\eta(|\mathbf{x}|) = \pi^{-1}(2 - |\mathbf{x}|^2)\, \exp(-|\mathbf{x}|^2)$ provides approximate approximations with the order $\mathcal{O}(\mathcal{D}^2 h^4)$ on Γ if one chooses

$$V_{\mathbf{m}} = \sqrt{\mathrm{meas}(\phi(Q^2_{\mathbf{m}})) - \mathrm{meas}(\phi(Q^1_{\mathbf{m}}))/4}\,.$$

3. Generating functions for quasi–interpolation of high order

In fact any sufficiently smooth and decaying function η with $\mathcal{F}\eta(0) \neq 0$ can be used as generating function for quasi–interpolation processes. One interesting feature of approximate approximations is the possibility to construct from a given basis function η new generating functions η_N satisfying the assumptions A.2 on the moments for arbitrary given large N using different analytic and algebraic methods. Some of these methods are described in this section.

3.1. Examples of generating functions

The following table lists some examples of useful functions for which the Fourier transform is known. The first number in each row indicates the space dimension. In the next section it is shown how these functions can be modified to generate high order quasi–interpolants.

	$\eta(x)$	$\mathcal{F}\eta(\lambda)$
1	$\pi^{-1}\,\mathrm{sech}\,(x)$	$\mathrm{sech}\,(\pi^2\lambda)$
1	$2\,\pi^{-2}\,x\,\mathrm{cosech}\,(x)$	$\mathrm{sech}^2\,(\pi^2\lambda)$
n	$\pi^{-n/2}\exp(-\lvert\mathbf{x}\rvert^2)$	$\exp(-\pi^2\lvert\lambda\rvert^2)$
1	$\sqrt{e/\pi}\exp(-x^2)\cos\sqrt{2}x$	$\exp(-\pi^2\lambda^2)\cosh\sqrt{2}\pi\lambda$
2	$\dfrac{\exp(-2\pi a\sqrt{\lvert\mathbf{x}\rvert^2+b^2})}{\sqrt{\lvert\mathbf{x}\rvert^2+b^2}}$	$\dfrac{\exp(-2\pi b\sqrt{\lvert\lambda\rvert^2+a^2})}{\sqrt{\lvert\lambda\rvert^2+a^2}}$
3	$\pi^{-4}\,\mathrm{sech}\,(\lvert\mathbf{x}\rvert)$	$\dfrac{\sinh(\pi^2\lvert\lambda\rvert)\,\mathrm{sech}^2\,(\pi^2\lvert\lambda\rvert)}{\pi^2\lvert\lambda\rvert}$
n	$\dfrac{4}{3\pi^{n+1/2}}\,\dfrac{\Gamma(\frac{n+5}{2})}{(1+\lvert\mathbf{x}\rvert^2)^{(n+5)/2}}$	$(1+2\pi\lvert\lambda\rvert+\dfrac{4}{3}\pi^2\lvert\lambda\rvert^2)\,e^{-2\pi\lvert\lambda\rvert}$
n	$(-1)^k\dfrac{\pi^{(n+1)/2}}{\Gamma(k+\frac{n+1}{2})}\,\dfrac{\partial^k}{\partial\tau^k}\dfrac{1}{\sqrt{\tau}}e^{-2\pi\sqrt{\tau}\lvert\mathbf{x}\rvert}\Big\vert_{\tau=1}$	$(1+\lvert\lambda\rvert^2)^{-k-(n+1)/2}$
n	$\dfrac{\Gamma(k+1+\frac{n}{2})}{\pi^{n/2}\Gamma(k+1)}\,(1-\lvert\mathbf{x}\rvert^2)^k\,\chi(\mathbf{x})$	$\Gamma(k+1+\frac{n}{2})\,\dfrac{J_{k+n/2}(2\pi\lvert\lambda\rvert)}{(\pi\lvert\lambda\rvert)^{k+n/2}}$

Here χ denotes the characteristic function of the unit ball $B(\mathbf{0},1)$ and J_ν the Bessel function of the first kind.

3.2. A general formula

If $\mathcal{F}\eta(\mathbf{0}) \neq 0$ then the function

$$\eta_N(\mathbf{x}) = \sum_{|\alpha|=0}^{N-1} \frac{\partial^\alpha (\mathcal{F}\eta)^{-1}(0)}{\alpha!\,(2\pi i)^{|\alpha|}}\;\partial^\alpha\eta(\mathbf{x}) \tag{3.1}$$

satisfies the moment condition (2.5), where $\partial^\alpha(\mathcal{F}\eta)^{-1}(0) := \partial^\alpha\big(1/\mathcal{F}\eta(\lambda)\big)\big|_{\lambda=0}$.

For radial basis functions $\eta(\mathbf{x}) = \eta(|\mathbf{x}|)$ this construction leads to the function

$$\eta_{2M}(\mathbf{x}) = \Gamma\big(\frac{n}{2}\big)\sum_{j=0}^{M-1}\frac{\Delta^j(\mathcal{F}\eta)^{-1}(0)}{j!\,(4\pi)^{2j}\,\Gamma\big(j+\frac{n}{2}\big)}\,(-\Delta)^j\eta(\mathbf{x}) \tag{3.2}$$

satisfying the moment condition (2.5) with $N = 2M$ and possessing the Fourier transform

$$\mathcal{F}\eta_{2M}(\lambda) = \mathcal{F}\eta(\lambda)\,\Gamma\big(\frac{n}{2}\big) \sum_{j=0}^{M-1} \frac{\Delta^j\,(\mathcal{F}\eta)^{-1}(0)}{j!\,4^j\,\Gamma\big(j+\frac{n}{2}\big)}\,|\lambda|^{2j}\ .$$

The additive structure of formula (3.2) allows to increase the order of a given quasi–interpolant by adding a new formula of the form (2.2) with the next term of (3.2) as generating function.

An interesting example is provided by the Gaussian function $\eta(\mathbf{x}) = \exp(-|\mathbf{x}|^2)$, where the application of (3.2) leads to the generating function

$$\eta_{2M}(\mathbf{x}) = \frac{1}{\pi^{n/2}}\,L_{M-1}^{(n/2)}(|\mathbf{x}|^2)\,e^{-|\mathbf{x}|^2} \tag{3.3}$$

with the generalized Laguerre polynomial $L_{M-1}^{(n/2)}$.

3.3. Symmetric generating functions

Here we list some other formulas for basis functions η with $\mathcal{F}\eta(\mathbf{0}) = 1$ which are symmetric with respect to the coordinate planes $x_i = 0$,

$$\eta(x_1,\ldots,x_i,\ldots,x_n) = \eta(x_1,\ldots,-x_i,\ldots,x_n)\ ,\quad i = 1,\ldots,n\ .$$

For the resulting generating functions the moment condition (2.5) is valid with $N = 2M$.

(i) $\tilde\eta_{\mathbf{D}}(\mathbf{x}) := \displaystyle\sum_{j=1}^{M}\prod_{\substack{l=1\\ l\neq j}}^{M} \frac{\mathcal{D}_l\,{\mathcal{D}_j}^{-n/2}}{\mathcal{D}_l-\mathcal{D}_j}\,\eta\Big(\frac{\mathbf{x}}{\sqrt{\mathcal{D}_j}}\Big)$

for any M-tuple $\mathbf{D} = (\mathcal{D}_1,\ldots,\mathcal{D}_M)$, $\mathcal{D}_j > 0$, $\mathcal{D}_j \neq \mathcal{D}_l$, $j \neq l$.

