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# The convergence and stability of splitting finite difference schemes for nonlinear evolutionary type equations

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**Abstract**

A splitting finite difference scheme for an initial-boundary value problem for a two-dimensional nonlinear evolutionary type equation is considered. The problem is split into nonlinear and linear parts. The linear part is also split into locally one-dimensional equations. The convergence and stability of the scheme in $L_2$ and $C$ norms are proved.

# 1 Introduction.

We consider the initial-boundary value problem for multidimensional nonlinear evolutionary equation of the type

$$\frac{\partial u}{\partial t} = a\Delta u + f(u).$$

Here $a = a_1 + ia_2$ is a complex valued constant and $\Delta$ is the d-dimensional Laplacian. We consider the following cases:

1. If $a_1 = 0$ and $a_2 \neq 0$ we have the Schrödinger equation.
2. If $a_1 > 0$ and $a_2 \neq 0$ we have the Kuramoto–Tsuzuki equation.
3. If $a_1 > 0$ and $a_2 = 0$ we have a heat equation.

Such linear and nonlinear equations appear in many models of nonlinear optics, quantum mechanics, seismology, plasma physics, in the theory of turbulent flows and many other fields of science.

Although evolutionary problems with only one spatial dimension have been investigated during a long period, even recently there appear a lot of papers devoted to the numerical solution of these problems [1] – [4]. As a rule, the difficulties in solving such the problems arise due to the different kinds of nonlinearities.

Multi-dimensional nonlinear and linear evolutionary problems are even more complicated. Frequently, solving such problems during small time step, one splits the problem into nonlinear and linear parts [5], [6] or considers multidimensional linear part as locally one-dimensional problems [5], [8], [9].

An object of this paper is the finite difference scheme that approximates the evolutionary equation. There are many papers on finite difference schemes for initial–boundary linear and nonlinear evolutionary problems. There are two–layered schemes with weights [2], [5], [7], various three–layered schemes [2] – [4] and also splitting schemes [5], [6], [8].

In the paper [6] the splitting of the equation into nonlinear and linear parts

$$\begin{aligned} \frac{\partial u^{(nonl)}}{\partial t} &= f(u^{(nonl)}), \quad u^{(nonl)}_{initial} = u_{initial}, \\ \frac{\partial u^{(lin)}}{\partial t} &= a\,\Delta u^{(lin)}, \quad u^{(lin)}_{initial} = u^{(nonl)}_{final}, \quad u_{final} = u^{(lin)}_{final} \end{aligned}$$

for a short evolution time $\tau$ was used and a proof of the existence and stability of the solution was given.

In a series of papers, for example in [5], locally one-dimensional finite difference schemes for the solution of multidimensional linear evolutionary problems have been constructed. These schemes have allowed to compute a value of the unknown solution on the next time level using differential operators in different directions step by step, i.e., splitting the $d$-dimensional linear part of the equation into the one-dimensional problems:

$$\frac{\partial u^{(k)}}{\partial t} = a\frac{\partial^2 u^{(k)}}{\partial x_k^2}, \qquad k = 1, \dots, d,$$
$$u^{(1)}_{initial} = u_{initial}, \quad u^{(k)}_{initial} = u^{(k-1)}_{final}, \quad u^{(d)}_{final} = u_{final}.$$

The other method for splitting the linear $d$-dimensional heat equation into local one-dimensional problems was investigated in [8]. When looking for the unknown solution on the next time level, instead of solving step by step the one-dimensional problems, one can solve all these problems simultaneously and get the solution on the new time level from the obtained data afterwards. Such splitting can be described as follows:

$$\alpha_k\frac{\partial u^{(k)}}{\partial t} = a\frac{\partial^2 u^{(k)}}{\partial x_k^2}, \qquad k = 1, \dots, d,$$
$$u^{(k)}_{initial} = u_{initial}, \quad \sum_{k=1}^{d} \alpha_k = 1, \quad u_{final} = \sum_{k=1}^{d} \alpha_k u^{(k)}_{final}.$$

It seems that such a splitting of the multidimensional problem can be very useful in parallel computations.

A purpose of the present paper is to prove the convergence and stability of the locally one-dimensional difference scheme for a broad class of two-dimensional nonlinear evolutionary type equations. The evolutionary problem is split into nonlinear and linear parts, as in [6]. A linear part of this problem is also split into locally one-dimensional problems as it had been done in [8].

