Abstract: We consider the Gibbs-measures of continuous-valued height configurations on the d-dimensional integer lattice in the presence a weakly disordered potential. The potential is composed of Gaussians having random location and random depth; it becomes periodic under shift of the interface perpendicular to the base-plane for zero disorder. We prove that there exist localized interfaces with probability one in dimensions $d \geq 3+1$, in a 'low-temperature' regime. The proof extends the method of continuous-to-discrete single-site coarse graining that was previously applied by the author for a double-well potential to the case of a non-compact image space. This allows to utilize parts of the renormalization group analysis developed for the treatment of a contour representation of a related integer-valued SOS-model in [BoK1]. We show that, for a.e. fixed realization of the disorder, the infinite volume Gibbs measures then have a representation as superpositions of massive Gaussian fields with centerings that are distributed according to the infinite volume Gibbs measures of the disordered integer-valued SOS-model with exponentially decaying interactions.

I. Introduction

The study of interface models from statistical mechanics, continuous as well as discrete ones, with respect to their localization vs. fluctuation properties, is an interesting topic in probability theory. In this paper we study the problem of continuous SOS-interfaces in random potentials that are random perturbations of periodic ones and prove stability of the interface in dimensions $d \geq 3+1$ (as suggested by the heuristic Imry-Ma argument, known for long to theoretical physicists).

A related stability result has been proved before for the simpler discrete version of such a model with nearest neighbor interactions in [BoK1]. The proof uses a renormalization group (or spatial coarse-graining) procedure that was based on the technique of Bricmont and Kupiainen that was developed for the Random Field Ising Model [BK]. The issue of this note is thus to clarify what to do with additional (possibly destabilizing) fluctuations of the continuous degrees of freedom.

An analogous problem was investigated in a recent paper by the author [K4] in the simpler case of a random double-well potential, where ferromagnetic ordering was shown in $d \geq 3$ (under suitable 'low temperature' and 'weak anharmonicity' assumptions on the potential). The key point here is to construct a suitable stochastic mapping from continuous to discrete configurations and study the image measures under this mapping. In the double-well case this mapping is just a smoothed sign-field indicating what minimum the continuous spin is close to. The image measure could then be shown to be an Ising-measure for a suitable absolutely summable Hamiltonian. It can be controlled by the known renormalization group method of [BK]. This is clearly in favor of running a suitably devised renormalization group transformation on the (in this context unpleasantly rich) space of continuous configurations.

The purpose of the present paper is to study the difficulties of infinitely many minima in the potential. The stochastic mapping we will apply to the continuous spins will now be a mapping to integer- valued spin configurations. As opposed to [K4] the mapping will now also depend on the realization of the disorder.

To explain the method in the simplest non-trivial context, we have decided to choose a specific potential that is the log of sums of Gaussians. The treatment of this potential provides the basic building block of the analysis also for more general potentials in that it explains the occurrence of the phase transition and the structure of the contour models that will arise. It corresponds to having vanishing 'anharmonic corrections'; how those anharmonicities (that are present for more general potentials) can be treated by additional expansions is explained in detail for the double-well case in [K4], so that combining those methods with the ones from the present paper should yield stability for a larger class of continuous interface models.

This restriction also allows us to obtain particularly nice 'factorization-formulas' for the continuous-spin Gibbs-measures in finite and infinite volume. They have some probabilistic appeal and clarify the structure of the coarse graining transformation we use. In particular we can describe the infinite volume Gibbs measures in terms of the 'explicit' building blocks of random discrete height measures and well-understood (random) massive Gaussian fields (see [1.7]).

Here is the model. We are aiming to investigate the Gibbs measures on the state space $\mathbb{R}^{\mathbb{Z}^d}$ of the continuous spin model given by the Hamiltonians in finite volumes $\Lambda \subset \mathbb{Z}^d$

$$E_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(m_\Lambda) = \frac{q}{2}\sum_{\substack{\{x,y\}\subset\Lambda\\ d(x,y)=1}}(m_x - m_y)^2 + \frac{q}{2}\sum_{\substack{x\in\Lambda;y\in\partial\Lambda\\ d(x,y)=1}}(m_x-\tilde m_y)^2 + \sum_{x\in\Lambda}V_x(m_x) \tag{1.1}$$

for a configuration $m_\Lambda \in \mathbb{R}^\Lambda$ with boundary condition $\tilde m_{\partial\Lambda}$. We write $\partial\Lambda = \{x \in \Lambda^c; \exists y \in \Lambda : d(x,y) = 1\}$ for the outer boundary of a set Λ where $d(x,y) = \|x-y\|_1$ is the 1-norm on $\mathbb{R}^d$. $q \geq 0$ will be small.

The random potential we consider is given by

$$\begin{aligned} V_x(m_x) &= -\log\left[\sum_{l\in\mathbb{Z}} e^{-\frac{1}{2}(m_x - m_x^*(l))^2 + \eta_x(l)}\right] \quad \text{where} \\ m_x^*(l) &= m^*\left(l + d_x(l)\right) \end{aligned} \tag{1.2}$$

The disorder is modelled by the random variables $(\eta_x(h))_{x\in\mathbb{Z}^d;h\in\mathbb{Z}}$ and $(d_x(h))_{x\in\mathbb{Z}^d;h\in\mathbb{Z}}$, describing the random depths of the Gaussians and the random deviations of the centerings of the Gaussians from the lattice $m^*\mathbb{Z}$. The unperturbed potential thus takes its minima for $m \in m^*\mathbb{Z}$, the fixed parameter $m^* > 0$ being its period. Later it will have to be large enough. (Note that the curvature of the potential is of the order unity for large m^*; thus the curvature really has to be large on the rescaled lattice where the potential has period 1.)

We will simply take the $(\eta_x(h))_{x\in\mathbb{Z}^d;h\in\mathbb{Z}}$ as i.i.d. random variables with distribution $\mathbb{P}_\eta$ and the $(d_x(h))_{x\in\mathbb{Z}^d;h\in\mathbb{Z}}$ as i.i.d. random variables with distribution $\mathbb{P}_d$, independent of the η's. Furthermore we impose the smallness conditions

$$\text{(i)} \ \mathbb{P}\left[|\eta_x(h)| \geq t\right] \leq e^{-\frac{t^2}{2\sigma_\eta^2}}, \qquad \text{(ii)} \ \ |\eta_x(h)| \leq \delta_\eta$$

$$\text{(iii)} \ \mathbb{P}\left[d_x(h)^2 \geq t\right] \leq e^{-\frac{t^2}{2\sigma_d^2}}, \qquad \text{(iv)} \ \ |d_x(h)| \leq \delta_d \leq \tfrac{1}{4}$$

where $\sigma_d^2, \sigma_\eta^2 \geq 0$ will be sufficiently small.

An assumption of the type (iv) is natural, since it just states that the shifted wells stay away from each other and don't merge or even cross. The assumption (iii) is less natural (and not really essential). Moreover we will need in the proof that δ_d, δ_η be sufficiently small, which is just to simplify the structure of the contour representation we will derive later and could be bypassed, see below.

To make explicit the local dependence of various quantities on the disorder variables we write $\omega_x = (d_x(h), \eta_x(h))_{h\in\mathbb{Z}}$ for the 'disorder variables at site x' and put $\omega_\Lambda = (\omega_x)_{x\in\Lambda}$.

A more general setting could of course be to consider V_x that are stationary w.r.t. a discrete shift in the height-direction and satisfy some mixing condition. Also the i.i.d. assumption in x could be weakened.

We use the following notation for the objects of interest, the finite volume Gibbs-measures $\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}$, defined in terms of their expectations:

$$\begin{aligned} &\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(f) = \frac{1}{Z_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}} \int_{\mathbb{R}^\Lambda} dm_\Lambda\, f\left(m_\Lambda, \tilde m_{\Lambda^c}\right) e^{-E_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(m_\Lambda)} \quad \text{where} \\ &Z_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda} = \int_{\mathbb{R}^\Lambda} dm_\Lambda\, e^{-E_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(m_\Lambda)} \end{aligned} \tag{1.3}$$

for any bounded continuous f on $\mathbb{R}^{\mathbb{Z}^d}$ (continuity is meant w.r.t. product topology). Most of the times we will put zero boundary conditions $\tilde m_x = 0$ (for all $x \in \mathbb{Z}^d$), writing simply $\mu_\Lambda^{0,\omega_\Lambda}$. (Due to stationarity that's the same as putting $\tilde m_x = m^* l$ for any fixed $l \in \mathbb{Z}$.)

Then we have the stability result

Theorem 1: *Let $d \geq 3$ and assume the conditions (i)-(iv) on the disorder variables. Then, there exist $q_0 > 0$ (small enough), $\delta_0 > 0$ (small enough), $\tau_0 < \infty$ (large enough), $\sigma_0^2 > 0$ (small* *enough) such that, whenever $\delta_\eta, \delta_d^2 \le \delta_0$, $q(m^*)^2 \ge \tau_0$, and $\sigma^2_{\text{eff.}} := \sigma_d^2 + \sigma_\eta^2 \le \sigma_0^2$, the following is true.*

There exists an infinite volume random Gibbs-measure μ^ω that can be obtained as the weak limit $\mu^\omega = \lim_{N\uparrow\infty} \mu_{\Lambda_N}^{0,\omega_{\Lambda_N}}$ along a non-random sequence of cubes Λ_N. The measure describes a continuous interface localized around the base plane; its 'roughness' is bounded by

$$I\!E\mu^\omega\left(m_{x_0}^2\right) \le 1 + (m^*)^2\left(e^{-const\,\tilde\beta} + e^{-\frac{1}{\sigma^\kappa_{\text{eff.}}}}\right) \tag{1.4}$$

for any x_0, with $\tilde\beta = const \times \min\left\{\log\frac{1}{q}, q m^{*2}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d\right\}$ and an exponent $\kappa > 0$.

Remark: So, measured on the scale of the period m^*, the roughness is in fact a very small number. The term 1 has to be present since it describes the true fluctuations of a continuous spin in an individual well of V_x. The quantity $q(m^*)^2$ gives the true order of magnitude of the minimal energetic contribution of a pair of nearest neighbor heights in neighboring potential wells, so it can be viewed as some basic temperature variable that has to be large enough. It appears in the definition of $\tilde\beta$, with some minor logarithmic deterioration that we need for technical reasons. The $\log\frac{1}{q}$-contribution comes from a high-temperature expansion, to be explained later.

Essential for the analysis is the following local transition kernel $T_x\left(\cdot\,\middle|\,\cdot\right)$ describing a single site coarse graining from a continuous height $m_x \in I\!R$ to an integer height $h_x \in Z\!\!Z$. It is defined by keeping the Gaussian for $l = h_x$,

$$T_x^{\omega_x}\left(h_x\middle|m_x\right) = \frac{e^{-\frac{1}{2}(m_x - m_x^*(h_x))^2 + \eta_x(h_x)}}{\text{Norm.}(m_x)}, \tag{1.5}$$

where $\text{Norm.}(m_x)$ is chosen to make $T_x^{\omega_x}\left(h_x\middle|m_x\right)$ a probability measure on $Z\!\!Z$, for fixed m_x. Note that this object depends on the disorder through ω_x (as opposed to our proceeding in the double-well case where a simpler analogous kernel was non-random). So, for fixed m_x, the random probability weights $\left(T_x\left(h\middle|m_x\right)\right)_{h\in Z\!\!Z}$ are integer samples of a randomly perturbed Gaussian. The reader may also note the complementary fact that, for fixed h_x, the probability $T_x\left(h_x\middle|m_x\right)$ is bounded by Gaussians from below and above and has a unique absolute maximum as a function of m_x. The maximizer is close to $m^* h_x$. [See Lemma 2.1 for these statements]. Now, the simplicity of our basic log-sum-of-Gaussian potential (1.2) lies in the fact that $\text{Norm.}(m_x) = e^{-V_x(m_x)}$! [For more general potentials $V_x(m_x)$ this equality will acquire error terms ('anharmonicities') that lead to additional expansions.] We use the same symbol, $T\left(dh_\Lambda\middle|m_\Lambda\right) = \prod_{x\in\Lambda} T_x^{\omega_x}(h_x|m_x)$, for the kernel from $I\!R^\Lambda$ to $Z\!\!Z^\Lambda$, and also in the infinite volume.

Before we put down more results in a precise way, let us describe in an informal way in symbolic notation what we are about to do. Starting from a (finite volume) Gibbs measure $\mu(dm)$

we look at the joint distribution $M(dh,dm) := T(dh|m)\mu(dm)$ on integer heights configurations h and continuous heights m. Then we analyse μ with the use of Bayes' formula: We have $\mu(dm) := \int \nu(dh) M(dm|h)$ where $\nu(dh) = \int \mu(dm) T(dh|m)$ is the h-marginal. It is of course a completely general (and a-priori empty) idea to look at distributions in suitably extended space that can only be useful for natural choices of this space. In our case we succeed with the control of $\nu(dh)$ since we can obtain a contour representation that can be treated by the spatial renormalization group. The conditional probability $M(dm|h)$ is nice for the specific choice of the potential; it is just a Gaussian distribution. ($M(dm|h)$ would be more complicated for perturbations of the potential, but the above decomposition would still be a successful one.) Our results, to be described below, will then concern the approach of the thermodynamic limit of ν. We will also have to clarify the interplay of the thermodynamic limit with the above formulas.