(ii) $\tilde\eta_M(\mathbf{x}) := \dfrac{(-1)^{M-1}}{(M-1)!}\Big(\dfrac{d}{d\tau}\Big)^{M-1}\Big(\tau^{-1-n/2}\eta\Big(\dfrac{\mathbf{x}}{\sqrt{\tau}}\Big)\Big)\Big|_{\tau=1}$

(iii) $\hat\eta(\mathbf{x}) = \displaystyle\int_1^\infty \eta\Big(\frac{\mathbf{x}}{\sqrt{\tau}}\Big)\,\frac{d\mu(\tau)}{\tau^{n/2}}$ with $\displaystyle\int_1^\infty \tau^k\,d\mu(\tau) = \delta_{0k}\ ,\quad k = 0,\ldots,M-1\ .$

3.4. Linear combinations of translates

Suppose that η is symmetric and $\mathcal{F}\eta(0) \neq 0$. From the solution $\{a_\beta(\mathcal{D})\}$ of the linear system

$$\sum_{|\beta|<M} a_\beta(\mathcal{D})\beta^{2\alpha} = \Big(-\frac{\mathcal{D}}{4\pi^2}\Big)^{|\alpha|}\partial^{2\alpha}(\mathcal{F}\eta)^{-1}(0)\ ,\quad |\alpha| < M\ .$$

one obtains a generating function satisfying condition A.2 with $N = 2M$ by

$$\tilde\eta_M(\mathbf{x},\mathcal{D}) = \sum_{[\beta]<M} 2^{-\sigma(\beta)}a_\beta(\mathcal{D}) \sum_{\{|\mathbf{k}|\}=\beta} \eta\Big(\mathbf{x}-\frac{\mathbf{k}}{\sqrt{\mathcal{D}}}\Big) \tag{3.4}$$

Here $\{|\mathbf{k}|\}$ denotes the vector $(|k_1|,\ldots,|k_n|)$ and $\sigma(\beta)$ is the number of nonzero components of β. Since

$$\partial^\alpha\mathcal{F}\big(\tilde\eta_M(\cdot,\mathcal{D})\big)(\sqrt{\mathcal{D}}\nu) = \sum_{\beta\le\alpha}\frac{\alpha!}{\beta!\,(\alpha-\beta)!}\partial^\beta\mathcal{F}\eta(\sqrt{\mathcal{D}}\nu)\,\partial^{\alpha-\beta}(\mathcal{F}\eta)^{-1}(0)\ ,$$

the saturation error of the quasi–interpolant generated by $\tilde\eta_M(\mathbf{x},\mathcal{D})$ satisfies assumption A.3. From (3.4) follows immediately, that it can be written in the form

$$\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\Big(\sum_{[\beta]<M}2^{-\sigma(\beta)}a_\beta\sum_{\{|\mathbf{k}|\}=\beta}u(h(\mathbf{m}-\mathbf{k}))\Big)\ ,$$

i.e. a linear combination of the translates of the basis function η can provide any given approximation order up to the saturation error.

4. Semi–analytic cubature formulas

The numerical treatment of potentials and other integral operators with singular kernels arises as a computational task in different fields. Since standard cubature methods are very time–consuming there is an ongoing research to develop new effective algorithms like panel clustering, multipole expansions or wavelet compression based on piecewise polynomial approximations of the density. The effective treatment of integral operators is also one of the main applications of approximate approximation. Here the density of a given integral operator

$$\mathcal{K}u(\mathbf{x}) = \int_{\Omega} g(\mathbf{x}-\mathbf{y})\, u(\mathbf{y})\, d\mathbf{y}\ . \tag{4.1}$$

is replaced by a high order quasi–interpolant with a specially chosen generating function for which the action of the integral operator can be effectively determined. For example, if η is chosen such that an analytic expression of the integral $\mathcal{K}\eta$ is available, which is in general impossible for piecewise polynomials or other finite–element functions, then

$$\mathcal{K}_h u(\mathbf{x}) := \mathcal{K}\mathcal{M}_h u(\mathbf{x}) = h^n \sum_{h\mathbf{m}\in\Omega} u(h\mathbf{m}) \int_{\mathbf{R}^n} g\Big(\sqrt{\mathcal{D}}h\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}} - \mathbf{y}\Big)\Big)\, \eta(\mathbf{y})\, d\mathbf{y}$$

is a semi–analytic cubature formula for the integral operator.

Consider for example the Newton potential

$$\mathcal{N}_3 u(\mathbf{x}) := \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y}\ .$$

If the density u is replaced by the quasi–interpolant with the generating function $\eta_{2M}(\mathbf{x}) = \pi^{-3/2}\, L_{M-1}^{(3/2)}(|\mathbf{x}|^2)\, e^{-|\mathbf{x}|^2}$ one obtains the semi–analytic formula

$$\mathcal{N}_{3,h} u(\mathbf{x}) := \frac{\mathcal{D}h^2}{4(\pi\mathcal{D})^{3/2}} \sum_{\mathbf{m}\in\mathbf{Z}^3} u(h\mathbf{m}) \left(\frac{\sqrt{\pi}}{|\mathbf{r_m}|}\, \mathrm{erf}(|\mathbf{r_m}|) + e^{-|\mathbf{r_m}|^2} \sum_{j=0}^{M-2} \frac{L_j^{(1/2)}(|\mathbf{r_m}|^2)}{j+1} \right),$$

with $\mathbf{r_m} = (\mathbf{x}/h - \mathbf{m})/\sqrt{\mathcal{D}}$ and the error function erf.

Error estimates for these approximations can be obtained by estimating the quasi–interpolation in weak Sobolev norms, for example. Here the saturation error, which is caused by fast oscillating functions, tends to zero. Therefore the cubature formulas for smoothing integral operators, for example pseudodifferential operators of negative order, become converging. For the particular case of the Newton potential the following estimate holds.

Theorem 4.1. *If $u \in W_p^N(\mathbf{R}^3) \cap W_q^N(\mathbf{R}^3)$, $1 < p < 3/2$, $q = 3p/(3-2p)$, $2 \le L \le 2M$, then for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that*

$$\|\mathcal{N}_3 u - \mathcal{N}_{3,h} u\|_{L_q(\mathbf{R}^3)} \le\ c_\eta\, (\sqrt{\mathcal{D}}h)^L \|u\|_{W_p^L(\mathbf{R}^3)} + \varepsilon h^2 \|u\|_{W_p^{L-1}(\mathbf{R}^3)}$$

and

$$\|\nabla\mathcal{N}_3 u - \nabla\mathcal{N}_{3,h} u\|_{L_q(\mathbf{R}^3)} \le\ c_\eta\, (\sqrt{\mathcal{D}}h)^L \|u\|_{W_p^L(\mathbf{R}^3)} + \varepsilon h \|u\|_{W_p^{L-1}(\mathbf{R}^3)}\ .$$

There exist several other important integral operators of mathematical physics and appropriate basis functions leading to semi–analytic cubature formulas or to approximations which require only simple one–dimensional quadratures of well–behaved functions (see [11], [15]). Here only two simple examples are mentioned.

The action of the diffraction potential

$$\mathcal{K}\eta_{2M}(\mathbf{x}) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\, \eta_{2M}(\mathbf{y})\, d\mathbf{y}\ , \quad \mathrm{Im}\ k \ge 0\ ,\ \mathbf{x}\in\mathbf{R^3}\ , \tag{4.2}$$

is given by the analytic formula

$$\mathcal{K}\eta_{2M}(\mathbf{x}) = \frac{e^{-k^2/4}}{4\pi|\mathbf{x}|}\Big(i\sin(k|\mathbf{x}|) + \mathrm{Re}\Big(e^{ik|\mathbf{x}|}\mathrm{erf}\,\big(|\mathbf{x}| + \frac{ik}{2}\big)\Big)\Big)\sum_{j=0}^{M-1}\frac{(k^2/4)^j}{j!}$$
$$+\frac{e^{-|\mathbf{x}|^2}}{\pi^{3/2}}\sum_{j=0}^{M-2}\frac{(-1)^j\,H_{2j+1}(|\mathbf{x}|)}{2^{2j+3}\,|\mathbf{x}|}\sum_{r=0}^{M-2-j}\frac{(k^2/4)^r}{(j+r+1)!}\,,$$

where $H_j(x)$ denotes the Hermite polynomials of order j.

