Very singular solutions to a nonlinear parabolic equation with absorption. I – Existence

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Abstract. We prove the existence of a very singular solution to

$$u_t - \Delta u + |\nabla u|^p = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^N,$$

when $1 < p < (N+2)/(N+1)$.

1. Introduction

We investigate the existence of a *very singular solution* at the origin to the following viscous Hamilton-Jacobi equation

$$u_t - \Delta u + |\nabla u|^p = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^N. \tag{1.1}$$

A *very singular solution* at the origin to (1.1) is a non-negative solution to (1.1) which is smooth in $(0, +\infty) \times \mathbb{R}^N$ and fulfills the following two conditions

$$\lim_{t \to 0+} \int_{\{|x| \le r\}} u(t,x)\ dx = +\infty,$$

$$\lim_{t \to 0+} \int_{\{|x| \ge r\}} u(t,x)\ dx = 0,$$

for every $r \in (0, +\infty)$. The name *very singular solution* has been introduced by Brezis, Peletier and Terman [9] who proved the existence and uniqueness of a self-similar very singular solution W to

$$u_t - \Delta u + u^p = 0 \quad \text{in} \quad (0, +\infty) \times \mathbb{R}^N, \tag{1.2}$$

when $1 < p < 1 + 2/N$. Such a name is justified by the fact that the singularity of W in $(t,x) = (0,0)$ is stronger than the singularity in $(t,x) = (0,0)$ of the fundamental solutions to (1.2), that is the solutions to (1.2) whose initial data is $c\delta$, where $c \in (0, +\infty)$ and δ denotes the Dirac mass centered at $x = 0$. Indeed, when $1 < p < 1 + 2/N$ and $c \in (0, +\infty)$, (1.2) has a unique non-negative solution W_c such that $W_c(0) = c\delta$ [8] and W_c satisfies

$$\lim_{t \to 0+} \int_{\{|x| \le r\}} W_c(t,x)\ dx = c < +\infty,$$

while the very singular solution W satisfies

$$\lim_{t \to 0+} \int_{\{|x| \le r\}} W(t,x)\ dx = +\infty.$$

In fact, if $1 < p < 1 + 2/N$, Oswald has proved in [20] that the following alternative holds : consider a non-negative solution u to (1.2) which is smooth in $\left([0, +\infty) \times \mathbb{R}^N\right) \setminus \{(0,0)\}$ and singular in $(t,x) = (0,0)$ with $u(0,x) = 0$ if $x \ne 0$. Then either $u \equiv W$ or there exists $c \in (0, +\infty)$ such that $u \equiv W_c$. A complete classification of the possible isolated singularities in $(t,x) = (0,0)$ of solutions to (1.2) is thus available.

Since the pioneering work of Brezis, Peletier and Terman [9], the existence, uniqueness and non-existence of non-negative very singular solutions have been extensively investigated for nonlinear parabolic equations with absorption of the form

$$u_t - \mathcal{A}u + u^p = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N,$$

where $\mathcal{A}u = \Delta u$ [9, 13], $\mathcal{A}u = \Delta u^m$, $m > 1$ [22, 16, 19], $\mathcal{A}u = \Delta u^m$, $(1-2/N)^+ < m < 1$ [21, 18], $\mathcal{A}u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $m > 2$ [23, 11, 15] and $\mathcal{A}u = \operatorname{div}(|\nabla u|^{m-2}\nabla u)$, $2N/(N+1) < m < 2$ [12]. Besides the description of the isolated singularities in $(t,x) = (0,0)$ the very singular solutions (when they exist) also play an important role in the description of the large time behaviour of the solutions to (1.2) (see, e.g., the survey paper [25]).

To our knowledge the existence of very singular solutions has not been considered for parabolic equations with absorption when the absorption term is a non-negative function of ∇u instead of being a non-negative function of u, as it is the case for (1.1). Before stating our main result let us make more precise the definition of a very singular solution to (1.1) we will use in this paper.

Definition 1.1. A very singular solution to (1.1) is a function

$$u \in \mathcal{C}((0,+\infty); L^1(\mathbb{R}^N))$$

such that

$$u(t) \geq 0 \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{and} \quad u \in L^p((s,t); W^{1,p}(\mathbb{R}^N))$$

for each $t \in (0,+\infty)$ and $s \in (0,t)$ which satisfies

$$u(t) = G(t-s)u(s) - \int_s^t G(t-\sigma)\ \left(|\nabla u(\sigma)|^p\right)\ d\sigma, \tag{1.3}$$

$$\lim_{t\to 0+} \int_{\{|x|\leq r\}} u(t,x)\ dx = +\infty, \quad r \in (0,+\infty), \tag{1.4}$$

$$\lim_{t\to 0+} \int_{\{|x|\geq r\}} u(t,x)\ dx = 0, \quad r \in (0,+\infty). \tag{1.5}$$

Here, $G(t)$ denotes the linear heat semigroup in $\mathbb{R}^N$.

Our result then reads as follows.

Theorem 1.2. *Assume that* $1 < p < (N+2)/(N+1)$ *and put* $a = (2-p)/(p-1)$. *There is at least one very singular solution* U *to (1.1). More precisely, there is a non-negative and non-increasing function*

$$f \in L^1((0,+\infty); r^{N-1}dr) \cap \mathcal{C}^\infty((0,+\infty))$$

such that

$$U(t,x) = t^{-a/2}\ f\left(|x|t^{-1/2}\right), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N, \tag{1.6}$$

and f is a solution to the ordinary differential equation

$$f''(r) + \left(\frac{N-1}{r} + \frac{r}{2}\right)\ f'(r) + \frac{a}{2}\ f(r) - |f'(r)|^p = 0, \qquad r \in (0,+\infty), \tag{1.7}$$

with the boundary conditions

$$f'(0) = 0 \quad \textit{and} \quad \lim_{r\to+\infty} r^a\ f(r) = 0. \tag{1.8}$$

Remark 1.3. Notice that the very singular solution to (1.1) we construct is self-similar by (1.6).

Remark 1.4. Let us mention at this point the related work [10] where solutions to (1.1) with homogeneous Dirichlet boundary conditions on an open bounded subset Ω of $\mathbb{R}^N$ are constructed with initial data taking the value $+\infty$ on a closed subset of Ω with non-empty interior when $p \in (1,2)$. Theorem 1.2 shows that it is also possible to construct solutions to (1.1) with initial data taking the value $+\infty$ at only one point. Indeed the very singular solution to (1.1) we constructed in the above theorem formally satisfies $U(0,x) = 0$ if $x \neq 0$ and $U(0,0) = +\infty$.