The following theorem describes how the control of Gibbs-measures on the integer heights carries over to the control of the Gibbs-measures on the continuous heights, under some harmless additional condition.

Theorem 2: *Suppose that the discrete height measures $\nu_\Lambda^\omega := T\left(\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}\right)$ converge locally along a sequence of cubes Λ_N, centered at the origin, to a limiting measure $\nu^\omega(dh_{\mathbb{Z}^d})$ for a sequence of boundary conditions which is uniformly bounded, i.e. we have $\sup_{x\in\mathbb{Z}^d;N}|\tilde m_{\partial\Lambda_N;x}| \le M$, for some $M<\infty$. (That is, convergence takes place for expectations of all bounded local observables.) Assume moreover that we have the site-wise summability*

$$\sup_N \sum_{y\in\Lambda_N} (1-q\Delta_{\mathbb{Z}^d})^{-1}_{x,y}\, \nu^\omega_{\Lambda_N}|h_y| =: K_x(\omega) < \infty \tag{1.6}$$

for a sequence of increasing cubes Λ_N, for all $x\in\mathbb{Z}^d$ for a.e. configurations of the disorder ω. Then the measures $\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}$ converge locally to the infinite linear combination of Gauss measures given by

$$\mu^\omega := \int \nu^\omega(dh_{\mathbb{Z}^d}) \mathcal{N}\left[(1-q\Delta_{\mathbb{Z}^d})^{-1} m^*_{\mathbb{Z}^d}(h_{\mathbb{Z}^d}); (1-q\Delta_{\mathbb{Z}^d})^{-1}\right] \tag{1.7}$$

That is, convergence takes place for sequences of expectations of bounded measurable $f(m_V)$ that depend only on spins in the finite volume V.

The symbol $\mathcal{N}\left[a;(1-q\Delta_{\mathbb{Z}^d})^{-1}\right]$ denotes the massive Gaussian field on the infinite lattice $\mathbb{Z}^d$, centered at $a\in I\!R^{\mathbb{Z}^d}$ with covariance matrix given by the second argument (so that we have e.g. $\int \mathcal{N}\left[a;(1-q\Delta_{\mathbb{Z}^d})^{-1}\right](dm)(m_x-a_x)(m_y-a_y) = \left[(1-q\Delta_{\mathbb{Z}^d})^{-1}\right]_{x,y}$).

Remark: Note that the random quantities $K_x(\omega)$ will typically not be bounded uniformly in x. In fact, even a localized interface will have unbounded fluctuations around regions of exceptionally large fluctuations when considered in the infinite lattice. Of course, (1.6) is implied by $\sup_{N,y} I\!E\nu_\Lambda^\omega |h_y| < \infty$.

Remark: We stress that Theorem II does not only apply to the 'flat' interfaces that we investigate here but also to more 'exotic' Gibbs-measures. So, e.g. the (supposed) existence of Dobrushin-type integer-height Gibbs-measures (that are perturbations of a flat interface at height 0 in one half-space and a flat interface at height H in the complement) would imply the existence of corresponding continuous spin Gibbs-measures.

The organization of the paper is as follows. In Chapter II we prove the 'factorization' Theorem 2, starting from its finite volume version Lemma 2.1. In Chapter III we derive the contour representation of the integer height model (see Proposition 1), starting from the finite volume Hamiltonian (3.1). In Chapter IV we conclude to prove Theorem I from these results applying the spatial renormalization group construction from [BoK1], [K1] on the contour model representation.

II. The Joint distributions of continuous and integer heights

Before we get started, let us make explicit some (concentration-) properties of the random transition kernel to get some intuition for it. The elementary proof is given at the end of the chapter.

Lemma 2.1: *For any realization of the disorder satisfying the bounds (ii) and (iv) below (1.2) the fixed-h_x (random) probability $m_x \to T_x^{\omega_x}\left(h_x \middle| m_x\right) = \frac{e^{-\frac{1}{2}(m_x - m_x^*(h_x))^2 + \eta_x(h_x)}}{\sum_{l\in Z} e^{-\frac{1}{2}(m_x - m_x^*(l))^2 + \eta_x(l)}}$ has a unique absolute maximum. It has no other local maxima. The maximizer lies in the symmetric interval about $m^* h_x$ with radius $\frac{m^*\left[4\delta_d + \delta_d^2\right]}{4(1-\delta_d)} + \frac{1}{m^* 2(1-\delta_d)}\left\{\log\left[\frac{1+2\delta_d}{1-2\delta_d}\right] + 2\delta_\eta\right\}$. We have that*

$$
const\, e^{-\frac{1}{2}(m_x - m_x^*(h_x))^2} \le T_x\left(h_x \middle| m_x\right) \le Const\, e^{-\frac{1}{2}(m_x - m_x^*(h_x))^2} \tag{2.1}
$$

with const > 0, Const > 0 depending only on a, m^*, δ_d, δ_η.

Now, the simplicity of our choice of the log-sum-of-Gaussian potential lies in the fact that the joint distribution on continuous heights and integer height can be written in the form

$$
\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(dm_\Lambda) \prod_{x\in\Lambda} T_x(h_x | m_x) = \frac{1}{Z_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}} e^{-H_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda}(m_\Lambda)} dm_\Lambda \tag{2.2}
$$

where

$$
\begin{aligned}
&H_V^{\tilde m_{\partial V},\omega_V,h_V}(m_V) \\
&= \frac{q}{2}\sum_{\substack{\{x,y\}\subset V\\ d(x,y)=1}} (m_x - m_y)^2 + \frac{q}{2}\sum_{\substack{x\in V;\, y\in\partial V\\ d(x,y)=1}} (m_x - \tilde m_y)^2 + \frac{1}{2}\sum_{x\in V} (m_x - m_x^*(h_x))^2 - \sum_{x\in V} \eta_x(h_x)
\end{aligned} \tag{2.3}
$$

is quadratic in m, for fixed h. This is due to the cancellation of the normalization in the transition kernel against the exponential of the potential. We remark that, for potentials that can be viewed as perturbations of our specific log-sum-of-Gaussians the formula would acquire error terms and the present formula is the main contribution of a further expansion.

We will now rewrite the joint distribution as a product of the marginal in the integer heights and the conditional distribution of the continuous heights given the h. We see that the m-distribution conditioned on a fixed value of h is Gaussian. The h-marginals on the other hand can be computed by a Gaussian integration over m_Λ: Since the quadratic terms of the above integral are h-independent this Gaussian integration yields

$$\int_{\mathbb{R}^\Lambda} dm_\Lambda e^{-H_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda}(m_\Lambda)} = C_\Lambda \times e^{-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda}(m_\Lambda)} \tag{2.4}$$

with a constant C_Λ that does not depend on h_Λ (and ω_Λ).

By multiplying and dividing the r.h.s. of (2.1) by (2.3) we get after a little rewriting of the Gaussian density, conditional on the h:

Lemma 2.2: *The finite volume joint distribution of continuous and integer heights can be written as*

$$\begin{aligned}
&\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(dm_\Lambda)\prod_{x\in\Lambda}T_x(h_x|m_x) = \nu_\Lambda^{\omega_\Lambda}(h_\Lambda)\,\mathcal{N}\left[m_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda};\left(1-q\Delta_\Lambda^D\right)^{-1}\right](dm_\Lambda) \quad \textit{where}\\
&\nu_\Lambda^{\omega_\Lambda}(h_\Lambda) = T\left(\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}\right)(h_\Lambda) = \frac{e^{-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda}(m_\Lambda)}}{\sum_{\tilde h_\Lambda} e^{-\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,\tilde h_\Lambda}(m_\Lambda)}}
\end{aligned} \tag{2.5}$$

with the random centering

$$m_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda} = (1-q\Delta_\Lambda)^{-1}\left(m_\Lambda^*(h_\Lambda)+q\partial_{\Lambda,\partial\Lambda}\tilde m_{\partial\Lambda}\right) \tag{2.6}$$

Here $\partial_{\Lambda,\partial\Lambda}$ is the matrix having entries $(\partial_{\Lambda,\partial\Lambda})_{x,y} = 1$ iff $x\in\Lambda$ and $y\in\partial\Lambda$ are nearest neighbors and zero otherwise. Δ_Λ is the Lattice Laplacian (with Dirichlet boundary condition) in the volume Λ.

Side-remark: We like to point out that the enlargement of the probability space by the introduction of auxiliary integration variables and conditioning on the latter ones can be found in various places in statistical mechanics:

1) In the renormalization group analysis one studies the Gibbs-distributions $\mu(\Gamma)$ of the variables Γ of the system with the aid of a mapping to spatially coarse-grained variables Γ'

by means of a transition kernel $T(d\Gamma'|\Gamma)$. This gives rise to a joint distribution $M(d\Gamma, d\Gamma') = \mu(d\Gamma)T(d\Gamma'|\Gamma)$. Reversing the order of conditioning gives $M(d\Gamma, d\Gamma') = \nu(\Gamma')M(d\Gamma|\Gamma')$, the idea being that the 'renormalized' measures $\nu(\Gamma')$ are easier to study than the measures $\mu(\Gamma)$. (Of coarse, even if this is true, the conditional distribution $M(d\Gamma|\Gamma')$ must also be controlled.)

2) The introduction of artificial integration variables is a commonly used trick also for the analysis in quadratic mean-field models (known here as Hubbard-Stratonovitch transformation). In fact, the analogue of formulas (2.5) looks as follows for the simplest candidate, the usual mean-field Ising ferromagnet. Its Gibbs distribution on the spins $(\sigma_i)_{i=1,\dots,N} \in \{-1,1\}^N =: \Omega$ is given by $\mu_N(\sigma) = e^{\frac{\beta N}{2} m_N(\sigma)^2}/\text{Norm.}$ where $m_N(\sigma) = \frac{1}{N}\sum_{i=1}^N \sigma_i$. Consider the variables $(\sigma, \tilde m)$ in the enlarged probability space $\Omega \times I\!R$ (with a new smoothed out magnetization variable $\tilde m$) that are distributed according to the joint distribution $M(\sigma, d\tilde m) = e^{\beta N m(\sigma)\cdot \tilde m} e^{-\frac{\beta N}{2}\tilde m^2} d\tilde m/\text{Norm.}$. Then the marginal distribution on the σ is the desired Gibbs-distribution; in fact we have $M(\sigma, d\tilde m) = \mu(\sigma)T(\tilde m|\sigma)$ with $T(d\tilde m|\sigma) = e^{-\frac{\beta N}{2}(\tilde m - m(\sigma))^2} d\tilde m/\text{Norm.}$ describing the smoothed out averaging over the whole lattice. To analyse the Gibbs-distribution the joint distribution is then written by reversing the order of conditioning in the form $M(\sigma, d\tilde m) = \nu(d\tilde m)M(\sigma|\tilde m)$ where $\nu(d\tilde m)$ is the marginal on the $\tilde m$ and $M(\sigma|\tilde m) = \prod_i M_i(\sigma_i|\tilde m)$ where $M_i(\sigma_i|\tilde m) = e^{\beta \tilde m \sigma_i - \log 2\cosh(\beta \tilde m)}$. The latter kernel is clearly trivial, the distribution $\nu(d\tilde m)$ is treated by a saddle-point method.

Likewise, our strategy in the present problem will now be to control the ν-distribution in the thermodynamic limit and get the Gibbs-measures of the continuous spins by summing (2.5) over h. Assuming this control over the integer heights we must however also control what happens to (the h-average over) equation (2.5) under the thermodynamic limit if we apply it to a local function $f(m_V)$, depending on continuous heights m_x only for $x \in V$, V being a fixed finite volume. Note that the Gaussian describing the conditional distribution of the continuous heights, given the integer heights, has some Λ- dependence both through its centering and the covariance matrix. Further, its dependence on h is not finite range (albeit strongly decaying unless the integer-heights are getting very large.) So we need some extra condition on the convergence of ν_Λ-measure and a little work to deduce the implication of the desired convergence of the μ_Λ-measure.

The precise result of this is given in Theorem 2 (see Introduction), that we are going to prove now. While doing so, we will also prove the following

Addition to Theorem 2: *Assume the site-wise existence of all exponential moments*

$$\sup_\Lambda \nu_\Lambda^\omega \left[e^{s\sum_{y\in\Lambda}\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}_{x,y}|h_y|} \right] =: \tilde K_x(\omega, s) < \infty, \tag{2.7}$$

Then the convergence $\lim_{\Lambda\uparrow\infty}\mu_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda}(f_V) = \mu^\omega(f_V)$ *takes place also for all local observables*

$f(m_V)$ that do not increase faster than exponentially; i.e. there exists a constant $\lambda \geq 0$ s.t. $|f(m_V)| \leq e^{\lambda \|m_V\|_2}$.