The nonlinear evolutionary equations which are considered in this paper represent a much broader class of equations than only linear heat equations that were considered in [8]. The difference schemes require a more precise investigation due to nonlinearity as well as to some specific properties of the Schrödinger and the Kuramoto-Tsuzuki equations.

Therefore, our paper presents new results proving convergence and stability of the splitting scheme for nonlinear evolutionary type equations. Doing so, we also show convergence of another splitting scheme, where the fully explicit difference scheme for the two-dimensional linear part is used.

It should be mentioned that to prove the convergence and stability of difference schemes we use a new type of *a priori* estimates that were introduced and developed in [6], [7]. This approach allows us to avoid any restrictions on the time and space grid steps.

In Section 2 the differential problem and the corresponding finite difference schemes are formulated. In Section 3 some grid embedding and multiplicative inequalities are proved, and some formulae for the finite difference differentiation are derived. In Section 4 some estimates for the nonlinear part of the equation are obtained. Section 5 contains various properties of the difference schemes. Sections 6 and 7 are devoted to a proof of the convergence of the two finite difference schemes. Section 8 contains an investigation of the stability for both schemes.

## 2 Formulation of the problem.

Let us consider an initial-boundary value problem with Dirichlet boundary conditions for the nonlinear Schrödinger, Kuramoto–Tsuzuki or heat equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} = a\,\Delta u + f(u,u^*), && (x,t)\in Q; \\ &u(x,0) = u_0(x), \quad x\in\bar{\Omega}; && u(x,t)=0, \quad (x,t)\in\partial\Omega\times[0,T]. \end{aligned} \tag{1}$$

Here $x=(x_1,x_2)$; $u(x,t)$ is a complex-valued function; $u^*$ is the complex conjugate function; $i=\sqrt{-1}$; $\Delta = \frac{\partial}{x_1^2}+\frac{\partial}{x_2^2}$ is the two-dimensional Laplacian; $a=a_1+ia_2$, $a_1\geq 0$, $|a|>0$; $\bar{\Omega}=[0,1]\times[0,1]$, $\bar{Q}=\bar{\Omega}\times[0,T]$.

Assume that the nonlinear function $f(u,u^*)$ satisfies the following conditions:

The partial derivatives of the function $f(u,u^*)$ with respect to $u$ and $u^*$ are continuous up to the second order and

$$\Big|\frac{\partial^{|k|} f(u,v^*)}{\partial u^{k_1}\partial v^{*k_2}}\Big| \leq \varphi(\max\{|u|,|v|\}), \qquad k_1+k_2=|k|\in\{0,1,2\}, \tag{2}$$

where $\varphi$ is a continuous nondecreasing function;

$$\operatorname{Re}(f(u,u^*)u^*)\leq 0. \tag{3}$$

Condition (2) is necessary for the evaluation of the differentiated nonlinear function. Estimate (3) allows to obtain the integral dissipation property $\|u(t)\|_{L_2}\leq\|u(0)\|_{L_2}$, $t\in[0,T]$. This inequality can be proved multiplying both sides of (1) by $2u^*$, integrating the obtained equation over $\Omega$, taking a real part and using the condition $a_1=\operatorname{Re}a\geq 0$.

Note that in the case of the Schrödinger equation $a=ia_2$ and condition (3) reads as $\operatorname{Re}(f(u,u^*)u^*)=0$, the integral conservation law holds: $\|u(t)\|_{L_2}=\|u(0)\|_{L_2}$, $t\in[0,T]$.

It should be also mentioned that more general *a priori* estimates hold.

**Remark 1** *Suppose (2) and (3) are satisfied,* $u_0 \in W_2^2(\Omega)$ *and there exists a solution* $u \in C(\bar{Q})$ *of problem (1). Then the following estimate holds:*

$$\|u(t)\|_{W_2^2} \le c_1 \|u_0\|_{W_2^2}, \quad t \in [0,T], \qquad c_1 = c_1(\varphi(\|u\|_{C(\bar{Q})}), \|u_0\|_{W_2^1}, T).$$

**Proof.** The proof of this remark can be found in [7].

We assume that there exists a solution of (1) which satisfies the following condition:

$$\max_{t\in[0,T]} \left\{ \Big\|\frac{\partial^2 u(t)}{\partial t^2}\Big\|_C, \Big\|\frac{\partial^4 u(t)}{\partial x_1^4}\Big\|_C, \Big\|\frac{\partial^4 u(t)}{\partial x_2^4}\Big\|_C \right\} \le C < \infty. \tag{4}$$

This smoothness is required to get a good approximation of the difference schemes. Note that the estimate $\max_{t\in[0,T]} \|u(t)\|_C \le C$ also follows from (4).