Consider the integral operator of the screen problem

$$\mathcal{V}u(\mathbf{x}) := \frac{1}{\pi}\int_{\mathbf{R}^2}\frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d\mathbf{y}\,,\quad \mathbf{x} = (x', x_3) \in \mathbf{R}^3\,. \tag{4.3}$$

The action of this operator on the function $\eta_{2M}(\mathbf{x}) = \pi^{-1}\,L_{M-1}^{(1)}(|\mathbf{x}|^2)\,e^{-|\mathbf{x}|^2}$ can be computed with the quadrature of the one–dimensional integral

$$\mathcal{V}\eta_{2m}(\mathbf{x}) = \frac{2}{\pi}\int_0^\infty J_0(|x'|\rho)\,\exp(-\rho^2 - |x_3|\rho)\sum_{j=0}^{M-1}\frac{\rho^{2j}}{j!}\,d\rho\,,$$

where J_0 is the Bessel function.

5. Cubature of integral operators over bounded domains

Often the computation of potentials is required for finding in some domain a particular solution of a partial differential equation with inhomogeneous right–hand side u. If this function can be extended smoothly outside the domain then the cubature method considered in the previous section provides a high order approximation of such a solution. Here we briefly describe how this method can be extended to the computation of integral operators over a given bounded domain Ω. Since in general the quasi–interpolant $\mathcal{M}_h u$ with uniformly distributed nodes does not approximate u near $\partial\Omega$ one has to refine the mesh towards the boundary. In view of the simplicity of the formulas for $\mathcal{M}_h$ it is useful to take a sequence of uniform meshes in certain boundary layers. In the joint paper with T. Ivanov [3] such a multiscale approximation formula was constructed which is numerically stable for Lipschitz domains. The approximant behaves like $\mathcal{M}_h u$ with the same basis function and compactly supported, smooth functions u if the error is measured in L_p, $1 \le p < \infty$, or in negative Sobolev–norms. This method was modified in [4] for polyhedral domains where anisotropic mesh refinement leads to a considerable reduction of the numerical costs.

5.1. Approximate refinement equation

The construction of the boundary layer approximants is based on the following

Theorem 5.1. ([16]) *Suppose that the Fourier transform of $\eta \in \mathcal{S}(\mathbf{R}^n)$ does not vanish, $\mathcal{F}\eta \neq 0$, and that for given $\mu \in (0,1)$ the function $\tilde\eta = \mathcal{F}^{-1}\big(\mathcal{F}\eta(\cdot)/\mathcal{F}\eta(\mu\cdot)\big)$ belongs also to $\mathcal{S}(\mathbf{R}^n)$. Then for any $\varepsilon > 0$ there exists $\mathcal{D} > 0$ such that*

$$\eta\Big(\frac{\mathbf{x}}{\sqrt{\mathcal{D}}}\Big) = \mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\tilde\eta\Big(\frac{\mu\mathbf{m}}{\sqrt{\mathcal{D}}}\Big)\eta\Big(\frac{\mathbf{x}-\mu\mathbf{m}}{\sqrt{\mathcal{D}}\mu}\Big) + R_{\eta,\mu,\mathcal{D}}(\mathbf{x})\,, \tag{5.1}$$

where $R_{\eta,\mu,\mathcal{D}} \in \mathcal{S}(\mathbf{R}^n)$ fulfills $|R_{\eta,\mu,\mathcal{D}}| < \varepsilon$.

Relation (5.1) is the approximate counterpart of the two–scale or refinement equation which is well-known from wavelet theory. In the next section the approximate refinement equation is

used to perform an approximate multiresolution decomposition of linear spaces generated by the translates of η. It leads also to an approximate factorization of quasi–interpolation operators which is the basis of multiscale approximation schemes.

Corollary 5.2. ([3]) *If* $\mu^{-1} \in \mathbf{N}$ *then* (5.1) *implies the factorization of the quasi–interpolation operator* (2.2)

$$\mathcal{M}_h = \mathcal{M}_{\mu h}\,\widetilde{\mathcal{M}}_h + \mathcal{R}_h\ , \quad \|\mathcal{R}_{h,\mu}\| < \varepsilon\ , \tag{5.2}$$

with the quasi-interpolant

$$\widetilde{\mathcal{M}}_h u(\mathbf{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\,\tilde{\eta}\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\ .$$

Since $\tilde{\eta}$ satisfies together with η the assumptions A.1–3 the quasi–interpolation $\widetilde{\mathcal{M}}_h$ provides the same approximation properties as $\mathcal{M}_h$.

5.2. Isotropic boundary layer approximate approximation

The isotropic boundary layer approximation is derived by the iterative application of the factorization (5.2) in boundary layers $\mathcal{Q}_k \subset \Omega$ where the quasi–interpolation on the given uniform mesh of size $h_{k-1} = \mu^{k-1}h$, $\mu^{-1} \in \mathbf{N}$, becomes worse. Here we switch to the finer uniform mesh of size $h_k = h_{k-1}/\mu$ and approximate $u - \mathcal{M}_{h_{k-1}}u$ by a linear combination of the functions $\eta((\cdot - h_k\mathbf{m})/h_k\sqrt{\mathcal{D}})$, where the nodes $h_k\mathbf{m} \in \mathcal{Q}_k$. Since in view of (5.2) $\mathcal{M}_{h_{k-1}}u$ can be expressed in terms of these functions with the accuracy ε the approximation error is determined by the difference of $u - \mathcal{M}_{h_k}u$, which is large only at some sublayer $\mathcal{Q}_{k+1}$ near the boundary. Thus one obtains a boundary layer approximant of the form

$$\mathcal{B}_L u(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{k=0}^{L} \sum_{h_k\mathbf{m}\in\mathcal{Q}_k} c_{k,\mathbf{m}}\,\eta\left(\frac{\mathbf{x}-h_k\mathbf{m}}{h_k\sqrt{\mathcal{D}}}\right), \tag{5.3}$$

where the coefficients $c_{k,\mathbf{m}}$ are given by

$$c_{k,\mathbf{m}} = \begin{cases} u(h_0\mathbf{m}), & k = 0, \\ u(h_k\mathbf{m}) - (\widetilde{\mathcal{M}}_{h_{k-1}}u)(h_k\mathbf{m}), & k \geq 1. \end{cases} \tag{5.4}$$

Note that $c_{k,\mathbf{m}}$ is a linear combination of values of u near the point $h_k\mathbf{m}$ with coefficients not depending on h and the number of the boundary layer. Indeed,

$$(\widetilde{\mathcal{M}}_{h_{k-1}}u)(h_k\mathbf{m}) = \mathcal{D}^{-n/2} \sum_{|\mu\mathbf{m}-\mathbf{j}|\leq\widetilde{N}_s} u(h_{k-1}\mathbf{j})\,\tilde{\eta}\Big(\frac{\mu\mathbf{m}-\mathbf{j}}{\sqrt{\mathcal{D}}}\Big),$$

therefore the coefficients $\tilde{\eta}\big((\mu\mathbf{m}-\mathbf{j})/\sqrt{\mathcal{D}}\big)$ can be precomputed and used for any initial mesh size h and $k \geq 1$. Furthermore, the width of the boundary layers $\mathcal{Q}_k$ is proportional to h_k with some factor depending only on the basis function η.

A detailed analysis of this multiscale approximation together with numerical results is contained in [3], where the following theorem has been proven.