Proof of Theorem 2: We start with the proof of the theorem under the 'site-wise summability assumption' (1.5). We must control large realizations of the h's (that are however improbable w.r.t ν, under this assumption.) Let f denote any measurable function of m_V, we assume for simplicity that f is uniformly bounded by 1. To produce a local observable (of the integer heights) we cut off the long range dependence of the Gaussians on the integer height h outside some volume Λ_2 that satisfies $V \subset \Lambda_2 \subset \Lambda$. We use an $\epsilon/3$-trick to decompose

$$\begin{aligned}
&\left|\nu_\Lambda^\omega \mathcal{N}\left[m_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda}; \left(1-q\Delta_\Lambda^D\right)^{-1}\right](f) - \nu_\infty^\omega \mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1} m^*_{\mathbb{Z}^d}\left(h_{\mathbb{Z}^d}\right); \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f)\right| \\
&\leq \nu_\Lambda^\omega \Bigg| \mathcal{N}\left[m_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda}; \left(1-q\Delta_\Lambda^D\right)^{-1}\right](f) \\
&\qquad - \mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m^*_{\Lambda_2}\left(h_{\Lambda_2}\right), 0_{\Lambda_2^c}\right); \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f)\Bigg| \\
&+ \left|\left(\nu_\Lambda^\omega - \nu_\infty^\omega\right)\mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m^*_{\Lambda_2}\left(h_{\Lambda_2}\right), 0_{\Lambda_2^c}\right); \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f)\right| \\
&+ \nu_\infty^\omega \Bigg| \mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1} m^*_{\mathbb{Z}^d}\left(h_{\mathbb{Z}^d}\right); \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f) \\
&\qquad - \mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m^*_{\Lambda_2}\left(h_{\Lambda_2}\right), 0_{\Lambda_2^c}\right); \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f)\Bigg|
\end{aligned} \tag{2.8}$$

We will show that the r.h.s. can be made arbitrarily small be choosing at first the auxiliary Λ_2 and then Λ large enough. Indeed, the middle term on the r.h.s. converges to zero with $\Lambda \uparrow \infty$, for any fixed Λ_2, due to the assumption of weak convergence of the ν_Λ^ω. The remaining task is to control the two error-terms; for this we need the condition (1.5) [resp. (2.7)].

We look at the first term on the r.h.s. more carefully. The last term is treated in a similar fashion. We need some continuity properties of $|V|$-dimensional Gaussian expectations considered as functions of their means and covariances. The following estimate will do, both for bounded observables and observables that are only exponentially bounded.

Lemma 2.3: *Let $\mathcal{N}[a,\Sigma]$, $\mathcal{N}[a',\Sigma']$ denote two $|V|$-dimensional non-degenerate Gaussians with mean $a, a' \in \mathbb{R}^V$ and covariance $\Sigma, \Sigma' \in \mathbb{R}^{V\times V}$. Assume that $f(m_V)$ is an observable that doesn't increase faster than exponentially, i.e. $|f(m_V)| \leq e^{\lambda\|m_V\|_2}$ for some $\lambda \geq 0$. Then we* *have the following estimate*

$$
\begin{aligned}
&\left|\int \mathcal{N}\left[a;\Sigma\right](dm_V)f(m_V) - \int \mathcal{N}\left[a';\Sigma'\right](dm_V)f(m_V)\right| \\
&\leq 2^{|V|}e^{\lambda\|a\|_2}e^{\frac{\lambda^2 \mathrm{Tr}\Sigma}{2}}\left[\left|1-\left(\frac{\det\Sigma}{\det\Sigma'}\right)^{\frac{1}{2}}\right|\right. \\
&\left.+\left(\frac{\det\Sigma}{\det\Sigma'}\right)^{\frac{1}{2}} g\left(\left(2S+\|a\|_2+\|a'\|_2\right)\|\Sigma^{-1}\|_2\,\|a-a'\|_2+2\left(S^2+\|a'\|_2^2\right)\times\|\Sigma^{-1}-{\Sigma'}^{-1}\|_2\right)\right] \\
&+2^{|V|}e^{\lambda S-\frac{(S-\|a\|_2)^2}{2\mathrm{Tr}\Sigma}}+2^{|V|}e^{\lambda S-\frac{(S-\|a'\|_2)^2}{2\mathrm{Tr}\Sigma'}}
\end{aligned}
\tag{2.9}
$$

where $g(x)=xe^x$, for any $S\geq\max\{\|a\|_2+\lambda\,Tr\Sigma,\|a'\|_2+\lambda\,Tr\Sigma'\}$.

Proof: To show the Lemma we decompose the range of integration into a ball of radius S and its complement; to control the corresponding integrals we need a simple application of the exponential Markov inequality in the form of

Lemma 2.4: *Let $\mathcal{N}[a,\Sigma]$ denote a Gaussian with mean $a\in I\!R^V$ and covariance $\Sigma\in I\!R^{V\times V}$ (i.e. $\int\mathcal{N}[a,\Sigma](dm_V)\,(m_x-a_x)\,(m_y-a_y)=\Sigma_{x,y}$). Then we have*

$$
\begin{aligned}
&\int \mathcal{N}\left[a;\Sigma\right](dm_V)e^{\lambda\|m_V\|_2}\leq 2^{|V|}e^{\lambda\|a\|_2}e^{\frac{\lambda^2\mathrm{Tr}\Sigma}{2}} \quad \textit{and} \\
&\int \mathcal{N}\left[a;\Sigma\right](dm_V)e^{\lambda\|m_V\|_2}1_{\|m_V\|_2\geq S}\leq 2^{|V|}e^{\lambda S-\frac{(S-\|a\|_2)^2}{2\mathrm{Tr}\Sigma}} \quad \textit{for}\quad S\geq\|a\|_2+\lambda\,Tr\Sigma
\end{aligned}
\tag{2.10}
$$

For $S=\|a\|_2+\lambda\,Tr\Sigma$ the two r.h.s. coincide.

Proof: Write $\int\mathcal{N}\left[a;\Sigma\right](dm_V)e^{\lambda\|m_V\|_2}\leq e^{\lambda\|a\|_2}\int\mathcal{N}\left[a;\Sigma\right](dm_V)e^{\lambda\|m_V-a\|_2}$ and denote by σ_i^2 the eigenvalues of Σ and by e_i the corresponding eigenvectors. Then we may write

$$
\int\mathcal{N}\left[a;\Sigma\right](dm_V)e^{\lambda\|m_V-a\|_2}\leq\left(\prod_{i=1,\ldots,|V|}\int\mathcal{N}\left[0;\sigma_i^2\right](d\hat{m}_i)e^{\lambda|\hat{m}_i|}\right)\leq 2^{|V|}e^{\frac{\lambda^2\sum_i\sigma_i^2}{2}}
\tag{2.11}
$$

which proves the first estimate. The second one is a corollary: Write, for nonnegative λ_1,

$$
\begin{aligned}
&\int\mathcal{N}\left[a;\Sigma\right](dm_V)e^{\lambda\|m_V\|_2}1_{\|m_V\|_2\geq S}\leq e^{-\lambda_1 S}\int\mathcal{N}\left[a;\Sigma\right](dm_V)e^{(\lambda+\lambda_1)\|m_V\|_2}1_{\|m_V\|_2\geq S} \\
&\leq e^{-\lambda_1 S}2^{|V|}e^{(\lambda+\lambda_1)\|a\|_2}e^{\frac{(\lambda+\lambda_1)^2\mathrm{Tr}\Sigma}{2}}
\end{aligned}
\tag{2.12}
$$

where the last inequality follows from the first claim. Minimizing the r.h.s. yields the claim (the optimal value of λ_1 being $\lambda_1=\frac{S-\|a\|_2}{\mathrm{Tr}\Sigma}-\lambda$). The requirement that λ_1 be positive leads to the given range of allowed S. $\diamondsuit$

We continue with the proof of the Lemma 2.3 writing

$$\begin{aligned}
&\left|\int \mathcal{N}[a;\Sigma](dm_V)f(m_V) - \int \mathcal{N}[a';\Sigma'](dm_V)f(m_V)\right| \\
&\leq \left|\left(\int \mathcal{N}[a;\Sigma] - \int \mathcal{N}[a';\Sigma']\right)(dm_V)f(m_V)1_{|m_V|\leq S}\right| \\
&\quad + \int \mathcal{N}[a;\Sigma]\, e^{\lambda\|m_V\|}1_{|m_V|\geq S} + \int \mathcal{N}[a';\Sigma']\, e^{\lambda\|m_V\|}1_{|m_V|\geq S}
\end{aligned} \tag{2.13}$$

The last two terms are estimated with the help of the above lemma, leading to the last line of (2.9). The first term can be estimated simply in terms of differences of the Gaussian densities:

$$\begin{aligned}
&\left|\int dm_V \left(\frac{e^{-\frac{1}{2}<(m_V-a),\Sigma^{-1}(m_V-a)>}}{(2\pi)^{\frac{|V|}{2}}(\det\Sigma)^{\frac{1}{2}}} - \frac{e^{-\frac{1}{2}<(m_V-a'),\Sigma'^{-1}(m_V-a')>}}{(2\pi)^{\frac{|V|}{2}}(\det\Sigma')^{\frac{1}{2}}}\right) f(m_V)1_{|m_V|\leq S}\right| \\
&\leq \sup_{|m_V|\leq S}\left|1 - e^{\frac{1}{2}\left[<(m_V-a),\Sigma^{-1}(m_V-a)>-<(m_V-a'),\Sigma'^{-1}(m_V-a')>\right]}\left(\frac{\det\Sigma}{\det\Sigma'}\right)^{\frac{1}{2}}\right| \\
&\quad\times \int dm_V \frac{e^{-\frac{1}{2}<(m_V-a),\Sigma^{-1}(m_V-a)>}}{(2\pi)^{\frac{|V|}{2}}(\det\Sigma)^{\frac{1}{2}}} f(m_V)1_{|m_V|\leq S}
\end{aligned} \tag{2.14}$$

The last term is estimated by dropping the characteristic function and applying the first statement in Lemma 2.4. The last sup is estimated from above by

$$\left|1 - \left(\frac{\det\Sigma}{\det\Sigma'}\right)^{\frac{1}{2}}\right| + \left(\frac{\det\Sigma}{\det\Sigma'}\right)^{\frac{1}{2}}\sup_{|m_V|\leq S}\left|1 - e^{\frac{1}{2}\left[<(m_V-a),\Sigma^{-1}(m_V-a)>-<(m_V-a'),\Sigma'^{-1}(m_V-a')>\right]}\right| \tag{2.15}$$

where, using the simple estimate $|e^x - 1| \leq |x|e^{|x|} =: g(|x|)$ the sup in the last expression can be estimated by $g\left[\frac{1}{2}\sup_{|m_V|\leq S}\left|< (m_V - a), \Sigma^{-1}(m_V - a) > - < (m_V - a'), \Sigma'^{-1}(m_V - a') >\right|\right]$. For this last sup in the argument of g we use the upper estimates in terms of two-norms

$$\begin{aligned}
&2\sup_{|m_V|\leq S}\left|< m_V, \Sigma^{-1}(a-a') >\right| + \left|< a, \Sigma^{-1}a > - < a', \Sigma^{-1}a' >\right| \\
&\quad + \sup_{|m_V|\leq S}\left|< (m_V - a'), \left[\Sigma^{-1} - \Sigma'^{-1}\right](m_V - a') >\right| \\
&\leq \left(2S + \|a\|_2 + \|a'\|_2\right)\|\Sigma^{-1}\|_2\,\|a - a'\|_2 + 2\left(S^2 + \|a'\|_2^2\right)\times\|\Sigma^{-1} - \Sigma'^{-1}\|_2
\end{aligned} \tag{2.16}$$

Collecting our results gives Lemma 2.3. $\diamondsuit$

To apply the Lemma we just use the short notation

$$\begin{aligned}
a &:= a(\Lambda) := \Pi_V m_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda} = \Pi_V\left(1 - q\Delta_\Lambda\right)^{-1}\left(m_\Lambda^*(h_\Lambda) + q\partial_{\Lambda,\partial\Lambda}\tilde m_{\partial\Lambda}\right) \\
a' &:= a'(\Lambda_2) := \Pi_V\left(1 - q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m_{\Lambda_2}^*(h_{\Lambda_2}), 0_{\mathbb{Z}^d\setminus\Lambda_2}\right)
\end{aligned} \tag{2.17}$$

for the expectations of the $|V|$-dimensional Gaussians under consideration. We also denote by $\Sigma := \Pi_V\left(1 - q\Delta_{Z^d}\right)^{-1}\Pi_V$ the infinite volume covariance matrix restricted to V, and correspondingly $\Sigma' := \Sigma'(\Lambda) := \Pi_V\left(1 - q\Delta_\Lambda^D\right)^{-1}\Pi_V$.

