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Asymptotical mean stability of numerical solutions with multiplicative noise

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Abstract

As an extent of asymptotically absolute stability of numerical methods in deterministic situation, in this report the asymptotically absolute mean stability of the null solution for stochastic differential equations with respect to different criterions will be examined, both for the exact solution and for its numerical approximations. Among the considered criterions the mean square stability plays the main role in the examinations. For the class of scalar, bilinear, complex-valued stochastic differential equations, comparison studies for different numerical schemes are provided and show their different stability features. However the balanced implicit methods have proved to be rich enough to possess appropriately large stability domains. Finally, experiments for the Kubo oscillator indicate how efficient the asymptotical mean stability examinations could be for the reality.

Contents

1 Introduction

After numerous publications on numerical methods for the treatment of stochastic differential equations (SDE's), see Pardoux & Talay ([21], 1985), Milstein ([17], 1988) or Kloeden & Platen ([13], 1992), it is necessary to examine their qualitative behaviour, to distinguish between the schemes, not only w.r.t. given error criterions and orders, and to describe further qualitative characteristics of them. From practical interest, the question of stability of these schemes has to be solved. If one obtains a stable numerical method for the considered equation then one knows the range of time step sizes to be used to achieve control on the numerical behaviour of the approximation and one knows appropriate parameters involved in these methods. Stable numerical methods guarantee a kind of 'reasonable' behaviour, even for so–called 'real time step sizes', which replicates qualitative features of the exact solution to be approximated. Especially, in stiff systems where one observes rapidly varying time scales (velocities) it is very important to know assertions about stability, both for the exact and the numerical solution. Furthermore stability investigations are necessary to develope adaptive and intelligent algorithms up to their practical implementation, for instance algorithms which themselves select step size and some order (in deterministics such as LSODE solver).
We are going to restrict ourselves to the notion of asymptotically absolute stability (A–stability) formulated via the moments and analogously to Dahlquist and others who investigated it in deterministic numerics. So we will touch upon on briefly the problem of suitable test equations which still has to be solved in stochastic numerics. For our investigations we are modelling with multiplicative noise, see model 2.1. In some extent, this test equation may be considered to be representative for autonomous stochastic differential equations (time–independent) due to Haszminskij's results [8] on stability and the connection of the linearized and the original equation, compare theorems 1.1 and 1.2 in chapter VII at page 299–300, if the first derivatives of the drift and diffusion are bounded. However, still it must be clarified which systems, in general, possess such linearizations of type 2.2 and their applications. It is apparent that the investigation includes the one-dimensional situation at least. Besides we remark the model 2.2 presents a class of processes with their linearization having commutative, but nondiagonalizable drift and diffusion matrices. This paper can be considered as a trial to gain a little more insight of the stability appearance of exact and numerical solutions for stochastic systems. Discussion on the topic of test equations and simultanous transformations we will provide in a later paper.
There is a nonneglectable difference between the notion of stability in probability and the herein considered p-th mean stability. A corresponding theorem concerning their relation for given linear systems, especially for constant coefficients, has been stated in Haszminskij ([8],chapter VI.4, 1969). The additive noise case has already been examined by Milstein ([17], 1988, page 60–67) and Kloeden & Platen ([13],[14], 1992). There the investigations draw back to the deterministic part of the schemes, e.g. those parts influenced decisively by the drift of the considered model equation. Later we will see that this kind of asymptotical stability corres-

ponds to asymptotically weak mean stability with small noise. In multiplicative noise cases, particulary in those cases where the stochastic part plays the decisive role (compare Milstein et al. ([19], 1992)), one cannot simply drop out the stochastic parts of the schemes. First stability examinations for multiplicative noise have been done by Mitsui & Saito ([20], 1992), where they reduced it to the mean square stability (p = 2) under other restrictions to the model equation (There the complex model is equivalent to a real model with noncommuting drift and diffusion matrices and the investigation is reducable to only one dimension.), and they do not include the pure–stochastic equation class ($\lambda = 0, \gamma <> 0$, non-mean square, but weak mean stable case). Of course, in practice, it seems that this is the way to plot stability domains at all, e.g. to restrict to parameter simplifications. We note that the model 2.2 will also include the one-dimensional bilinear model they considered. Furthermore, for a special scheme class (weak approximations with simplified random variables having finite extremas), Hofmann [10] has studied the stability of the corresponding numerical solutions, but it is used the essup criterion to classify the schemes w.r.t. their stability. Although this work is very useful, thereby this notion and formalism is not applicable to general and, in particular, strong approximations. Now we are dealing with asymptotically absolute stability in the mean sense, the weak and p-th mean stability of both strong and weak approximations and will stay within the Itô calculus.

2 Mean Stability for Stochastic Processes

In this section it is discussed the asymptotical mean stability for stochastic processes governed by special SDE's. The conditions guaranteeing mean stability for the exact solutions will ensure the sense of further stability investigations for the numerical solutions which should replicate the stability behaviour of the exact solution. Analogously to Sasagawa & Willems ([19], 1991) we introduce the notion of p-th mean stability. For this purpose we consider the d-dimensional real–valued stochastic process $X = \{X(t) : t \geq 0\}$ satisfying the bilinear Itô equation

$$\begin{aligned} dX(t) &= AX(t)dt + \sum_{j=1}^{m} B^j X(t) dW^j(t) \qquad (2.1) \\ X(0) &= x_0 \in \mathbb{R}^d \end{aligned}$$

driven by a m-dimensional standard Wiener process

$$W = \{(W^1(t), \cdots, W^m(t));\ t \geq 0\}$$

where A and B^j are dxd-matrices with constant elements.

Suppose x_0 is a d-dimensional random vector, independent of the σ - algebra $\mathcal{F}_t = \sigma\{W^j(s);\ j = 1, \cdots, m,\ 0 \leq s \leq t\},\ t \in \mathbb{R}^+$. Furthermore, $\|x\|$ denotes the Euclidean norm of a vector x and $< \cdot, \cdot >$ the inner scalar product of two vectors inscribed. In the following we assume that $\mathbb{E}\|x_0\|^p < +\infty\, (p = 1, 2, \ldots)$.

Definition: (p-th mean stability) , $p \in \mathbb{N}^+$

The null solution $x(t; 0) \equiv 0$ of 2.1 is called *asymptotically p-th mean stable* for the process $X = \{X(t),\, t \geq 0\}$ satisfying 2.1 or shorter *p-th mean stable* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned} I\!E\,\|X(t;x_0)\|^p &< \epsilon \quad \forall t \in I\!R^+ \wedge \|x_0\| < \delta \\ \text{and}\quad I\!E\,\|X(t;x_0)\|^p &\longrightarrow 0 \quad \text{as } t \to +\infty \text{ for sufficiently small } \|x_0\|\,. \end{aligned}$$

To cover the stability investigation for the 'deterministic part' taking part in the dynamics of the stochastic processes we additionally introduce the notion of *asymptotically weak mean stability*, although it is a very 'weak' notion for stochastics.

Definition : (weak mean stability)
The null solution $x(t;0) \equiv 0$ of 2.1 is called *asymptotically weak mean stable* for the process $X = \{X(t),\, t \geq 0\}$ satisfying 2.1 or shorter *weak mean stable* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned} \|I\!E\,X(t;x_0)\| &< \epsilon \quad \forall t \in I\!R^+ \wedge \|x_0\| < \delta \\ \text{and}\quad \|I\!E\,X(t;x_0)\| &\longrightarrow 0 \quad \text{as } t \to +\infty \text{ for sufficiently small } \|x_0\|\,. \end{aligned}$$

In the special case $p = 1$ we also say *strong mean stable* and for $p = 2$ *mean square stable*. Furthermore, we make the convention that in the whole paper we are only talking about asymptotically absolute stability and leave out these two adjectives mostly unless we stress it explicitly. Stochastic processes w.r.t. their stability were studied by the school of Haszminskij ([8], 1969), Arnold ([1], 1974), Sasagawa ([24], 1981), Sasagawa & Willems ([25], 1991) and many others. By the definition above one immediately obtains this assertion :

Lemma 1 : $p \in I\!N^+$
If the null solution is p-th mean stable for the process $X = \{X(t),\, t \geq 0\}$, then it is weak and (p–1)–th mean stable too.

which can be justified by the triangular and Lyapunov's inequality, respectively. Note that this stability is understood as a characterization of the asymptotically stable behaviour w.r.t. the null solution. A simple example which demonstrates that there is a basic difference between p-th and weak mean stability, and that it makes sense to differ between these notions, is provided by the one-dimensional stochastic process $Z = \{Z(t) : t \geq 0\}$ defined by $Z(t) = t^{\varepsilon} W(t)$ $(\varepsilon \in I\!R,\ t \geq t_0 > 0)$ satisfying the nonautonomous SDE

$$dZ(t) = \varepsilon \cdot \frac{Z(t)}{t} dt + t^{\varepsilon} dW(t).$$

From this process it can be easily concluded that

1. $Z(t)$ is weak mean stable, as well as the Wiener process itself, but not p-th mean stable for each $\varepsilon \geq -\frac{1}{2}$, $p \in I\!N^+$.

2. $Z(t)$ is p-th mean stable iff $\varepsilon < -\frac{1}{2}$ because of

$$I\!E\|Z(t)\|^p = t^{p(\varepsilon+\frac{1}{2})} \cdot I\!E\|t^{-1/2}W(t)\|^p$$

$$\text{and } \mathbb{E}\|t^{-\frac{1}{2}}W(t)\|^p = \begin{cases} \sqrt{\frac{2}{\pi}} \cdot \prod_{i=1}^{q}(2i) & \text{if } p = 2q+1 \\ \sqrt{\frac{2}{\pi}} \cdot \prod_{i=1}^{q}(2i-1) & \text{if } p = 2q \quad \text{for } p \in \mathbb{N}^+ \end{cases}$$

Moreover, one can state stochastic processes for which the null solution is p-th mean stable, but not (p+1)-th mean stable for any fix $p \in \mathbb{N}^+$ (compare Lemma 2 below where the inequality $0 \leq \lambda_r + \frac{1}{2}(\gamma_i^2 + \gamma_r^2 p) < \frac{1}{2}\gamma_r^2$ is fulfilled).
For simplicity and to demonstrate the gist of asymptotical stability examinations we chose the test class of two-dimensional equations of structure 2.1 with one Wiener noise, e.g. $d = 2$ and $m = 1$. To ensure for which parameters of model 2.1 it makes sense to examine the stability behaviour of the approximations we are firstly dealing with stability for the exact solutions of our test class considered. Explicit solutions of 2.1 are only known if matrices A and B^j commute, e.g. $AB^j = B^j A$. In such situations, for our two-dimensional system with one Wiener noise, we obtained as its explicit solution

$$X(t) = \exp\{(A - \frac{1}{2}B^2)t + BW(t)\} \cdot X(0)$$

and hence in componentwise description, if

$$A = \begin{pmatrix} \lambda_r & -\lambda_i \\ \lambda_i & \lambda_r \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \gamma_r & -\gamma_i \\ \gamma_i & \gamma_r \end{pmatrix}$$

with $\lambda = \lambda_r + \imath\lambda_i$ and $\gamma = \gamma_r + \imath\gamma_i \in \mathbb{C}$, the corresponding system

$$\begin{aligned} X^1(t) &= \Big(X^1(0)\cos((\gamma_r\gamma_i - \lambda_i)t + \gamma_i W_t) - X^2(0) \cdot \sin((\gamma_r\gamma_i - \lambda_i)t + \gamma_i W_t)\Big) \\ &\quad \cdot \exp\{(\lambda_r - \frac{1}{2}(\gamma_r^2 - \gamma_i^2))t + \gamma_r W_t\} \end{aligned}$$

and

$$\begin{aligned} X^2(t) &= \Big(X^1(0)\sin((\gamma_r\gamma_i - \lambda_i)t + \gamma_i W_t) + X^2(0) \cdot \cos((\gamma_r\gamma_i - \lambda_i)t + \gamma_i W_t)\Big) \\ &\quad \cdot \exp\{(\lambda_r - \frac{1}{2}(\gamma_r^2 - \gamma_i^2))t + \gamma_r W_t\} \end{aligned}$$

which is equivalent to $X(t) = \exp\{(\lambda - \gamma^2/2)t + \gamma W(t)\}X(0)$ in complex notation, representing the solution of the system

$$dX(t) = \lambda X(t)dt + \gamma X(t)\, dW(t) \tag{2.2}$$

where λ, γ complex parameters, $X(t) \in \mathbb{C}$ and $W(t)$ real-valued standard Wiener process. With $\imath$ the imaginary unit of $\mathbb{C}$ is denoted. For this system we are able to state some results concerning the mean stability of its null solution.

Lemma 2 :
Suppose it is given a system of the form 2.2 with its solution $X(t)$.
Then it holds
(i) : The null solution is weak mean stable iff $\lambda_r < 0$.
(ii) : The null solution is p-th mean stable iff $\lambda_r + \frac{1}{2}(\gamma_i^2 + \gamma_r^2(p-1)) < 0$.

Proof : The first assertion can be concluded immediately from the deterministic differential equation for the first moment $d\mathbb{E}(X(t)|X(0)) = \lambda_r \mathbb{E}(X(t)|X(0))\, dt$. The second follows from

$$\begin{aligned} \mathbb{E}(\|X(t)\|^p|X(0))) &= \|X(0)\|^p \cdot \exp\{\Big(\lambda_r - \frac{1}{2}(\gamma_r^2 - \gamma_i^2)\Big)\, pt\} \cdot \mathbb{E}\exp\{p\gamma_r W(t)\} \\ &= \|X(0)\|^p \cdot \exp\{\Big(\lambda_r + \frac{1}{2}\left(\gamma_i^2 + \gamma_r^2(p-1)\right)\Big)\, pt\} \end{aligned}$$

We note that

$$\lambda(x_0;p) := \lim_{t\to\infty} \sup \frac{1}{t} \log \mathbb{E}\|X(t,x_0)\|^p = p(\lambda_r + \frac{1}{2}(\gamma_i^2 + \gamma_r^2(p-1)))$$

which is called the p-th mean Lyapunov exponent of $X(t)$. Thereby, one of the results (proposition 1, Sasagawa & Willems ([25], 1991)) has confirmed which also says that the system 2.1 (2.2) is p-th mean stable iff the p-th mean Lyapunov exponent is negative via the notion of exponential p-stability and its relation to the considered system (see Lemma 4.3 in Haszminskij ([8], page 255, 1969). With these results in mind, it only makes sense to examine weak and p-th mean stability for the approximations of system 2.1 in those cases where the null solution is mean stable, resp., too. Thus, from now on we assume $\lambda_r < 0$ or

$$\lambda_r + \frac{1}{2}\gamma_i^2 + \frac{1}{2}\gamma_r^2(p-1) < 0 \tag{2.3}$$

for systems as 2.2 whenever $p \in \mathbb{N}^+$. To illustrate graphically how such region determined by 2.3 looks like we add the figure 1. In this figure the border plane for $p = 2$ (mean square stability domain) is plotted. The region 2.3 establishes the set of all tripels $(\lambda_r, \gamma_r, \gamma_i)$ which are located under the grid hyperplane. Note that the values of λ_i plays no role for the stability in the mean sense.