Theorem 5.3. *Suppose that* Ω *is a bounded domain with Lipschitz boundary and let* $u \in W_p^N(\Omega)$ *with* $N > n/p$. *If* $\eta \in \mathcal{S}(\mathbf{R}^n)$ *satisfies the assumptions of Theorem* 5.1 *and the moment condition* A.2 *then the boundary layer approximant* $\mathcal{B}_L$ *approximates* u *with the error*

$$\|u - \mathcal{B}_L u\|_{L_p(\mathbf{R}^n)} \leq c_1(\mathcal{D}h)^N \|\nabla_N u\|_{L_p(\Omega)} + c_2(\mu^L h)^{1/p}\|u\|_{L_\infty(\Omega)} + \varepsilon\|u\|_{W_p^N(\Omega)}\ .$$

Thus, by choosing the number of boundary layers L so that $\mu^L = O(h^{Np-1})$ the error for the boundary layer approximation $\mathcal{B}_L u$ is of the form $\mathcal{O}((\mathcal{D}h)^N)) + \varepsilon$.

The cubature formula $\mathcal{K}_h u$ for an integral operator over the bounded domain Ω

$$\mathcal{K}u(\mathbf{x}) = \int_{\Omega} k(\mathbf{x}-\mathbf{y})u(\mathbf{y})d\mathbf{y}\,,$$

is obtained from the boundary layer approximation of the density u by

$$\mathcal{K}_h u(\mathbf{x}) := \mathcal{K}\mathcal{B}_L u(\mathbf{x}) = \mathcal{D}^{-n/2}\sum_{k=0}^{L}\sum_{h_k\,\mathbf{m}\in\mathcal{Q}_k} c_{k,\mathbf{m}} \int_{\mathbf{R}^n} k(\mathbf{x}-\mathbf{y})\eta\left(\frac{\mathbf{y}-h_k\,\mathbf{m}}{h_k\sqrt{D}}\right)d\mathbf{y}\,.$$

Theorem 5.4. ([3]) *Let $u \in W_p^N(\Omega)$ with $N > n/p$ and the integral operator $\mathcal{K}$ maps $L_p(\Omega)$ boundedly into $W_p^m(\Omega_1)$. Under the assumptions made above for any $\varepsilon > 0$ there exists $D > 0$ such that*

$$\|\mathcal{K}u - \mathcal{K}_h u\|_{W_p^m(\Omega_1)} \le c_1(Dh)^N \|\nabla_N u\|_{L_p(\Omega)} + c_2(\mu^L h)^{1/p}\|u\|_{L_\infty(\Omega)} + \varepsilon\|u\|_{W_p^{N-1}(\Omega)}\,.$$

If additionally $\mathcal{K} \in \mathcal{L}((W^m_{p/(p-1)}(\Omega))', L_p(\Omega_1))$ then

$$\|\mathcal{K}u - \mathcal{K}_h u\|_{L_p(\Omega_1)} \le (c_1(Dh)^N + c_2(\mu^L h)^{1/p+r})\|u\|_{W_p^N(\Omega)} + \varepsilon\, h^m\, \|u\|_{W_p^{N-1}(\Omega)}\,,$$

where $0 < r < m/n,\ r \le (p-1)/p$.

5.3. Anisotropic boundary layer approximate approximation

The cubature method based on isotropic boundary layer approximations has the advantage, that one gets immediately effective formulas for integrals over Lipschitz domains if only $\mathcal{K}\eta$ is known. On the other hand, the refinement in all directions may lead to a large number of summands in these formulas. But the approximation error increases in the direction towards the boundary, thus one actually needs refinement only in this direction. Therefore a considerable reduction of data points in the boundary layer approximation (5.3) can be derived by anisotropic mesh refinement. This is discussed in [4] for the example of a polyhedral domain Ω in $\mathbf{R}^3$.

Using a partition of unity the function u is decomposed into the sum of functions with support in the interior, near corners, near edges and near faces of Ω, respectively. The approximation of the function with compact support in the interior of Ω can be performed by the usual quasi–interpolant (2.2) with a suitable generating function, whereas for the approximation of the functions supported near the corners of Ω the approximation operator (5.3) can be used. The reduction of data points is reached by constructing the boundary layer approximants for the functions near edges and faces using mesh refinement only in the direction normal to the boundary. Consider for example in $\mathbf{R}^3_+ = \{\mathbf{x} = (x', x_3) \in \mathbf{R}^3 : x_3 \ge 0\}$ a sufficiently smooth function u with bounded support and $u(x', 0) \not\equiv 0$. The construction of high order approximants to u with respect to the sequence of anisotropically distributed mesh points $\{(hm', \mu^k h m_3) : (m', m_3) \in \mathbf{Z}^3_+\}$ is based on a generating function of the tensor product form

$$\eta_3(x', x_3) = \eta_2(x')\,\eta_1(x_3) \tag{5.5}$$

where η_2 is a suitable generating function and η_1 satisfies the conditions of Theorem 5.1. The approximate refinement equation for this function results in the factorization

$$\mathcal{M}_{h,\mu} = \mathcal{M}_{h,\mu^2}\,\widetilde{\mathcal{M}}_{h,\mu} + \mathcal{R}_{h,\mu}\,,\quad \|\mathcal{R}_{h,\mu}\| < \varepsilon\,,$$

where now

$$\mathcal{M}_{h,\mu}u(\mathbf{x}) := \mathcal{D}^{-3/2}\sum_{\mathbf{m}\in\mathbf{Z}^3} u(h\mathbf{m})\,\eta_3\Big(\frac{x'-hm'}{h\sqrt{\mathcal{D}}}, \frac{x_3-\mu h m_3}{\mu h\sqrt{\mathcal{D}}}\Big)$$

$$\widetilde{\mathcal{M}}_{h,\mu}u(\mathbf{x}) := \mathcal{D}^{-3/2}\sum_{\mathbf{m}\in\mathbf{Z}^3} u(h\mathbf{m})\,\eta_2\Big(\frac{x'-hm'}{\sqrt{\mathcal{D}}h'}\Big)\,\tilde{\eta}_1\Big(\frac{x_3-\mu h m_3}{\mu h\sqrt{\mathcal{D}}}\Big)$$

with the dual function $\tilde{\eta}_1$ from the refinement equation for η_1. Analogously to the isotropic case the approximant is defined as

$$\mathcal{D}^{-3/2} \sum_{k=0}^{L} \sum_{(hm', \mu^k h m_3) \in \mathcal{Q}_k} c_{k,\mathbf{m}}\, \eta_3\Big(\frac{x' - hm'}{h\sqrt{\mathcal{D}}}, \frac{x_3 - \mu^k h m_3}{\mu^k h \sqrt{\mathcal{D}}}\Big) \tag{5.6}$$

with the coefficients

$$c_{k,\mathbf{m}} = \begin{cases} u(hm', hm_3), & k = 0 \\ u(hm', \mu^k h m_3) - (\widetilde{\mathcal{M}}_{h,\mu^{k-1}} u)(hm', \mu^k h m_3), & k \geq 1 \end{cases}$$

where $\mathcal{Q}_k$ are layers in $\mathbf{R}^3$ parallel to the plane $\{x_3 = 0\}$. In [4] it is shown that the assertion of Theorem 5.3 remains true for the approximation (5.6) if η satisfies the moment condition (2.5) for given N.

The quasi–interpolation for edge supported functions with the order $O((\mathcal{D}h)^N)$ can be performed using generating functions of the form (5.5) with η_2 satisfying the conditions of Theorem 5.1. The wedge under consideration is affinely transformed to a wedge with right inner angle, and here the construction of the boundary layer approximant is a straightforward extension of the above construction.