The volume difference of the covariances is fairly harmless: Given $\epsilon > 0$ we can choose Λ_0 sufficiently large, s.t. for all $\Lambda \supset \Lambda_0$, we have that $\left|1-\left(\frac{\det\Sigma}{\det\Sigma'(\Lambda)}\right)^{\frac{1}{2}}\right| \leq \epsilon$ and $\|\Sigma^{-1}-\Sigma'(\Lambda)^{-1}\|_2 \leq \epsilon$. Further, all matrix elements of $\Sigma'(\Lambda)$ are bounded from above by the corresponding infinite volume expression Σ. In particular, we can use the upper bound $\mathrm{Tr}\Sigma'(\Lambda) \leq \mathrm{Tr}\Sigma$. (All of this can be explicitly seen from the random walk representation of the resolvent, see e.g. [K4])

Assuming these choices we get with Lemma 2.3

$$
\begin{aligned}
&\left|\int \mathcal{N}[a;\Sigma](dm_V)f(m_V) - \int \mathcal{N}[a';\Sigma'](dm_V)f(m_V)\right| \\
&\leq 2^{|V|}\left[\epsilon + (1+\epsilon)g\left((2S+\|a\|_2+\|a'\|_2)\|\Sigma^{-1}\|_2\|a-a'\|_2 + 2\epsilon\left(S^2+\|a'\|_2^2\right)\right)\right] \\
&+ 2^{|V|}e^{-\frac{(S-\|a\|_2)^2}{2\mathrm{Tr}\Sigma}} + 2^{|V|}e^{-\frac{(S-\|a'\|_2)^2}{2\mathrm{Tr}\Sigma}}
\end{aligned}
\tag{2.18}
$$

for $\Lambda \supset \Lambda_0(\epsilon)$ where the Σ-terms appear now as fixed constants.

To estimate the ν_Λ^ω-expectation of this bound we will decompose the space of the integer heights into a 'regular set' $\mathcal{H} := \mathcal{H}(\Lambda_2,\Lambda) := \mathcal{H}^{(1)}(\Lambda_2,\Lambda) \cap \mathcal{H}^{(2)}(\Lambda_2,\Lambda)$ where

$$
\begin{aligned}
\mathcal{H}^{(1)}(\Lambda_2,\Lambda) &:= \{h_{\mathbb{Z}^d}; \|a(\Lambda)\|_2 \leq B, \|a'(\Lambda_2)\|_2 \leq B\} \\
\mathcal{H}^{(2)}(\Lambda_2,\Lambda) &:= \{h_{\mathbb{Z}^d}; \|a(\Lambda)-a'(\Lambda_2)\|_2 \leq \epsilon_2\}
\end{aligned}
\tag{2.19}
$$

and its complement. We get from this (with $\|f\|_\infty \leq 1$) that

$$
\begin{aligned}
&\nu_\Lambda^\omega\left|\mathcal{N}\left[m_\Lambda^{\tilde m_{\partial\Lambda},\omega_\Lambda,h_\Lambda};\left(1-q\Delta_\Lambda^D\right)^{-1}\right](f) - \mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m^*_{\Lambda_2}(h_{\Lambda_2}),0_{\Lambda_2^c}\right);\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f)\right| \\
&\leq \nu_\Lambda^\omega\left[\mathcal{H}(\Lambda_2,\Lambda)^c\right] + 2^{|V|}\left[\epsilon + (1+\epsilon)g\left((2S+B+B)\|\Sigma^{-1}\|_2\,\epsilon_2 + 2\epsilon\left(S^2+B^2\right)\right)\right] \\
&+ 2^{|V|}e^{-\frac{(S-B)^2}{2\mathrm{Tr}\Sigma}} + 2^{|V|}e^{-\frac{(S-B)^2}{2\mathrm{Tr}\Sigma}}
\end{aligned}
\tag{2.20}
$$

To estimate the exceptional set of integer heights we will prove below

Lemma 2.5:

(i) For all (arbitrarily small) δ there exists a $B<\infty$ (sufficiently large) s.t. $\nu_\Lambda^\omega\left[\mathcal{H}^{(1)}(\Lambda_2,\Lambda)^c\right] \leq \delta$ for all sufficiently large Λ_2,Λ.

(ii) For all (arbitrarily small) δ,ϵ_2, there exist choices of volumes $\Lambda_2 \subset \tilde\Lambda_0$ (suff. large) s.t. $\nu_\Lambda^\omega\left[\mathcal{H}^2(\Lambda_2,\Lambda)^c\right] \leq \delta$ whenever $\Lambda \supset \tilde\Lambda_0$.

But assuming this property, (2.20) can be made smaller than any given δ for sufficiently large Λ in the following way:

1) Choose $B = S/2$ large enough that a) the sum of the last two terms is smaller than $\delta/3$ and b) $\nu_\Lambda^\omega \left[\mathcal{H}^{(1)}(\Lambda_2,\Lambda)^c\right] \le \delta/6$, according to Lemma 2.5(i). (So we must have that both Λ_2, Λ are large enough.)

2) Given these choices of S, B, choose ϵ, ϵ_2 small enough s.t. the middle term is smaller than $\delta/3$. (Then the estimates hold true, after possibly enlarging Λ.)

3) Finally there are then choices of $\Lambda_2 \subset \tilde\Lambda_0$ s.t. $\nu_\Lambda^\omega \left[\mathcal{H}^{(2)}(\Lambda_2,\Lambda)^c\right] \le \delta/6$ for all $\Lambda \supset \tilde\Lambda_0$, according to Lemma 2.5(ii).

This finishes our discussion of the proof of Theorem 2; it remains however to give the ***Proof of Lemma 2.5:*** In fact, the Lemma holds under the following two weaker conditions of

(a) Uniform integrability $\lim_{B\uparrow\infty} \sup_\Lambda \nu_\Lambda^\omega \left[\sum_{y\in\mathbb{Z}^d} R_{\infty;x,y}|h_y| \ge B\right] = 0$

(b) Uniform summability $\lim_{R\uparrow\infty} \sup_\Lambda \nu_\Lambda^\omega \left[\sum_{y\in\mathbb{Z}^d;|y|\ge R} R_{\infty;x,y}|h_y| \ge \epsilon\right] = 0$

for each $x \in \mathbb{Z}^d$. (These condition are implied from the hypothesis by Chebycheff, e.g. $\sup_\Lambda \nu_\Lambda^\omega \left[\sum_{y\in\mathbb{Z}^d} R_{\infty;x,y}|h_y| \ge B\right] \le \frac{1}{B}\sum_{y\in\mathbb{Z}^d} R_{\infty;x,y} \sup_\Lambda \nu_\Lambda^\omega \left[|h_y|\right]$.)

(i): Note that, due to the exponential decay of the resolvent we have for uniformly bounded boundary conditions that $\lim_{\Lambda\uparrow\infty} \Pi_V \left(1-q\Delta_\Lambda\right)^{-1} q\partial_{\Lambda,\partial\Lambda}\tilde m_{\partial\Lambda} = 0$. It suffices to look at one matrix element, say $a_x(\Lambda)$ of the vector $a(\Lambda)$. Using that $R_{\Lambda;x,y} \le R_{\infty;x,y}$ uniform in the volume, the form of $m^*(h)$, and the uniform boundedness of the random shift in the continuous spin Hamiltonian, it is immediate to see that what we need is implied by the above uniform integrability condition.

(ii): The differences are estimated as follows. For $x \in V$ we have

$$
\begin{aligned}
&\left| \left[\left(1-q\Delta_\Lambda\right)^{-1} \left(m^*_\Lambda(h_\Lambda) + q\partial_{\Lambda,\partial\Lambda}\tilde m_{\partial\Lambda}\right)\right]_x - \left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m^*_{\Lambda_2}(h_{\Lambda_2}), 0_{\mathbb{Z}^d\setminus\Lambda_2}\right)\right]_x \right| \\
&\le \sum_{y\in\Lambda_2} \left| \left(1-q\Delta_\Lambda\right)^{-1}_{x,y} - \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}_{x,y}\right| \left|m^*_y(h_y)\right| \\
&+ \sum_{y\in\Lambda\setminus\Lambda_2} \left(1-q\Delta_\Lambda\right)^{-1}_{x,y} \left|m^*_y(h_y)\right| + qM \sum_{y\in\partial\Lambda} \left(1-q\Delta_\Lambda\right)^{-1}_{x,y} \\
&\le \sup_{y\in\Lambda_2} \left| \frac{\left(1-q\Delta_\Lambda\right)^{-1}_{x,y}}{\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}_{x,y}} - 1\right| \sum_{y\in\Lambda_2} \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}_{x,y} \left|m^*_y(h_y)\right| \\
&+ \sum_{y\in\Lambda\setminus\Lambda_2} \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}_{x,y} \left|m^*_y(h_y)\right| + qM \sum_{y\in\partial\Lambda} \left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}_{x,y}
\end{aligned}
\tag{2.21}
$$

We need to show that the ν_Λ^ω-probability of the event that the r.h.s. is bigger than some $\tilde\epsilon_2$ can be made small by choosing the volumes in a useful way. The last (deterministic) term converges to zero; so we assume that Λ is large enough s.t. it is smaller than $\tilde{\epsilon}_2/3$. Now, from (b) we know that, given any δ, we have for all sufficiently large Λ_2 that $\sum_{y\in\Lambda\setminus\Lambda_2}(1-q\Delta_{\mathbb{Z}^d})^{-1}_{x,y}\left|m^*_y(h_y)\right|\leq\tilde{\epsilon}_2/3$ with (say) ν^ω_Λ-probability bigger than $1-\delta/2$. We fix such a Λ_2. What we have just seen in the proof of (i) ensures that, for given δ we can find a B such that the sum of $y\in\Lambda_2$ on the r.h.s. of the last inequality is bounded by B, uniformly in Λ_2, with (say) ν^ω_Λ-probability bigger than $1-\delta/2$. Now it remains to choose Λ is large as we want to make the sup over y's in the fixed Λ_2 as small as we want, and thus the first line on the r.h.s. smaller than $\tilde{\epsilon}_2/3$ to finish the proof of Lemma 2.5.$\diamondsuit$

Let us finally give the modifications needed to get the

Proof of the Addition to Theorem 2:. Let us look again at first at the first term of the decomposition (2.8) where f is now a local observable that is only exponentially bounded. We introduce the same type of exceptional set $\mathcal{H}(\Lambda_2,\Lambda)$. Then, after using the Lemma 2.4 for the Gaussian expectation *on* the exceptional set, the analogue of (2.20) becomes

$$\begin{aligned}
&\nu^\omega_\Lambda\left|\mathcal{N}\left[m_\Lambda^{\bar{m}_{\partial\Lambda},\omega_\Lambda,h_\Lambda};\left(1-q\Delta^D_\Lambda\right)^{-1}\right](f)-\mathcal{N}\left[\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\left(m^*_{\Lambda_2}\left(h_{\Lambda_2}\right),0_{\Lambda_2^c}\right);\left(1-q\Delta_{\mathbb{Z}^d}\right)^{-1}\right](f)\right|\\
&\leq 2^{|V|}e^{\frac{\lambda^2\mathrm{Tr}\Sigma}{2}}\nu^\omega_\Lambda\left[e^{\lambda\|a\|_2}1_{\mathcal{H}(\Lambda_2,\Lambda)^c}\right]+2^{|V|}e^{\lambda B}e^{\frac{\lambda^2\mathrm{Tr}\Sigma}{2}}\\
&\times\left[\epsilon+(1+\epsilon)g\left((2S+B+B)\left\|\Sigma^{-1}\right\|_2\epsilon_2+2\epsilon\left(S^2+B^2\right)\right)\right]+2^{|V|}e^{\lambda S-\frac{(S-B)^2}{2\mathrm{Tr}\Sigma}}+2^{|V|}e^{\lambda S-\frac{(S-B)^2}{2\mathrm{Tr}\Sigma}}
\end{aligned}\tag{2.22}$$

To treat the ν-integral over the exceptional set, use the Schwartz inequality $\nu^\omega_\Lambda\left[e^{\lambda\|a\|_2}1_{\mathcal{H}(\Lambda_2,\Lambda)^c}\right]\leq\left(\nu^\omega_\Lambda\left[e^{2\lambda\|a\|_2}\right]\right)^{\frac12}\nu^\omega_\Lambda\left[\mathcal{H}(\Lambda_2,\Lambda)^c\right]^{\frac12}$. Now, given the assumption of the existence of exponential moments, the first term on the r.h.s. can easily seen to be bounded by a constant independent of Λ. But after this, we are essentially in the same situation as after (2.20) and the way of choosing the parameters stays the same as before. $\diamondsuit$

We are still due the

Proof of Lemma 2.1: By joint shift in height-direction we can assume that $h_x=0$. We write the kernel in the form $T_x\left(h_x=0\middle|m_x\right)=\left[1+\sum_{h\in\mathbb{Z};h\neq0}f_h(m_x)\right]^{-1}$ with $f_h(m_x):=e^{(m^*_x(h_x)-m^*_x(h_x=0))m-\frac12\left[m^*_x(h_x)^2-m^*_x(h_x=0)^2\right]+\eta_x(h_x)-\eta_x(h_x=0)}$. To prove unicity of the local maximum of the kernel we note that $m_x\mapsto\sum_{h\in\mathbb{Z};h\neq0}f_h(m_x)$ (being a sum of strictly convex functions) is a strictly convex function; hence it has a unique local minimum which is the global minimum.