There are basically two possible approaches to study the existence (and uniqueness) of a very singular solution to (1.1) and both of them have actually been employed for (1.2). The first approach relies on the fact that (1.2) is invariant by a rescaling in both space and time. Such a property then ensures that, if there is a unique very singular solution V to (1.2), it has to have a self-similar form and to be radially symmetric as well. Therefore V shall be of the form $V(t,x) = t^{-\alpha}\ v\left(|x|t^{-\beta}\right)$, where α and β are positive real numbers depending only on N, m and p. Inserting this specific form of V into the equation (1.2) yields an ordinary differential equation for the profile v which is similar to (1.7) with boundary conditions similar to (1.8). Shooting methods are then used to prove the existence of the profile v [9, 22, 23, 18, 19] and the uniqueness of the profile may be studied by ordinary differential equations methods [9, 11]. Another possible approach is to construct a very singular solution to (1.2) as the limit of the fundamental solutions to (1.2) (i.e. the solutions to (1.2) with initial data $c\delta$) as the initial mass c increases to infinity (when this limit exists) [13, 16, 21, 15, 12]. We will use this second approach to prove Theorem 1.2. The main step in this method is to obtain an L^∞-estimate for the fundamental solutions which does not depend on the initial mass. For (1.2) such an estimate follows from the existence of a super-solution to (1.2) which depends only on time $t \mapsto ((p-1)t)^{-1/(p-1)}$.

Such a super-solution is not available for (1.1) and we have to proceed in a different way. Namely we derive an L^∞-estimate for the fundamental solutions which do not depend on the initial mass with the help of an L^∞-estimate of $\nabla u^{(p-1)/p}$ obtained in [3] and a stationary super-solution to (1.1). This is done in Section 2. Section 3 is devoted to the proof of Theorem 1.2. In the last section of the paper we prove that there is no non-negative very singular solution to (1.1) when $p=1$ (though there are fundamental solutions in that case [6]). We finally mention that some of the results presented above have been announced in [4]. Furthermore in a paper which is yet to be completed we study the uniqueness of very singular solutions to (1.1) [5].

2. Preliminaries

We first recall the well-posedness of (1.1) in the space of non-negative and bounded measures $\mathcal{M}_b^+(\mathbb{R}^N)$ [3, Theorem 1 & 3].

Theorem 2.1. *Consider $p \in (1,(N+2)/(N+1))$ and $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. There is a unique non-negative function*

$$u \in \mathcal{C}((0,T);L^1(\mathbb{R}^N)) \cap L^p((0,T);W^{1,p}(\mathbb{R}^N)), \quad T \in (0,+\infty),$$

satisfying

$$u(t) = G(t-s)u(s) - \int_s^t G(t-\sigma)\left(|\nabla u(\sigma)|^p\right)\, d\sigma, \quad 0 < s \le t,$$

$$\lim_{t\to 0+} \int u(t,x)\ \psi(x)\ dx = \int \psi(x)\ du_0(x), \quad \psi \in \mathcal{BC}(\mathbb{R}^N),$$

and

$$\left\{\begin{array}{l} \sup\limits_{t\in(0,+\infty)} t^{N/2}\ \|u(t)\|_{L^\infty} \le C_H, \\[2ex] \sup\limits_{t\in(0,+\infty)} t^{(p(N+1)-N)/2p}\ \left\|\nabla u^{(p-1)/p}(t)\right\|_{L^\infty} \le C_H. \end{array}\right. \tag{2.1}$$

Here $\mathcal{BC}(\mathbb{R}^N)$ denotes the space of bounded and continuous functions in $\mathbb{R}^N$ and C_H is a positive real number depending only on N, p and $\|u_0\|_{\mathcal{M}_b}$.

In addition, there holds

$$\sup_{t\in(0,+\infty)} t^{1/p}\ \left\|\nabla u^{(p-1)/p}(t)\right\|_{L^\infty} \le (p-1)^{1-1/p}\ p^{-1} := C_{HJ}. \tag{2.2}$$

We now derive additional estimates for solutions to (1.1) with non-negative and compactly supported smooth initial data. Recall that (1.1) has a unique non-negative classical solution when u_0 is a non-negative function in $\mathcal{D}(\mathbb{R}^N)$ (see, e.g., [17, 2]).

For $p \in (1,2)$ we put

$$\Gamma_p(r) = \gamma_p \; r^{-a}, \quad r \in (0,+\infty), \tag{2.3}$$

where

$$\gamma_p = (p-1)^{(p-2)/(p-1)} \; (2-p)^{-1}. \tag{2.4}$$

Recall that $a = (2-p)/(p-1)$.

Lemma 2.2. *Let $p \in (1,(N+2)/(N+1))$ and consider a non-negative function u_0 in $L^1(\mathbb{R}^N)$. We define*

$$R(u_0) = \inf \{ R > 0 \; , \; |x|^a \; u_0(x) \le \gamma_p \;\; a.e. \;\; in \;\; \{|x| \ge R\}\}, \tag{2.5}$$

($R(u_0) \in [0,+\infty]$), and denote by u the non-negative solution to (1.1) with initial datum u_0 given by Theorem 2.1. If $R(u_0) < +\infty$ there holds

$$0 \le u(t,x) \le \Gamma_p \left(|x| - R(u_0) \right), \quad x \in \mathbb{R}^N, \quad |x| > R(u_0). \tag{2.6}$$

Proof. We first assume that $u_0 \in \mathcal{D}(\mathbb{R}^N)$, so that u is the unique non-negative classical solution to (1.1) with initial datum u_0. Note that as u_0 is compactly supported we have $R(u_0) < +\infty$. Consider $\omega \in S^{N-1}$ where S^{N-1} denotes the $N-1$-dimensional unit sphere and put

$$D_\omega = (0,+\infty) \times \left\{ x \in \mathbb{R}^N , x.\omega > R(u_0) \right\}$$
$$\vartheta_\omega(x) = \Gamma_p(x.\omega - R(u_0)), \quad x \in \mathbb{R}^N, \quad x.\omega > R(u_0).$$