Let us note that it is possible to find tensor product generating functions (5.5) where the action of potentials can be computed by simple one–dimensional quadrature (see [4]).

6. Approximate wavelets

The application of wavelet methods to the representation of integral and differential operators is one of the actual research topics in the numerical analysis of solution methods for corresponding operator equations. These methods are based on the multiresolution analysis for fine scale spaces. In numerical procedures one starts with a finite sequence of nested closed subspaces

$$V_0 \subset V_1 \subset \ldots \subset V_d \subset L_2(\mathbf{R}^n) \tag{6.1}$$

with the properties

(i) $f(\mathbf{x}) \in V_0$ if and only if $f(\mathbf{x} - \mathbf{m}) \in V_0$ for any $\mathbf{m} \in \mathbf{Z}^n$;

(ii) $f(\mathbf{x}) \in V_j$ if and only if $f(2\mathbf{x}) \in V_{j+1}$ for any $j = 0, \ldots, d-1$;

(iii) there exists ϕ such that $\{\phi(\cdot - \mathbf{m})\}_{\mathbf{m} \in \mathbf{Z}^n}$ is an L_2–stable basis in V_0.

Then the spaces V_j are spanned by the dilated shifts $\phi(2^j \cdot - \mathbf{m})$, $\mathbf{m} \in \mathbf{Z}^n$, of the scaling function ϕ. The main goal of the multiresolution is to determine a new basis of the space V_d corresponding to the finest grid. To this end V_d is decomposed into the orthogonal sum

$$V_n = V_0 \bigoplus_{j=1}^{d} W_j \ , \tag{6.2}$$

where the wavelet space W_j is the orthogonal complement $W_j = V_j \ominus V_{j-1}$. There exist $2^n - 1$ functions $\psi_v \in W_1$, called prewavelets, with the property that the shifts $\{\psi_v(2^j \cdot - \mathbf{x}), \mathbf{m} \in \mathbf{Z}^n, v \in \mathcal{V}'\}$ form an L_2–stable basis in the space W_{j+1}. Here the prewavelets ψ_v are indexed by the set $\mathcal{V}' = \mathcal{V} \backslash \{\mathbf{0}\}$ with $\mathcal{V}$ denoting the set of vertices of the cube $[0,1]^n$. Thus one obtains a basis of V_d consisting of

$$\{\phi(\cdot - \mathbf{m}), \mathbf{m} \in \mathbf{Z}^n\} \quad \text{and} \quad \{\psi_v(2^j \cdot - \mathbf{m}), \mathbf{m} \in \mathbf{Z}^d, v \in \mathcal{V}', j = 0, \ldots, d-1\} \ .$$

The elements of V_d and operators acting on them are now expanded into the new basis and the computations take place in this system of coordinates, where one hopes to achieve that the computation is faster than in the original system. Since in view of (ii) the scaling function ϕ has to satisfy an exact refinement equation as a rule the approximating functions for the above mentioned applications of wavelet based numerical methods are piecewise polynomials.

The generating functions considered in approximate approximations do not fulfil an exact refinement equation. But in view of Theorem 5.1 a large class of these functions satisfy approximate refinement equations and the error can be controlled by the parameter $\mathcal{D}$. Therefore it is possible to perform an approximate multiresolution analysis of fine scale spaces spanned by the translates of theses functions, i.e. to decompose a fine scale space within some given tolerance into a direct sum of a coarse scale space and wavelet spaces. Then elements of the fine scale space and operators acting on them can be represented within the given tolerance in this new base. The advantage of this approximate wavelet approach lies in the possibility to get efficiently computable sparse approximations for important integral operators, as shown in [16].

6.1. Approximate multiresolution analysis

Consider the closed linear subspaces of $L_2(\mathbf{R}^n)$

$$\mathbf{V}_j := \Big\{ \sum_{\mathbf{m}\in\mathbf{Z}^n} a_{\mathbf{m}}\, \eta_{\mathcal{D}}(2^j \cdot -\mathbf{m}),\ \{a_{\mathbf{m}}\} \in \ell_2(\mathbf{Z}^n) \Big\},$$

where $\eta_{\mathcal{D}} := \eta(\cdot/\sqrt{\mathcal{D}})$ satisfies the assumptions of Theorem 5.1. Since $\mathcal{F}\eta_{\mathcal{D}} \neq 0$ the family of translates $\{\eta_{\mathcal{D}}(2^j \cdot -\mathbf{m})\}_{\mathbf{m}\in\mathbf{Z}^n}$ is an L_2–stable basis in $\mathbf{V}_j$. From the approximate refinement equation (5.1) it is clear that for any $l < j$ the space $\mathbf{V}_l$ is almost included in $\mathbf{V}_j$. In particular, if $\mu = 1/2$ then any element

$$\varphi_j = \sum_{\mathbf{m}\in\mathbf{Z}^n} a_{\mathbf{m}} \eta_{\mathcal{D}}(2^j \mathbf{x} - \mathbf{m}) \in \mathbf{V}_j$$

can be perturbed such that

$$\varphi_j - \sum_{\mathbf{m}\in\mathbf{Z}^n} a_{\mathbf{m}} R_{\eta,2,\mathcal{D}}(2^j \cdot -\mathbf{m}) \in \mathbf{V}_{j+1}\,.$$

The norm of the perturbation is small for $\mathcal{D}$ large enough, hence for any $\varepsilon > 0$ there exists $\mathcal{D}$ such that

$$\|\varphi_j - P_{j+1}\varphi_j\|_2 \le \varepsilon \,\|\varphi_j\|_2\,, \tag{6.3}$$

where P_{j+1} denotes the L_2–orthogonal projection onto $\mathbf{V}_{j+1}$.

Furthermore, we allow also a small perturbation of the wavelet basis which spans the closed subspace of all functions in $\mathbf{V}_{j+1}$ which are orthogonal to $\mathbf{V}_j$. We assume the existence of functions $\psi_{\mathcal{D},v}$, $v \in \mathcal{V}'$, such that the space $\mathbf{W}_1$ spanned by their integer translates has the property that if $\varphi_1 \in \mathbf{V}_1$ and $\varphi_1 \perp \mathbf{V}_0$ then

$$\|\varphi_1 - Q_1\varphi_1\|_2 \le \varepsilon \,\|\varphi_1\|_2\,, \tag{6.4}$$

where Q_1 is the L_2–orthoprojection onto $\mathbf{W}_1$.

Then it can be easily seen that for fixed d any $\varphi_d \in \mathbf{V}_d$ can be represented within some prescribed tolerance as an element of the multiresolution structure

$$\widetilde{\mathbf{V}}_d := \mathbf{V}_0 \dot{+} \mathbf{W}_1 \dot{+} \ldots \dot{+} \mathbf{W}_d\,. \tag{6.5}$$

More precisely, under the assumptions (6.3) and (6.4) the following estimate holds:

$$\Big\| \varphi_d - P_0\varphi_d - \sum_{j=1}^{d} Q_j\varphi_d \Big\|_2 \le d\,\frac{3\varepsilon - \varepsilon^2}{1-\varepsilon}\,\|\varphi_d\|_2\,. \tag{6.6}$$

Hence the approximate multiresolution analysis allows to combine the flexible choice of basis functions used in approximate approximations with the interesting features of wavelets, as the localization in both space and frequency domains and vanishing moment properties.