To prove the bounds on the minimizer we look at the individual minimizers $\bar{m}(h)$ of each of the pairs f_h+f_{-h} for $h=1,2,\dots$. We will show that $|\bar{m}(h)|\leq A$, for all h. This implies that the minimizer of $m_x \mapsto \sum_{h\in \mathbb{Z};h\neq 0} f_h(m_x)$ satisfies the same bound (since all terms in the sum are strictly decreasing [increasing] for $m_x \geq A$ [$\leq -A$].) Now, a computation gives

$$
\begin{aligned}
\bar{m}(h) = &\frac{1}{m^*\left(2h + d_x(h) - d_x(-h)\right)}\Biggl\{\log\left[\frac{h - d_x(-h) + d_x(h=0)}{h + d_x(-h) - d_x(h=0)}\right]\\
&+\frac{(m^*)^2}{2}\left[2h(d_x(h)+d_x(-h)) + d_x(h)^2 - d_x(-h)^2\right] - \eta_x(h) + \eta_x(-h)\Biggr\}
\end{aligned}
\tag{2.23}
$$

Substituting the a-priori bounds of the random quantities $|d_x(h)| \leq \delta_d$ and $|\eta_x(h)| \leq \delta_\eta$ we get from this

$$
\begin{aligned}
|\bar{m}(h)| &\leq \frac{1}{m^* 2h(1-\delta_d)}\left\{\log\left[\frac{h+2\delta_d}{h-2\delta_d}\right] + \frac{(m^*)^2}{2}\left[4h\delta_d + \delta_d^2\right] + 2\delta_\eta\right\}\\
&\leq \frac{m^*\left[4\delta_d + \delta_d^2\right]}{4(1-\delta_d)} + \frac{1}{m^* 2(1-\delta_d)}\left\{\log\left[\frac{1+2\delta_d}{1-2\delta_d}\right] + 2\delta_\eta\right\}
\end{aligned}
\tag{2.24}
$$

To see the last estimates, just look at the nominator Norm.(m_x) (see 1.4): It is simple to check that this sum converges and it is bounded from above as well as bounded from below away from 0. From this the bounds (2.1) follow. $\diamondsuit$

III. A useful contour representation for discrete heights

In this chapter we will treat the measures ν_Λ^ω for the discrete height model that are given by (2.5) with (2.3). The effective finite volume Hamiltonian for the integer heights can be simply computed as a minimum of a quadratic expression in continuous variables: For any boundary condition $\tilde{m}$ it reads

$$
\begin{aligned}
&\inf_{m_\Lambda \in \mathbb{R}^\Lambda} H_\Lambda^{\tilde{m}_{\partial\Lambda},\omega_\Lambda,h_\Lambda}(m_\Lambda) = -\frac{1}{2q} < m^*_\Lambda(h_\Lambda), R_\Lambda m^*_\Lambda(h_\Lambda) >_\Lambda + \frac{1}{2}\sum_{x\in\Lambda}(m^*_x(h_x))^2 - \sum_{x\in\Lambda}\eta_x(h_x)\\
&-\frac{1}{q} < \tilde{\eta}_{\partial(\Lambda^c)}(q\tilde{m}), R_\Lambda m^*_\Lambda(h_\Lambda) >_\Lambda\\
&\quad-\frac{1}{2q} < \tilde{\eta}_{\partial(\Lambda^c)}(q\tilde{m}), R_\Lambda\left(\tilde{\eta}_{\partial(\Lambda^c)}(q\tilde{m})\right) >_\Lambda + \frac{q}{2}\sum_{\substack{x\in\Lambda; y\in\partial\Lambda\\ d(x,y)=1}} \tilde{m}_y^2 \quad \text{with}\\
&R_\Lambda = \left(q^{-1} - \Delta_\Lambda\right)^{-1}
\end{aligned}
\tag{3.1}
$$

with $\tilde{\eta}_{\partial(\Lambda^c)}(\tilde{m}) := \partial_{\Lambda,\partial\Lambda}\tilde{m}_{\partial\Lambda}$ denoting the field created by the boundary condition. We also note that the continuous-spin minimizer is given by (2.6).

We will deduce a contour representation for the ν-measures. Let us give the commonly used notion of 'contour', adapted to this model:

Definitions: *A* **contour** Γ *in the volume* Λ *is a pair composed of a* **support** $\underline{\Gamma}\subset\Lambda$ *and a 'height configuration'* $h_\Lambda \in \mathbb{Z}^\Lambda$*, such that the extended configuration* $(h_\Lambda, 0_{\mathbb{Z}^d\setminus\Lambda})$ *is constant on* *connected components of* $\mathbb{Z}^d \backslash \underline{\Gamma}$. *A* **contour model representation** *for a probability measure* ν *on the space* $\mathbb{Z}^\Lambda$ *of integer height- configurations in* Λ *is a probability measure* Q *on the space of contours in* Λ *whose height-marginal reproduces* ν, *i.e.* $\nu(\{h_\Lambda\}) = \sum_{\substack{\Gamma: \\ h_\Lambda(\Gamma) = h_\Lambda}} Q(\{\Gamma\})$. *The* **connected components** *of a contour* Γ *are the contours* γ_i *whose supports are the connected components* $\underline{\gamma}_i$ *of* $\underline{\Gamma}$ *and whose sign is determined by the requirement that it be the same as that of* Γ *on* $\overline{\underline{\gamma}_i}$.

The result of this chapter is

Proposition 1: *Suppose that* q *is sufficiently small,* $q(m^*)^2$ *sufficiently large and* $\delta_d \leq \frac{1}{4}$. *Then there is a* h_Λ*-independent constant* $K_\Lambda(\omega_\Lambda)$ *s.t. we have the representation*

$$e^{-\inf_{m_\Lambda \in \mathbb{R}^\Lambda} H_\Lambda^{\tilde{m}_{\partial\Lambda}=0,\omega_\Lambda,h_\Lambda}(m_\Lambda)} = K_\Lambda(\omega_\Lambda) \times e^{-<S(\omega),V(h_\Lambda)>} \sum_{\substack{\Gamma \\ h_\Lambda(\Gamma) = h_\Lambda}} \rho_0(\Gamma; \omega_{\underline{\Gamma}}) \tag{3.2}$$

for any $h_\Lambda \in \mathbb{Z}^\Lambda$. *The quantities in the above representation are as follows:*

(i) Small fields: S_C *is a random variable for each* $h \in \mathbb{Z}$ *and* $C \subset \mathbb{Z}^d$ *connected and we have used the notation*

$$< S(\omega), V(h_\Lambda) > := \sum_{h \in \mathbb{Z}} \sum_{C \subset \Lambda \cap V_h} S_C(h) \quad \textit{where} \quad V_h(h_\Lambda) := \{x \in \Lambda, h_x = h\} \tag{3.3}$$

The $S_C(h)$ *are functions of the random centerings* $d_C(h) = (d_x(h))_{x \in C}$ *for* $|C| \geq 2$. *Up to a boundary term (see below), the single-site part* $\beta S_x(h)$ *is the 'random depth'* $\eta_x(h)$.

We have the smallness properties, for all realizations of the disorder,

$$\begin{aligned} &|S_C(h)| \leq Const\, \delta_d^2 e^{-const\, \alpha |C|}, \quad \textit{for} \quad |C| \geq 3 \quad \textit{with} \quad \alpha = \frac{1}{2}\log\left[1 + 1/(2dq)\right] \sim \frac{1}{2}\log\frac{1}{q} \\ &|S_C(h)| \leq Const\, \left(\delta_d^2 + \delta_\eta\right) \quad \textit{for} \quad |C| \leq 2 \end{aligned} \tag{3.4}$$

with $const$, $Const$ *being of the order unity, depending only on the dimension.*

(ii) Contour-Activities: *The activity* $\rho_0(\Gamma; \omega_{\underline{\Gamma}})$ *is non-negative. It factorizes over the connected components of* Γ, *i.e. we have* $\rho_0(\Gamma; \omega_{\underline{\Gamma}}) = \prod_{\gamma_i \text{ conn cp. of } \Gamma} \rho_0(\gamma_i; \omega_{\gamma_i})$.

For $\underline{\Gamma}$ *not touching the boundary (i.e.* $\partial_{\partial\Lambda}\underline{\Gamma} = \emptyset$*) the value of* $\rho_0(\Gamma; \omega_{\underline{\Gamma}})$ *is independent of* Λ. *We then have the 'infinite volume properties' of invariance under joint lattice shifts of spins and random fields, as well as under joint shift in the height-direction.*

Peierls-type bounds: *There exist positive constants* $\tilde{\beta}, \beta$ *s.t. we have the upper bounds*

$$0 \leq \rho_0(\Gamma; \omega_{\underline{\Gamma}}) \leq e^{-\beta E_s(\Gamma) - \tilde{\beta}|\underline{\Gamma}|} \tag{3.5}$$

with the 'nearest-neighbor contour energy'

$$E_s(\Gamma) := \sum_{\substack{\{x,y\}\subset\underline{\Gamma} \\ |x-y|=1}} |\tilde{h}_x - \tilde{h}_y| \quad \textit{for a contour} \quad \Gamma = \left(\underline{\Gamma}, (\tilde{h}_x)_{x\in\Lambda}\right) \tag{3.6}$$

In the above estimates the 'Peierls-constants' can be chosen like

$$\beta = q(m^*)^2 \frac{(1-2\delta_d)^2}{12[(1+2dq)^2 - q^2]}, \quad \tilde{\beta} = Const \times \min\left\{\log\frac{1}{q}, qm^{*2}\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d\right\} \quad (\leq \alpha, \beta) \tag{3.7}$$

Probabilistic bounds for the small field: *For* $|C| = 1, 2$ *we have* $I\!E S_C = 0$ *and*

$$I\!P\left[|S_C(h)| \geq t\right] \leq e^{-\frac{t^2}{2\sigma^2}} \quad \textit{with} \quad \sigma^2 = const\,(\sigma_\eta^2 + \sigma_d^2) \tag{3.8}$$

Proof: To produce a sum of the type (3.2) we need to decompose the terms in the Hamiltonian in such a way as to exhibit a low-temperature part, a non-local field part and high-temperature parts that can be expanded. We will produce 'support of contours' from all these various sources. As we will see there we will have to introduce some type of support that occurs only for unbounded spin models with interactions having no finite range.

Remember that we put zero boundary conditions. First of all it is now convenient to rewrite the integer-height Hamiltonian in the following form that makes explicit that it has purely ferromagnetic couplings:

$$\begin{aligned}
&\inf_{m_\Lambda \in I\!R^\Lambda} H_\Lambda^{\tilde{m}_{\partial\Lambda}=0,\omega_\Lambda,h_\Lambda}(m_\Lambda) \\
&= \sum_{\{x,y\}\subset\overline{\Lambda}} J_{\Lambda;x,y}\left(\hat{h}_x - \hat{h}_y - \left[\hat{d}_x(h_x) - \hat{d}_y(h_y)\right]\right)^2 - \sum_{x\in\Lambda}\eta_x(h_x) + K_1(\omega_\Lambda) \quad \text{with} \\
&\hat{h}_x = h_x 1_{x\in\Lambda}, \quad \hat{d}_x(h) = d_x(h)1_{x\in\Lambda}
\end{aligned} \tag{3.9}$$

where $K_1(\omega_\Lambda)$ is a constant that is independent of the height-configuration h_Λ. The J_Λ's are positive and their nearest neighbor parts satisfy the lower bound for nearest neighbors and and upper bound for the decay of the form

$$\begin{aligned}
&\min_{\{x,y\}\subset\overline{\Lambda},|x-y|=1} J_{\Lambda;x,y} \geq qm^{*2}\left[4\left((1+2dq)^2 - q^2\right)\right]^{-1} \\
&J_{\Lambda;x,y} \leq \frac{m^{*2}(1+2dq)}{4}\left((2dq)^{-1}+1\right)^{-|x-y|}
\end{aligned} \tag{3.10}$$

where $|x-y|$ is the 1-norm. Assuming this upper bound we have that

$$\begin{aligned} &J_{\Lambda;x,y} \le e^{-\alpha|x-y|} \quad \text{for} \quad |x-y| \ge r(m^*,q) \quad \text{where} \\ &\alpha = \frac{1}{2}\log\left[1+1/(2dq)\right], \quad r(m^*,q) := \left[2\frac{\log(m^{*2}(1+2dq)/4)}{\log\left(1+1/(2dq)\right)}\right]+1 \end{aligned} \tag{3.11}$$

where square brackets means integer part in the definition of the integer range $r = r(m^*,q)$.