Let us introduce a uniform grid in the domain $\bar{Q}$:

$$\begin{aligned}
\omega_\tau &= \{t_j = \tau j; j = 0,\dots,M-1, M\tau = T\}, \qquad \bar{\omega}_\tau = \{t_j; j = 0,\dots,M\},\\
\omega_x &= \{x = (x_{1l_1}, x_{2l_2}); x_{jl_j} = l_j h_j, l_j = 1,\dots,N_j - 1,\ j = 1,2,\ N_j h_j = 1\},\\
\bar{\omega}_x &= \{x = (x_{1l_1}, x_{2l_2}); l_j = 0,\dots,N_j,\ j = 1,2\}.
\end{aligned}$$

Denote $Q_{h\tau} = \omega_x \times \omega_\tau$, $\bar{Q}_{h\tau} = \bar{\omega}_x \times \bar{\omega}_\tau$. Let $\partial\omega_x = \bar{\omega}_x \setminus \omega_x$ be the set of grid boundary points on $\partial\Omega$. We require $0 < 1/d \le h_1/h_2 \le d < \infty$ for $h_1, h_2 \to 0$. For simplicity we assume $h = h_1 \le h_2 = dh$.

In the sequel we use the notation

$$u = u(x,t_j) = u(x_1,x_2,t_j), \quad \hat{u} = u(x,t_{j+1}), \quad \dot{u} = (\hat{u} + u)/2, \quad u_t = (\hat{u} - u)/\tau.$$

$I$ is the identity operator, $T_k^+$ and $T_k^-$ are shift operators, that is $Iu = u$ and $T_k^\pm u = u(x \pm h_k e_k, t_j)$, where $e_1 = (1,0)$ and $e_2 = (0,1)$ are unit vectors. Let $D_{h_k}$ be a first order grid differentiation operator and

$$\begin{aligned}
&u_{\bar{x}_k} = (I - T_k^-)u/h_k = D_{h_k} u, \qquad u_{x_k} = (T_k^+ - I)u/h_k = D_{h_k} T_k^+ u,\\
&u_{\bar{x}_k x_k} = D_{h_k} u_{x_k} = D_{h_k}^2 T_k^+ u, \qquad \Delta_h u = u_{\bar{x}_1 x_1} + u_{\bar{x}_2 x_2}.
\end{aligned}$$

We associate problem (1) with two different splitting finite difference schemes.

The finite difference equation for the nonlinear part and the initial conditions are the same for both of these schemes:

$$z_t = f(\dot{z}, \dot{z}^*), \quad (x,t) \in \bar{\omega}_x \times \omega_\tau; \quad z = p, \qquad p(x,0) = u_0(x), \quad x \in \bar{\omega}_x. \tag{5}$$

In general, the equation is nonlinear and we can solve it using iterations:

$$\frac{z^{(n+1)} - z}{\tau} = f\Big(\frac{z^{(n)} + z}{2}\Big), \qquad z^{(0)} = z, \qquad n \to \infty.$$

For the linear part of (1) we have two different finite difference equations.

First, we approximate the two-dimensional linear problem:

$$g_t = a\,\Delta_h \hat{g},\ (x,t) \in Q_{h\tau};\quad g = \hat{z},\ \hat{p} = \hat{g};\quad \hat{g} = 0,\ (x,t) \in \partial\omega_x \times \omega_\tau. \qquad (6)$$

Second, we introduce the locally one-dimensional equations:

$$\begin{aligned} &\frac{g^{(k)} - g}{2\tau} = a\Big(\frac{g^{(k)} + g}{2}\Big)_{\bar{x}_k x_k}, && (x,t) \in Q_{h\tau}, \\ &g^{(k)} = 0,\quad (x,t) \in \partial\omega_x \times \omega_\tau, && g = \hat{z},\quad \hat{p} = \hat{g} = (g^{(1)} + g^{(2)})/2. \end{aligned} \qquad (7)$$