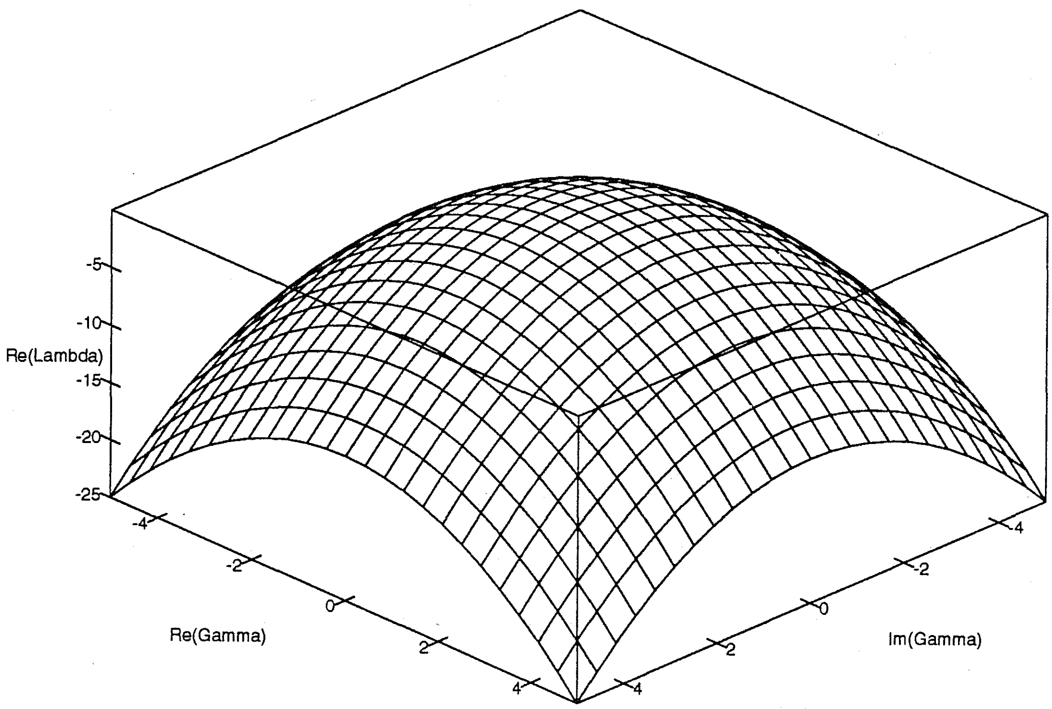

Figure 1 : Hyperplane for the mean square stability domain 2.3

3 Numerical Solutions of Model 2.1

Before we will start with the stability investigation for numerical solutions we declare what is meant by discrete time approximations with weak and strong order $q \in \mathbb{R}^+$ called shortly weak and strong approximations in Kloeden & Platen [9] in order to classify additionally the approximations, their schemes or methods w.r.t. their convergence order against the exact solution. For this purpose we consider an equidistant time discretization of a given time interval $[0, T]$:

$$0 = \tau_0 \le \tau_1 \le \ldots \le \tau_{n_T} = T$$

Suppose $Y^\Delta = (Y_n) = (Y(\tau_n))_{n=0,1,\ldots,n_T}$ is a sequence of random values corresponding to the discretization points τ_n where $\Delta = \tau_{n+1} - \tau_n$.

Definition : (strong approximation, numerical solution)
Such Sequences Y^Δ with step size Δ are called *discrete time approximations* of the solution or *numerical solutions* of 2.1 *with strong order* $q \ge 0$ if it holds

$$\mathbb{E}(\|X(\tau_n) - Y_n^\Delta\|) \le K \cdot \Delta^q \quad \forall n : n = 0, 1, \ldots, n_T \tag{3.1}$$

where the constant K is only depending on T.

Approximations of this form can be settled by numerical schemes of the form

$$Y_{n+1} = \Phi(\tau_n, \Delta, Y_n, Y_{n+1}),\ n = 0, 1, \ldots, n_T - 1.$$

Analogously, the notion of *discrete time approximations (numerical solutions) with weak order* $q \ge 0$ is introduced via the requirement

$$\|\mathbb{E}(g(X(\tau_n)) - g(Y_n^\Delta))\| \le K \cdot \Delta^q \tag{3.2}$$

w.r.t a class of sufficiently smooth functions g bounded by polynomial growth ($g \in C_p^\infty$). Simple examples of such numerical solutions are provided by the family of implicit Euler schemes with weak order 1.0 and strong order 0.5 which takes the form for model 2.2 stated in Kloeden & Platen ([9], [10]) for $\alpha \in [0, 1]$

$$Y_{n+1} = Y_n + \{\alpha A Y_{n+1} + (1-\alpha) A Y_n\}\Delta + B Y_n \Delta W_n, \tag{3.3}$$

the family of implicit Milstein schemes with weak order 1.0 and strong order 1.0 evaluated for model 2.2

$$Y_{n+1} = Y_n + \{\alpha A Y_{n+1} + (1-\alpha) A Y_n\}\Delta + B Y_n \Delta W_n + \frac{1}{2} B^2 Y_n ((\Delta W_n)^2 - \Delta) \tag{3.4}$$

for $\alpha \in [0, 1]$ or the balanced implicit methods with weak and strong order 0.5 introduced by Milstein et al.([14], 1992) and, for model 2.2, described by

$$Y_{n+1} = Y_n + A Y_n \Delta + B Y_n \Delta W_n + C(\tau_n, Y_n)(Y_n - Y_{n+1}) \tag{3.5}$$

where $C(\tau_n, Y_n) = c^0(\tau_n, Y_n)\Delta + c^1(\tau_n, Y_n)|\Delta W_n|$ with 2 x 2 matrices c^0 and c^1 which may be chosen as stochastically bounded matrices such that the inverse of $I + C(t, x)$ always exist and is uniformly bounded w.r.t the pair (t, x). In 3.3 - 3.5 we supposed one noise source ($m = 1$) and identified $\Delta W_n = W(\tau_{n+1}) - W(\tau_n)$ with the increment of the standard Wiener noise where $n = 0, 1, \ldots, n_T - 1$. For further methods, see [4], [6], [12], [13], [17], [21], [26] or [27].

4 Weak Mean Stability for Numerical Solutions

Analogously to the definition of weak mean stability for stochastic processes X we introduced the asymptotically absolute stability in the weak mean sense for numerical solutions. For this purpose we substituted $\|\mathbb{E}X(t;x_0)\|$ by $\|\mathbb{E}Y(\tau_n;y_0)\|$ at each time point τ_n and as $\tau_n \to \infty$, resp., in the corresponding definition of chapter 2. For simplicity we only considered equidistant approximations. Suppose it is given an approximation of system 2.1 which provides a numerical solution (Y_n) at the time points $\tau_n = n\Delta$ (Y_n as described in section 3).

Definition : (weak mean stability of numerical solutions)
Assume that the step size Δ of the numerical solution is fixed. Then the null solution $x(t;0) \equiv 0$ of 2.1 is called *asymptotically weak mean stable* for the numerical solution (Y_n) (method, scheme, approximation) of system 2.1 or shorter the numerical solution (Y_n) is *weak mean stable* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{array}{rrcl} & \|\mathbb{E}\,Y(\tau_n;y_0)\| & < & \epsilon \quad \forall n \in \mathbb{N} \wedge \|y_0\| < \delta \\ \text{and} & \|\mathbb{E}\,Y(\tau_n;y_0)\| & \longrightarrow & 0 \quad \text{as } \tau_n \to +\infty \text{ for sufficiently small } \|y_0\|\,. \end{array}$$

Consider model 2.2. If in the above definition the requirements are true for all step sizes Δ and parameter $\lambda_r < 0$ then we call the numerical solution (method, scheme, approximation) *weak mean A-stable.* This corresponds to a stochastic version of the notion of A-stability as known through Dahlquist in deterministic numerics. In modells with small noise intensities, weak mean stability for numerical solutions described in the previous section corresponds to asymptotical stability of their 'deterministic parts', e.g. those parts being left if one averages out the stochastic influence in them (Nonlinear modells with large noise require further special examinations). For the solutions using one of the implicit Euler 3.3 or Milstein schemes 3.4, these investigations coincide with the stability investigations of their deterministic counterparts determined only by the drift part of model 2.2. If the weight function $c^1(t,x) \equiv 0$ for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$ in the balanced methods 3.5 then one could also point to the stability behaviour of their deterministic counterparts. That's why we will draw the attention to deduce recommendations for nonzero matrices $c^1(t,x)$ in the balanced methods.

4.1 The Notion of Stability Function and Domain

Before we will start with investigating of stability we declare what is meant by a *stability function* of a numerical solution for model 2.2 and its corresponding *stability domain.* Suppose that it is given a scheme form $Y_{n+1} = \Phi(\lambda\Delta, \gamma\sqrt{\Delta}, \xi_n)Y_n$ where Φ is a 2 x 2 real-valued matrix or a random complex-valued function mapping on $\mathbb{C} \times \mathbb{C}$ *to* $\mathbb{C}$ and ξ_n represents the noise influence involved in the scheme at time τ_n (for Euler, this is only the current Wiener noise increment). Note that schemes 3.3 - 3.5 can be written in this form for model 2.2. Then we call the real-valued function

$$R(\mu,\nu) := \|\mathbb{E}\Phi(\mu,\nu,\xi)\| \quad \text{for } \mu,\nu \in \mathbb{C} \tag{4.1}$$

the *weak mean stability function.* Obviously, it holds $\|\mathbb{E}Y_n\| \to 0$ as $\tau_n \to \infty$ iff $\|\mathbb{E}\Phi(\mu,\nu,\xi)\| < 1$. Thereby, one obtains assertions about the weak mean stability of the numerical solution Y^Δ via the investigation of the positive real-valued function $R(\mu,\nu)$ depending on complex variables $\mu,\nu \in \mathbb{C}$. More precisely speaking, the deterministic complex region

$$\Gamma := \{(\mu,\nu) \in \mathbb{C} \times \mathbb{C} \,:\, R(\mu,\nu) < 1\} \tag{4.2}$$

corresponding to a given numerical solution and its scheme with random function $\Phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ provides us just the domain where the numerical solution behaves weak mean stable. It can be calculated using the connection between the parameters and step size of the numerical solution and the complex pairs $(\mu,\nu) \in \Gamma$. Such a region Γ is called *weak mean stability domain of the numerical solution.* Now we are interested in the description and structure of such stability domains Γ for the numerical solutions 3.3 - 3.5.

4.2 Weak Mean Stability of Euler and Milstein Schemes

It is easy to see that these scheme families 3.3 and 3.4 with $\alpha \in [0,1]$ even possess the same weak mean stability behaviour. Obviously, this fact is due to the property of the expectation of Itô-processes taking out the diffusion influence and hence the difference between these schemes. For the stability investigation we rewrote formula 3.3 to

$$\begin{aligned} Y_{n+1} &= (I-\alpha\Delta A)^{-1}(I+(1-\alpha)\Delta A + B\Delta W_n)\, Y_n \\ &= (1-\alpha\lambda\Delta)^{-1}(1+(1-\alpha)\lambda\Delta + \gamma\sqrt{\Delta}\xi_n)\, Y_n \\ &=: \Phi^\alpha(\lambda\Delta, \gamma\sqrt{\Delta}, \xi_n)\, Y_n \end{aligned} \tag{4.3}$$

where $\Delta W_n = \sqrt{\Delta}\xi_n$ and ξ is the current standard Gaussian noise. Thereby we obtained for $\mu = \lambda\Delta, \nu = \gamma\sqrt{\Delta} \in \mathbb{C}$

$$\begin{aligned} R^\alpha(\mu,\nu) &= \|\mathbb{E}\Phi^\alpha(\mu,\nu,\xi)\| \\ &= \left|\frac{1+(1-\alpha)\mu}{1-\alpha\mu}\right| =: R^\alpha(\mu). \end{aligned}$$

Furthermore we set $\mu = \mu_r + \imath\mu_i, \nu = \nu_r + \imath\nu_i$ and splitted up the term inside the stability function $R^\alpha(\mu)$ in real and imaginary parts. This leads to

$$R^\alpha(\mu) = R^\alpha(\mu_r + \imath\mu_i) = \left|\frac{1+(1-\alpha)\mu_r + \imath(1-\alpha)\mu_i}{1-\alpha\mu_r - \alpha\mu_i\imath}\right| \tag{4.4}$$

where $\imath$ represents the imaginary unit in $\mathbb{C}$. Therefore $R^\alpha(\mu) < 1$ is fulfilled iff

$$|1+(1-\alpha)\mu_r + \imath(1-\alpha)\mu_i| < |1-\alpha\mu_r - \alpha\mu_i\imath|$$

By quadrating this inequality and rearranging terms one immediately gains this assertion.

Lemma 3 :
Suppose it is given a system of the form 2.2.
Then the numerical solutions based on the implicit Euler 3.3 and Milstein schemes 3.4 are weak mean stable iff

$$2\mu_r + (\mu_r^2 + \mu_i^2)(1 - 2\alpha) \;<\; 0 \tag{4.5}$$

for $\mu = \lambda\Delta$ and $\lambda \in \mathbb{C}$, or in another words, they possess the weak mean stability domain

$$\Gamma^\alpha := \{(\mu, \nu) \in \mathbb{C} \;:\; 2\mu_r + (\mu_r^2 + \mu_i^2)(1 - 2\alpha) \;<\; 0\,\}\,.$$

Analysing 4.5, for $0 \leq \alpha < \frac{1}{2}$ one can rewrite this as

$$\frac{2}{1-2\alpha}\mu_r + \mu_r^2 \;<\; 0$$

$$\Longleftrightarrow (\mu_r + \frac{1}{1-2\alpha})^2 + \mu_i^2 \;<\; (\frac{1}{1-2\alpha})^2 \tag{4.6}$$

Thus, the weak mean stability domain of implicit Euler 3.3 and Milstein schemes 3.4 is the interrior of the circle of radius $(1-2\alpha)^{-1}$ which is centered at $((2\alpha-1)^{-1}, 0)$, when $0 \leq \alpha < \frac{1}{2}$. Otherwise, for $\frac{1}{2} \leq \alpha \leq 1$, inequality 4.5 is trivially satisfied for all complex μ with $Re(\mu) = \mu_r \;<\; 0$. Therefore the corresponding numerical solutions 3.3 and 3.4 are weak mean A-stable for model 2.2. Figure 2 shows the complement of the weak mean stability domain of the both methods using implicitness $\alpha = 1$. This domain is bounded by the plotted curve and excludes all values belonging to the curve itself and its included area (hatched in figure 2).

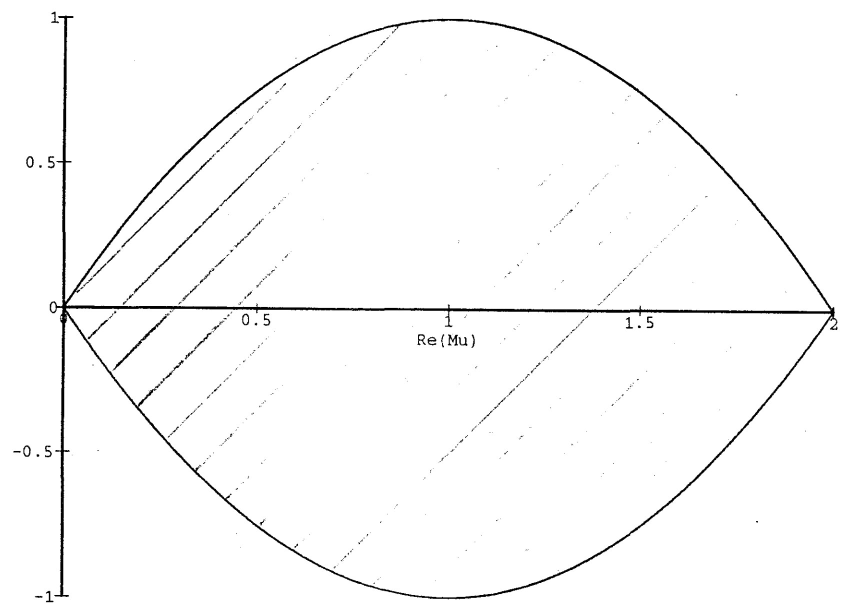

Figure 2 : Complementary set of the weak mean stability domain Γ^1

For these scheme families 3.3 and 3.4, these results coincide with the results of Kloeden & Platen [14] in the additive noise case. So, here the weak mean stability investigation for the families of implicit Euler and Milstein schemes does not depend on the additivity or multiplicativity of the noise coming up in the considered model equation. Obviously, not suprisingly, this fact is due to the given simple linear model 2.1, the special criterion structure of weak mean stability and its property projecting to the deterministic case. Moreover, as it has been indicated at the beginning, there is no need to provide the weak mean stability investigation for small noise or such linearizations as model 2.1 because these cases are covered by the extensive stability examinations in deterministic numerics, but we let them in this paper as a supplement, to remind the reader or to give occasion and a tool for examinations in the 'large noise case'.