On the one hand a straightforward computation yields that the function ϑ_ω is a stationary solution to (1.1) on D_ω.
On the other hand it follows from the definition of $R(u_0)$ that

$$u(0,x) = u_0(x) \le \gamma_p \; |x|^{-a} = \left(\frac{x.\omega - R(u_0)}{|x|} \right)^a \; \vartheta_\omega(x) \le \vartheta_\omega(x)$$

for every $x \in \mathbb{R}^N$ such that $x.\omega > R(u_0)$. Also for $t \in (0,+\infty)$ and $x \in \mathbb{R}^N$ satisfying $x.\omega = R(u_0)$ there holds

$$u(t,x) < +\infty = \vartheta_\omega(x).$$

Consequently $u \le \vartheta_\omega$ on the parabolic boundary of D_ω and the comparison principle entails

$$u(t,x) \le \vartheta_\omega(x), \quad (t,x) \in D_\omega. \tag{2.7}$$

Now take $t \in (0,+\infty)$, $x \in \mathbb{R}^N$ satisfying $|x| > R(u_0)$ and put $\omega(x) = x/|x|$. Then $(t,x) \in D_{\omega(x)}$ and (2.7) yields

$$u(t,x) \le \Gamma_p(x.\omega(x) - R(u_0)) = \Gamma_p(|x| - R(u_0)),$$

and the proof of the lemma is complete for $u_0 \in \mathcal{D}(\mathbb{R}^N)$.

We next consider a non-negative function $u_0 \in L^1(\mathbb{R}^N)$ such that $R(u_0)$ defined by (2.5) is finite. We then construct a sequence of non-negative functions $(u_{0,n})_n$ in $\mathcal{D}(\mathbb{R}^N)$ such that $(u_{0,n})_n$ converges to u_0 in $L^1(\mathbb{R}^N)$ and $R(u_{0,n}) \le R(u_0)+2/n$. Denoting by u_n the unique non-negative classical solution to (1.1) with initial datum $u_{0,n}$ we proceed as in the proof of [3, Theorem 3] to show that $(u_n)_n$ converges towards u in $\mathcal{C}([0,T];L^1(\mathbb{R}^N))$ for every $T \in (0,+\infty)$. We now take $x \in \mathbb{R}^N$ with $|x| > R(u_0)$. For n large enough we have $|x| > R(u_{0,n})$ hence, as Lemma 2.2 holds true for (u_n)

$$0 \le u_n(t,x) \le \Gamma_p\left(|x| - R(u_{0,n})\right) \le \Gamma_p\left(|x| - R(u_0) - \frac{2}{n}\right).$$

The lemma then follows by letting $n \to +\infty$ in the above inequality. □

Remark 2.3. Let us mention at this point that the idea of using a stationary solution to (1.1) to obtain (2.6) is borrowed from [15].

We now combine (2.2) and (2.6) to obtain temporal decay estimates for the solutions to (1.1) with initial data in $L^1(\mathbb{R}^N)$.

Proposition 2.4. *Let $p \in (1,(N+2)/(N+1))$ and consider a non-negative function u_0 in $L^1(\mathbb{R}^N)$. If u denotes the non-negative solution to (1.1) with initial datum u_0 given by Theorem 2.1 and $R(u_0) < +\infty$ there holds*

$$\|u(t)\|_{L^1} \le C_1\ t^{-((N+2)-p(N+1))/(2(p-1))}, \tag{2.8}$$

$$\|u(t)\|_{L^\infty} \le C_1\ t^{-a/2}, \tag{2.9}$$

$$\|\nabla u(t)\|_{L^\infty} \le C_1\ t^{-1/(2(p-1))}, \tag{2.10}$$

for each $t > \tau(u_0)$, where C_1 is a positive real number depending only on N and p and

$$\tau(u_0) = \left(\frac{(N+2)-p(N+1)}{(N+1)p-N}\right)^{1-p} R(u_0)^2. \tag{2.11}$$

Recall that $R(u_0)$ is defined in (2.5).

Proof. In the following we denote by C any positive real number depending only on N and p. We fix $t \in (\tau(u_0),+\infty)$.

By (2.6) we have for $R \ge R(u_0)$

$$\|u(t)\|_{L^1} \le \int_{\{|x|\le 2R\}} u(t,x)\ dx + \int_{\{|x|>2R\}} u(t,x)\ dx$$

$$\|u(t)\|_{L^1} \le C\ R^N\ \|u(t)\|_{L^\infty} + \int_{\{|x|>2R\}} \Gamma_p(|x| - R(u_0))\ dx. \tag{2.12}$$

On the one hand we infer from the Gagliardo-Nirenberg inequality [17, Theorem II.2.2] and (2.2) that

$$\begin{aligned} \|u(t)\|_{L^\infty} &\leq C \ \left\|\nabla u^{(p-1)/p}(t)\right\|_{L^\infty}^{(Np)/((N+1)p-N)} \\ &\quad \times \ \left\|u^{(p-1)/p}(t)\right\|_{L^{p/(p-1)}}^{p^2/((p-1)((N+1)p-N))} \end{aligned}$$

$$\|u(t)\|_{L^\infty} \leq C \ \|u(t)\|_{L^1}^{p/((N+1)p-N)} \ t^{-N/((N+1)p-N)}. \tag{2.13}$$

On the other hand, since Γ_p is a non-increasing function, $R > R(u_0)$ and $p \in (1,(N+2)/(N+1))$ we have

$$\begin{aligned} &\int_{\{|x|>2R\}} \Gamma_p(|x| - R(u_0)) \ dx \\ &= \ C \ \int_{2R}^{\infty} \Gamma_p(r - R(u_0)) \ r^{N-1} \ dr \\ &\leq \ C \ \left(\sup_{r\in[2R,+\infty)} \frac{r}{r-R}\right)^{N-1} \ \int_{2R}^{\infty} \Gamma_p(r-R) \ (r-R)^{N-1} \ dr \end{aligned}$$

Consequently

$$\int_{\{|x|>2R\}} \Gamma_p(|x| - R(u_0)) \ dx \leq C \ R^{((N+1)p-(N+2))/(p-1)}. \tag{2.14}$$

Combining (2.12)-(2.14) then yields

$$\begin{aligned} \|u(t)\|_{L^1} &\leq \ C \ R^N \ \|u(t)\|_{L^1}^{p/((N+1)p-N)} \ t^{-N/((N+1)p-N)} \\ &\quad + \ C \ R^{((N+1)p-(N+2))/(p-1)} \end{aligned}$$

hence, thanks to the Young inequality,

$$\|u(t)\|_{L^1} \leq C \ \left(t^{-1/(p-1)} \ R^{((N+1)p-N)/(p-1)} + R^{((N+1)p-(N+2))/(p-1)}\right).$$