It appears that we can exclude the functions $g^{(1)}$ and $g^{(2)}$ in (7) and obtain the following equivalent equation:

$$g_t = a\,\Delta_h \hat{g} - 2a^2\tau \dot{g}_{\bar{x}_1 x_1 \bar{x}_2 x_2},\ (x,t) \in Q_{h\tau};\ \ g = \hat{z},\ \hat{p} = \hat{g};\ \ \hat{g} = 0,\ (x,t) \in \partial\omega_x \times \omega_\tau. \ (8)$$

**Lemma 1** *The finite difference equations (7) and (8) are equivalent.*

**Proof.** Adding the equations (7) for $k = 1, 2$ and using the expression for $\hat{g}$ via $g^{(k)}$ we obtain

$$\frac{\hat{g} - g}{\tau} = \frac{a}{2}\Big(\Delta_h g + g^{(1)}_{\bar{x}_1 x_1} + g^{(2)}_{\bar{x}_2 x_2}\Big) = \frac{a}{2}\Big(\Delta_h g + 2\Delta_h \hat{g} - g^{(2)}_{\bar{x}_1 x_1} - g^{(1)}_{\bar{x}_2 x_2}\Big).$$

Applying the finite difference operators $(\cdot)_{\bar{x}_2 x_2}$ and $(\cdot)_{\bar{x}_1 x_1}$ respectively to the equations in (7) where $k = 1$ and $k = 2$ one can find that

$$g_t = \frac{a}{2}\Big(\Delta_h g + 2\Delta_h \hat{g}\Big) - \frac{a}{2}\Big(\Delta_h g + a\tau(g^{(1)} + g^{(2)} + 2g)_{\bar{x}_1 x_1 \bar{x}_2 x_2}\Big).$$

It is exactly the same equation as (8). The lemma is proved.

Due to the equivalence of the schemes, all the results obtained for (8) are also valid for (7).

# 3 Some properties of the grid functions.

Let us introduce grid analogues of some functional spaces. We shall say that the grid function $v$ belongs to some functional space if there exists $h_0$ such that for all positive $h_1, h_2 \le h_0$ the corresponding norm of the function $v$ is bounded by a constant which does not depend on grid steps.

Let $l$ be any operator acting on some grid function. Denote by $l(\bar{\omega}_x)$ the subset of the grid $\bar{\omega}_x$ where the operator $l$ is defined.

We shall define the $L_p = L_p(\omega_x)$ norm of the grid functions $u$ or $lu$, and an inner product in $L_2$ as follows:

$$\|u\|_p = \Big(h_1 h_2 \sum_{x \in \bar{\omega}_x} |u|^p\Big)^{1/p},\ \ \|lu\|_p = \Big(h_1 h_2 \sum_{x \in l(\bar{\omega}_x)} |lu|^p\Big)^{1/p},\ \ (u,v) = h_1 h_2 \sum_{x \in \omega_x} uv^*.$$

For simplicity, we define $\|\cdot\|_2 = \|\cdot\|$.

Let us introduce the grid analogues of the Sobolev spaces $W_p^l = W_p^l(\omega_x)$ with norms

$$\|u\|_{l,p} = \Big( \sum_{0\le|k|\le l} \|D_h^k u\|_p^p \Big)^{1/p}, \qquad 1 \le p < \infty, \quad l = 1, 2, \ldots$$

Here $D_h^k = D_h^{(k_1,k_2)} = D_{h_1}^{k_1} D_{h_2}^{k_2}$ are finite difference operators.

In the sequel we also use the shift operator $T^k = T^{(k_1,k_2)} = (T_1^-)^{k_1}(T_2^-)^{k_2}$. It is easy to check that $T^l(vw) = T^l v T^l w$, $T^l(v+w) = T^l v + T^l w$. The operators $T^l$ and $D_h^k$ are commutative. Note that, due to a definition of the $L_p(\omega_x)$ norm for the grid function $u$, the estimate $\|T^k D_h^l u\|_p \le \|D_h^l u\|_p$ holds.

Denote by $\overset{\circ}{W}{}_2^1$ a subset of functions of the space $W_2^1$ which have zeroes on the boundary $\partial\omega_x$, $W = \overset{\circ}{W}{}_2^1 \cap W_2^2$. $\overset{\circ}{C} = \overset{\circ}{C}(\bar\omega_x)$ is the space of the grid functions having zeroes on the boundary and with norm $\|u\|_C = \max_{x\in\bar\omega_x}\{|u|\}$.