4.3 Weak Mean Stability of Balanced Implicit Methods

Although one has already obtained weak mean stable schemes (implicit Euler or Milstein schemes for $\alpha \in [\frac{1}{2}, 1]$), now we are going to examine the weak mean stability of balanced implicit schemes, just for completeness. For this purpose we rewrote formula 3.5 and got

$$Y_{n+1} = (I + c^0\Delta + c^1\sqrt{\Delta}|\xi_n|)^{-1}(I + (c^0 + A)\Delta + (c^1|\xi_n| + B\xi_n)\sqrt{\Delta})\, Y_n \quad (4.7)$$

using matrices c^0 and c^1 such that the inverse of $I + c^0\Delta + c^1|\xi_n|$ always exists. Thereby we obtained

$$\Phi^{\hat{c}^0,\hat{c}^1}(\lambda\Delta, \gamma\sqrt{\Delta}, \xi) = \frac{1 + (\hat{c}^0 + \lambda)\Delta + (\hat{c}^1|\xi| + \gamma\xi)\sqrt{\Delta}}{1 + \hat{c}^0\Delta + \hat{c}^1\sqrt{\Delta}|\xi|}$$

as a random complex-valued mapping for $\hat{c}^0, \hat{c}^1, \lambda, \gamma \in \mathbb{C}$ where ξ is standard Gaussian distributed random variable. To make this description possible we restrict to the special choice of such weight matrices c^0 and c^1 in the balanced methods that the real parts of $\hat{c}^0$ and $\hat{c}^1$ are nonnegative. Another choice, for example the choice of positive definite 2 x 2 matrices C, within this subclass would require that these real parts are positive and the matrices are of diagonal form (purely real case). This can be easily checked by examination under which conditions $< Cx, x > \geq 0$ holds when $x \in \mathbb{R}^2$. Once again we set $\mu = \lambda\Delta = \mu_r + \imath\mu_i \in \mathbb{C}$, $\nu = \gamma\sqrt{\Delta} = \nu_r + \imath\nu_i \in \mathbb{C}$ and identified $z^0 = \hat{c}^0\Delta = z_r^0 + \imath z_i^0 \in \mathbb{C}$ and $z^1 = \hat{c}^1\sqrt{\Delta} = z_r^1 + \imath z_i^1 \in \mathbb{C}$. Then the weak mean stability function of the balanced methods comes to

$$R^{z^0,z^1}(\mu,\nu) = \left|\mathbb{E}\Phi^{z^0,z^1}(\mu,\nu,\xi)\right| = \left|\mathbb{E}\Big(\frac{1 + z^0 + \mu + z^1|\xi|}{1 + z^0 + z^1|\xi|}\Big)\right| =: \left|\mathbb{E}\Big(\frac{\varphi_r + \imath\varphi_i}{Q}\Big)\right| \quad (4.8)$$

$$\text{with } \varphi_r = 1 + \mu_r(1 + z_r^0 + z_r^1|\xi|) + \mu_i(z_i^0 + z_i^1|\xi|)$$
$$\text{and } \varphi_i = \mu_r(z_i^0 + z_i^1|\xi|) + \mu_i(1 + z_r^0 + z_r^1|\xi|)\,.$$

With $Q = (1 + z_r^0 + z_r^1|\xi|)^2 + (z_i^0 + z_i^1|\xi|)^2$ it is denoted the lower denominator in 4.8. Obviously, it is hard to calculate exactly such expressions for any complex parameters, but by analysing the real and imaginary part inside the stability function 4.8 and a suitable choice of z^0 and $z^1 \in \mathbb{C}$ we will find estimates for the expectation value in 4.8. At first we examined the real part in $R^{z^0,z^1}(\mu,\nu)$ and obtained

$$\begin{aligned} 0 \le (\varphi_r)^2 &= (1+\mu_r(1+z_r^0+z_r^1|\xi|)+\mu_i(z_i^0+z_i^1|\xi|))^2 \\ &\le 1+\mu_r^2(1+z_r^0+z_r^1|\xi|)^2+\mu_i^2(z_i^0+z_i^1|\xi|)^2+2\mu_i(z_i^0+z_i^1|\xi|) \end{aligned}$$

$$\text{if} \quad \text{sign}(z_i^0)=\text{sign}(z_i^1)=\text{sign}(\mu_i) \quad \text{or} \quad \mu_i=0 \quad \text{or} \quad z_i^0=z_i^1=0 \tag{4.9}$$

Note that $-\mu_r$, z_r^0 and z_r^1 are supposed to be nonnegative. Analogously one examines the imaginary part in $R^{z^0,z^1}(\mu,\nu)$:

$$\begin{aligned} 0 \le (\varphi_i)^2 &= (\mu_r(z_i^0+z_i^1|\xi|)+\mu_i(1+z_r^0+z_r^1|\xi|))^2 \\ &\le \mu_r^2(z_i^0+z_i^1|\xi|)^2+\mu_i^2(1+z_r^0+z_r^1|\xi|)^2 \end{aligned}$$

under the condition 4.9. Hence, for the sum of these squares one receives

$$\begin{aligned} 0 &\le (\varphi_r)^2+(\varphi_i)^2 \\ &\le 1+\|\mu\|^2(1+z_r^0+z_r^1|\xi|)^2+\|\mu\|^2(z_i^0+z_i^1|\xi|)^2+2\mu_i(z_i^0+z_i^1|\xi|) \\ &< (1+\|\mu\|^2)(1+z_r^0+z_r^1|\xi|)^2+2(1+\|\mu\|^2/2)(z_i^0+z_i^1|\xi|)^2 \\ &\le Q^2 \end{aligned}$$

$$\text{if} \quad |\mu_i| \le |z_i^0| \quad \text{and} \quad 1+\|\mu\|^2 \le (1+z_r^0+z_r^1|\xi|)^2 \tag{4.10}$$

Thereby, we are able to estimate the absolute amount of the weak mean stability function 4.8 under the conditions 4.9 and 4.10 such that $R^{z^0,z^1}(\mu,\nu) < 1$ for all $\mu,\nu \in \mathbb{C}$ with $\mu_r < 0$. Analogously one considers $\text{sign}(z_i^0) = \text{sign}(z_i^1) = -\text{sign}(\mu_i)$ and obtains finally

Lemma 4 :
Suppose the null solution is weak mean stable for the system 2.2.
If one of the conditions

$$\begin{aligned} (i) &: \ \text{sign}(c_{12}^0)=\text{sign}(c_{12}^1) \ \text{and} \ c_{11}^0 \ge \|\lambda\| \ \text{and} \ |c_{12}^0| \ge 2|\lambda_i| \\ (ii) &: \ \lambda_i=0 \ \text{and} \ c_{11}^0 \ge \|\lambda\| \\ (iii) &: \ c_{12}^0=c_{12}^1=0 \ \text{and} \ c_{11}^0 \ge \|\lambda\| \end{aligned}$$

is satisfied then the balanced implicit methods using matrices

$$c^0 = \begin{pmatrix} c_{11}^0 & -c_{12}^0 \\ c_{12}^0 & c_{11}^0 \end{pmatrix} \quad \text{and} \quad c^1 = \begin{pmatrix} c_{11}^1 & -c_{12}^1 \\ c_{12}^1 & c_{11}^1 \end{pmatrix} \tag{4.11}$$

with nonnegative constants c_{11}^0 and c_{11}^1 are weak mean stable.

With the help of the Lemma 4 one immediately finds suitable matrices c^0 and c^1 for the balanced methods to be weak mean stable. It turns out that they are even weak mean A-stable using these recommendations. So we suggested, for example, to use

$$c^0 = \begin{pmatrix} \|\lambda\| & -\lambda_i \\ \lambda_i & \|\lambda\| \end{pmatrix} \quad \text{and} \quad c^1 = \begin{pmatrix} c_{11}^1 & -\text{sign}(\mu_i)\,c_{12}^1 \\ \text{sign}(\mu_i)\,c_{12}^1 & c_{22}^1 \end{pmatrix} \tag{4.12}$$

with nonnegative coefficients c^1_{ij} $(i,j=1,2)$. Of course, the simplest choice for the weights in the balanced methods would be to take c^1 as the zero matrix which also guarantees weak mean stable numerical solutions based on these balanced methods, and hence it shows there is no need to introduce stochastic corrections by the weights c^1 provided that the system has a weak mean stable null solution as in 2.2. Moreover, following the steps above, one can find such recommendations for the balanced methods such that their weak mean stability domain for model 2.2 includes the domain Γ^α with $\alpha \in [\frac{1}{2},1]$ stated in chapter 4.2, the weak mean stability domain of methods 3.3 and 3.4 with implicitness parameter α. This situation is achieved by the use of the zero matrix c^1. Under this assumption we obtained the recommendation to take the constant matrix

$$c^0 = \begin{pmatrix} c^0_1 & -c^0_2 \\ c^0_2 & c^0_1 \end{pmatrix} \qquad \text{with} \qquad c^0_1,\, c^0_2 \geq \|\lambda\|$$

where $c^0_1 \geq 2|\lambda_r|$ or $c^0_2 \geq 2|\lambda_i|$ in the balanced methods. By the way, then this methods even possess the whole complex plane $\mathbb{C} \times \mathbb{C}$ as their weak mean stability domain covering obviously the domain Γ^α for any α. Moreover, the simplest choice is to take $c^1 \equiv 0, c^0_1 = -\lambda_r$ and $c^0_2 = -\lambda_i$. Such a balanced method is weak mean A-stable what one immediately concludes from the structure of the corresponding mapping Φ stated under the formula 4.7. This recommendation coincides with that of the implicit Euler method with $\alpha = 1$ for model 2.2. The only drawback of all these methods as well as the methods 3.3 and 3.4 with implicitness $\alpha > 0.5$, they do not replicate the unstable behaviour of the exact solutions of 2.2 w.r.t. the null solution, e.g. if $\lambda_r > 0$, but this we did not want to be guaranteed. Besides, in these cases one should prefer other weight matrices for the balanced methods which preserve the unstable behaviour of the numerical solution, too. In any case, this would require further investigations dealing with instability what we omit here. Finally, we saw each of the three methods 3.3 - 3.5 possesses a suitable subclass of numerical schemes being weak mean A-stable.

5 P-th Mean Stability for Numerical Solutions

At first we state the notion of p-th mean stability of numerical solutions. Suppose it is given an equidistant approximation started in the deterministic point $y_0 \in \mathbb{R}^d$ and providing a numerical solution (Y_n) for the system 2.1 at the time points $\tau_n = n\Delta$ (Y_n as described in section 3). For a given $p \in \mathbb{N}^+$ we define

Definition : (p-th mean stability of numerical solutions)
Assume that the step size Δ of the numerical solution is fixed. Then the null solution $x(t;0) \equiv 0$ of 2.1 is called *asymptotically p-th mean stable* for the numerical solution (Y_n) (method, scheme, approximation) of system 2.1 or shorter the numerical solution (Y_n) is *p-th mean stable* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned} \mathbb{E}\,\|Y(\tau_n;y_0)\|^p &\;<\; \epsilon \quad \forall n \in \mathbb{N} \wedge \|y_0\| < \delta \\ \text{and} \quad \mathbb{E}\,\|Y(\tau_n;y_0)\|^p &\;\longrightarrow\; 0 \quad \text{as } \tau_n \to +\infty \text{ for sufficiently small } \|y_0\|\,. \end{aligned}$$

Consider model 2.2. If in the above definition the requirements are true for all step sizes Δ and complex parameters (λ, γ) satisfying the condition 2.3 then we call the numerical solution (method, scheme, approximation) *p-th mean A-stable.* This corresponds to a further extention of the notion of deterministic A-stability to stochastic numerics and is a much stronger condition as in the case of weak mean A-stability. To explore the p-th mean stability for numerical solutions it is very helpful to introduce the notions of p-th mean stability function and domain. Suppose that it is given a scheme form $Y_{n+1} = \Phi(\lambda\Delta, \gamma\sqrt{\Delta}, \xi_n)Y_n$ where Φ is a 2 x 2 real-valued matrix or a random complex-valued function mapping on $\mathbb{C} \times \mathbb{C}$ *to* $\mathbb{C}$ and ξ_n represents the noise at the time τ_n. Then we call the positive real-valued function

$$R^p(\mu, \nu) := \mathbb{E}\|\Phi(\mu, \nu, \xi)\|^p \quad \text{for } \mu, \nu \in \mathbb{C} \tag{5.1}$$

the *p-th mean stability function.* Obviously, it holds $\mathbb{E}\|Y_n\|^p \rightarrow 0$ as $\tau_n \rightarrow \infty$ iff $\mathbb{E}\|\Phi(\mu, \nu, \xi)\|^p < 1$. Thereby, one obtains assertions about the p-th mean stability of the numerical solution Y^Δ via the investigation of the positive real-valued function $R^p(\mu, \nu)$ depending on complex variables $\mu, \nu \in \mathbb{C}$. The deterministic complex region

$$\Gamma^p := \{(\mu, \nu) \in \mathbb{C} \times \mathbb{C} \ : R^p(\mu, \nu) < 1\} \tag{5.2}$$

corresponding to a given numerical solution and its scheme with random function $\Phi : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ provides us just the domain where the numerical solution behaves p-th mean stable. Such a region is called *p-th mean stability domain* of the numerical solution. In practise, it is very difficult to calculate the p-th mean stability domain of a given numerical solution based on general numerical methods for any $p \in \mathbb{N}^+$. Already assertions on bounds for their p-th mean stability function and subregions included in their stability domain have proved to be very useful. Those functions $R^p_d(\mu, \nu)$ which estimate the original p-th mean stability function for fixed $p \in \mathbb{N}^+$ in such a way that $R^p(\mu, \nu) \leq R^p_d(\mu, \nu)$ for each $(\mu, \nu) \in M \subseteq \mathbb{C} \times \mathbb{C}$ we call *dominating p-th mean stability functions* for $R^p(\mu, \nu)$ and given numerical solution. In general, it is sufficient to find a function R^p_d for R^p in the sense such that from $R^p_d(\mu, \nu) < 1$ it follows $R^p(\mu, \nu) < 1$ on M. Thus, then we will also make use of the title '*dominating stability function*'. Hence, if one requires $R^p_d(\mu, \nu) < 1$ for fixed $p \in \mathbb{N}^+$, one obtains a subregion $\Gamma_d \subseteq M$ of the p-th mean stability domain Γ. At least, this allows to formulate sufficient conditions for p-th mean stability of numerical solutions, e.g. to say for which parameters laying in $\Gamma_d \subseteq \Gamma$ the numerical solution based on the considered method is p-th mean stable. Naturally we have to require the validity of inequality 2.3 for that $p \in \mathbb{N}^+$ we use in the stability investigations in the p-th mean sense. In this section we will see that the case $p = 2$, the mean square stability, plays a special role in our examinations. This is just the case where stability domains can be exactly calculated in an easier way (also the case where p is even). Besides, if $R(\mu, \nu) > 1$ for $p = 2$ then it follows $R^p(\mu, \nu) > 1$ for all $p \in \mathbb{N}\backslash\{0, 1\}$, e.g. the p-th mean instability of the numerical solution (compare Lemma 1). Therefore we will be able to formulate necessary and sufficient conditions for the p-th mean stability.

5.1 P-th Mean Stability of Implicit Euler Schemes

At first we recall the function $\Phi^\alpha(\mu,\nu,\xi)$ stated in chapter 4.2 which determines the numerical solutions based on the implicit Euler scheme with parameter $\alpha \in [0,1]$ for the model 2.2. There we have received

$$\Phi^\alpha(\mu,\nu,\xi) = (1-\alpha\mu)^{-1}(1+(1-\alpha)\mu+\nu\xi) \tag{5.3}$$

for any complex μ,ν and standard Gaussian random variable ξ (see at mark 4.3). Now we are specially interested in the structure and for estimates of $||\Phi^\alpha(\mu,\nu,\xi)||^2$. Using this expression allows to estimate the p-th mean stability function $R^p(\mu,\nu)$ $= \mathbb{E}\,(||\Phi^\alpha(\mu,\nu,\xi)||^2)^{\frac{p}{2}}$ by $R^2_d(\mu,\nu) = R^2(\mu,\nu)$ for any $p \in \mathbb{N}^+$. In case of strong mean stability $R^2(\mu,\nu)$ can be used as dominating stability function and to formulate sufficient conditions, in case of p-th mean stability with p larger than 2, for providing of necessary conditions to have a p-th mean stable numerical solution. The answere concerning mean square stability is given by the following lemma.