6.2. Example

To illustrate the above approach we consider the example of the one–dimensional Gaussian function $\eta_{\mathcal{D}}(x) = \exp(-x^2/\mathcal{D})$ providing the approximate refinement equation

$$\eta_{\mathcal{D}}(x) = \frac{2}{\sqrt{3\pi\mathcal{D}}} \sum_{m=-\infty}^{\infty} e^{-m^2/3\mathcal{D}}\, \eta_{\mathcal{D}}(2x-m) - \eta_{\mathcal{D}}(x) \sum_{\nu\neq 0} e^{-3\pi^2\mathcal{D}\nu^2/4} e^{3\pi i x\nu}\,.$$

To find an approximate wavelet we start with the element of V_1

$$\sum_{m=-\infty}^{\infty} (-1)^{m-1} \mu_{m-1}\, \eta_{\mathcal{D}}(2x-m) \quad \text{with} \quad \mu_m = \int_{\mathbf{R}} \eta_{\mathcal{D}}(x)\, \eta_{\mathcal{D}}(2x+m)\, dx\,, \tag{6.7}$$

which is orthogonal to all integer shifts of the scaling function $\eta_{\mathcal{D}}$ and has the Fourier transform

$$c_{\mathcal{D}}\, e^{-\pi i\lambda} e^{-\pi^2\mathcal{D}\lambda^2/4} \sum_{k=-\infty}^{\infty} \exp(-5\pi^2\mathcal{D}(\lambda+2k+1)^2/4)$$

with some factor $c_{\mathcal{D}}$. It can be easily seen that for $\mathcal{D}$ sufficiently large the function with the Fourier transform

$$c_{\mathcal{D}}\, e^{-\pi i\lambda} e^{-\pi^2\mathcal{D}\lambda^2/4} \Big(\exp(-5\pi^2\mathcal{D}(\lambda+1)^2/4) + \exp(-5\pi^2\mathcal{D}(\lambda-1)^2/4) \Big)$$

provides a very accurate smooth approximation of (6.7). Therefore the approximate wavelet can be defined as

$$\psi_{\mathcal{D}}(x) := e^{-(2x-1)^2/6\mathcal{D}} \cos\frac{5\pi}{6}(2x-1)\,, \tag{6.8}$$

which perturbs the corresponding function in V_1 orthogonal to V_0 by

$$R_{\mathcal{D}}(x) = e^{-(2x-1)^2/6\mathcal{D}} \sum_{k=1}^{\infty} \cos\frac{5\pi}{6}(2k+1)(2x-1)\, e^{-5\pi^2\mathcal{D}(k^2+k)/6} = O(e^{-5\pi^2\mathcal{D}/3})\,.$$

Hence, any element $\varphi_d \in V_d$, which approximates some function up to a prescribed accuracy with the order $\mathcal{O}(2^{-dN})$, can be expanded within the same accuracy into the new basis

$$\{\eta_{\mathcal{D}}(\cdot - m), m \in \mathbf{Z}\} \quad \text{and} \quad \{\psi_{\mathcal{D}}(2^j \cdot - m),\ m \in \mathbf{Z},\ j = 0, \ldots, d-1\}\,.$$

Analytic representations of the projection operators P_0 and Q_1 are given in [16].

Besides the good localization the approximate wavelets have an interesting feature. In general its power moments do not vanish, but they are very small and decrease as $\mathcal{D}$ increases. This implies in particular fast decay of integral operators $\mathcal{K}$ applied to them if the kernel $k(\mathbf{x},\mathbf{y})$ satisfies

$$|\partial_{\mathbf{y}}^{\alpha} k(\mathbf{x},\mathbf{y})| \le c_\alpha\, |\mathbf{x}-\mathbf{y}|^{-(\gamma+|\alpha|)} \quad \text{for some } \gamma > 0\,.$$

This effect is shown in Figure 1 giving the graphs of the approximate wavelet ψ_3 and of the Hilbert transform of ψ_3

$$\begin{aligned}\mathcal{H}\psi_{\mathcal{D}}(x) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\psi_{\mathcal{D}}(y)}{y-x}\, dy \\ &= \frac{ie^{-25\pi^2\mathcal{D}/24}}{4} \left(W\Big(\frac{5\pi\mathcal{D} - 2i(2x-1)}{2\sqrt{6\mathcal{D}}}\Big) - W\Big(\frac{5\pi\mathcal{D} + 2i(2x-1)}{2\sqrt{6\mathcal{D}}}\Big)\right),\end{aligned}$$

where the function W is defined by

$$W(z) := e^{z^2}\, (\mathrm{erfc}(z) - \mathrm{erfc}(-z))\,.$$

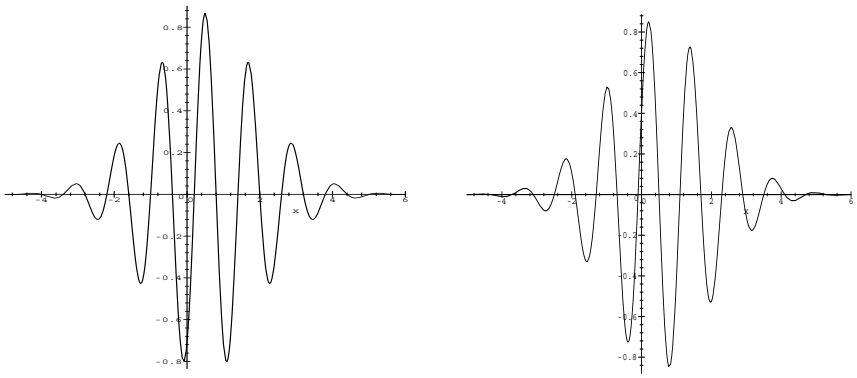


Figure 1: Graph of the approximate wavelet ψ_3 (left) and its Hilbert transform (right)

6.3. The multivariate case

As mentioned above in d dimensions the wavelet space W_1 is spanned by the integer shifts of $2^d - 1$ prewavelets ψ_v. If the scaling function $\eta_{\mathcal{D}}$ used in the approximate resolution analysis is a tensor product of one-dimensional functions

$$\eta_{\mathcal{D}}(\mathbf{x}) = \prod_{j=1}^{d} \phi_{\mathcal{D}}(x_j) \ ,$$

as for example the Gaussian, then the approximate wavelets are obtained as the tensor products

$$\psi_{\mathcal{D},v}(\mathbf{x}) = w_{v_1}(x_1) \cdots w_{v_d}(x_d) \ , \ v = (v_1, \cdots, v_d) \in \mathcal{V}' \ ,$$

where $w_0(x) = \phi_{\mathcal{D}}(x)$ and $w_1(x) = \psi_{\mathcal{D}}(x)$ denotes the one–dimensional approximate wavelet function associated with $\phi_{\mathcal{D}}$. Note that for different $v \in \mathcal{V}$ the principal shift invariant spaces $X_{\mathbf{v}} := \{\psi_{\mathcal{D},v}(\cdot - \mathbf{m}),\ \mathbf{m} \in \mathbf{Z}^d\}$ are nearly orthogonal. Thus the space of approximate wavelets is the direct sum of the spaces $X_{\mathbf{v}}$, $v \in \mathcal{V}'$. For more general basis functions the approximate wavelets can be determined by using the extension of formula (6.7)

$$\sum_{\mathbf{m} \in \mathbf{Z}^d} (-1)^{(\mathbf{m},v)} \mu_{\mathbf{m}} \, \eta_{\mathcal{D}}(2\mathbf{x} - \mathbf{m} - v) \ , \quad v \in \mathcal{V}' \ ,$$

$$\text{where} \ \ \mu_{\mathbf{m}} = \int_{\mathbf{R}^d} \eta_{\mathcal{D}}(\mathbf{x}) \, \eta_{\mathcal{D}}(2\mathbf{x} + \mathbf{m}) \, d\mathbf{x} \, ,$$

which gives elements from V_1 orthogonal to V_0. Again, appropriate truncation of the infinite series representations of the Fourier transform of these functions leads to small perturbations. The resulting functions, the multivariate approximate wavelets, have compact analytic representations.