We can prove some relations concerning different norms of the grid functions.

**Lemma 2** *Suppose that $u$ is the grid function defined on $\bar\omega_x$ with zero boundary conditions and that $\|D_h^k u\| \le C < \infty$ for all positive $h_1, h_2 \le h_0$. Then the following estimates hold:*

$$\|D_h^{k-e_j} u\| \le c_2 \|D_h^k u\|, \qquad k_j \ge 1, \quad 0 \le k_1, k_2 \le 2. \tag{9}$$

**Proof.** Let us denote by $w_{l_j}(x_j)$ the functions $\sqrt{2}\sin(l_j \pi x_j)$, $j = 1, 2$. Here $x = (x_1, x_2)$ is some point of the grid $\bar\omega_x$.

Since zero boundary conditions hold for the grid function $u$ we can define the function $u$ on the grid $\bar\omega_x$ as follows:

$$u(x_1, x_2) = \sum_{l_1=1}^{N_1-1} \sum_{l_2=1}^{N_2-1} a_{l_1,l_2} w_{l_1}(x_1) w_{l_2}(x_2).$$

Due to completeness and orthogonality of the functions $w_{l_1}(x_1)w_{l_2}(x_2)$ in the space $\overset{\circ}{W}{}_2^1$, the Fourier coefficients $a_{l_1,l_2}$ can be found as

$$a_{l_1,l_2} = h_1 h_2 \sum_{x_1=h_1}^{1} \sum_{x_2=h_2}^{1} u(x_1, x_2) w_{l_1}(x_1) w_{l_2}(x_2).$$

Due to Parsevall's equality, the $L_{2h}$ norm of the grid function $u$ can be expressed as follows:

$$\|u\|^2 = h_1 h_2 \sum_{x\in\bar\omega_x} |u(x)|^2 = h_1 h_2 \sum_{x_1=h_1}^{1} \sum_{x_2=h_2}^{1} |u(x)|^2 = \sum_{l_1=1}^{N_1-1} \sum_{l_2=1}^{N_2-1} |a_{l_1,l_2}|^2.$$

Let us introduce the eigenvalues of the operator $-D^2_{h_j}T^+_j$ corresponding to the eigenfunctions $w_{l_1}(x_1)w_{l_2}(x_2)$:

$$-(w_{l_1}(x_1)w_{l_2}(x_2))_{\bar{x}_j x_j}=\lambda^{(j)}_{l_j}(w_{l_1}(x_1)w_{l_2}(x_2)), \quad \lambda^{(j)}_{l_j}=\frac{4}{h_j^2}\sin^2(l_j\pi h_j/2),\ j=1,2.$$

Note that for $h_j$ small enough, say, $h_j \leq 2/\pi$, we have

$$\min_{l_j}\{\lambda^{(j)}_{l_j}\} = \frac{4}{h_j^2}\sin^2(\pi h_j/2) \geq \frac{\pi^2}{2}.$$

Using the formula of summation by parts for the grid functions and the zero boundary conditions, we have

$$\|D_{h_1}u\|^2 = h_1h_2\sum_{x_1=h_1}^{1}\sum_{x_2=0}^{1}|D_{h_1}u(x)|^2 = -h_1h_2\sum_{x_1=h_1}^{1-h_1}\sum_{x_2=0}^{1}u_{\bar{x}x}(x)u^*(x)$$
$$= h_1h_2\sum_{x_1=h_1}^{1}\sum_{x_2=h_2}^{1}\Big(\sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}\lambda^{(1)}_{l_1}a_{l_1,l_2}w_{l_1}(x_1)w_{l_2}(x_2)\Big)u^*(x) = \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}\lambda^{(1)}_{l_1}|a_{l_1,l_2}|^2.$$

Similarly, we can find the expressions for some other differentiated function norms via the Fourier coefficients and eigenvalues:

$$\|D^k_h u\|^2 = \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}(\lambda^{(1)}_{l_1})^{k_1}(\lambda^{(2)}_{l_2})^{k_2}|a_{l_1,l_2}|^2, \quad 0 \leq k_1, k_2 \leq 2.$$

Note that for other values of $k$ this equality can be not satisfied, since $D^k_h(\bar{\omega}_x)$ is just a subset of the grid $\omega_x$.