Lemma 5 :
Suppose it is given a system of the form 2.2 with $(\lambda_r,\lambda_i) <> ((\alpha\Delta)^{-1},0)$.
Then the family of implicit Euler schemes with implicitness parameter $\alpha \in [0,1]$ is mean square stable iff it holds

$$2\mu_r + ||\nu||^2 + (1-2\alpha)||\mu||^2 < 0 \tag{5.4}$$

$$\text{for} \quad \mu(=\lambda\Delta) = \mu_r + \imath\mu_i \quad \textit{and} \quad \nu(=\gamma\sqrt{\Delta}) = \nu_r + \imath\nu_i \ \in \mathbb{C}.$$

Proof : Identify $\mu = \mu_r + \imath\mu_i$ and $\nu = \nu_r + \imath\nu_i \in \mathbb{C}$. Thereby we obtained

$$\Phi^\alpha(\mu,\nu,\xi) = \frac{1+(1-\alpha)\mu_r+\nu_r\xi+\imath((1-\alpha)\mu_i+\nu_i\xi)}{1-\alpha\mu_r-\imath\alpha\mu_i} =$$

$$\frac{1+(1-2\alpha)\mu_r-\alpha(1-\alpha)||\mu||^2+\xi(\nu_r-\alpha\mu_r\nu_r-\alpha\mu_i\nu_i)}{1-2\alpha\mu_r+\alpha^2||\mu||^2}$$
$$+\imath\frac{\mu_i+\xi(\nu_i-\alpha\mu_r\nu_i+\alpha\mu_i\nu_r)}{1-2\alpha\mu_r+\alpha^2||\mu||^2}$$

for any random ξ. This leads to

$$||\Phi^\alpha(\mu,\nu,\xi)||^2\cdot(1-2\alpha\mu_r+\alpha^2||\mu||^2)^2 = (1+(1-2\alpha)\mu_r-\alpha(1-\alpha)||\mu||^2)^2+\mu_i^2+2\xi\cdot$$
$$\Big((\nu_r-\alpha\mu_r\nu_r-\alpha\mu_i\nu_i)(1+(1-2\alpha)\mu_r-\alpha(1-\alpha)||\mu||^2)+\mu_i(\nu_i-\alpha\mu_r\nu_i+\alpha\mu_i\nu_r)\Big)$$
$$+\xi^2\left((\nu_r-\alpha\mu_r\nu_r-\alpha\mu_i\nu_i)^2+(\nu_i-\alpha\mu_r\nu_i+\alpha\mu_i\nu_r)^2\right)$$

So $R^2(\mu,\nu) = \mathbb{E}\,||\Phi^\alpha(\mu,\nu,\xi)||^2 < 1$ is satisfied for Gaussian white noise ξ iff the complex pair (μ,ν) fulfills the inequality

$$(1+(1-2\alpha)\mu_r-\alpha(1-\alpha)||\mu||^2)^2+\mu_i^2+(\nu_r(1-\alpha\mu_r)-\alpha\mu_i\nu_i)^2$$
$$+(\nu_i(1-\alpha\mu_r)-\alpha\nu_r\mu_i)^2 \ < \ (1-2\alpha\mu_r+\alpha^2||\mu||^2)^2$$

which is equivalent to

$$2(\mu_r - \alpha||\mu||^2)(1 - 2\alpha\mu_r + \alpha^2||\mu||^2) + (\mu_r - \alpha||\mu||^2)^2$$
$$+\mu_i^2 + ||\nu||^2(1 - 2\alpha\mu_r + \alpha^2||\mu||^2) \ < 0 \ .$$

After rearranging this expression one comes across the equivalent inequality

$$\left(2(\mu_r - \alpha||\mu||^2) + ||\mu||^2 + ||\nu||^2\right)(1 - 2\alpha\mu_r + \alpha^2||\mu||^2) < 0.$$

Consequently, because of $1 - 2\alpha\mu_r + \alpha^2||\mu||^2 > 0$ (which is guaranteed by $\lambda_r <> (\alpha\Delta)^{-1}$ or $\lambda_i <> 0$), we received that the Euler schemes 3.3 for the system 2.2 are mean square stable iff

$$2(\mu_r - \alpha||\mu||^2) + ||\mu||^2 + ||\nu||^2 \ < \ 0$$

which is equivalent to the inequality 5.4 of Lemma 4.

Conclusions : From special interest, in the case $\alpha = 0$, the nonimplicit Euler scheme, 5.4 simplifies to

$$2\mu_r + ||\nu||^2 + ||\mu||^2 \ < \ 0 \tag{5.5}$$

Therefore the amount of μ_i plays a role for the mean square stability of that scheme, in contrast to the situation for the exact solution where $\lambda_i = \mu_i/\Delta$ has no influence on the stability ($p = 2$) of the null solution. Furthermore 5.5 is equivalent to

$$(\mu_r + 1)^2 + \mu_i^2 + \nu_r^2 + \nu_i^2 \ < \ 1 \tag{5.6}$$

Obviously this inequality describes the interior of the unit ball in $I\!R^4$ with midpoint $(-1, 0, 0, 0)$ and hence the mean square stability domain Γ_0^2. On the other hand, for the mean square stability domain of the numerical solutions 3.3 with implicitness $\alpha = 1$, the class of 'deterministic fully implicit' Euler schemes, the inequality 5.4 takes the form

$$2\mu_r + ||\nu||^2 - ||\mu||^2 \ < \ 0 \tag{5.7}$$

Consequently, the equivalent relation

$$(\mu_r - 1)^2 + \mu_i^2 - ||\nu||^2 \ > \ 1 \tag{5.8}$$

must be fulfilled to have mean square stable solutions of the form 3.3 with $\alpha = 1$. The domain Γ_1^2 defined by 5.8 can be considered as the outside of a hyperboloid in $I\!R^3$ shifted from the origin to the point $(1, 0, 0)$. To visualize the stability domains Γ_0^2 and Γ_1^2 established by 5.6 and 5.8 it is drawn surfaces w.r.t. the tripel $(\mu_r, \mu_i, ||\nu||)$ which present the boundary of these domains. The corresponding regions are visible in figure 3 and 4. Note that the domain for the 'deterministic half-implicit' Euler schemes has already been plotted by the figure 1 (multiply scales with $\Delta, \sqrt{\Delta}$). In the plane $||\nu|| = 0$, the well-known results from the deterministic numeric are confirmed by the both figures. In the asymptotical sense of step sizes used, for very small step sizes, the inequality 5.4 is almost identical with the inequality 2.3. For any parameter $\alpha \in [0, 1]$, thereby the Euler schemes 3.3 asymptotically replicate the mean square stability behaviour of the null solution.

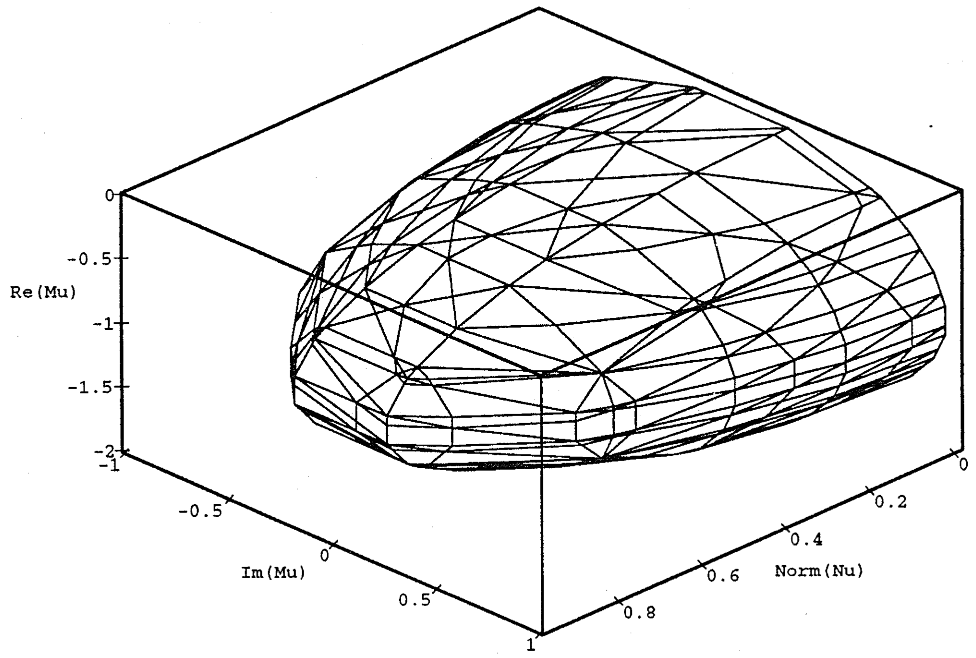

Figure 3 : Boundary of the mean square stability domain Γ_0^2 of methods 3.3 with implicitness $\alpha = 0$

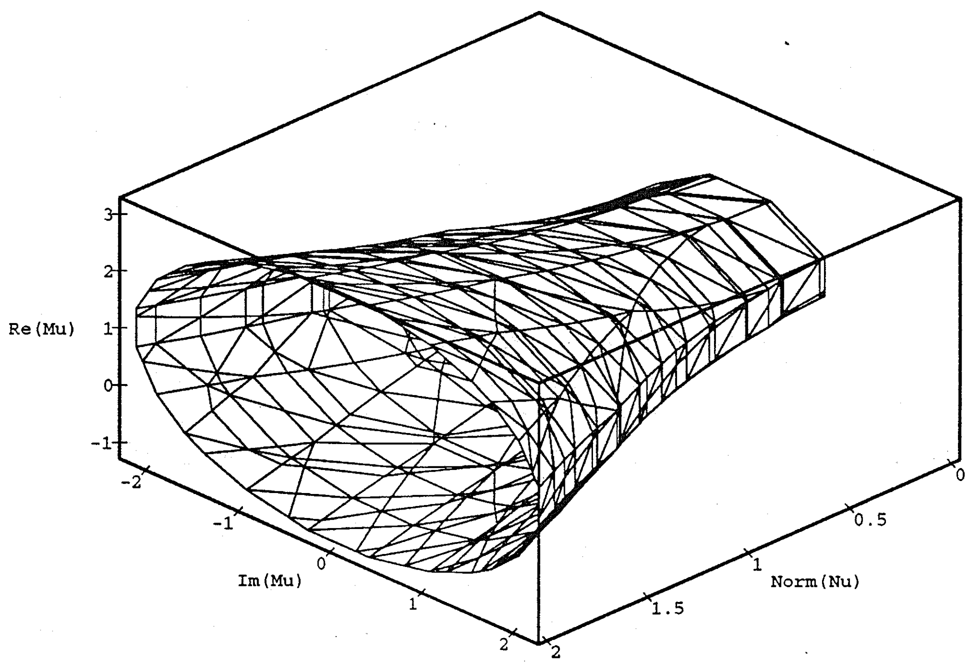

Figure 4 : Boundary of the mean square stability domain Γ_1^2 of methods 3.3 with implicitness $\alpha = 1$

Moreover, looking once again at the condition 5.4, it turns out that one finds the following relation between the inequalities of the type 5.4 for different implicitness parameters $\alpha \in [0,1]$:

$$2\mu_r + \|\nu\|^2 + \|\mu\|^2 \geq 2\mu_r + \|\nu\|^2 + (1-2\alpha)\|\mu\|^2 \geq 2\mu_r + \|\nu\|^2 - \|\mu\|^2$$

or the equivalent expression in terms of the system parameters (λ, γ)

$$2\lambda_r + \|\gamma\|^2 + \|\lambda\|^2\Delta \geq 2\lambda_r + \|\gamma\|^2 + (1-2\alpha)\|\lambda\|^2\Delta \geq 2\lambda_r + \|\gamma\|^2 - \|\lambda\|^2\Delta .$$

Thereby it is possible to state necessary and sufficient conditions for the mean square stability of the family of implicit Euler schemes. The domain $\bar{\Gamma}_1^2$ defined by

$$\bar{\Gamma}_1^2 := \{(\mu,\nu) \in \mathbb{C} \times \mathbb{C} \ : 2\mu_r + \|\nu\|^2 - \|\mu\|^2 \ \leq \ 0\} \tag{5.9}$$

is a 'necessary mean square stability domain' for this family, the domain Γ_0^2 fixed by

$$\Gamma_0^2 := \{(\mu,\nu) \in \mathbb{C} \times \mathbb{C} \ : 2\mu_r + \|\nu\|^2 + \|\mu\|^2 \ < \ 0\} \tag{5.10}$$

a 'sufficient mean square stability domain'. By the way, it is not hard to recognize that for the family of the mean square stability domains Γ_α^2 of the schemes 3.3 it holds

$$\Gamma_{\alpha_1}^2 \ \subseteq \ \Gamma_{\alpha_2}^2 \qquad \text{if } \alpha_1 \ \leq \ \alpha_2 \ \text{ and } \ \alpha_1, \alpha_2 \in [0,1],$$

e.g. we obtained the 'monotonous inclusion property' of the stability domains.
In practise, among the implicit Euler schemes, we suggest to prefer the scheme with implicitness $\alpha = 0.5$ because for this scheme the stability behaviour of the numerical solution exactly coincides with that of the null solution for the system 2.2. Only then the conditions 5.4 and 2.3 are equivalent, only then the amount of μ_i plays no role in the mean square stability examination for the numerical solution.
Furthermore, another interesting conclusion we have achieved for the pure-stochastic model equation

$$dX(t) = \sigma X(t) dW(t) \tag{5.11}$$

where σ is any positive real parameter and its numerical solutions of type 3.3. Both for the exact solution and the numerical solutions 3.3, the null solution is not mean square stable (also not p-th mean stable for $p \geq 2$), hence these numerical solutions replicate in the mean square sense the stability behaviour of the exact solution of equation 5.11.

Relation to p-th mean stability :
The result of Lemma 5 provides us necessary and sufficient conditions for p-th mean stability. If the inequality 5.4 is satisfied then one knows that the numerical solutions 3.3 are strong mean stable ($p = 1$) too. Thus 5.4 is sufficient for strong mean stability, the strong mean stability domain must include the p-th mean stability domains Γ^p for $p \in \mathbb{N}^+$. Otherwise, if the left part in 5.4 is larger than zero then we also obtained that the numerical solutions 3.3 are not p-th mean stable for $p \geq 2 \, (p \in \mathbb{N}^+)$, e.g. the validity of inequality 5.4 is necessary for p-th mean

stability of these numerical solutions ($p \geq 2$). This can be easily concluded from Lyapunov's inequality which says

$$\left(\mathbb{E}\,\|\Phi(\cdot)\|^p\right)^{1/p} \leq \left(\mathbb{E}\,\|\Phi(\cdot)\|^{p+1}\right)^{1/(p+1)}$$

for any L^{p+1}-integrable function Φ. Further investigations could be done for the p-th mean stability with p larger than 2, and especially for $p = 2q$. Of course, $\bar{\Gamma}_1^2$ is also a 'necessary domain' for the p-th mean stability ($p > 2$), and Γ_0^2 describes a 'sufficient domain' for the strong mean stability. In detail, further examinations concerning the p-th mean stability we omit here.

5.2 P-th Mean Stability of Implicit Milstein Schemes

In contrast to the investigations for weak mean stability, the p-th mean stability behaviour of the Euler 3.3 and Milstein schemes 3.4 do not coincide for the model equation 2.2. Here the corresponding investigations are quite more complicated. For the model 2.2 the implicit Milstein schemes with implicitness $\alpha \in [0, 1]$ can be written in the following scheme form

$$\begin{aligned} Y_{n+1} &= (I - \alpha\Delta A)^{-1}(I + (1-\alpha)\Delta A + B\sqrt{\Delta}\xi_n + \tfrac{1}{2}B^2(\xi_n^2 - 1)\Delta)\, Y_n \\ &= \Phi^\alpha(\lambda\Delta, \gamma\sqrt{\Delta}, \xi_n)\, Y_n \end{aligned}$$

with the complex mapping

$$\Phi^\alpha(\lambda\Delta, \gamma\sqrt{\Delta}, \xi) = (1 - \alpha\lambda\Delta)^{-1}(1 + (1-\alpha)\lambda\Delta + \gamma\sqrt{\Delta}\xi + \frac{1}{2}\gamma^2\Delta(\xi^2 - 1)) \quad (5.12)$$

where $\Delta W_n = \sqrt{\Delta}\xi_n$ corresponds to the current Wiener noise increment. Identify $\mu = \mu_r + \imath\mu_i = \lambda\Delta$ and $\nu = \nu_r + \imath\nu_i = \gamma\sqrt{\Delta}$, and hence as complex numbers. To formulate a corresponding assertion about the stability of the implicit Milstein schemes 3.4 we used the abbreviation

$$F(\mu, \nu, \alpha) = F(\mu_r, \mu_i, \nu_r, \nu_i, \alpha) = 2\mu_r + \|\nu\|^2 + (1 - 2\alpha)\|\mu\|^2 + \frac{1}{2}\|\nu\|^4 . \quad (5.13)$$

Once again the case $p = 2$ plays a special role in our examinations. By means of the function F it is possible to decide whether the corresponding implicit Milstein scheme is mean square stable or not. The following lemma gives the answere on the mean square stability of numerical solutions 3.4 and hence necessary conditions for p-th mean stability of them for $p \geq 2$, as well as sufficient conditions for their strong mean stability.