7. Numerical algorithms based upon approximate approximations

The real power of approximate approximations is in the capability to treat multi–dimensional integral operators very efficiently. Therefore it is natural to use it as underlying approximation method in numerical algorithms for solving problems with integro–differential equations. Another very important application of approximate approximations is in the large field of integral equations methods for solving initial and boundary value problems for partial differential equations. V. Maz'ya developed numerical algorithms for some typical examples, which can be found in [11], [10], [12], [5], [6]. Here I briefly describe the approach of Maz'ya and Karlin to solve evolution equations with nonlocal operators based upon approximate approximations

7.1. Nonlocal evolution equations

In [6] V. Maz'ya and V. Karlin considered the Cauchy-Problem for equations of the general form

$$u_t - P_1(D_{\mathbf{x}})u = P_2(D_{\mathbf{x}})F(\mathbf{x}, t, u, P_3(D_{\mathbf{x}})u) \ , \ t > 0 \ , \ \mathbf{x} \in \mathbf{R}^d \ , \quad u(\mathbf{x}, 0) = \varphi(\mathbf{x}) \ ,$$

where $D_{\mathbf{x}} = (-i\partial/\partial x_1, \ldots, -i\partial/\partial x_n)$. The operators $P_j(D_{\mathbf{x}})$ are convolutions with the symbol $P_j(2\pi\xi)$ and F is supposed to be a smooth function. The equation is discretized in time by a two–parameter finite–difference approximation with time step τ. Then $u(\mathbf{x}, t)$ is approximated by a sequence of functions $u_n(\mathbf{x}) = u(\mathbf{x}, n\tau)$, $n = 0, 1, 2, \ldots$, satisfying

$$\begin{aligned} &\tau^{-1}(u_n - u_{n-1}) - \theta_1 P_1(D_{\mathbf{x}})u_n - (1-\theta_1)P_1(D_{\mathbf{x}})u_{n-1} \\ &= P_2(D_{\mathbf{x}})[((1+\theta_2)F_{n-1}(\mathbf{x}) - \theta_2 F_{n-2}(\mathbf{x})] \ , \end{aligned}$$

where $F_n(\mathbf{x}) = F(\mathbf{x}, n\tau, u_n, P_3(D_{\mathbf{x}})u_n)$ and $0 < \theta_i \leq 1$. For $\theta_1 = \theta_2 = 1/2$ the scheme is of second order accuracy. With the notations

$$\begin{aligned} &\mu = \tau\theta_1 \ , \ y = u_n + (\theta_1^{-1} - 1)u_{n-1} \ , \\ &f = \theta_1^{-1}u_{n-1} \ , \ g = \tau[(1+\theta_2)F_{n-1} - \theta_2 F_{n-2}] \end{aligned}$$

one derives the following linear problem

$$-\mu P_1(D_{\mathbf{x}})y + y = f + P_2(D_{\mathbf{x}})g \ , \ \mathbf{x} \in \mathbf{R}^d \ ,$$

providing the solution

$$y = f + (R - I)f + P_2 Rg \qquad \text{with} \quad R = (I - \mu P_1)^{-1} \ .$$

Replacing f and g by its quasi–interpolants (2.2) one obtains the approximate solution

$$\begin{aligned} y_h &= f + (R-I)\mathcal{M}_h f + P_2 R\mathcal{M}_h g \\ &= f + \mathcal{D}^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} f(h\mathbf{m})[(R\eta)\big(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\big) - \eta\big(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\big)] \\ &\quad + \mathcal{D}^{-d/2} \sum_{\mathbf{m}\in\mathbf{Z}^d} g(h\mathbf{m})(P_2 R\eta)\big(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}h}\big) \ . \end{aligned}$$

Thus the values of y_h at the grid points $x_{\mathbf{k}} = h\mathbf{k}$ are linear combinations of

$$(R\eta)\big(\frac{\mathbf{k} - \mathbf{m}}{\sqrt{\mathcal{D}}}\big) \quad \text{and} \quad (P_2 R\eta)\big(\frac{\mathbf{k} - \mathbf{m}}{\sqrt{\mathcal{D}}}\big) \ ,$$

which may be effectively computed if the generating function η is suitably chosen. By this way Maz'ya and Karlin derive the following explicit scheme to compute the approximate solution $u_{n,h}$ of the Cauchy problem:

$$u_{n,h} = u_{n-1,h} + \theta_1^{-1}(R - I)\mathcal{M}_h u_{n-1,h} + \tau P_2 R\mathcal{M}_h[(1+\theta_2)F_{n-1} - \theta_2 F_{n-2}] \ .$$

Compared with other explicit schemes for solving time–dependent problems the proposed method is very robust with respect to variations of the ratio between time and spatial discretization. Numerous numerical tests for different equations have shown, that of course the non–linearity in the original equation imposes restrictions to the time step τ, but there exists no strict connection between τ and the mesh size h. In these tests the present method provides an accuracy of $\mathcal{O}(\tau^2 + \tau h^N)$ at each time step, where N is the approximation order of the approximate quasi–interpolation with the generating function η. For $\theta_1 = \theta_2 = 1/2$ the numerical accuracy increases to $\mathcal{O}(\tau^3 + \tau h^N)$. However, the rigorous error analysis of this quite general method remains open. The estimation of the saturation errors, which occur at each time step, is rather involved and understood at present only for some special equations. Here again for $\mathcal{D}$ sufficiently large the saturations errors can be kept below a given error level, as expected also from the numerical experiments.

7.2. Examples

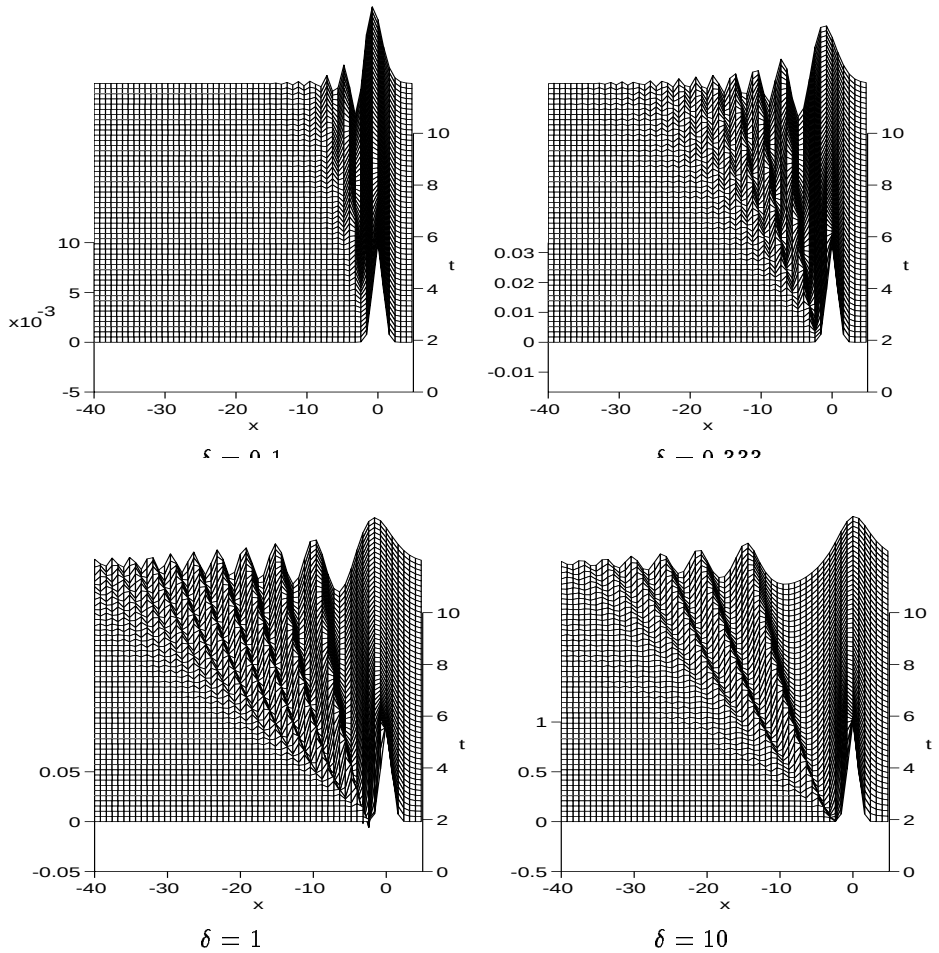
In [6] the method is applied to different nonlinear nonlocal evolution equations, here two examples will be mentioned. The first concerns the Joseph equation