Indeed, these couplings $J_{\Lambda;x,y}$ can be conveniently read off from the following rewriting of the quadratic form appearing in the minimum of the continuous-spin Hamiltonian using the random walk representation of the resolvent: We decompose $R_{\Lambda;x,y} = \sum_{C\subset\Lambda}\mathcal{R}\left(x\to y\,;C\right)$ with $\mathcal{R}\left(x\to y\,;C\right) = \sum_\gamma \left(q^{-1}+2d\right)^{-(|\gamma|+1)}$ where the sum is over all nearest neighbor paths γ on the lattice from x to y that visit precisely the connected set C (see e.g. [K4]A.11 ff.) We have then

$$\begin{aligned} &-< m_\Lambda, R_\Lambda m_\Lambda >_\Lambda + q\sum_{x\in\Lambda} m_x^2 \\ &= \sum_{x,y\in\Lambda}\sum_{C\subset\Lambda}\mathcal{R}\left(x\to y\,;C\right)\frac{1}{2}\left[\left(m_x-m_y\right)^2 - m_x^2 - m_y^2\right] + \sum_{x\in\Lambda}\sum_{C\subset\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d}\mathcal{R}\left(x\to y\,;C\right)m_x^2 \\ &= \frac{1}{2}\sum_{x,y\in\Lambda} R_{\Lambda;x,y}\left(m_x-m_y\right)^2 + \sum_{x\in\Lambda}\left[\sum_{\substack{C:C\cap(\mathbb{Z}^d\setminus\Lambda)\neq\emptyset \\ y:y\in C}}\mathcal{R}\left(x\to y\,;C\right)\right]m_x^2 \end{aligned} \tag{3.12}$$

Indeed, this gives immediately the closely related form

$$\begin{aligned} &\inf_{m_\Lambda\in\mathbb{R}^\Lambda} H_\Lambda^{\tilde m_{\partial\Lambda}=0,\omega_\Lambda,h_\Lambda}\left(m_\Lambda\right) \\ &= \sum_{\{x,y\}\subset\Lambda} J_{\Lambda;x,y}\left(h_x-h_y-\left[d_x(h_x)-d_y(h_y)\right]\right)^2 - \sum_{x\in\Lambda}\eta_x(h_x) + \sum_{x\in\Lambda}K_{\Lambda;x}\left(h_x-d_x(h_x)\right)^2 + K_1(\omega_\Lambda) \end{aligned} \tag{3.13}$$

with

$$J_{\Lambda;x,y} = \frac{m^{*2}}{4q}\sum_{C:x,y\in C\subset\Lambda}\mathcal{R}\left(x\to y\,;C\right), \quad K_{\Lambda;x} = \frac{m^{*2}}{2q}\sum_{\substack{C:C\cap(\mathbb{Z}^d\setminus\Lambda)\neq\emptyset \\ y:y\in C}}\mathcal{R}\left(x\to y\,;C\right) \tag{3.14}$$

To get the form (3.9) we can of course look at the boundary term as a coupling to a boundary condition $\tilde h_x \equiv 0$ for $x\in\partial\Lambda$. Note that $K_{\Lambda;x}$ falls off exponentially as a function of the distance from x to $\partial\Lambda$. Now, for every site $x\in\Lambda$ we pick a site $y(x)\in\partial\Lambda$ that has minimal distance to x (with some arbitrary deterministic prescription to make this choice unique.) Then we extend the definition of J to all pairs in $\mathbb{Z}^d\times\mathbb{Z}^d$ by $J_{\Lambda;x,y} := K_{\Lambda;x}$ for $y=y(x)$, $J_{\Lambda;x,y} := 0$ for $y\in\left(\overline{\Lambda}\right)^c$ or $x,y\in(\Lambda)^c$. The lower bounds in (3.10) follow from the fact that $\mathcal{R}\left(x\to x+e\,;C\right) = \left[(q^{-1}+2d)^2-1\right]^{-1}$ for nearest neighbors $x, x+e$. The upper bound follows from the fact that

$$\sum_{y\in\mathbb{Z}^d}\mathcal{R}\left(x\to y\,;C\right) \le q\left(\frac{2d}{q^{-1}+2d}\right)^{|C|-1} \tag{3.15}$$

(see Appendix of [K4]). Now, for a given configuration h_Λ there are pair interaction terms $J_{\Lambda;x,y}\left(\hat{h}_x - \hat{h}_y - [d_x(h_x) - d_y(h_y)]\right)^2$ that are *big* (they will make up the essential contributions to the 'low-temperature contours'), *small* (they will be expanded and make up high-temperature contours) and *intermediate* (they cannot be expanded and will be adjoined to the low-temperature contributions). The difficulty about these intermediate contributions is that we need to find conditions that ensure that their existence implies the existence of low-temperature contours nearby that will actually dominate them.

Below we will introduce a set of pairs of 'big and intermediate interactions', $\mathcal{E}_{\overline{\Lambda}}(\hat{h}) \subset \overline{\Lambda} \times \overline{\Lambda}$. In particular the pairs that make up the low-temperature contributions will be contained in this set of 'dangerous edges'. For fixed height configuration we decompose the exponential of (3.1):

$$\begin{aligned}
&e^{-\sum_{\{x,y\}\subset\overline{\Lambda}} J_{\Lambda;x,y}\left(\hat{h}_x-\hat{h}_y-\left[\hat{d}_x(h_x)-\hat{d}_y(h_y)\right]\right)^2+\sum_{x\in\Lambda}\eta_x(h_x)} \\
&= e^{-\sum_{\{x,y\}\in\mathcal{E}_{\overline{\Lambda}}(\hat{h})} J_{\Lambda;x,y}\left(\hat{h}_x-\hat{h}_y-\left[\hat{d}_x(h_x)-\hat{d}_y(h_y)\right]\right)^2 1_{\hat{h}_x\neq\hat{h}_y}} \\
&\quad\times e^{-\sum_{\{x,y\}\notin\mathcal{E}_{\overline{\Lambda}}(\hat{h})} J_{\Lambda;x,y}\left(\hat{h}_x-\hat{h}_y-\left[\hat{d}_x(h_x)-\hat{d}_y(h_y)\right]\right)^2 1_{\hat{h}_x\neq\hat{h}_y}} \\
&\quad\times e^{\sum_{h\in\mathbf{Z}}\left[-\left(\sum_{\{x,y\}\subset\Lambda\cap V_h}+\sum_{\substack{x\in\Lambda\cap V_h\\ y\in\partial\Lambda}}\right)J_{\Lambda;x,y}\left(\hat{d}_x(h)-\hat{d}_y(h)\right)^2+\sum_{x\in V_h}\eta_x(h)\right]}
\end{aligned} \tag{3.16}$$

The rest of the proof is a careful treatment the three exponentials on the r.h.s.

(i): First exponential in (3.16): Low temperature-contours (nearest neighbor parts, large fluctuation long range parts)

Given h_Λ, our aim is to define a support of a contour $\underline{\Gamma}^{\mathrm{LT}}(h_\Lambda)$, s.t. the first exponential can be written as a contour activity $\rho^{\mathrm{LT}}(\underline{\Gamma}^{\mathrm{LT}}(h_\Lambda), h_\Lambda)$ that satisfies a Peierls-type estimate in terms of the n.n. surface energy *and* the volume. We will have two contributions to the set of dangerous edges, $\mathcal{E}_{\overline{\Lambda}}(\hat{h}) := \mathcal{E}^{(1)}_{\overline{\Lambda}}(\hat{h}) \cup \mathcal{E}^{(2)}_{\overline{\Lambda}}(\hat{h})$. For the first, the **short range part** we put

$$\mathcal{E}^{(1)}_{\overline{\Lambda}}(\hat{h}) := \{\{x,y\} \in \overline{\Lambda} \times \overline{\Lambda}; d(x,y) \leq r \text{ where } \hat{h}_x \neq \hat{h}_y\} \tag{3.17}$$

with the range $r = r(q, m^*) \geq 1$ given above in (3.11). These interactions provide the 'main-part' of the low-temperature Peierls constant. Looking at these is in perfect analogy to what one would do in the case of an Ising model. The corresponding ingredient of the support of the LT-Peierls contours will be the connected components of the corresponding vertex set, i.e.

$$\underline{\Gamma}^{(1)}_{\overline{\Lambda}}(\hat{h}) := \{x \in \overline{\Lambda}; \exists y \in \overline{\Lambda} \text{ s.t. } d(x,y) \leq r \text{ where } \hat{h}_x \neq \hat{h}_y\} \tag{3.18}$$

Assuming that $\delta \leq \frac{1}{4}$ it is readily seen that we have an energetic suppression in terms of the n.n. surface energy:

$$\sum_{\{x,y\}\in\mathcal{E}^{(1)}_{\overline{\Lambda}}(\hat{h})} J_{\Lambda;x,y}\left(\hat{h}_x-\hat{h}_y-\left[\hat{d}_x(h_x)-\hat{d}_y(h_y)\right]\right)^2 \geq \tau_{\text{n.n.}} \sum_{<x,y>\in\overline{\Lambda}} |\hat{h}_x-\hat{h}_y| \quad \text{with}$$
$$\tau_{\text{n.n.}} := (1-2\delta_d)^2 \min_{\{x,y\}\subset\overline{\Lambda},|x-y|=1} J_{\Lambda;x,y} \tag{3.19}$$

Now we come to the second LT-part of the support of the contour, the **large fluctuation long range part**. It is unavoidable since the height variables can in principle have unbounded fluctuations: Indeed, if the difference between heights at far away sites is extremely large, it becomes impossible to treat the corresponding terms in the Hamiltonian by an high-temperature expansion. It is however intuitively clear that such an event should be very unlikely since it implies a large energy cost due to the short range parts. To turn this into a contour representation we look at the following set of 'dangerous' bonds,

$$\mathcal{E}^{(2)}_{\overline{\Lambda}}(\hat{h}) := \{\{x,y\}\subset\overline{\Lambda}\times\overline{\Lambda}; d(x,y)\geq r;\ \text{where } |\hat{h}_x-\hat{h}_y| \geq e^{\frac{\alpha}{2}|x-y|}\} \tag{3.20}$$

whose interactions can not be treated by a high-temperature expansion, with α being the estimate on the decay-rate of the J's, as given in (3.11). The corresponding part of the LT-support will have to contain the vertex sets of the connected components of the corresponding graph. Our aim is then to show the Peierls-type estimate in terms of the n.n. interface energy *and* the volume of the contour. The problem with the volume-estimate is that there is of coarse a gap between high-temperature and low-temperature expansions: if the interaction term $J_{\Lambda;x,y}\left(\hat{h}_x-\hat{h}_y-\left[\hat{d}_x(h_x)-\hat{d}_y(h_y)\right]\right)^2$ is not small enough for a high-temperature expansion, the term itself need not be large enough to provide a low-temperature Peierls constant that is big enough. So, what might happen is that the large fluctuation long range parts just glue together connected components of the $\underline{\Gamma}^{(1)}$-parts without contributing much energy themselves. We will however show that in such a case the n.n. surface energy will be at least as large as the resulting volume of the total contour. Indeed, if the interaction term is not very small, each nearest neighbor path from x to y contributes a nearest neighbor interface energy of $e^{\frac{\alpha}{2}|x-y|}$. Patching together those paths along with safety cubes around them we will get the second part of the contour with a useful Peierls constant. The details are as follows.