Lemma 6 :
Suppose it is given a system of the form 2.2 with $(\lambda_r, \lambda_i) <> ((\alpha\Delta)^{-1}, 0)$.
Then the family of implicit Milstein schemes with implicitness parameter $\alpha \in [0, 1]$ is mean square stable iff it holds

$$F(\mu, \nu, \alpha) < 0 \quad (5.14)$$

for the mapping $F : \mathbb{C} \times \mathbb{C} \times [0,1] \rightarrow \mathbb{R}$ defined by 5.13 for complex $\mu = \lambda\Delta, \nu = \gamma\sqrt{\Delta}$ and $\alpha \in [0,1]$. Moreover, the mean square stability of the implicit Euler scheme with $\alpha \in [0,1]$ is necessary for the mean square stability of the corresponding implicit Milstein scheme using the same implicitness α.

Proof : Identify $\mu = \mu_r + \imath\mu_i$ and $\nu = \nu_r + \imath\nu_i \in \mathbb{C}$. By rearranging terms one obtains for the random mapping Φ^α

$$\Phi^\alpha(\mu,\nu,\xi) = \frac{1 + (1-\alpha)\mu + \nu\xi + \frac{1}{2}\nu^2(\xi^2-1)}{1-\alpha\mu} = \tag{5.15}$$

$$\frac{1 + (1-\alpha)\mu_r + \nu_r\xi + \frac{1}{2}(\nu_r^2 - \nu_i^2)(\xi^2-1) + \imath((1-\alpha)\mu_i + \nu_i\xi + \nu_r\nu_i(\xi^2-1))}{1-\alpha\mu_r - \imath\alpha\mu_i}$$

After multiplying this expression with its conjugate complex denominator $1-\alpha\mu_r + \imath\alpha\mu_i$ in order to receive a real denominator and splitting up in the real and imaginary part of Φ^α one encounters with the real part $Re(\Phi^\alpha)$ which takes the relation

$$Re(\Phi^\alpha)\cdot(1-2\alpha\mu_r + \alpha^2\|\mu\|^2) = 1 + (1-2\alpha)\mu_r - \alpha(1-\alpha)\|\mu\|^2$$

$$+\xi(\nu_r(1-\alpha\mu_r) - \alpha\mu_i\nu_i) + (\xi^2-1)(\frac{1}{2}(\nu_r^2-\nu_i^2)(1-\alpha\mu_r) - \alpha\mu_i\nu_r\nu_i)$$

Therefore we have the identity that

$$\mathbb{E}\,(Re(\Phi^\alpha))^2\cdot(1-2\alpha\mu_r+\alpha^2\|\mu\|^2)^2 = \left(1+(1-2\alpha)\mu_r - \alpha(1-\alpha)\|\mu\|^2\right)^2$$

$$+\,(\nu_r(1-\alpha\mu_r)-\alpha\mu_i\nu_i)^2 + 2(\frac{1}{2}(\nu_r^2-\nu_i^2)(1-\alpha\mu_r) - \alpha\mu_i\nu_r\nu_i)^2 \tag{5.16}$$

In analogous way, the part of $\mathbb{E}\|\Phi^\alpha\|^2$ determined by the imaginary part of Φ^α is derived and comes up in the relation

$$\mathbb{E}\,(Im(\Phi^\alpha))^2\cdot(1-2\alpha\mu_r+\alpha^2\|\mu\|^2)^2 = \mu_i^2 + (\nu_i(1-\alpha\mu_r)+\alpha\mu_i\nu_r)^2$$

$$+\,2(\frac{1}{2}(\nu_r^2-\nu_i^2)\alpha\mu_i + (1-\alpha\mu_r)\nu_r\nu_i)^2 \tag{5.17}$$

Denoting with $R^2_{\alpha,E}$ the mean square stability function of the implicit Euler scheme and with $R^2_{\alpha,M}$ the mean square stability function of the implicit Milstein scheme using the same implicitness $\alpha \in [0,1]$ one can conclude that $R^2_{\alpha,M}$ =

$$R^2_{\alpha,E} + 2\frac{(\frac{1}{2}(\nu_r^2-\nu_i^2)(1-\alpha\mu_r)-\alpha\mu_i\nu_r\nu_i)^2 + (\frac{1}{2}(\nu_r^2-\nu_i^2)\alpha\mu_i + (1-\alpha\mu_r)\nu_r\nu_i)^2}{(1-2\alpha\mu_r+\alpha^2\|\mu\|^2)^2}$$

$$= R^2_{\alpha,E} + \frac{\|\nu\|^4/2}{1-2\alpha\mu_r+\alpha^2\|\mu\|^2}, \tag{5.18}$$

and hence the necessity that the corresponding implicit Euler scheme must be mean square stable (the stability functions only differ through a quadratic term). Furthermore, the condition

$$R^2_{\alpha,M}(\mu,\nu) = \mathbb{E}\|\Phi^\alpha(\mu,\nu,\xi)\|^2 < 1$$

on the stability function $R^2_{\alpha,M}(\mu,\nu)$ is fulfilled iff

$$\mathbb{E}\ Re^2(\Phi^\alpha) + \mathbb{E}\ Im^2(\Phi^\alpha) < 1 \tag{5.19}$$

is valid. Naturally, this equivalence is also true for the mean square stability of other numerical solutions. Exploiting the equalities 5.16, 5.17, 5.18, using steps of the proof of Lemma 5 and rearranging terms occured in the inequality 5.19 it follows that the inequalities 5.14 and 5.19 are equivalent, what confirms the assertion of Lemma 6.

Remarks and asymptotical sufficiency : Concerning mean square stability, as the result of this lemma, one cannot recommend the family of implicit Milstein schemes 3.4 in order to provide mean square stable numerical solutions. Any scheme of the Euler family 3.3 does it more efficiently. However, the stability function $R^2_{\alpha,M}$ can be used as a dominating mean square stability function of the corresponding Euler scheme, e.g. $R^2_{\alpha,M}$ dominates $R^2_{\alpha,E}$. These statements are proved by the equality 5.18. Otherwise it also means, if one rises the convergence order then one looses stability of the approximation. Anyway, the Milstein family 3.4 is not mean square A-stable. If one desires to use them though, for computer simulations with fixed parameters λ and γ in the model 2.2, one has to check the validity of the condition 5.14 and find suitable step sizes for the approximation. The domains which describe the region of these suitable step sizes Δ for the Milstein family 3.4 are given by the mean square stability domains

$$\Gamma^2_{\alpha,M} := \{(\mu,\nu) \in \mathbb{C} \times \mathbb{C}\ : F(\mu,\nu,\alpha) < 0\}$$

Note that $\mu = \lambda\Delta$ and $\nu = \gamma\sqrt{\Delta}$. In general, here it is also possible to deduce the 'monotonous inclusion' of the stability domains $\Gamma^2_{\alpha,M}$ like for the Euler family in chapter 5.1, e.g. within the family of implicit Milstein schemes 3.4 for the model 2.2 the Milstein scheme with implicitness parameter $\alpha = 1$ provides the most stable numerical solution in the mean square sense. To visualize these domains we identified $||\nu||$ as one variable. Then the boundaries of $\Gamma^2_{0,M}$ and $\Gamma^2_{1,M}$ are represented in figure 5 and 6. In figure 5 the domain $\Gamma^2_{0,M}$ can be identified with the interior of the drawn surface, and in figure 6 $\Gamma^2_{1,M}$ is described by the outside of the plotted tube. We remark that the domains $\Gamma^2_{\alpha,M}$ must be included in the domains $\Gamma^2_{\alpha,E}$, the mean square stability domains of the corresponding Euler schemes. Naturally, in the plane $||\nu|| = 0$, the deterministic situation, the results coincide with those of the implicit Euler family, and hence the well-known stability domains in deterministic numerics are confirmed by these two figures too. The stability domain $\Gamma^2_{0,M}$ seems to be almost identical with the domain Γ^2_0 of the nonimplicit Euler scheme plotted in subsection 5.1, not suprisingly, their stability functions only differ from the amount of $||\nu||^4/2 \geq 0$. Analogously to subsection 5.1 one finds necessary as well as sufficient p-th mean stability domains, just by simplification and estimation of the *'stability indicator'* $F(\mu,\nu,\alpha)$ in 5.14, e.g. by lower and upper bounds for the function F. This work is omitted here. Nevertheless, another interesting property of these schemes consists in the *'asymptotical sufficiency'* for their mean square stability. That is we want to examine the asymptotical behaviour of the 'stability indicator' $F(\mu,\nu,\alpha)$.

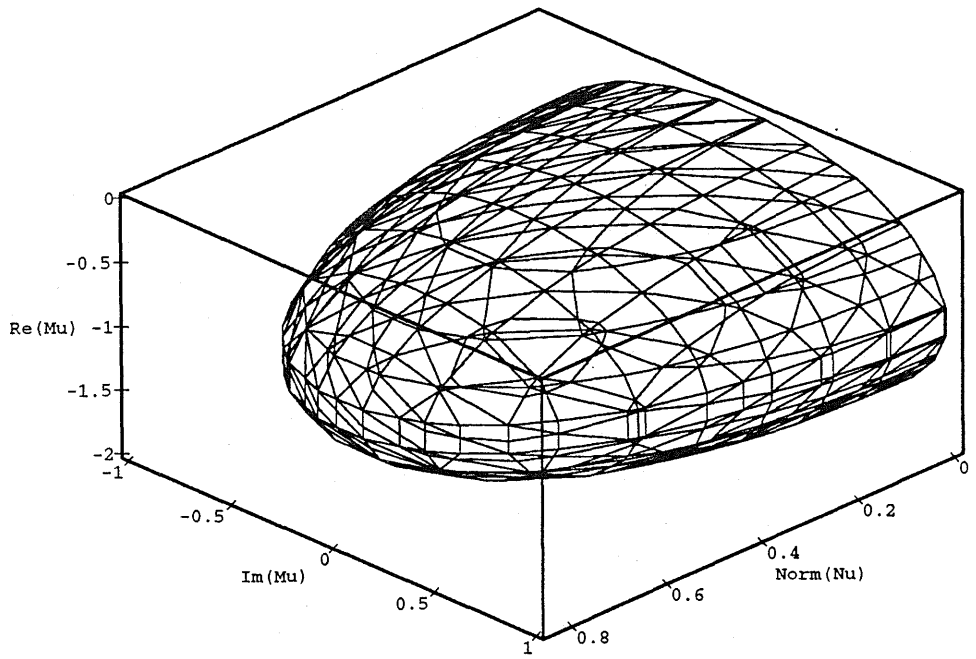

Figure 5 : Boundary of the mean square stability domain $\Gamma^2_{0,M}$ of methods 3.4 with implicitness $\alpha = 0$

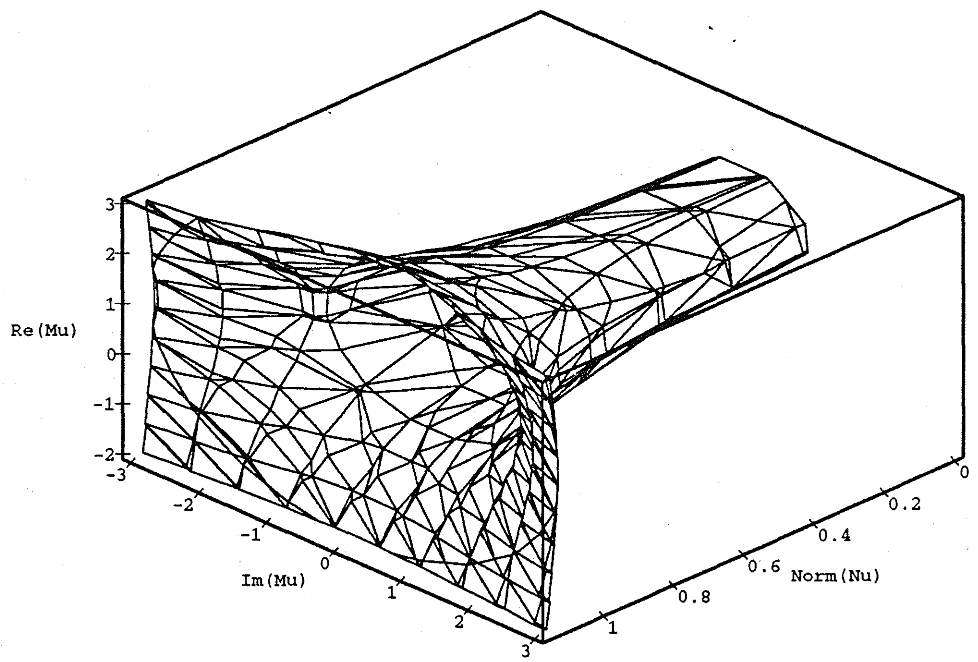

Figure 6 : Boundary of the mean square stability domain $\Gamma^2_{1,M}$ of methods 3.4 with implicitness $\alpha = 1$

For this purpose we investigated the following expression

$$s(\mu,\nu,\alpha) := \lim_{\Delta\to 0} \frac{F(\mu,\nu,\alpha)}{\Delta} \tag{5.20}$$

which surely exists because of $F(\mu,\nu,\alpha) = \mathcal{O}(\Delta)$. Such mappings $s : \mathbb{C} \times \mathbb{C} \times [0,1] \to \mathbb{R}$ we call *asymptotical sufficiency indicators* for the p-th mean stability of the numerical solution considered. For the implicit Milstein schemes 3.4 it turns out that

$$\begin{aligned} s(\mu,\nu,\alpha) &= s(\lambda_r,\lambda_i,\gamma_r,\gamma_i,\alpha) = 2\mu_r + \nu_r^2 + \nu_i^2 \\ &= (2\lambda_r + \gamma_r^2 + \gamma_i^2)\,\Delta \ . \end{aligned} \tag{5.21}$$

Particularly in more complicated situations, with the help of this mapping one is able to decide whether the numerical solution is p-th mean stable in asymptotical sense of step sizes, e.g. p-th mean stability for small enough step sizes Δ, or not. Of course, this property is a mixture between the assertion of stability and an asymptotical behaviour for very small step sizes, but numerical solutions without the property $s(\mu,\nu,\alpha) < 0$ do not possess stability domains including environments of the zero point $(0,0)$ in $\mathbb{C} \times \mathbb{C}$ which are sufficiently small. Such cases replicate a poor numerical stability behaviour. For the implicit Milstein schemes, as we have already seen for the implicit Euler schemes in subsection 5.1, $s(\mu,\nu,\alpha) = (2\lambda_r + \|\gamma\|^2)\,\Delta < 0$ coincides with the condition 2.3 for the exact solution. Thereby

$$\lambda_r < \frac{-\|\gamma\|^2}{2} \tag{5.22}$$

must be fulfilled for the asymptotical sufficiency of the numerical solutions 3.4. That's why the family of implicit Milstein schemes 3.4 is asymptotically sufficient for mean square stable numerical solutions at least. Analogously one could investigate the remainder term $\frac{F(\mu,\nu,\alpha)}{\Delta} - s(\mu,\nu,\alpha)$ to formulate further results for asymptotical sufficiency of the higher order terms in $F(\mu,\nu,\alpha)$, but this we omit here. The asymptotical sufficiency indicators form a further possibility to characterize the numerical stability behaviour and are effective to check necessary stability conditions, although one should expect that the 'asymptotical sufficiency of first order' is necessary for the convergence of the numerical solution to the exact solution.