The above inequality being valid for every $R \in (R(u_0),+\infty)$ we finally obtain

$$\|u(t)\|_{L^1} \leq C \ \inf_{R>R(u_0)} \mathcal{F}(R,t), \tag{2.15}$$

$$\mathcal{F}(R,t) = R^{((N+1)p-N)/(p-1)} \ \left(t^{-1/(p-1)} + R^{-2/(p-1)}\right).$$

Now, since $t > \tau(u_0)$ we have

$$R(u_0) < \left(\frac{(N+2)-p(N+1)}{(N+1)p-N} \ t^{1/(p-1)}\right)^{(p-1)/2} := \mathcal{R}(t),$$

and we may take $R = \mathcal{R}(t)$ in (2.15). We thus obtain

$$\|u(t)\|_{L^1} \leq C \ t^{-((N+2)-p(N+1))/(2(p-1))},$$

hence (2.8). Next, (2.9) follows at once from (2.13) and (2.8). Finally since

$$\nabla u(t) = \frac{p}{p-1}\ u(t)^{1/p}\ \nabla u^{(p-1)/p}(t),$$

(2.10) is a consequence of (2.2) and (2.9). □

Remark 2.5. As $p \in (1, (N+2)/(N+1))$ we have

$$\frac{a}{2} > \frac{N}{2}.$$

Consequently the L^∞-norm of the non-negative solutions to (1.1) with non-negative initial data in $L^1(\mathbb{R}^N)$ decays faster than the L^∞-norm of the non-negative solutions to the linear heat equation with the same initial data.

Remark 2.6. The temporal decay estimate (2.8) of the L^1-norm of the solutions to (1.1) with initial data satisfying $R(u_0) < +\infty$ is in some sense optimal : indeed it has been shown in [7, Corollary 3.5] that the L^1-norm of a non-zero and integrable solution to (1.1) cannot decay as $t^{-\alpha}$ for $\alpha > ((N+2) - p(N+1))/(2(p-1))$.

3. Existence of a very singular solution

In this section we assume that $p \in (1, (N+2)/(N+1))$ and we denote by $(C_i)_{i\geq 2}$ any positive real number depending only on p and N. Let $M \in (0, +\infty)$. Since $M\delta$ belongs to $\mathcal{M}_b^+(\mathbb{R}^N)$ it follows from Theorem 2.1 that (1.1) has a unique weak solution with initial datum $M\delta$ which we denote by u_M. In the next lemma we gather some useful properties enjoyed by the family $\{u_M, M \in (0, +\infty)\}$.

Lemma 3.1. *There is a constant C_2 depending only on p and N such that for every $M \in (0, +\infty)$ and $t \in (0, +\infty)$ there holds*

$$\|u_M(t)\|_{L^1} \leq C_2\ t^{-((N+2)-p(N+1))/(2(p-1))}, \tag{3.1}$$

$$\|u_M(t)\|_{L^\infty} \leq C_2\ t^{-a/2}, \tag{3.2}$$

$$\|\nabla u_M(t)\|_{L^\infty} \leq C_2\ t^{-1/(2(p-1))}, \tag{3.3}$$

$$0 \leq u_M(t,x) \leq \Gamma_p(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}, \tag{3.4}$$

$$\int u_M(t,x)\ \varrho(x)\ dx \leq \exp\left(C_2 t\right) - 1, \tag{3.5}$$

where

$$\varrho(x) = |x|^{2\alpha}\ \left(1+|x|^2\right)^{-\alpha}, \quad x \in \mathbb{R}^N, \tag{3.6}$$

and

$$\alpha = \frac{p}{2(p-1)} \in (1, +\infty).$$

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^N)$ be a non-negative and radially symmetric function with support in $\{x \in \mathbb{R}^N, |x| \le 1\}$ and $\|\varphi\|_{L^1} = 1$. For $n \ge 1$ we put

$$\varphi_n(x) = n^N\ \varphi(nx), \quad x \in \mathbb{R}^N.$$

We fix $M \in (0,+\infty)$ and denote by $v_{n,M}$ the non-negative classical solution to (1.1) with initial datum $M\varphi_n$. It follows from the analysis of [3, Section 3] that for every $t \in (0,+\infty)$ and $s \in (0,t)$

$$\left\{ \begin{array}{lll} v_{n,M} \longrightarrow u_M & \text{in} & \mathcal{C}([s,t];L^1(\mathbb{R}^N)), \\ & & \\ \nabla v_{n,M} \longrightarrow \nabla u_M & \text{in} & L^p((s,t)\times\mathbb{R}^N). \end{array} \right. \tag{3.7}$$

As φ is radially symmetric the rotation-invariance of (1.1) and the uniqueness of classical solutions to (1.1) ensure that

$$v_{n,M}(t) \quad \text{is radially symmetric for each} \quad t \in (0,+\infty). \tag{3.8}$$

By (2.8), (2.9) and (2.10) we have

$$\|v_{n,M}(t)\|_{L^1} \le C_3\ t^{-((N+2)-p(N+1))/(2(p-1))}, \tag{3.9}$$

$$\|v_{n,M}(t)\|_{L^\infty} \le C_3\ t^{-a/2}, \tag{3.10}$$

$$\|\nabla v_{n,M}(t)\|_{L^\infty} \le C_3\ t^{-1/(2(p-1))}, \tag{3.11}$$

$$0 \le v_{n,M}(t,x) \le \Gamma_p(|x| - R(M\varphi_n)), \quad |x| > R(M\varphi_n), \tag{3.12}$$

for every $n \ge 1$ and $t \in [t_{n,M},+\infty)$, where

$$t_{n,M} = \left(\frac{(N+2)-p(N+1)}{(N+1)p-N} \right)^{1-p}\ R(M\varphi_n)^2. \tag{3.13}$$

Now, as the support of φ_n is included in $\{x \in \mathbb{R}^N, |x| \le 1/n\}$ we have

$$\lim_{n\to+\infty} t_{n,M} = \lim_{n\to+\infty} R(M\varphi_n) = 0,$$

and we infer from (3.7), (3.9)-(3.12) and the continuity of Γ_p that u_M enjoys the properties (3.1)-(3.4).