It is convenient to work in *all* of $\mathbb{Z}^d$ and define sets that will be the essential part of the low-temperature contours which are not necessarily subsets of Λ. The final supports of contours will then be obtained as intersections. Given our extended configuration $\hat{h}$ it is convenient to extend the above definition writing $\mathcal{E}^{(2)}_{\mathbb{Z}^d}(\hat{h}) := \{\{x,y\}\subset\mathbb{Z}^d\times\mathbb{Z}^d; d(x,y)\geq r;\ \text{where } |\hat{h}_x-\hat{h}_y| \geq e^{\alpha|x-y|}\}$. This is a finite set. To each pair $\{x,y\}\in\mathcal{E}^{(2)}_{\mathbb{Z}^d}(\hat{h})$ we associate a cube $Q(\{x,y\})\subset\mathbb{Z}^d$ among the cubes containing the points x and y with the smallest side-length. (If the line between x and y

doesn't happen to be a diagonal, we choose some arbitrary deterministic tie-breaking procedure to make this choice unique.) Then we put $\underline{\Gamma}^{(2)}_{\mathbf{Z}^d}(\hat{h}) := \bigcup_{\{x,y\}\in\mathcal{E}^{(2)}_{\mathbf{Z}^d}(\hat{h})} Q\left(\{x,y\}\right)$. Finally we define the LT-support $\underline{\Gamma}^{\text{LT}}(h)\subset\Lambda$ of the contour by

$$\underline{\Gamma}^{\text{LT}}(h) := \left(\underline{\Gamma}^{(1)}_{\Lambda}(\hat{h}) \cup \underline{\Gamma}^{(2)}_{\mathbf{Z}^d}(\hat{h})\right) \cap \Lambda \tag{3.21}$$

Then the desired lower bound of the n.n. surface energy in terms of the volume of the long-range parts is given by the following

Lemma 3.1: *Suppose that $\underline{\gamma}$ is a connected component of the set $\underline{\Gamma}^{(2)}_{\mathbf{Z}^d}(\hat{h})$. Then the nearest neighbor surface energy satisfies a Peierls estimate of the form*

$$\sum_{<x,y>\in\underline{\gamma}} |\hat{h}_x - \hat{h}_y| \geq K\,|\underline{\gamma}| \quad \textit{with} \quad K := \inf_{L'\geq r}\left(\frac{e^{\frac{\alpha L'}{2}}}{(3L')^d}\right) \geq \left(\frac{e\alpha}{6d}\right)^d \tag{3.22}$$

Proof: From the set of cubes $Q(\{x,y\})$ whose union makes up $\underline{\gamma}$ we will consider in the following only the maximal ones w.r.t. inclusion. Let us denote them by Q_i, $i = 1,\dots,N$. (That is we discard those that are contained in a strictly bigger one.) We prove the Lemma by induction over the number N of such cubes of a connected component. In the case of one cube, say $Q(x,y)$ of side-length L, we have that

$$\begin{aligned}&\sum_{<i,j>\in Q(x,y)} |h_i - h_j| \geq |h_x - h_y| \\ &\geq e^{\frac{\alpha}{2}|x-y|} \geq e^{\frac{\alpha}{2}L} \geq L^d \inf_{L'\geq r}\left(\frac{e^{\frac{\alpha}{2}L'}}{L'^d}\right) = 3^d K^d L^d\end{aligned} \tag{3.23}$$

where the first inequality is the triangle inequality, the second the definition of the 'dangerous bonds'. This proves the single-cube case even with a constant $3^d K$.

Now, given a contour $\underline{\gamma} = \bigcup_{i=1}^N Q_i$, we define a smaller contour $\underline{\Gamma}'$ in the following way. Pick one of the cubes Q_i with the largest side-length, call this cube Q_{i_0} (with corresponding side-length L_{i_0}). Define

$$\underline{\Gamma}' := \bigcup_{\substack{i\in\{1,\dots,N\};i\neq i_0\\ Q_i\cap Q_{i_0}=\emptyset}} Q_i \tag{3.24}$$

The resulting contour might be connected or not. Since we took away the *biggest* cube and its 'contact partners' we have $|\underline{\Gamma}'| \geq |\underline{\gamma}| - 3^d L_{i_0}^d$. For the nearest neighbor surface energy we have

$$\sum_{<x,y>\in\underline{\gamma}} |h_x - h_y| \geq \sum_{<x,y>\in\underline{\Gamma}'} |h_x - h_y| + \sum_{<x,y>\in Q_{i_0}} |h_x - h_y| \tag{3.25}$$

Using the induction hypothesis on the connected components of $\underline{\Gamma}'$ and the estimate on the single cube from above we get the desired

$$\sum_{<x,y>\in\underline{\gamma}} |h_x - h_y| \geq K\left|\underline{\Gamma}'\right| + 3^d K L_{i_0}^d \geq K\left|\underline{\gamma}\right| \tag{3.26}$$

$\diamondsuit$

Now we are done with the treatment of the first exponential in (3.16): Indeed, the sum $\sum_{\{x,y\}\in\mathcal{E}_{\overline{\Lambda}}(\hat{h})} J_{\Lambda;x,y}\left(\hat{h}_x - \hat{h}_y - \left[\hat{d}_x(h_x) - \hat{d}_y(h_y)\right]\right)^2 1_{\hat{h}_x\neq\hat{h}_y}$ decomposes over connected components $\underline{\gamma}_i$ of $\underline{\Gamma}^{\mathrm{LT}}(h)$. Denote the corresponding pairs by $\mathcal{E}_{\overline{\Lambda};i}(\hat{h})$ (i.e. the pairs with vertices in $\underline{\gamma}\cup\left(\overline{\underline{\gamma}}\cap\partial\Lambda\right)$). Denote by γ_i the contour whose support is $\underline{\gamma}_i$ and whose height configuration is $\hat{h}_{\overline{\Lambda}}$ on $\overline{\underline{\gamma}_i}$. We define the LT-activities by

$$\rho^{\mathrm{LT}}(\gamma_i) := e^{-\sum_{\{x,y\}\in\mathcal{E}_{\overline{\Lambda};i}(\hat{h})} J_{\Lambda;x,y}\left(\hat{h}_x-\hat{h}_y-\left[\hat{d}_x(h_x)-\hat{d}_y(h_y)\right]\right)^2 1_{\hat{h}_x\neq\hat{h}_y}} \tag{3.27}$$

The desired LT surface-energy/volume Peierls-type estimate is obvious: Denote $\underline{\Gamma}^{(2)} := \gamma^{(i)}\cap\underline{\Gamma}_{\overline{\Lambda}}^{(2)}(\hat{h})$ the parts of the connected component that are due to the large-fluctuation-long-range part. Then we have

$$\begin{aligned}
&\sum_{\{x,y\}\in\mathcal{E}_{\overline{\Lambda};i}(\hat{h})} J_{\Lambda;x,y}\left(\hat{h}_x - \hat{h}_y - \left[\hat{d}_x(h_x) - \hat{d}_y(h_y)\right]\right)^2 1_{\hat{h}_x\neq\hat{h}_y} \geq \tau_{n.n.}E_s(\gamma_i)\\
&\geq \frac{\tau_{n.n.}}{3}E_s(\gamma_i) + \frac{\tau_{n.n.}}{3}\frac{\left|\underline{\gamma}_i\cap\underline{\Gamma}_{\overline{\Lambda}}^{(1)}(\hat{h})\right|}{(2r+1)^d} + \frac{\tau_{n.n.}}{3}K\left|\underline{\gamma}_i\cap\underline{\Gamma}_{\overline{\Lambda}}^{(2)}(\hat{h})\right|\\
&\geq \beta E_s(\gamma_i) + \tau_1\left|\underline{\gamma}_i\right|
\end{aligned} \tag{3.28}$$

with $\beta = \frac{\tau_{n.n.}}{3}$ and $\tau_1 = \frac{\tau_{n.n.}}{3}\times\min\{(2r+1)^{-d}, K\}$. Let us look at the large m^*, small q-asymptotics with $\tau_{n.n.}\sim q{m^*}^2/4$ sufficiently large, $\alpha\sim\frac{1}{2}\log\frac{1}{q}$, $r\sim 4\frac{\log m^*}{\log\frac{1}{q}}$. From this we get that $\beta\sim q{m^*}^2/12$, $\tau_1\sim Const\, q{m^*}^2\left(\frac{\log\frac{1}{q}}{\log m^*}\right)^d$. (Note that the contributions of the K-term will actually obtain a better Peierls constant that hence won't be visible.)

(ii): Second exponential in (3.16): High-temperature expansion

Let us just write $\underline{\Gamma} := \underline{\Gamma}^{\mathrm{LT}}(h)$. We find it convenient to use a little rewriting of the exponent. Since we have exponential decay of the interaction, we can just 'fill the space between the endpoints' to define contours that obey Peierls estimates: To each pair $\{x,y\}\in\left[\overline{\Lambda}\times\overline{\Lambda}\right]\setminus\mathcal{E}_{\overline{\Lambda}}(\hat{h})$ of sites we associate a 'one-dimensional' polymer $g = g(x,y)\subset\overline{\Lambda}$ that is the set of sites of one of the nearest-neighbor paths from x to y (with some prescription to make the choice of this path unique.) Then we have for the number of sites that $|g| = |x-y|_1 + 1$. We use this notation to denote the terms in the last sum by sets g and put

$$S_g(h_g) := 1_{\{x,y\}\not\in \mathcal{E}_{\overline{\Lambda}}(\hat h)} \times J_{\Lambda;x,y} \left(\hat h_x - \hat h_y - \left[\hat d_x(h_x) - \hat d_y(h_y)\right]\right)^2 \; 1_{\hat h_x \neq \hat h_y} \tag{3.19}$$

We note that, due to the decay of the resolvent (with α), the uniform boundedness of $|d_x| \leq \delta_d$ and the definition of the 'dangerous bonds' (with $\alpha/2$!) we have $0 \leq S_g(h_g) \leq e^{-const\,\alpha|g|}$. Note that there are only non-vanishing terms for $g \cap \underline{\Gamma} \neq \emptyset$. (Indeed, in the case that both x and y are not in $\underline{\Gamma}$, they must lie in different connected components of the complement of $\underline{\Gamma}$.) The exponential can now be treated by the subtraction of bounds trick: We have

$$e^{-\sum_{g:g\cap\underline{\Gamma}\neq\emptyset} S_g(h_g)} = \prod_{\underline{\gamma} \text{ conn.cp. of } \underline{\Gamma}} e^{-\sum_{g:g\cap\underline{\gamma}\neq\emptyset} e^{-const\,\alpha|g|}} \times e^{\sum_{g:g\cap\underline{\Gamma}\neq\emptyset}\left(n(\underline{\Gamma},g)e^{-const\,\alpha|g|} - S_g(h_g)\right)} \tag{3.30}$$

where $n(\underline{\Gamma}, g)$ counts the number of connected components of $\underline{\Gamma}$ that are connected to g. The term under the first product defines a non-negative quantity $r(\underline{\gamma})$ that is h-independent and satisfies $1 \geq r(\underline{\gamma}) \geq e^{-|\underline{\gamma}|e^{-const'\alpha}}$ (for suff. large α). The last exponential can be polymer-expanded and written as a sum $\sum_{G:G\cap\underline{\Gamma}\neq\emptyset} \rho^{HT1}_{\underline{\Gamma}}(G, h_G)$ with a nonnegative activity $0 \leq \rho_{\underline{\Gamma}}(G, h_G) \leq e^{-const'\alpha|G|}$ (which is one for empty G).

(iii): Third exponential in (3.16): Non-local small fields

The last exponential would not be present in the corresponding model without randomness. It describes the random modulations of the 'vacuum-energy' caused by the d-variables in the flat pieces outside the LT-contours (where height-fluctuations are occuring). To get the decomposition into S_C's we use the decomposition of the resolvent and define for all C's

$$\beta \tilde S_C(h_C) := \frac{{m^*}^2}{4q} \sum_{\substack{x,y\in C:\\ h_x = h_y}} \mathcal{R}\left(x \to y\,; C\right)\left(d_x(h_x) - d_y(h_y)\right)^2 \tag{3.31}$$

From the bound (3.15) we get

$$\left|\beta \tilde S_C(h_C)\right| \leq (2\delta_d)^2 q (m^*)^2 \frac{2d}{1+2dq} e^{-\alpha(|C|-2)} \tag{3.32}$$

This is always fine for $|C| \geq 3$; for $|C| = 2$ we see that δ_d really needs to be small enough to get a useful bound. We put $S_C := \tilde S_C$ for $|C| \geq 3$ and $S_C := \tilde S_C - I\!E \tilde S_C$ for $|C| = 2$. [We remark that we could relax the assumption of smallness of δ_d by the introduction of large $d_x(h) = d_x(h) 1_{|d_x(h)|\geq\delta_1} + d_x(h) 1_{|d_x(h)|<\delta_1} =: d^{l}_x(h) + d^{s}_x(h)$. Then we would have to introduce a control field $N_x(h)$ that would contain a contribution of the type $Const\ |d_x(h)|\, 1_{|d_x(h)|\geq\delta_1}$ as well as a contribution from the large $\eta_x(h)$'s. Since this is simple but would obscure the structure of the contour-model we don't present the details here.]