5.3 P-th Mean Stability of Balanced Implicit Methods

Finally, we want to formulate a fully-implicit scheme for model 2.2 with its stability domain covering 'as much as possible' the region

$$\Gamma^p = \{(\mu,\nu) \in \mathbb{C} \times \mathbb{C} \ : \ \mu_r + \frac{1}{2}\nu_i^2 + \frac{1}{2}\nu_r^2(p-1) < 0\} \tag{5.23}$$

which is just the domain where the system 2.2 behaves p-th mean stable w.r.t. its null solution (see section 2). At least, for p $\geq$ 2, in the case of mean square stability it will turn out that the balanced implicit methods 3.5 possess a subclass which is p-th mean A-stable, hence its stability domain includes the domain Γ^2. For this purpose we examined the function $\Phi : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$ with

$$\Phi(\mu,\nu,\xi) = (1 + z^0 + z^1|\xi|)^{-1}(1 + z^0 + \mu + \nu\xi + z^1|\xi|) \tag{5.24}$$

for any $\mu, \nu \in \mathbb{C}$ and standard Gaussian distributed random variable ξ. Here, with the choice of parameters z^0 and $z^1 \in \mathbb{C}$, we can achieve control on the total amount of Φ. Then for the convergence of the balanced implicit methods we required that both z^0 and z^1 have positive real parts (compare Milstein et a([14]) as in the case of weak mean stability. In general, it is sufficient to require $z_r^1 \geq 0$, but then one restricts oneselves to small enough step sizes Δ in order to achieve invertibility of the correction term (hence no mean A-stability). At first we state a result concerning mean square stability and providing us suitable parameters involved in the proposed balanced implicit methods.

Lemma 7 :
Suppose it is given a system of the form 2.2, and the complex pair (z^0, z^1) is nonrandom with $z^0 = \mathcal{O}(\Delta)$ and $z^1 = \mathcal{O}(\Delta^{1/2})$ and nonnegative real parts. Then the balanced implicit methods with matrices

$$c^0 = \begin{pmatrix} z_r^0 & -z_i^0 \\ z_i^0 & z_r^0 \end{pmatrix} \cdot \Delta^{-1} \quad \text{and} \quad c^1 = \begin{pmatrix} z_r^1 & -z_i^1 \\ z_i^1 & z_r^1 \end{pmatrix} \cdot \Delta^{-1/2} \tag{5.25}$$

are mean square stable iff it holds

$$(\,F(\lambda\Delta, \gamma\sqrt{\Delta}) \;=)\quad F(\mu,\nu) \;<\; 0 \qquad \text{with} \tag{5.26}$$

$$F(\mu,\nu) = \int_0^{+\infty} \frac{\|\mu\|^2 + \|\nu\|^2 x^2 + 2(1 + z_r^0 + z_r^1 x)\mu_r + 2(z_i^0 + z_i^1 x)\mu_i}{(1 + z_r^0 + z_r^1 x)^2 + (z_i^0 + z_i^1 x)^2} \exp\{-\frac{x^2}{2}\} dx\,.$$

Proof : Identifying the real and imaginary parts of μ, ν, z^0 and z^1 with those forms as in subsection 4.3 the real part in formula 5.24 is rewritten to $Re(\Phi) =$

$$\frac{(1 + z_r^0 + \mu_r + \nu_r\xi + z_r^1|\xi|)(1 + z_r^0 + z_r^1|\xi|) + (z_i^0 + \mu_i + \nu_i\xi + z_i^1|\xi|)(z_i^0 + z_i^1|\xi|)}{(1 + z_r^0 + z_r^1|\xi|)^2 + (z_i^0 + z_i^1|\xi|)^2}$$

and the imaginary part in 5.24 to $Im(\Phi) =$

$$\frac{(1 + z_r^0 + z_r^1|\xi|)(z_i^0 + \mu_i + \nu_i\xi + z_i^1|\xi|) - (1 + z_r^0 + \mu_r + \nu_r\xi + z_r^1|\xi|)(z_i^0 + z_i^1|\xi|)}{(1 + z_r^0 + z_r^1|\xi|)^2 + (z_i^0 + z_i^1|\xi|)^2}$$

For the further investigation we introduced the following abbreviations

$$a = 1 + z_r^0 + z_r^1|\xi| \quad \text{and} \quad b = z_i^0 + z_i^1|\xi|\,.$$

Note that a must be positive and b can have any values. In the next step we investigate $\|\Phi(\mu,\nu,\xi)\|^2$ for any complex μ, ν and random variable ξ. Thereby one obtains

$$\begin{aligned}
\|\Phi(\mu,\nu,\xi)\|^2 &= Re^2(\Phi(\mu,\nu,\xi)) + Im^2(\Phi(\mu,\nu,\xi)) \\
&= \frac{(a+\mu_r+\nu_r\xi)^2a^2 + 2(a+\mu_r+\nu_r\xi)a(b+\mu_i+\nu_i\xi)b + (b+\mu_i+\nu_i\xi)^2b^2}{(a^2+b^2)^2} \\
&+ \frac{a^2(b+\mu_i+\nu_i\xi)^2 - 2a(b+\mu_i+\nu_i\xi)(a+\mu_r+\nu_r\xi)b + (a+\mu_r+\nu_r\xi)^2b^2}{(a^2+b^2)^2} \\
&= \frac{(a+\mu_r+\nu_r\xi)^2 + (b+\mu_i+\nu_i\xi)^2}{a^2+b^2} \qquad (5.27)
\end{aligned}$$

This expression is equivalent to

$$\begin{aligned}
& \frac{(a+\mu_r)^2 + (b+\mu_i)^2 + \|\nu\|^2\xi^2 + 2(a+\mu_r)\nu_r\xi + 2(b+\mu_i)\nu_i\xi}{a^2+b^2} \\
= \ & 1 + \frac{\|\mu\|^2 + \|\nu\|^2\xi^2 + 2(a\mu_r + b\mu_i) + 2(a+\mu_r)\nu_r\xi + 2(b+\mu_i)\nu_i\xi}{a^2+b^2}
\end{aligned}$$

Assuming that the complex pair (z^0, z^1) is nonrandom and using the symmetry property of Gaussian random variables the mean square stability function $R_B^2(\mu,\nu)$ takes the values

$$\begin{aligned}
R_B^2(\mu,\nu) &= \mathbb{E}\|\Phi(\mu,\nu,\xi)\|^2 \\
&= 1 + \mathbb{E}\left(\frac{\|\mu\|^2 + \|\nu\|^2\xi^2 + 2(a\mu_r + b\mu_i)}{a^2+b^2}\right) \qquad (5.28)
\end{aligned}$$

So $R_B^2(\mu,\nu) < 1$, and hence the mean square stability of the given balanced methods based on the complex pair (z^0, z^1), is satisfied iff

$$F(\mu,\nu) = \sqrt{\frac{\pi}{2}} \cdot \mathbb{E}\left(\frac{\|\mu\|^2 + \|\nu\|^2\xi^2 + 2((1+z_r^0+z_r^1|\xi|)\mu_r + (z_i^0+z_i^1|\xi|)\mu_i)}{(1+z_r^0+z_r^1|\xi|)^2 + (z_i^0+z_i^1|\xi|)^2}\right)$$

is negative. After rewriting this expression the equivalence of $F(\mu,\nu)$ with the integral form 5.26 and Lemma 7 is confirmed.

Note that $F(\mu,\nu) \geq -\sqrt{\pi/2}$ for all $\mu,\nu \in \mathbb{C}$. This follows from formula 5.28 directly.

Conclusions : Integrals of the form 5.26 can be exactly calculated very rarely. Mostly one has to integrate them numerically by appropriate approximation procedures. Only in some special cases, there are nice conclusions, but sufficiently enough to make stability assertions. Nevertheless, we will find a subclass of the balanced implicit methods 3.5 where $F(\mu,\nu)$ is negative. For this pupose we have to estimate the integral term in 5.26. Obviously, in the inequality 5.26 the integrand has both positive and negative values and cannot be simply bounded for all μ and ν. From more practical interest, it turns out to investigate both $\Phi(\mu,\nu,\xi)$ and $F(\mu,\nu)$ under appropriate assumptions further. Therefore we looked at formula 5.27 once again and obtained immediately the special structure

$$\|\Phi(\mu,\nu,\xi)\|^2 = \frac{(1+z_r^1|\xi|+\nu_r\xi)^2 + (z_i^1|\xi|+\nu_i\xi)^2}{(1+|\mu_r|+z_r^1|\xi|)^2 + b^2} \qquad (5.29)$$

$$\text{where}\quad z_r^0 = |\mu_r|,\ z_i^0 = -\mu_i \quad \text{and} \quad b = -\mu_i + z_i^1|\xi| \tag{5.30}$$

for the complex pairs $(\mu,\nu)(=(\lambda\Delta,\gamma\sqrt{\Delta}))$ satisfying the condition 2.3. Under these assumptions the corresponding stability function $R_B^2(\mu,\nu)$ has the form

$$\begin{aligned} R_B^2(\mu,\nu) &= \mathbb{E}\,\|\Phi(\mu,\nu)\|^2 \\ &= \mathbb{E}\,\frac{(1+z_r^1|\xi|)^2+(z_i^1)^2\xi^2+\|\nu\|^2\xi^2}{(1+z_r^1|\xi|)^2+(z_i^1)^2\xi^2+2|\mu_r|+\|\mu\|^2+2(z_r^1|\mu_r|-z_i^1\mu_i)|\xi|}\,. \end{aligned}$$

Assuming

$$z_r^1|\mu_r| - z_i^1\mu_i = (c_{11}^1|\lambda_r| - c_{12}^1\lambda_i)\sqrt{\Delta} \geq 0 \qquad \forall\,\Delta \tag{5.31}$$

then the expression $R_B^2(\mu,\nu)$ can be estimated by $R_d^2(\mu,\nu)$ with

$$R_d^2(\mu,\nu) = \mathbb{E}\,\frac{(1+z_r^1|\xi|)^2+(z_i^1)^2\xi^2+\|\nu\|^2\xi^2}{1+2|\mu_r|+\|\mu\|^2} = \frac{1+\|z^1\|^2+\|\nu\|^2+2z_r^1\sqrt{\frac{2}{\pi}}}{1+2|\mu_r|+\|\mu\|^2}\,.$$

Thereby, if $R_d^2(\mu,\nu)<1$, for example one of the conditions

$$\begin{aligned} &\text{(i)} &:&\ z_r^1=0,\ |z_i^1|\leq\|\mu\| \ \text{ and } \ \text{sign}(z_i^1)=-\text{sign}(\mu_i) \\ &\text{(ii)} &:&\ z_i^1=0 \ \text{ and } \ 0\leq\Big(2\sqrt{\frac{2}{\pi}}+z_r^1\Big)z_r^1\leq\|\mu\|^2 \\ &\text{(iii)} &:&\ z_r^1=z_i^1=0\,, \end{aligned}$$

or equivalently written in terms of the system parameters (λ,γ,Δ)

$$\begin{aligned} &\text{(i)'} &:&\ c_{11}^1=0,\ |c_{12}^1|\leq\|\lambda\|\sqrt{\Delta} \ \text{ and } \ \text{sign}(c_{12}^1)=-\text{sign}(\lambda_i) \\ &\text{(ii)'} &:&\ c_{12}^1=0 \ \text{ and } \ 0\leq\Big(2\sqrt{\frac{2}{\pi}}+c_{11}^1\sqrt{\Delta}\Big)c_{11}^1\leq\|\lambda\|^2\Delta^{3/2} \\ &\text{(iii)'} &:&\ c_{11}^1=c_{12}^1=0\,, \end{aligned}$$

is fulfilled, the balanced methods using the matrices

$$c^0=\begin{pmatrix} |\lambda_r| & \lambda_i \\ -\lambda_i & |\lambda_r| \end{pmatrix} \quad \text{and} \quad c^1=\begin{pmatrix} c_{11}^1 & -c_{12}^1 \\ c_{12}^1 & c_{11}^1 \end{pmatrix}$$

with nonnegative coefficient c_{11}^1 provide mean square stable numerical solutions for the model 2.2. These conditions are direct conclusions from 5.30 and 5.31. Consequently, under the assumption that $c^1\not\equiv 0$ and

$$c^0=\begin{pmatrix} |\lambda_r| & \lambda_i \\ -\lambda_i & |\lambda_r| \end{pmatrix},$$

to obtain a good replication of the mean square stability behaviour of the system 2.2 by the numerical solutions 3.5, we recommend to use the balanced methods where just $c_{12}^1 = -\text{sign}(\lambda_i)\|\lambda\|\sqrt{\Delta}$ and $c_{11}^1 = 0$, hence a special form satisfying (i)'. Then this balanced method is mean square A-stable. Besides, its stability domain is 'relatively close' to the region defined by 2.3 with $p=2$, at least for small enough step sizes. Summarizing these conditions (i) - (iii), an 'optimal recommendation' leads more or less to the choice c^1 as the zero matrix. However, the trivial choice $c^1 \equiv 0$ can be concluded immediately from the stability indicator $F(\mu,\nu)$ given by 5.26. Returning to that formula we obtained

$$F_{z^1=0}(\mu,\nu) \;=\; \sqrt{2\pi}\,\frac{\|\mu\|^2+\|\nu\|^2+2(1+z_r^0)\mu_r+2z_i^0\mu_i}{(1+z_r^0)^2+(z_i^0)^2}\,. \tag{5.32}$$

Then it holds $F_{z^1=0}(\mu,\nu) \;<\; 0$ iff

$$2\mu_r+\|\nu\|^2+\|\mu\|^2+2(z_r^0\mu_r+z_i^0\mu_i) \;<\; 0$$

which is equivalent to

$$2\lambda_r+\|\gamma\|^2+\Big(\|\lambda\|^2+2(c_{11}^0\lambda_r+c_{12}^0\lambda_i)\Big)\,\Delta \;<\; 0 \tag{5.33}$$

With the condition 5.33 we received a very effective recommendation for the balanced methods. Firstly, we achieved mean square stability of the numerical solutions. Secondly, mean square A-stability of them is exactly guaranteed by the requirement $\|\lambda\|^2+2(c_{11}^0\lambda_r+c_{12}^0\lambda_i) \le 0$. Thirdly, an 'optimal choice' is to take the coefficients of c^0 such that

$$\|\lambda\|^2+2(c_{11}^0\lambda_r+c_{12}^0\lambda_i) \;=\; 0\,. \tag{5.34}$$

Just with the recommendation $c^1 \equiv 0$ and 5.34 the balanced methods are mean square stable iff the null solution is mean square stable for the system 2.2. Thereby, one could use

$$c^0=\begin{pmatrix}-\frac{\lambda_r}{2} & +\frac{\lambda_i}{2}\\ -\frac{\lambda_i}{2} & -\frac{\lambda_r}{2}\end{pmatrix} \quad\text{and}\quad c^1 \equiv 0 \tag{5.35}$$

or, if $\lambda_r \neq 0$, the 'pure-diagonal choice'

$$c^0=\begin{pmatrix}-\left(\frac{\lambda_r}{2}+\frac{\lambda_i^2}{2\lambda_r}\right) & 0\\ 0 & -\left(\frac{\lambda_r}{2}+\frac{\lambda_i^2}{2\lambda_r}\right)\end{pmatrix} \quad\text{and}\quad c^1 \equiv 0 \tag{5.36}$$

or, if $\lambda_i \neq 0$, the 'pure-codiagonal choice'

$$c^0=\begin{pmatrix}0 & +\left(\frac{\lambda_i}{2}+\frac{\lambda_r^2}{2\lambda_i}\right)\\ -\left(\frac{\lambda_i}{2}+\frac{\lambda_r^2}{2\lambda_i}\right) & 0\end{pmatrix} \quad\text{and}\quad c^1 \equiv 0\,. \tag{5.37}$$

For these recommendations there is no need to visualize the corresponding mean square stability domain Γ_B^2. It has already been plotted in the figure 1 (multiply the scales with Δ and $\sqrt{\Delta}$, resp.), because the corresponding mean square stability domains of the methods 3.5 with one of the choices 5.35 - 5.37 are identical with Γ^2, the domain for which the null solution is mean square stable. This appearence is guaranteed by the requirement 5.34. By the way, the recommendation 5.35 coincides with that of the Euler scheme using implicitness $\alpha = 0.5$. Moreover, the balanced methods include the implicit Euler family what we have already seen in

Milstein et al.[19], too. Obviously, the class of balanced methods is much more general and permits to choose a variety of appropriate numerical methods in order to treat mean square stabilities and instabilities for the system 2.2 (for an illustrative example see section 6). For mean square stability one is not forced to use the 'stochastic control term' c^1 in them! In contrast to that one has to correct stochastically the numerical solutions by the weight matrices c^j $(j = 1, 2, ..., m)$ in the instable situation, compare Milstein et al. [19].