We next check (3.5). Recalling that $p < 2$ we have $\alpha > 1$ and $\varrho \in \mathcal{C}^2(\mathbb{R}^N)$ defined by (3.6) satisfies

$$\Delta\varrho(x) \le C_4\ \varrho(x)\ |x|^{-2}, \quad x \in \mathbb{R}^N. \tag{3.14}$$

Let $t \in (0,+\infty)$ and $s \in (0,t)$. Also let ξ be a function in $\mathcal{D}(\mathbb{R}^N)$ satisfying $0 \le \xi \le 1$,

$$\xi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \xi(x) = 0 \quad \text{if} \quad |x| \ge 2,$$

and put $\xi_n(x) = \xi(x/n)$ for $x \in \mathbb{R}^N$ and $n \geq 1$. It follows from (1.1) that

$$\begin{aligned}\int u_M(t)\ \varrho\ \xi_n\ dx \ &\leq\ \int u_M(s)\ \varrho\ \xi_n\ dx \\ &+\ \int_s^t \int u_M(\sigma)\ \Delta\left(\varrho\ \xi_n\right)\ dx d\sigma.\end{aligned}$$

Since $\Delta\left(\varrho\ \xi_n\right)$ converges pointwisely to $\Delta\varrho$ as $n \to +\infty$ and is uniformly bounded with respect to $n \geq 1$ we infer from the integrability (3.1) of u_M on $(s,t) \times \mathbb{R}^N$ and the Lebesgue dominated convergence theorem that

$$\lim_{n\to+\infty} \int_s^t \int u_M(\sigma)\ \Delta\left(\varrho\ \xi_n\right)\ dx d\sigma = \int_s^t \int u_M(\sigma)\ \Delta\varrho\ dx d\sigma.$$

Since ξ_n converges pointwisely to 1 as $n \to +\infty$ we may pass to the limit in the previous inequality and use again (3.1) and the Lebesgue dominated convergence theorem to obtain

$$\begin{aligned}\int u_M(t)\ \varrho\ dx \ &\leq\ \int u_M(s)\ \varrho\ dx \\ &+\ \int_s^t \int u_M(\sigma)\ \Delta\varrho\ dx d\sigma.\end{aligned}$$

It then follows from (3.14) that

$$\begin{aligned}\int u_M(t)\ \varrho\ dx \ &\leq\ \int u_M(s)\ \varrho\ dx \\ &+\ C_4 \int_s^t \int u_M(\sigma)\ \varrho\ |x|^{-2}\ dx d\sigma.\end{aligned} \tag{3.15}$$

By (3.4) we have

$$\begin{aligned}&\int u_M(\sigma,x)\ \varrho(x)\ |x|^{-2}\ dx \\ \leq\ &\int_{\{|x|\geq 1\}} u_M(\sigma,x)\ \varrho(x)\ dx + \int_{\{|x|\leq 1\}} \Gamma_p(|x|)\ \varrho(x)\ |x|^{-2}\ dx \\ \leq\ &\int u_M(\sigma,x)\ \varrho(x)\ dx + C_5.\end{aligned}$$

Recalling (3.15) we obtain

$$\begin{aligned}\int u_M(t)\ \varrho\ dx \ &\leq\ \int u_M(s)\ \varrho\ dx \\ &+\ C_6 \int_s^t \left(1 + \int u_M(\sigma)\ \varrho\ dx\right)\ d\sigma.\end{aligned}$$

The Gronwall lemma then yields

$$\int u_M(t)\ \varrho\ dx \le \left(1+\int u_M(s)\ \varrho\ dx\right)\ \exp\left(C_6(t-s)\right)-1.$$

Finally, since $\varrho \in \mathcal{BC}(\mathbb{R}^N)$ with $\varrho(0)=0$ we may let $s\to 0$ in the above inequality and obtain (3.5). ☐

An obvious consequence of (3.7) and (2.2) is the following result.

Lemma 3.2. *For each $M\in(0,+\infty)$ and $t\in(0,+\infty)$ there holds*

$$\sup_{t\in(0,+\infty)} t^{1/p}\ \left\|\nabla u_M^{(p-1)/p}(t)\right\|_{L^\infty} \le C_{HJ}. \tag{3.16}$$

We next prove that the sequence (u_M) is monotonic with respect to M.

Lemma 3.3. *For each $M\in(0,+\infty)$ and $t\in(0,+\infty)$, $u_M(t)$ is a radially symmetric and non-increasing function and*

$$M_1\le M_2 \Longrightarrow u_{M_1}\le u_{M_2}. \tag{3.17}$$

Proof. We fix $M\in(0,+\infty)$. For $n\ge 1$ we again denote by $v_{n,M}$ the non-negative classical solution to (1.1) with initial datum $M\varphi_n$ defined at the beginning of the proof of Lemma 3.1. A straightforward consequence of (3.7) and (3.8) is that

$$u_M(t) \quad \text{is radially symmetric for each} \quad t\in(0,+\infty).$$

We next check that $u_M(t)$ is non-increasing with respect to the space variable. Let $n\ge 1$ and consider $(y,z)\in\mathbb{R}\times\mathbb{R}$ such that $|y|+2/n\le|z|$. We then introduce the function $w_{n,M}$ defined by

$$w_{n,M}(t,x_1,x_2,\ldots,x_N)=v_{n,M}(t,y+z-x_1,x_2,\ldots,x_N)$$

for $(t,x_1,x_2,\ldots,x_N)\in(0,+\infty)\times\mathbb{R}^N$. We also put

$$\mathcal{E}=(0,+\infty)\times\left\{x\in\mathbb{R}^N,\ \left(x_1-\frac{y+z}{2}\right)(z-y)\le 0\right\}.$$

We first observe that $v_{n,M}$ and $w_{n,M}$ are solutions to (1.1) on $\mathcal{E}$ and enjoy the following properties on the parabolic boundary of $\mathcal{E}$:

$$w_{n,M}(0,x)=0\le v_{n,M}(0,x) \quad \text{if} \quad \left(x_1-\frac{y+z}{2}\right)(z-y)\le 0,$$
$$w_{n,M}(t,x)=v_{n,M}(t,x) \quad \text{if} \quad x_1=\frac{y+z}{2}.$$

We may then apply the comparison principle and conclude that

$$v_{n,M}(t,x)\ge w_{n,M}(t,x), \quad (t,x)\in\overline{\mathcal{E}}.$$

In particular, taking $x = (y, 0, \ldots, 0)$ we obtain

$$v_{n,M}(t, y, 0, \ldots, 0) \geq v_{n,M}(t, z, 0, \ldots, 0). \tag{3.18}$$

Now take $(x, \bar{x}) \in \mathbb{R}^N \times \mathbb{R}^N$ satisfying $|\bar{x}| \geq |x| + 2/n$. Owing to (3.8) and (3.18) there holds

$$v_{n,M}(t, x) = v_{n,M}(t, |x|, 0, \ldots, 0) \leq v_{n,M}(t, |\bar{x}|, 0, \ldots, 0) = v_{n,M}(t, \bar{x}).$$

Consequently, for every $(x, \bar{x}) \in \mathbb{R}^N \times \mathbb{R}^N$ there holds

$$|\bar{x}| \geq |x| + 2/n \Longrightarrow v_{n,M}(t, x) \geq v_{n,M}(t, \bar{x}), \quad t \in (0, +\infty). \tag{3.19}$$

We may then let $n \to +\infty$ in (3.19) and use (3.7) to conclude that $u_M(t)$ is non-increasing with respect to the space variable for each $t \in (0, +\infty)$.