$$u_t + \delta^{-1} u_x + (2\delta)^{-1} \int_{-\infty}^{\infty} u_{yy} \coth \frac{\pi(y-x)}{2\delta}\, dy = -(u^2)_x \ , \tag{7.1}$$

which describes the unidirectional propagation of small–amplitude, nonlinear, dispersive, long waves in stratified fluids. Note that the shallow water approximation ($\delta \ll 1$) of (7.1) is the Korteweg–de–Vries equation, whereas the deep water approximation ($\delta \gg 1$) is the Benjamin–Ono equation

$$u_t + \mathcal{H} u_{xx} = -(u^2)_x$$

with the Hilbert transform $\mathcal{H}$. In Figure 2 the computational results for Joseph's equation with the initial data $\varphi(x) = \delta \exp(-x^2)$ for $\delta = 0.1$, 0.333, 1, and 10, corresponding to shallow, intermediate, and deep water, are shown. In these computations η was the Gaussian function, $\mathcal{D} = 3$, $h = 0.1$ and $\tau = 0.001$.



$\delta = 1$ $\qquad$ $\delta = 10$

Figure 2: Solution of the Joseph equation for different δ

Another interesting example which is difficult to solve by using finite–difference or finite–element methods represents the two–dimensional equation of flame front propagation

$$u_t + a\Delta^2 u + \epsilon \Delta u + b\Delta \int_{\mathbf{R}^2} \frac{u(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\, d\mathbf{y} + cu = -\frac{1}{2}(\nabla u)^2 \ , \ t > 0 \ , \ \mathbf{x} \in \mathbf{R}^2. \tag{7.2}$$

Here

$$P_1(\xi) = -a|\xi|^4 + \epsilon|\xi|^2 + b|\xi| - c,\ P_2(\xi) = 1,\ P_3(\xi) = i\xi,\ F(\mathbf{x}, t, u, \mathbf{v}) = -|\mathbf{v}|^2/2\,.$$

With the two–dimensional generating functions (3.3) the occuring integrals are transformed to the zero–order Hankel transform with smooth and rapidly decaying integrands and may be done by using a standard quadrature procedure. In Figure 3 the results of the numerical solution of (7.2) with the initial data

$$\varphi(\mathbf{x}) = 1 + \frac{1}{2}\sin 2x_1 \sin 2x_2$$

and the parameters $a = 10^{-4}$, $\epsilon = 0.05$, $b = 0.005$ and $c = 1/6$ are given for different time values. The 2π–periodic solution was computed for the discretization parameters $h = \pi/32$ and $\tau = 10^{-3}$.

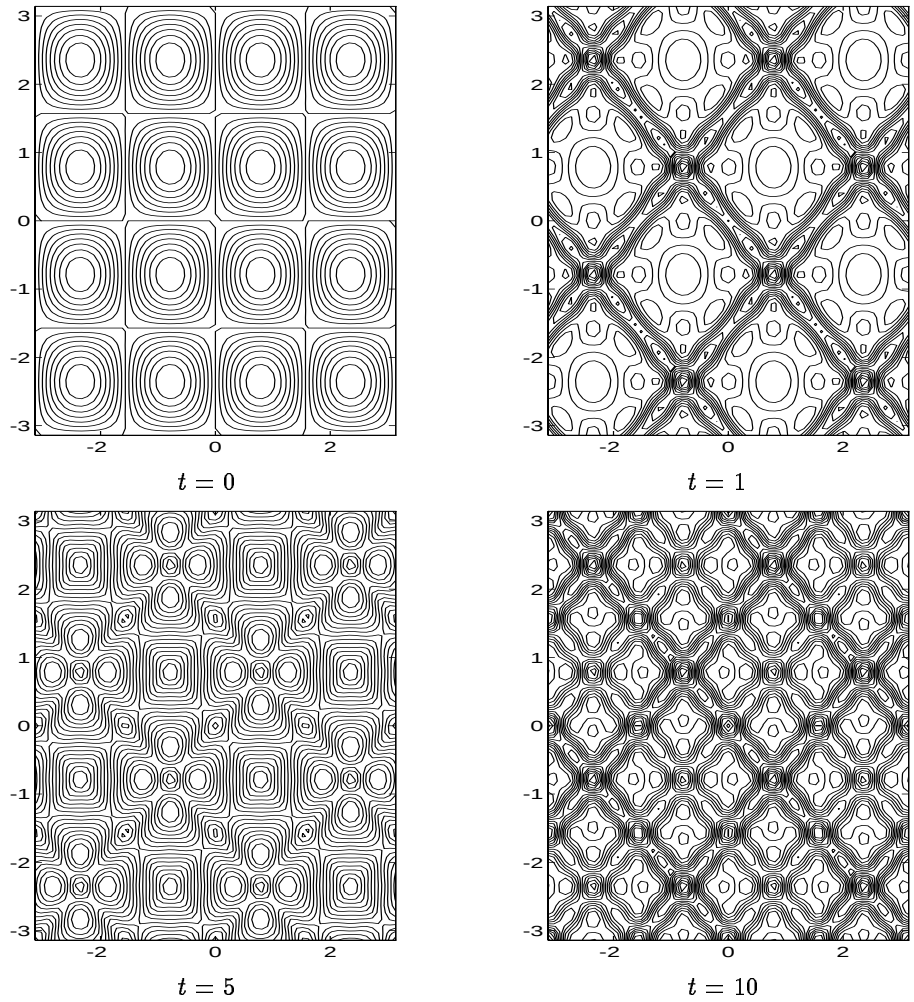


Figure 3: Level lines of the surface of the flame front for different t

The given parameters correspond to a linear flame instability and the computations result in a corrugated flame front.

7.3. Further applications

Finally I mention some other problems for which algorithms based on approximate approximations have been applied successfully. The algorithm described above was used to solve non–linear parabolic differential equations. The papers [11], [12], [5] contain also numerical results for different model equations, among them the two–dimensional Navier–Stokes problem.

Approximate approximations were successfully used in [7] to solve hypersingular integral equations of the Peierls type

$$\int_{-\infty}^{\infty} K(x-y)\,u(y)\,dy = F(u(x))\,, \quad K(x) = -x^{-2} + \kappa(x)\,,$$

with κ being smooth and the integral defined in the finite part Hadamard sense. Integral equations of this type occur in dislocation theory. Due to the efficient cubature formulas critical Peierls stresses were calculated at very high accuracy for a variety of dislocations.

A further application of approximate approximations is an extension of the Boundary Element Method. While usually the numerical solution of boundary integral equations is sought in form of piecewise polynomial functions the so–called Boundary Point Method uses suitable basis functions for which the boundary integrals can be effectively computed. Representing the numerical solution as linear combination of translates of this basis function similar to the quasi–interpolant (2.8) the discretization of the boundary integral equations requires only the computation of boundary potentials whose densities are the basis functions. In [10], [11] the Boundary Point Method is applied to two– and three–dimensional potential problems with smooth boundary, where the surface integration is replaces by the integration over the tangent plane. The obtained second order approximation rate corresponds to the theoretically expected results.

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