Relation to p-th mean stability : Instead of a discussion on conclusions for p-th mean stability domains as in the previous subsections we are going to concentrate on the construction of p-th mean stable numerical solutions with p larger than 2. It seems to be a hard task to construct a numerical method which provides p-th mean stable solutions as long as it is valid for the null solution of the system 2.2 for a given $p \in \mathbb{N} \backslash \{0, 1, 2\}$. In the following we will discuss this subject, but will not give a completely satisfactory answere. By the way, during the discussion it also rises the interesting question whether do higher order methods which are p-th mean A-stable for the system 2.2 with $p > 2$ really exist or not? Consider the special class of balanced methods 3.5 using the weight matrices

$$c^0 = \begin{pmatrix} g_1(\lambda, \|\gamma\|, p) & -g_2(\lambda, \|\gamma\|, p) \\ +g_2(\lambda, \|\gamma\|, p) & g_1(\lambda, \|\gamma\|, p) \end{pmatrix} \quad \text{and} \quad c^1 \equiv 0 . \tag{5.38}$$

where g_1 and g_2 are real-valued functions to be chosen and g_1 is nonnegative (to ensure convergence one needs boundedness of these functions), e.g. not to make use of 'stochastic correction terms' in them. For the model 2.2, such methods are also characterized by the mapping Φ stated in formula 5.24. To use that formalism for the proposed methods with 5.38 we identified the complex pair (z^0, z^1) with $z^0 = (g_1\Delta, g_2\Delta)$ and $z^1 = 0$. Then one investigates the function Φ_0^B with

$$\Phi_0^B(\mu, \nu, \xi) = (1 + z^0)^{-1}(1 + z^0 + \mu + \nu\xi) \tag{5.39}$$

for any $(\mu, \nu) \in \Gamma^p$ and standard Gaussian distributed random variable ξ. Thus, as a climax of this article, it is possible to state numerical methods which provide p-th mean stable numerical solutions. Unfortunately, it turns out that they are not p-th mean A-stable for the general system 2.2 due to a remarkable loss of accuracy. Only under some restrictions on the model class 2.2 we achieve p-th mean stability of them for any step size Δ. Because these investigations seem to be so difficult, even for this simple class 2.2, we suspect it does not exist any implicit Euler scheme 3.3 which is p-th mean A-stable for this model class with p larger than 2. With the above preparations in mind, we will make use of the complex function R_B^p mapping on $\Gamma^p \subset \mathbb{C} \times \mathbb{C}$ to $\mathbb{R}^+$ and defined by

$$R_B^p(\mu, \nu) = \mathbb{E}\,\|\Phi_0^B(\mu, \nu)\|^p = |1 + z^0|^{-p}\,\mathbb{E}\,|1 + z^0 + \mu + \nu\xi|^p \tag{5.40}$$

as the p-th mean stability function for the balanced implicit methods using a recommendation of type 5.38.

Lemma 8 :
Suppose it is given a system of the form 2.2 for which its null solution is p-th mean stable for a fixed $p \in \mathbb{N} \backslash \{0,1\}$, e.g. condition 2.3 is valid for this p, and the nonrandom complex number $z^0 = \mathcal{O}(\Delta)$ has nonnegative real part.
Then the balanced implicit method 3.5 with matrices

$$c^0 = \begin{pmatrix} z_r^0 & -z_i^0 \\ +z_i^0 & z_r^0 \end{pmatrix} \cdot \Delta^{-1} \quad \text{and} \quad c^1 \equiv 0 \tag{5.41}$$

is p-th mean stable too if it holds

$$F_0^B(\mu,\nu,p) := \|\mu\|^2 + 2(z_r^0\mu_r + z_i^0\mu_i) + 2\mu_r + (p-1)\|\nu\|^2 < 0 \tag{5.42}$$

whereas $(\mu,\nu) = (\lambda\Delta, \gamma\sqrt{\Delta}) \in \Gamma^p$, the 'p-th mean stability domain 5.23 of the null solution'.

Such expressions $F_0^B(\mu,\nu,p)$ we call '*dominating p-th mean stability indicators*' for the proposed balanced method. By the help of this lemma we will be able to state some 'appropriately' balanced methods. At least, under the conditions $\nu_i = 0$ or $2|\mu_r| \geq \|\nu\|^2(p-1)$, the recommendations should behave reasonably and accurately enough (a kind of 'conditionally p-th mean A-stability').

Proof : Suppose $(\mu,\nu) = (\lambda\Delta, \gamma\sqrt{\Delta}) \in \Gamma^p$ for a given $p \geq 2$. Because of the symmetry property of standard Gaussian distributed random variables ξ one comes up the identity

$$\mathbb{E}\,\|\Phi_0^B(\mu,\nu,\xi)\|^p = \frac{1}{2}\left(\mathbb{E}\,\|\Phi_0^B(\mu,\nu,\xi)\|^p + \mathbb{E}\,\|\Phi_0^B(\mu,\nu,-\xi)\|^p\right).$$

Furthermore, we assumed that $1 + z^0 + \mu \neq 0$. This is justified as it makes no sense to take $z_r^0 = -1 - \mu_r \neq \mathcal{O}(\Delta)$, and it would contradict to the requirements for the convergence of any balanced method. Using these facts the p-th mean stability function R_B^p of the numerical methods proposed by 5.38 and given in 5.40 is rearranged and estimated in such a way that

$$\begin{aligned}
R_B^p(\mu,\nu) &= \left|\frac{1+z^0+\mu}{1+z^0}\right|^p \mathbb{E}\left|1 + \frac{\nu}{1+z^0+\mu}\xi\right|^p \\
&= \left|\frac{1+z^0+\mu}{1+z^0}\right|^p \cdot \frac{1}{2}\left(\mathbb{E}\left|1+\frac{\nu}{1+z^0+\mu}\xi\right|^p + \mathbb{E}\left|1-\frac{\nu}{1+z^0+\mu}\xi\right|^p\right) \\
&= \left|\frac{1+z^0+\mu}{1+z^0}\right|^p \cdot \frac{1}{2}\left(\mathbb{E}\left|1+\frac{1}{\sqrt{p-1}}\frac{\nu\sqrt{p-1}}{1+z^0+\mu}\xi\right|^p + \mathbb{E}\left|1-\frac{1}{\sqrt{p-1}}\frac{\nu\sqrt{p-1}}{1+z^0+\mu}\xi\right|^p\right) \\
&\leq \left|\frac{1+z^0+\mu}{1+z^0}\right|^p \cdot \left(\frac{1}{2}\mathbb{E}\left|1+\frac{\nu\sqrt{p-1}}{1+z^0+\mu}\xi\right|^2 + \frac{1}{2}\mathbb{E}\left|1-\frac{\nu\sqrt{p-1}}{1+z^0+\mu}\xi\right|^2\right)^{p/2} && (5.43) \\
&\leq \left|\frac{1+z^0+\mu}{1+z^0}\right|^p \cdot \left(\mathbb{E}\left|1+\frac{\nu\sqrt{p-1}}{1+z^0+\mu}\xi\right|^2\right)^{p/2}. && (5.44)
\end{aligned}$$

Note, for the estimate in 5.43, we used an inequality stated by the corollary 1.*e*.15 in Lindenstrauss & Tzafriri [15] (1979, at page 76) which is, in general, valid on

Banach spaces and due to Beckner [3]. Naturally, the occured inequalities above are equalities for $p = 2$. Because of the identity

$$\mathbb{E}\left|1+\frac{\nu\sqrt{p-1}}{1+z^0+\mu}\xi\right|^2 = 1 + (p-1)\frac{\|\nu\|^2}{\|1+z^0+\mu\|^2} \tag{5.45}$$

the relation 5.44 simplifies to

$$R_B^p(\mu,\nu) = \mathbb{E}\,\|\Phi_0^B(\mu,\nu,\xi)\|^p \le (f_B(\mu,\nu,z^0,p))^{p/2} \tag{5.46}$$

$$\text{where} \qquad f_B(\mu,\nu,z^0,p) = \frac{\|1+z^0+\mu\|^2+(p-1)\|\nu\|^2}{\|1+z^0\|^2}. \tag{5.47}$$

Consequently, it holds $\mathbb{E}\,\|\Phi_0^B(\mu,\nu,\xi)\|^p < 1$ if the condition $f_B(\mu,\nu,z^0,p) < 1$ is satisfied. The rearrangement of the latter relation leads to the equivalence

$$f(\mu,\nu,z^0,p) < 1 \qquad \Longleftrightarrow$$

$$F_0^B(\mu,\nu,p) = 2 < 1+z^0,\mu> +\|\mu\|^2+(p-1)\|\nu\|^2 < 0,$$

which coincides with the expression 5.42, hence the assertion of the Lemma 8 has been proved.

Some recommendations for the balanced methods : Of course, in the case $p = 2$ the obtained relation 5.42 is equivalent to the condition 5.33 derived itself as an application of the Lemma 7. For $p > 2$, concluding from the received sharp estimation in the proof of the Lemma 8, one may not expect to have simultanously the p-th mean A-stability and the theorectically predicted convergence order of the proposed balanced methods using the weights of the form 5.38 for the considered test model, although, it still has to be proved a corresponding 'if and only if' relation for the p-th mean stability of them. By the use of the transformation

$$(\hat{z}_r^0,\hat{z}_i^0) = (z_r^0+\frac{1}{2}\mu_r, z_i^0+\frac{1}{2}\mu_i) = z^0+\frac{1}{2}\mu \tag{5.48}$$

one obtains the equivalent expression

$$F_0^B(\mu,\nu,p) = 2 < \hat{z}^0,\mu> +2\mu_r+(p-1)\|\nu\|^2 \tag{5.49}$$

for the dominating stability indicator F_0^B given in 5.42. Thereby, on Γ^p it holds $F_0^B(\mu,\nu,p) < 0$ iff the requirement

$$2 < \hat{z}^0,\mu> +(p-2)\nu_i^2 \le 0 \tag{5.50}$$

is fulfilled. Thus, in general, evaluating this fact it is not possible to construct p-th mean A-stable balanced methods within the class 5.38 for p larger than 2. However, for large step sizes Δ, one can formulate an 'appropriate' recommendation for them in order to achieve p-th mean stability for a given $p > 2$. For example, take the transformation

$$(\hat{z}_r^0,\hat{z}_i^0) = -(\frac{1}{2}K(p-2)\mu_r, \frac{1}{2}K(p-2)\mu_i) = -\frac{1}{2}K(p-2)\mu \tag{5.51}$$

for a real number $K > 0$. Then the relation 5.50 comes to $\nu_i^2 \leq K\|\mu\|^2$, hence on the domain Γ^p we obtained a p-th mean stable numerical solution under this restriction, although it is useless to achieve stability for very small step sizes. In terms of the system tripel $(\lambda, \gamma, \Delta)$, summarizing the transformations 5.50 and 5.51 and the last conclusions, that is we suggested to take the balanced methods using the matrices

$$c^0 = \frac{1}{2}(1 + K(p-2))\begin{pmatrix} -\lambda_r & +\lambda_i \\ -\lambda_i & -\lambda_r \end{pmatrix} \quad \text{and} \quad c^1 \equiv 0 \tag{5.52}$$

where K is any nonnegative real constant. Furthermore, returning to the formula 5.49, it follows immediately the following statement. If the condition

$$2\mu_r + \|\nu\|^2(p-1) < 0 \tag{5.53}$$

is true then the balanced method using the matrices

$$c^0 = \frac{1}{2}(1 + K)\begin{pmatrix} -\lambda_r & +\lambda_i \\ -\lambda_i & -\lambda_r \end{pmatrix} \quad \text{and} \quad c^1 \equiv 0 \tag{5.54}$$

where K is a given real number with $K \geq 0$ is p-th mean stable for all step sizes Δ and exponents $p \geq 2$. In the latter recommendation one should prefer the choice $K = 0$ under the restriction 5.53, which coincides with that of the 'deterministically half-implicit' Euler scheme. So finally we received reasonable recommendations for balanced methods under the additional restriction $\|\nu\|^2(p-1) \leq 2|\mu_r|$, e.g. the parameter λ must be able to compensate uniformly the squared norm of the noise intensity $\sqrt{(p-1)/2}\gamma$ by its real part, roughly speaking, in contrast to the situation for the exact solution (compare with the condition 2.3). To summarize, the above discussion has once again shown the difference between the requirements of p-th mean stability ($p > 2$) and the convergence order (convergence at all) of the balanced methods, and hence for all up to now known numerical methods of lowest accuracy order. It still must be clarified whether there always is a p-th mean A-stable numerical method ($p > 2$) for systems of the form 2.2 or not. Similar discussions one could do for other interesting subclasses of the balanced methods, in concern with p-th mean stability, for example, for the additional use of stochastic corrections controlled by the choice of the matrix c^1 in them. Further examinations we leave out in this respect here.

6 Experiments for the Kubo Oscillator

The following example has been taken from Honerkamp ([11],1990). Driven by the one-dimensional Wiener noise $W(t)$, the system is given by a complex Stratonovich stochastic differential equation of the form ($\imath$ denotes the imaginary unit)

$$dX(t) \;=\; \imath X(t)\,dt \,+\, \imath\rho X(t) \circ dW(t) \tag{6.1}$$

for the variable $X(t)$ with $\|X(t)\| = 1$ and parameter $\rho \in I\!R^1$ on the time interval $[0,T]$. This system describes the movement of the Kubo-Oscillator on the unit circle. The equation 6.1 is explicitly solvable and has the solution

$$X(t) \;=\; \exp\{\imath\rho Z(t) + \imath t\} \;=\; \cos(\rho Z(t) + t) + \imath \sin(\rho Z(t) + t)$$

$$\text{where} \qquad Z(t) = \int_0^t W(s)\,ds \quad .$$

Another interesting fact occurs, with the Kubo oscillator we studied a system where all p-th mean Lyapunov exponents $\lambda(x_0; \mathrm{p}) = 0$ $(x_0 \neq 0)$. Because of

$$I\!E X(t) = \exp\{-\frac{1}{2}\rho^2 t + \imath t\}$$

one knows that the first moment is converging to zero as $t \to \infty$, e.g. in the weak mean sense the null solution is stable for this system. The Itô-version of 6.1 has the form

$$dX(t) \;=\; (\imath - \frac{1}{2}\rho^2)X(t)\,dt \,+\, \imath\rho X(t)\,dW(t)$$

or in componentwise description

$$\begin{aligned} dX^1(t) \;&=\; \left(-\frac{1}{2}\rho^2 X^1(t) - X^2(t)\right)\,dt \,-\, \rho X^2(t)\,dW(t) \\ dX^2(t) \;&=\; \left(X^1(t) - \frac{1}{2}\rho^2 X^2(t)\right)\,dt \,+\, \rho X^1(t)\,dW(t) \quad . \end{aligned} \tag{6.2}$$

That is in our system notation of 2.2 we have the drift and diffusion matrices

$$A \;=\; \begin{pmatrix} -\frac{1}{2}\rho^2 & -1 \\ +1 & -\frac{1}{2}\rho^2 \end{pmatrix} \quad \text{and} \quad B \;=\; \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix} \tag{6.3}$$

Obviously the condition of weak mean stability, here $\lambda_r = -\frac{1}{2}\rho^2 < 0$ $(\rho \neq 0)$, is always fulfilled. To check the condition 2.3 for strong and mean square stability one obtains

$$\lambda_r + \frac{1}{2}\gamma_i^2 + \frac{1}{2}\gamma_r^2(p-1) = -\frac{1}{2}\rho^2 + \frac{1}{2}\rho^2 = 0\,, \tag{6.4}$$

hence the null solution is not p-th mean stable for any p, although it holds

$$I\!E(X(t))^2 \;=\; \exp\{-2\rho^2 t + 2\imath t\}\,,$$

it can be easily concluded that $I\!E\|X(t)\| = I\!E\|X(t)\|^2 = 1$. For experiments we chose the following three schemes taken from the families 3.3 - 3.5 for the model

system 6.2 with $\rho \neq 0$:
(i) The Euler method with implicitness $\alpha = \frac{1}{2}$

$$\begin{aligned} Y_{n+1}^1 &= Y_n^1 - \frac{1}{2}\left(\frac{1}{2}\rho^2(Y_{n+1}^1 + Y_n^1) + Y_{n+1}^2 + Y_n^2\right)\Delta - \rho Y_n^2 \Delta W_n \\ Y_{n+1}^2 &= Y_n^2 + \frac{1}{2}\left(Y_{n+1}^1 + Y_n^1 - \frac{1}{2}\rho^2(Y_{n+1}^2 + Y_n^2)\right)\Delta + \rho Y_n^1 \Delta W_n \end{aligned}$$