Finally, if $M_1 \leq M_2$ we clearly have $M_1 \varphi_n \leq M_2 \varphi_n$ and the comparison principle entails $v_{n,M_1} \leq v_{n,M_2}$ for each $n \geq 1$. This fact and (3.7) at once yields (3.17). $\square$

We are now ready to prove Theorem 1.2. Let $t \in (0, +\infty)$. Owing to (3.1) and (3.17), $(u_M(t))_{M>0}$ is a non-decreasing sequence of functions in $L^1(\mathbb{R}^N)$ which is bounded in $L^1(\mathbb{R}^N)$. The monotone convergence theorem then entails that

$$U(t, x) = \sup_{M \in (0, +\infty)} u_M(t, x), \quad x \in \mathbb{R}^N, \tag{3.20}$$

belongs to $L^1(\mathbb{R}^N)$ and

$$\lim_{M \to +\infty} \|u_M(t) - U(t)\|_{L^1} = 0. \tag{3.21}$$

Now proceeding along the lines of [3, Section 3] and using (3.1)-(3.4) and (3.21) we prove that for each $t \in (0, +\infty)$ and $s \in (0, t)$, we have

$$U(t) \geq 0 \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{and} \quad U \in L^p((s, t); W^{1,p}(\mathbb{R}^N))$$

and U satisfies

$$U(t) = G(t - s)U(s) - \int_s^t G(t - \sigma) \ \left(|\nabla U(\sigma)|^p\right) \ d\sigma.$$

Also, by (3.2)-(3.3) we have that both U and ∇U belong to $L^\infty((s, t) \times \mathbb{R}^N)$. Therefore classical parabolic regularity results and a bootstrap argument entail that $U \in \mathcal{C}^{1,2}_{t,x}(\mathcal{K})$ for any compact subset $\mathcal{K}$ of $(0, +\infty) \times \mathbb{R}^N$. Furthermore we infer from Lemma 3.3 and (3.21) that

(3.22) $U(t)$ is radially symmetric and non-increasing for each $t \in (0, +\infty)$.

It remains to check that U has the expected behaviour as $t \to 0$. Fix $r \in (0,+\infty)$ and let $\zeta \in \mathcal{D}(\mathbb{R}^N)$ be a non-negative function such that $0 \le \zeta \le 1$ and

$$\zeta(x) = 0 \quad \text{if} \quad |x| \ge r \quad \text{and} \quad \zeta(x) = 1 \quad \text{if} \quad |x| \le r/2.$$

By (3.20) we have

$$\int_{\{|x|\le r\}} U(t,x)\ dx \ge \int u_M(t,x)\ \zeta(x)\ dx$$

for every $M \in (0,+\infty)$. Letting $t \to 0$ in the above inequality yields

$$\liminf_{t\to 0} \int_{\{|x|\le r\}} U(t,x)\ dx \ge M \quad \text{for every} \quad M \in (0,+\infty).$$

Therefore

$$\lim_{t\to 0} \int_{\{|x|\le r\}} U(t,x)\ dx = +\infty. \tag{3.23}$$

It next follows from (3.5) and (3.21) that

$$\int U(t,x)\ \varrho(x)\ dx \le \exp\left(C_2 t\right) - 1, \quad t \in (0,+\infty).$$

Consequently

$$\begin{aligned} \int_{\{|x|\ge r\}} U(t,x)\ dx &\le \left(\frac{1+r^2}{r^2}\right)^{\alpha} \int_{\{|x|\ge r\}} U(t,x)\ \varrho(x)\ dx \\ &\le \left(\frac{1+r^2}{r^2}\right)^{\alpha} \left(\exp\left(C_2 t\right) - 1\right), \end{aligned}$$

hence

$$\lim_{t\to 0} \int_{\{|x|\ge r\}} U(t,x)\ dx = 0. \tag{3.24}$$

Summing up we have proved that U defined by (3.20) is a very singular solution to (1.1) in the sense of Definition 1.1 and that $U(t)$ is radially symmetric and non-increasing for each $t \in (0,+\infty)$. In addition we infer from Lemma 3.2 and (3.21) that

$$\sup_{t\in(0,+\infty)} t^{1/p}\ \left\|\nabla U^{(p-1)/p}(t)\right\|_{L^\infty} \le C_{HJ}. \tag{3.25}$$

We now prove that U has the self-similar form (1.6). For $\lambda \in (0,+\infty)$ and $M \in (0,+\infty)$ we define

$$\begin{aligned} &u_M^\lambda(t,x) = \lambda^a\ u_M\left(\lambda^2 t, \lambda x\right), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N, \\ &\mu(\lambda, M) = \lambda^{((N+2)-p(N+1))/(p-1)}\ M. \end{aligned}$$

Recall that $a=(2-p)/(p-1)$.
It is straightforward to check that u_M^λ is a solution to (1.1) with initial datum $\mu(\lambda,M)\delta$ in the sense of Theorem 2.1. Such a solution being unique by Theorem 2.1 we conclude that

$$u_M^\lambda = u_{\mu(\lambda,M)}. \tag{3.26}$$

We then infer from (3.20) and (3.26) that

$$\lambda^a\ U\left(\lambda^2 t,\lambda x\right) = U(t,x).$$

As this equality is valid for every $(\lambda,t,x)\in(0,+\infty)^2\times\mathbb{R}^N$ it is easy to deduce from (3.22) that

$$U(t,x) = t^{-a/2}\ f\left(|x| t^{-1/2}\right), \quad (t,x)\in(0,+\infty)\times\mathbb{R}^N, \tag{3.27}$$

where

$$f(r) = U(1,r,0,\dots,0), \quad r\in(0,+\infty).$$

Observe that by (3.22)

$$f \quad \text{is a non-increasing function on} \quad (0,+\infty). \tag{3.28}$$

As U is a solution to (1.1) a standard computation shows that f is a solution to the ordinary differential equation (1.7) and (3.22) and the smoothness of U yield $f'(0)=0$. Also, owing to (3.27) and (3.28) we have

$$\begin{aligned}
\int_{\{|x|\ge 1\}} U(t,x)\ dx &= t^{(N-a)/2} \int_{\{|y|\ge t^{-1/2}\}} f(|y|)\ dy \\
&\ge t^{(N-a)/2} \int_{\{2t^{-1/2}\ge |y|\ge t^{-1/2}\}} f(|y|)\ dy \\
&\ge C_7\ t^{-a/2}\ f\left(2t^{-1/2}\right) \\
&\ge C_8\ \left(2t^{-1/2}\right)^a\ f\left(2t^{-1/2}\right).
\end{aligned}$$