For a flat height configuration on C (i.e. if $C \subset V_h$ for an $h \in \mathbb{Z}$) we just write $S_C(h) := S_C(h_C)$ the former defining the 'small field' appearing in the final contour-model representation. Then we have for the bulk term in the third exponential in (3.16)

$$\sum_{h\in\mathbb{Z}} \sum_{\{x,y\}\subset\Lambda\cap V_h} J_{\Lambda;x,y} \left(d_x(h) - d_y(h)\right)^2 = \beta \sum_{h\in\mathbb{Z}} \sum_{C\subset\Lambda\cap V_h} S_C(h) + \beta \sum_{\substack{C\subset\Lambda \\ h_C \not\equiv const}} S_C(h_C) + K'_\Lambda \tag{3.33}$$

with some obvious h_Λ-independent constant K'_Λ. The first term is the desired small-field contribution. The second term is attached to the LT-contours and can be expanded. Using subtraction-of-bounds as before, its expansion gives

$$e^{-\beta \sum_{\substack{C\subset\Lambda \\ h_C \not\equiv const}} S_C(h_C)} = \left[\prod_{\underline{\gamma} \text{ conn.cp. of } \underline{\Gamma}^{\mathrm{LT}}(h)} r_2(\underline{\gamma}) \right] \sum_{G: G\cap\underline{\Gamma}^{\mathrm{LT}}(h)\neq\emptyset} \rho^{HT2}_{\underline{\Gamma}^{\mathrm{LT}}(h)}(G, h_G) \tag{3.34}$$

with *const* α-decay. As far as for the bulk terms, the fields $\eta_x(h)$ simply make up the local contributions to the small field. Finally, we discuss the corrections due to boundary effects. We write the interaction with the boundary in the form

$$\begin{aligned} &\sum_{h\in\mathbb{Z}} \sum_{\substack{x\in\Lambda\cap V_h \\ y\in\partial\Lambda}} J_{\Lambda;x,y} \left(\hat{d}_x(h) - \hat{d}_y(h)\right)^2 = -\sum_{x\in\Lambda} \tilde{\eta}_x(h_x) + \tilde{K}_\Lambda \quad \text{where} \\ &\tilde{\eta}_x(h) := -K_{\Lambda;x} \left(d_x(h)^2 - \mathbb{E}\left[d_x(h)^2 \right] \right) \end{aligned} \tag{3.35}$$

We note that $|\tilde{\eta}_x(h)| \leq q m^{*2} \frac{d\delta_d^2}{1+2dq}$. The centered field $\tilde{\eta}_x(h)$ will just give a small volume dependent modification of the local small field that we finally define by

$$S_x(h) := \eta^s_x(h) + \tilde{\eta}_x(h) \tag{3.36}$$

We note that from this definition the probabilistic bounds for $|C| = 1$ are clear. Also, the probabilistic bounds for $|C| = 2$ are obvious.

Finally putting together our results from that (i)-(iii) we end up with the representation

$$\begin{aligned} &e^{-\sum_{\{x,y\}\subset\overline{\Lambda}} J_{\Lambda;x,y}\left(\hat{h}_x - \hat{h}_y - \left[\hat{d}_x(h_x) - \hat{d}_y(h_y)\right]\right)^2 + \sum_{x\in\Lambda} \eta_x(h_x)} \\ &= \prod_{\underline{\gamma} \text{ conn.cp. of } \underline{\Gamma}^{\mathrm{LT}}(h_\Lambda)} \left(r(\underline{\gamma}) r_2(\underline{\gamma}) \rho^{\mathrm{LT}}(\underline{\gamma}, h_{\underline{\gamma}}) \right) \sum_{\substack{G: G\cap\underline{\Gamma}^{\mathrm{LT}}(h_\Lambda)\neq\emptyset \\ G_2: G_2\cap\underline{\Gamma}^{\mathrm{LT}}(h_\Lambda)\neq\emptyset}} \rho^{HT1}_{\underline{\Gamma}^{\mathrm{LT}}(h_\Lambda)}(G, h_G) \rho^{HT2}_{\underline{\Gamma}^{\mathrm{LT}}(h_\Lambda)}(G_2, h_{G_2}) \\ &\times e^{-<S, V(h_\Lambda)>} \end{aligned} \tag{3.37}$$

Resumming over G, G_2's that have the same $\underline{\Gamma}^{\text{LT}}(h_\Lambda) \cup \underline{\Gamma}^D(h_\Lambda) \cup G \cup G_2 =: \underline{\Gamma}$ we get the desired form. This concludes the proof of the proposition. $\diamondsuit$

IV. Proof of Theorem 1

The proof is a direct consequence of the representation of Theorem 2, the contour-representation for the discrete-height model of Proposition 1, and the results from the renormalization group analysis for a discrete-height contour model from [BoK1], [K1]. It is crucial for this that the contour model constructed in Chapter III and given in Proposition 1 is 'renormalizable' with the procedure described in detail in [BoK1] in 'Chapter 4. The Gibbs State at Finite Temperature'. Indeed, it satisfies the inductive assumption of a contour model given in [BoK1] 4.1, p. 457, for the trivial choice of empty bad regions and vanishing control-field N. We must however have for this that the uniform bounds δ_η and δ_d are sufficiently small; otherwise we would have had to introduce bad regions and large fields, which is however mainly a notational inconvenience. (There is further a completely trivial difference in that we have exponential decay for the small fields only for $|C| \geq 3$; we could of course trivially cast the present contour representation into the one from [BoK1] by splitting the field S_C for $C = \{x, x+e\}$ into new local small fields at sites x and $x+e$ and producing a stochastic dependence up to distance 2.) The result of [BoK1] then gives that, for sufficiently large $\tilde{\beta}$ $(\leq \alpha, \beta)$, for sufficiently small $\sigma^2_{\text{eff.}}$ there exists a non-random subsequence of cubes Λ s.t. the measures ν^ω_Λ (obtained from the zero boundary continuous Gibbs-measures) converge weakly to an infinite-volume Gibbs-measure ν^ω, for a.e. ω [see [BoK1], p.417, Theorem 1]. To conclude that convergence of the μ-measures takes place also on local observables f that are only polynomially bounded, $|f(m_V)| \leq Const\,(1 + |m_V|)^p$, we would like to use the addition to Theorem 2 (2.7). Although its assumption on the convergence of the ν-measures on exponentially bounded observables is a very natural one that is believed to hold, it is unfortunately not a straightforward consequence of the RG-analysis. Along the lines of Chapter II, the reader will however have no difficulty to prove the analogous extension for polynomially bounded observables under the condition that $\sup_{\Lambda,y} I\!E \nu_\Lambda |h_y|^p < \infty$, for all exponents p. This assumption is in fact true; we even have (4.2), see below.

Let us use the short notation $\mathcal{N}[h_{\mathbb{Z}^d}] := \mathcal{N}\left[(1 - q\Delta_{\mathbb{Z}^d})^{-1} m^*_{\mathbb{Z}^d}(h_{\mathbb{Z}^d}); (1 - q\Delta_{\mathbb{Z}^d})^{-1}\right]$. Then we have for the second moment

$$
\begin{aligned}
I\!E\mu^\omega\left(m^2_{x_0}\right) &= I\!E\nu^\omega\left\{\mathcal{N}[h_{\mathbb{Z}^d}]\left(m^2_{x_0}\right) - \mathcal{N}[h_{\mathbb{Z}^d}]\left(m_{x_0}\right)^2\right\} + I\!E\nu^\omega \mathcal{N}[h_{\mathbb{Z}^d}]\left(m_{x_0}\right)^2 \\
&= (1 - q\Delta_{\mathbb{Z}^d})^{-1}_{0,0} + I\!E\nu^\omega\left(\sum_y (1 - q\Delta_{\mathbb{Z}^d})^{-1}_{x_0,y}\, m^*_y(h_y)\right)^2
\end{aligned}
\tag{4.1}
$$

With the Schwartz inequality we bound the last expectation by $(m^*)^2 \sup_y I\!E\nu^\omega\left[(|h_y| + \delta_d)^2\right]$. To estimate this last expectation, we utilize a corollary of the RG-analysis for the discrete height model of [BoK1] (whose complete proof can be found in [K1], see Theorem 1.2 therein) saying that, for any $q \in \mathbb{Z}$, we have

$$\mathbb{E}\nu^{\omega}\left[|h_y|^q\right] \le K(q)\left(e^{-\frac{1}{\sigma_{\text{eff.}}^{\kappa}}} + e^{-Const\,\tilde{\beta}}\right) \tag{4.2}$$

for any y, with q-dependent $K(q)$ and some positive exponent κ. This immediately proves (1.4).$\diamondsuit$

To get more estimates on the continuous infinite volume measures in terms of the discrete one might now utilize $\mu^{\omega}[B] \le \nu^{\omega}[A^c] + \sup_{h_{\mathbb{Z}^d}\in A} \mathcal{N}[h_{\mathbb{Z}^d}][B]$ for any good choice of an event A in integer height space.

References

[AW] M.Aizenman, J.Wehr, Rounding Effects of Quenched Randomness on First-Order Phase Transitions, Comm. Math.Phys **130**, 489-528 (1990)

[Ba1] Balaban, Tadeusz, A low temperature expansion for classical N-vector models. I. A renormalization group flow, Comm.Math.Phys. **167** no. 1, 103–154 (1995)

[Ba2] Balaban, Tadeusz, A low temperature expansion for classical N-vector models. II. Renormalization group equations, Comm.Math.Phys. **182** no. 3, 675–721 (1996)

[BCJ] C.Borgs, J.T.Chayes, J.Frölich, Dobrushin States for Classical Spin Systems with Complex Interactions, J.Stat.Phys. **89** no.5/6, 895-928 (1997)

[BI] E.Bolthausen, D.Ioffe, Harmonic crystal on the wall: a microscopic approach, Comm.Math.Phys. **187** no.3, 523-566 (1997)

[BK1] J.Bricmont, A.Kupiainen, Phase transition in the 3d random field Ising model, Comm. Math.Phys. **142**, 539-572 (1988)

[BKL] J.Bricmont, A.Kupiainen, R. Lefevere, Renormalization Group Pathologies and the Definition of Gibbs States, Comm. Math.Phys. **194** 2, 359-388 (1998)

[BoG] A.Bovier, V.Gayrard, Hopfield models as generalized random mean field models, in: Mathematical aspects of spin glasses and neural networks, Progr. Probab. 41, 3-89, Birkhäuser Boston, Boston, MA, 1998.

[BoK1] A.Bovier, C.Külske, A rigorous renormalization group method for interfaces in random media, Rev.Math.Phys. **6**, no.3, 413-496 (1994)

[BoK2] A.Bovier, C.Külske, There are no nice interfaces in $2+1$ dimensional SOS-models in random media, J.Stat.Phys. **83**, 751-759 (1996)

[B] D.Brydges, A short course on cluster expansions, in 'Critical phenomena, random systems, gauge theories' (Les Houches 1984) (K.Osterwalder, R.Stora, eds.), North Holland, Amsterdam (1986)

[Do1] R.L.Dobrushin, Gibbs states describing a coexistence of phases for the three-dimensional Ising model, Th.Prob. and its Appl. **17**, 582-600 (1972)

[DoZ] R.L.Dobrushin, M.Zahradnik, Phase Diagrams of Continuum Spin Systems, Math.Problems of Stat.Phys. and Dynamics, ed. R.L.Dobrushin, Reidel pp.1-123 (1986)

[EFS] A.C.D.van Enter, R. Fernández, A.Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. J.Stat.Phys. **72**, 879-1167 (1993)

[F] R. Fernandez, Measures for lattice systems, preprint, to appear in: Proceedings of Statphys 20, Paris (1998), available as preprint 98-567 at http://www.ma.utexas.edu/mp_arc

[FFS] R.Fernandez, J.Fröhlich, A.Sokal, Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory, Springer, Berlin, Heidelberg, New York (1992)

[Geo] H.O. Georgii, Gibbs measures and phase transitions, Studies in mathematics, vol. 9 (de Gruyter, Berlin, New York, 1988)

[K1] C.Külske, Ph.D. Thesis, Ruhr-Universität Bochum (1993)

[K2] C.Külske, A random energy model for size dependence: recurrence vs. transience, Prob.Theor.Rel.Fields **111**, 57-100 (1998)

[K3] C.Külske, Limiting behavior of random Gibbs measures: metastates in some disordered mean field models, in: Mathematical aspects of spin glasses and neural networks, Progr. Probab. **41**, 151-160, eds. A.Bovier, P.Picco, Birkhäuser Boston, Boston (1998)

[K4] C.Külske, The continuous spin random field model: Ferromagnetic ordering in $d \geq 3$, to be published in Rev.Math.Phys, available at http://www.ma.utexas.edu/mp_arc/, prepint 98-175 (1998)

[Z1] M.Zahradník, On the structure of low temperature phases in three dimensional spin models with random impurities: A general Pirogov-Sinai approach, Mathematical Physics of Disordered Systems, Proc. of a workshop held at the CIRM, ed. A.Bovier and F.Koukiou, IAAS-Report No.4, Berlin (1992)

[Z2] M.Zahradník, Contour Methods and Pirogov Sinai Theory for Continous Spin Models, preprint Prague (1998), to appear in the AMS volume dedicated to R.L.Dobrushin