(ii) The Milstein method with implicitness $\alpha = \frac{1}{2}$

$$\begin{aligned} Y_{n+1}^1 &= Y_n^1 - \frac{1}{2}\left(\frac{1}{2}\rho^2(Y_{n+1}^1 + Y_n^1) + Y_{n+1}^2 + Y_n^2\right)\Delta - \rho Y_n^2 \Delta W_n \\ &\quad -\frac{1}{2}\rho^2 Y_n^1((\Delta W_n)^2 - \Delta) \\ Y_{n+1}^2 &= Y_n^2 + \frac{1}{2}\left(Y_{n+1}^1 + Y_n^1 - \frac{1}{2}\rho^2(Y_{n+1}^2 + Y_n^2)\right)\Delta + \rho Y_n^1 \Delta W_n \\ &\quad -\frac{1}{2}\rho^2 Y_n^2((\Delta W_n)^2 - \Delta) \end{aligned}$$

(iii) The balanced method with matrices

$$c^0 = \begin{pmatrix} \frac{1}{4}\rho^2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4}\rho^2 \end{pmatrix} \quad \text{or} \quad c^0 = \begin{pmatrix} \frac{1}{4}\rho^2 + \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{4}\rho^2 + \frac{1}{\rho^2} \end{pmatrix} \quad \text{and} \quad c^1 \equiv 0$$

which possesses the scheme structure in matrix notation

$$Y_{n+1} = Y_n + (I + c^0\Delta + c^1|\Delta W_n|)^{-1}(A\,Y_n\,\Delta + B\,Y_n\,\Delta W_n)\,.$$

Thereby, the method (iii) using the first choice of c^0 is identical with the method (i) for this model. Thus, in the following simulations we will draw more attention to the second form of the balanced method stated in (iii), just the balanced method with the 'pure-diagonal correction' c^0. Although the three methods are weak mean A-stable (can be concluded from Lemma 3, formula 4.7 and Lemma 4), the methods (i) and (iii) do not provide both mean square stable and instable numerical solutions (check conditions 5.4 in Lemma 5 and 5.33 as a conclusion of Lemma 7), which replicate the behaviour of the null solution for the system 6.2 in this respect. In contrast to them the method (ii) produces mean square instable numerical solutions for all step sizes (except for one step size!) since 5.13 leads to

$$F(\mu, \nu, \frac{1}{2}) = \frac{1}{2}\rho^4\Delta^2 \quad > \quad 0$$

for the corresponding mean square stability indicator $F(\mu, \nu, \alpha)$. Even for the 'fully drift-implicit' Milstein scheme for the system 2.2 it is not getting better. Then we obtained for the corresponding stability indicator

$$\begin{aligned} F(\mu, \nu, 1) &= -(\frac{1}{4}\rho^4 + 1)\Delta^2 + \frac{1}{2}\rho^4\Delta^2 \\ &= \left(\frac{1}{4}\rho^4 - 1\right)\Delta^2 \quad > \quad 0 \end{aligned}$$

if the value $|\rho| > \sqrt[4]{4}$ is sufficiently large, for example $\rho = 10$. On the other hand, for smaller values $|\rho|$, for example $\rho = 1$, their stability indicator $F(\mu, \nu, 1)$ becomes negative, e.g. their stability indicator changes its sign, although this is not true for the exact solution where the (top) p-th mean Lyapunov exponent can be considered as its stability indicator ($\lambda(x_0; p) = \lambda(x_0; 2) = 0$ if $x_0 \neq 0$). Consequently, the implicit Milstein schemes 3.4 with fixed implicitness α do not replicate the mean square stability behaviour of the null solution for the system 6.2. To compare the methods (i) - (iii) experimentally we plotted estimates for the second moments $\mathbb{E}\|Y_n\|^2$ at the time points τ_n on the time interval $[0,1]$ and interpolated linearly the data to be visualized. The corresponding results are visible in figure 7. There the dotted line corresponds to the exact level to be expected trivially at the height 1.0. Distinctly, the methods (i) and (ii) provide better approximations concerning the mean square stability. They are able to control the second moment much longer than the method (iii). Moreover, the second moment of the implicit Milstein approximation even seems to 'explode'. Of course, the difference depends on the amount of the parameter ρ, but is still observable for a quite large range of these parameters. Note that these experiments bear more experimental character, demonstrating some numerical effects (explicit solution is known here, but it must be approximated the term $Z(t)$).

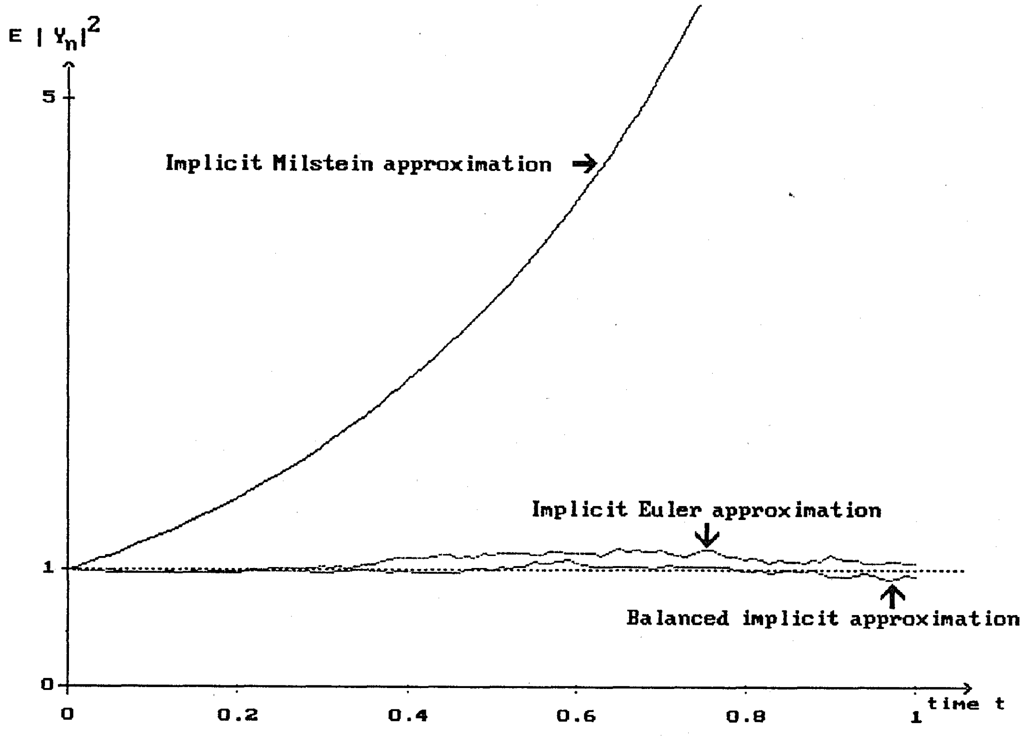

Figure 7 : The estimates of $\mathbb{E}\|Y_n\|^2$ of methods (i)-(iii) using the step size $\Delta = 10^{-2}$ with $\rho = 4$ started in $(1,0)$

Additionally, for the special model 6.1 one could be tricky. Concerning the information $\|X(t)\| = 1$ for all $t \in [0,T]$, by the use of $\hat{Y}_n := Y_n/\|Y_n\|$ as the approximation value for the Kubo oscillator at the time τ_n the condition $\|\hat{Y}(\tau_n)\| = 1$

would be trivially fulfilled, too. But it assumes that $\|X(t)\|$ is always a constant. Consequently, only in movements on the circle one can apply this trick. In practise, such a 'normalization' does not help in the general situation. Moreover, it would ruin the 'goodness' of the approximation.

Other applications for model 2.2 : In Markov chain filtering we found some models which possess the linearization of type 2.2. A plenty of further very interesting applications can be found in quantum mechanics. Among them, for example, Smith & Gardiner ([26],1989) investigated a model of a harmonic oscillator damped by both one photon and two photon absorption. They arrived at Itô SDE's where their linearization fits in our stability examinations, for example, the nonlinear Itô SDE for the intensity of the cavity mode $N(t)$

$$dN(t) \;=\; -(\frac{1}{2}\beta + N(t))\,N(t)\,dt \;+\; \imath\,N(t)\,dW(t)\,.$$

Here, the notion of weak mean stability has proved to be too 'weak' in order to treat numerically, accurately enough and hence successfully the problem of estimating the mean value function of the cavity mode $N(t)$. Although it is a statistical problem, we suggest to introduce and examine this model with the so-called 'explosion-stable' or 'spikes-stable' numerical solutions, with the hope of qualitative improvements in the estimates. A corresponding report is being worked out by Gerlach & Schurz. Recently Gardiner, Gilchrist & Drummond ([5],1993) offered several models possessing also the considered linerization form 2.2. Perhaps, one could explain the numerical problem of having 'spikes' in the plots for the photon numbers during the use of the positive P-representation. The field of laser equations seems to form a further area to deal with stability (see Schack & Schenzle, 1991), although it requires new approaches. Thus, it made use to study the numerical stability behaviour for that special model 2.2. However, we want to encourage the reader to go deeper in the stability problem of stochastic numerics, explore its application and help to answere on how reliable the approximations are in compare with the exact solution.

7 Summary

Of course, we know we are still quite far away from the final aim to provide numerical solutions satisfying the stability principle 'Small initial perturbations should cause small terminal errors' in stochastic numerics. However, some introductional work for multi-dimensional stochastic systems and the investigation of the mean stability of their numerical solution has been done in this paper. Systematically it could be introduced the notion and examined the world of stability depending on the moments. At least, the results are useful to provide a 'reasonable' approximation for the stochastic Kubo oscillator possessing both a kind of stability and instability. Further applications can be found in quantum mechanics. Naturally it is still the task to extend the examination to more general modells than 2.2.

At first one should stay in the class of SDE's with linearization of type 2.1 with more general matrices A and B^j, then even for nonautonomous systems. Furthermore, the studies for systems with multi-dimensional noise could be done straight forward, analogously to this paper, but consequently with much more parameters involved in them. Surely, the class of commutative matrices A and B^j, would be very interesting, but restricts the problem to perhaps a too small class. Within this field it is worth to get more insight of the problem of test equations. Therein the idea of simultanous transformations to systems which have an equivalent stability behaviour like the original system plays a special role. Concerning this work we point to a later paper on 'Test equations and simultanous transformations' which we are going to be published in this preprint series and will require more algebraic background. The herein introduced notions have proved to be effective, but maybe not efficiently enough to destinguish numerical methods with respect to their stability behaviour. The three considered scheme classes 3.3 - 3.5 possess subclasses providing both weak and mean square stable numerical solutions. The balanced methods seem to be the richest class at the moment in order to treat stabilities and instabilities in stochastic systems numerically and successfully. Under the assumption of having mean square stable null solution it turns out that to achieve also mean square stability of the numerical solution by the balanced methods it is not necessary to correct with stochastic weights controlled by the choice of the matrices c^j $(j = 1,2,..,m)$. Therefore the simple choice , $c^j \equiv 0$ in them can be prefered. By some further restrictions ($\|\gamma\|^2 \leq 2|\lambda_r|/(p-1)$ or $\gamma_i = 0$, see section 5.3) to the considered model class one also obtains reasonable recommendations for them such that their numerical solutions are even p-th mean A-stable. Furthermore, much more work must be done to treat more accurately and reliably instable systems, for example in such systems where the first moment behaves stable, but the second moment not.
The asymptotical mean stability is one notion to classify schemes with respect to qualitative features, another way would be to consider the Lyapunov exponents of the corresponding discrete system. Obviously, this is much more complicated and requires further extensive studies. Actually, in this respect, what we also did by this paper was the trial to use the sign of the top Lyapunov exponent of the discrete system governed by a given numerical method. Its sign has been made visible and exploited by the considered stability indicators $F(\mu, \nu)$. However, the knowledge about stability and instability of numerical solutions is still in its infancy.

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References

[1] Arnold, L. : Stochastic differential equations, Wiley, 1974

[2] Auslender, E.I. and Milstein, G.N. : Asymptotic expansion of the Lyapunov index for linear stochastic systems with small noise (in Russian), Prikl. Matem. Mekhan. **46**, 358-365, 1982

[3] Beckner, W. : Inequalities in Fourier analysis, Ann. of Math. **102**, 159-182, 1975

[4] Drummond, P.D. and Mortimer, I.K. : Computer simulation of multiplicative stochastic differential equations, J. Comput. Phys., 1991

[5] Gardiner, C.W., Gilchrist, A. and Drummond, P.D. : Using the positive P-representation, unpublished, 1993

[6] Greiner, A., Strittmatter, W. and Honerkamp, J. : Numerical integration of stochastic differential equations, J. Stat. Physics **51**, 95-108, 1987

[7] Hairer, E. and Wanner, G. : Solving differential equations II : Stiff Problems, Springer Series in Comp. Math. **8**, Springer, 1991

[8] Haszminskij, R.Z. : Stability of systems of differential equations with stochastic perturbations of their parameters (in Russian), 1969

[9] Hernandez, D.B. and Spigler, R. : Numerical stability of implicit Runge-Kutta methods for stochastic differential equations, Preprint, University of Padua, 1990

[10] Hofmann, N. : Stability of numerical schemes for stochastic differential equations with multiplicative noise, talk at conf. Probabilites Numeriques, Paris (March 1993), published as Preprint no. **39**, IAAS-Berlin, 1993

[11] Honerkamp, J. : Stochastic dynamical systems (in German), VCH New York, 1990

[12] Klauder, J.R. and Petersen, W.P. : Numerical integration of multiplicative-noise stochastic differential equations, SIAM J. Numer. Anal. **22**, 1153-1166, 1985

[13] Kloeden, P.E. and Platen, E. : Numerical solution of stochastic differential equations, Appl. of Mathem. **23**, Springer, 1992

[14] Kloeden, P.E. and Platen, E. : Higher-order implicit strong numerical schemes for stochastic differential equations, J. Statist. Phys. **66**, 283-314, 1992

[15] Lindenstrauss, J. and Tzafriri, L. : Classical Banach spaces II - function spaces, Modern Surveys in Mathem. **97**, Springer, 1979

[16] McNeil, K.J. and Craig, I.J.D. : An unconditional stable numerical scheme for nonlinear quantum optical systems – quantised limit cycle behaviour in second harmonic generation, Univ. Waikato, Research Report no. **166**, 1988

[17] Milstein, G.N. : The numerical integration of stochastic differential equations (in Russian), Uralski Univ. Press, Sverdlovsk (Ekaterinburg), 225 pp., 1988

[18] Milstein, G.N. : A theorem on the order of convergence of mean square approximations of solutions of systems of stochastic differential equations, Theor. Prob. Appl. **32**, 738-741, 1988

[19] Milstein, G.N., Platen, E. and Schurz, H. : Balanced implicit methods for stiff stochastic systems : An introduction and numerical experiments, Preprint no. **33**, IAAS-Berlin, 1992

[20] Mitsui, T. and Saito, Y. : Stability analysis of numerical schemes for stochastic differential equations, Report no. **9202**, Univ. Nagoya, 1992

[21] Pardoux, E. and Talay, D. : Discretization and simulation of stochastic differential equations, Acta Appl. Math., 23-47, 1985

[22] Pardoux, E. and Talay, D. : Stability of nonlinear differential systems with parametric excitation, In Nonlinear Stoch. Dynamic Engin. Systems, Proc. IUTAM Sympos. Innsbruck 1987, Springer, 1988

[23] Petersen, W.P. : Stability and accuracy of simulations for stochastic differential equations, IPS Research Report no. **90-02**, ETH Zürich, 1990

[24] Sasagawa, T. : On the exponential stability and instability of linear stochastic systems, Int. J. Control, **33**, 2, 363-370, 1981

[25] Sasagawa, T. and Willems, J.L. : P-th mean stability of linear systems with multiplicative noise, Report no. **247**, Univ. Bremen, 1991

[26] Smith, A.M. and Gardiner, C.W. : Simulation of nonlinear quantum damping using the positive P representation, Phys. Rev., **39**, 3511-3524, 1989

[27] Talay, D. : Analyse Numérique des Equations Différentielles Stochastiques, Thèse 3ème cycle, Univ. Provence, 1982

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