Consequently for $r\in(0,+\infty)$

$$r^a\ f(r) \le C_9 \int_{\{|x|\ge 1\}} U\left(4r^{-2},x\right)\ dx,$$

and (3.24) entails that

$$\lim_{r\to+\infty} r^a\ f(r) = 0. \tag{3.29}$$

Thus f fulfills the boundary conditions (1.8).

We finally check that $f \in \mathcal{C}^\infty((0,+\infty))$. Classical arguments first ensure that $f \in \mathcal{C}^2((0,+\infty))$. Also as U is not identically zero by (1.4) the function f is not identically zero. In fact we claim that

$$f(r) > 0, \quad r \in (0,+\infty). \tag{3.30}$$

Indeed arguing by contradiction we assume that $f(r_0) = 0$ for some $r_0 > 0$. Since f is non-increasing we obtain that f and thus f' vanish identically on $[r_0,+\infty)$. Then $r \mapsto (f(r_0-r), -f'(r_0-r))$ is a solution on $(0,r_0)$ to a second-order ordinary differential equation with Lipschitz continuous non-linearities and initial data $(0,0)$ which entails $f \equiv 0$ and a contradiction. We next prove that

$$f'(r) < 0, \quad r \in (0,+\infty). \tag{3.31}$$

Indeed define

$$\mathcal{S} = \{r \in (0,+\infty), f' < 0 \ \text{ in } \ [0,r)\}.$$

Since $f > 0$ on $(0,+\infty)$ and is non-increasing, $f(0) > 0$ and thus $f''(0) = -(a/2N)\ f(0) < 0$. Consequently $\mathcal{S}$ is non-empty and we put

$$\rho = \sup \mathcal{S}.$$

If $\rho < +\infty$ we necessarily have $f'(\rho) = 0$ and it follows from (1.7) and (3.30) that $f''(\rho) < 0$. But then f' has to be positive on some interval $(\rho - \eta, \rho)$ for some $\eta > 0$, hence a contradiction. Consequently $\rho = +\infty$ and the claim (3.31) is proved.

Now as $r \mapsto |r|^p$ is $\mathcal{C}^\infty$-smooth on compact subsets of $(0,+\infty)$ the smoothness of f follows from (1.7) and (3.31) by classical arguments.

4. Non-existence of very singular solutions for $p=1$

In this section we consider the case $p=1$, i.e.

$$u_t - \Delta u + |\nabla u| = 0 \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N. \tag{4.1}$$

Note that (4.1) is a nonlinear parabolic equation but has the same homogeneity as the linear heat equation. We first recall the well-posedness of (4.1) in $\mathcal{M}_b^+(\mathbb{R}^N)$ [6].

Proposition 4.1. *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$. There exists a unique non-negative function*

$$u \in \mathcal{C}([0,+\infty); \mathcal{M}_b^+(\mathbb{R}^N)) \cap L^1(0,+\infty; W^{1,1}(\mathbb{R}^N))$$

satisfying

$$u(t) = G(t)u_0 - \int_0^t G(t-s)\left(|\nabla u(s)|\right)\ ds, \quad t \in [0,+\infty). \tag{4.2}$$

Notice that since $u(t)$ belongs to $L^1(\mathbb{R}^N)$ for almost every $t \in (0,+\infty)$ we have in fact $u \in \mathcal{C}((0,+\infty);L^1(\mathbb{R}^N))$. We now supplement Proposition 4.1 with some further properties enjoyed by u.

Lemma 4.2. *Let $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$ and denote by u the solution to (4.1) with $u(0)=u_0$ given by Proposition 4.1. For each $T \in (0,+\infty)$ and $t \in (0,T)$ we have*

$$\nabla u \in L^\infty((t,T);L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$$

and there is a positive constant $C_1(T)$ depending only on N and T such that

$$\|\nabla u(t)\|_{L^1} \le C_1(T)\ \|u_0\|_{\mathcal{M}_b}\ t^{-1/2}, \tag{4.3}$$

$$\|\nabla u(t)\|_{L^\infty} \le C_1(T)\ \|u_0\|_{\mathcal{M}_b}\ t^{-(N+1)/2}. \tag{4.4}$$

Proof. Let $T \in (0,+\infty)$ and $t \in (0,T)$. By the Duhamel formula (4.2) we have

$$\|\nabla u(t)\|_{L^1} \le C_1\ t^{-1/2}\ \|u_0\|_{\mathcal{M}_b} + C_1\ \int_0^t (t-s)^{-1/2}\ \|\nabla u(s)\|_{L^1}\ ds,$$

and a singular Gronwall lemma (see, e.g., [1, Theorem II.3.3.1]) yields (4.3). We next use again the Duhamel formula (4.2) to obtain

$$\begin{aligned} \|\nabla u(t)\|_{L^\infty} &\le C_1\ t^{-(N+1)/2}\ \|u_0\|_{\mathcal{M}_b} \\ &+ C_1\ \int_0^{t/2} (t-s)^{-(N+1)/2}\ \|\nabla u(s)\|_{L^1}\ ds \\ &+ C_1\ \int_{t/2}^t (t-s)^{-1/2}\ \|\nabla u(s)\|_{L^\infty}\ ds \end{aligned}$$

Thanks to (4.3) we deduce

$$\|\nabla u(t)\|_{L^\infty} \le C_1\ t^{-(N+1)/2}\ \|u_0\|_{\mathcal{M}_b} + C_1\ \int_0^t (t-s)^{-1/2}\ \|\nabla u(s)\|_{L^\infty}\ ds.$$

We apply again a singular Gronwall lemma and obtain (4.4). □

We now state and prove the main result of this section.

Proposition 4.3. *There is no very singular solution to (4.1) in the sense of Definition 1.1.*

In order to prove Proposition 4.3 we shall follow the classical approach which is to prove that a very singular solution (if it exists) is necessarily above every fundamental solution. More precisely we have the following result.

Lemma 4.4. *Let* $M \in (0,+\infty)$ *and denote by* u_M *the non-negative solution to (4.1) with initial datum* $M\delta$ *which is given by Proposition 4.1. If* v *is a very singular solution to (4.1) there holds*

$$u_M(t,x) \le v(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N. \tag{4.5}$$

Proof. By (1.4)-(1.5) we have

$$\lim_{t\to 0+} \|v(t)\|_{L^1} = +\infty.$$

By a suitable truncation it is then possible to construct a sequence of non-negative functions $(v_{0,k})_{k\ge k_M}$ where k_M is a large integer such that

$$v_{0,k}(x) \le v(1/k,x), \quad x \in \mathbb{R}^N, \quad k \ge k_M \quad \text{and} \quad \|v_{0,k}\|_{L^1} = M. \tag{4.6}$$

We next denote by v_k the unique non-negative solution to (4.1) with initial datum $v_{0,k}$ given by Proposition 4.1. Owing to (4.6) and (4.4) we may proceed as in the proof of [3, Theorem 3] to show that there is a subsequence of (v_k) (not relabeled) and a non-negative function $w \in \mathcal{C}((0,+\infty);L^1(\mathbb{R}^N))$ such that

$$v_k \longrightarrow w \ \text{ in } \ \mathcal{C}((s,t);L^1(\mathbb{R}^N)) \cap L^1(s,t;W^{1,1}(\mathbb{R}^N)) \tag{4.7}$$

for every $t \in (0,+\infty)$ and $s \in (0,t)$. Passing to the limit in (4.2) then yields

$$w(t) = G(t-s)w(s) - \int_s^t G(t-\sigma)\left(|\nabla w(\sigma)|\right)\, d\sigma, \quad 0 < s < t.$$

It remains to identify the initial datum taken by w. For $\zeta \in \mathcal{D}(\mathbb{R}^N)$ and $t \in (0,1)$ we infer from (4.1), (4.3) and (4.6) that

$$\begin{aligned} & \left| \int v_k(t)\ \zeta\ dx - \int v_{0,k}\ \zeta\ dx \right| \\ & \le \|\zeta\|_{W^{2,\infty}} \int_0^t \int v_k\ dxds + \|\zeta\|_{L^\infty} \int_0^t \int |\nabla v_k|\ dxds \\ & \le \|\zeta\|_{W^{2,\infty}}\ \left(Mt + 2C_1(1)\ M\ t^{1/2}\right). \end{aligned} \tag{4.8}$$

It also follows from (4.6) that for each $r \in (0,+\infty)$ there holds

$$\begin{aligned} \left| \int v_{0,k}\ \zeta\ dx - M\ \zeta(0) \right| \ & \le \ M \sup_{|x|\le r} |\zeta(x) - \zeta(0)| \\ & \quad + 2\ \|\zeta\|_{L^\infty} \int_{\{|x|\ge r\}} v(1/k,x)\ dx. \end{aligned}$$

We first let $k \to +\infty$ in the above estimate and use (1.5) to obtain

$$\limsup_{k\to+\infty} \left| \int v_{0,k}\ \zeta\ dx - M\ \zeta(0) \right| \le M \sup_{|x|\le r} |\zeta(x) - \zeta(0)|\,.$$

As the above inequality is valid for every $r \in (0,+\infty)$ the continuity of ζ allows to conclude that

$$\lim_{k\to+\infty} \int v_{0,k}\ \zeta\ dx = M\ \zeta(0). \tag{4.9}$$

Owing to (4.7) and (4.9) we may let $k \to +\infty$ in (4.8) and obtain

$$\left| \int w(t)\ \zeta\ dx - M\ \zeta(0) \right| \le \|\zeta\|_{W^{2,\infty}}\ \left(Mt + 2C_1(1)\ M\ t^{1/2} \right).$$

Consequently

$$\lim_{t\to 0+} \int w(t,x)\ \zeta(x)\ dx = M\ \zeta(0) \tag{4.10}$$

for every $\zeta \in \mathcal{D}(\mathbb{R}^N)$. As w is a subsolution to the heat equation it can be proved that (4.10) actually holds for every $\zeta \in \mathcal{BC}(\mathbb{R}^N)$. It then follows from (4.10) and the properties of the linear heat semigroup that in fact

$$\lim_{t\to 0+} \|w(t) - M\delta\|_{\mathcal{M}_b} = 0,$$

which together with Proposition 4.1 yields $w = u_M$.

Finally, as $v_{0,k} \le v(1/k)$ by (4.6) the comparison principle entails

$$v_k(t,x) \le v(t+1/k, x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N.$$

As $v \in \mathcal{C}((0,+\infty); L^1(\mathbb{R}^N))$ by Definition 1.1 we may let $k \to +\infty$ in the above inequality and obtain (4.5). □

We may now prove Proposition 4.3. Assume for contradiction that there is a very singular solution v to (4.1) in the sense of Definition 1.1. By Lemma 4.4 there holds

$$u_M(t,x) \le v(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N$$

for every $M \in (0,+\infty)$. But a simple homogeneity argument and Proposition 4.1 yield that $u_M = M\ u_1$. Consequently on the one hand

$$M\ u_1(t,x) \le v(t,x), \quad (t,x) \in (0,+\infty) \times \mathbb{R}^N \tag{4.11}$$

for every $M \in (0,+\infty)$. On the other hand a maximum principle argument entails that $u_1(t,x) > 0$ for $(t,x) \in (0,+\infty) \times \mathbb{R}^N$ [24, p. 173]. We may then let $M \to +\infty$ in (4.11) and obtain that v is identically equal to $+\infty$ in $(0,+\infty)\times\mathbb{R}^N$, hence a contradiction. The proof of Proposition 4.3 is then complete.

Remark 4.5. We conjecture that if $p \ge (N+2)/(N+1)$ there does not exist very singular solutions to (1.1). At least the method we have used to prove Theorem 1.2 and which is based on suitable properties of the fundamental solutions u_M to (1.1) does not work. Indeed, if $p \ge (N+2)/(N+1)$ we have proved in

[3, Theorem 4] that there is no fundamental solution to (1.1) (i.e. solutions with initial data $M\delta$).

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