Finally, the estimates from below for the eigenvalues $\lambda^{(j)}_{l_j}$ imply (9) where $c_2 = \sqrt{2}/\pi$. The lemma is proved.

**Lemma 3** *Let $v_1, v_2$ be the grid functions defined on $\bar{\omega}_x$. Suppose that $l_1, l_2$ are two operators, $\tilde{\omega}_x = l_1(\omega_x) \cap l_2(\omega_x) = [l_1'h_1, l_1''h_1] \times [l_2'h_2, l_2''h_2] \subset \bar{\omega}_x$, $(l_j'' - l_j' + 1)h_j \geq 0.5$, $j = 1,2$ and $\|D_{h_1}l_1v_1\|, \|D_{h_2}l_2v_2\| \leq C < \infty$ for all positive $h_1, h_2 \leq h_0$. Then the following inequality holds:*

$$\|l_1v_1\, l_2v_2\|^2 \leq 4\|l_1v_1\|\|l_2v_2\|(\|l_1v_1\| + \|D_{h_1}l_1v_1\|)(\|l_2v_2\| + \|D_{h_2}l_2v_2\|) \tag{10}$$

**Proof.** For any fixed grid point coordinate $x_2$ we denote by $\tilde{x}_1(x_2)$ the value of $x_1$ such that $|l_1v_1(\tilde{x}_1(x_2), x_2)| = \min_{x_1\in[l_1'h_1, l_1''h_1]}\{|l_1v_1(x_1,x_2)|\}$. Therefore,

$$|l_1v_1(\tilde{x}_1(x_2),x_2)|^2 \leq 2h_1\sum_{x_1=l_1'h_1}^{l_1''h_1}|l_1v_1(x_1,x_2)|^2, \quad h_2\sum_{x_2=l_2'h_2}^{l_2''h_2}|l_1v_1(\tilde{x}_1(x_2),x_2)|^2 \leq 2\|l_1v_1\|^2.$$

Similarly, we define $\tilde{x}_2(x_1)$, $|l_2v_2(x_1,\tilde{x}_2(x_1))| = \min_{x_2\in[l_2'h_2, l_2''h_2]}\{|l_2v_2(x_1,x_2)|\}$, and have $h_1\sum_{x_1=l_1'h_1}^{l_1''h_1}|l_2v_2(x_1,\tilde{x}_2(x_1))|^2 \leq 2\|l_2v_2\|^2$.

Now we can have the following estimate:

$$\begin{aligned}\|l_1 v_1\, l_2 v_2\|^2 &\le h_2 \sum_{x_2=l_2' h_2}^{l_2'' h_2} \Big(|l_1 v_1(\tilde{x}_1(x_2), x_2)|^2 + h_1 \sum_{\zeta=(l_1'+1)h_1}^{l_1'' h_1} |D_{h_1}|l_1 v_1(\zeta, x_2)|^2|\Big) \\ &\times h_1 \sum_{x_1=l_1' h_1}^{l_1'' h_1} \Big(|l_2 v_2(x_1, \tilde{x}_2(x_1))|^2 + h_2 \sum_{\eta=(l_2'+1)h_2}^{l_2'' h_2} |D_{h_2}|l_2 v_2(x_1, \eta)|^2|\Big).\end{aligned}$$

Estimate (10) follows from here. The lemma is proved.

A direct consequence of the two previous lemmas is the following corollary:

**Corollary 1** *Suppose that $v_1, v_2$ are the grid functions defined on $\bar{\omega}_x$ with zero boundary conditions and that $\|D_h^{k'} v_1\|, \|D_h^{k''} v_2\| \le C < \infty$ for all positive $h_1, h_2 \le h_0$. Then the following inequalities hold:*

$$\begin{aligned}\|T^{r'} D_h^{k'-e_1} v_1\, T^{r''} D_h^{k''-e_2} v_2\| &\le c_3 \Big(\|D_h^{k'-e_1} v_1\| \|D_h^{k''-e_2} v_2\| \|D_h^{k'} v_1\| \|D_h^{k''} v_2\|\Big)^{1/2} \\ \le c_2 c_3 \|D_h^{k'} v_1\| \|D_h^{k''} v_2\|, &\qquad k_1', k_2'' \in \{1, 2\}, \quad k_2', k_1'' \in \{0, 1, 2\}.\end{aligned} \tag{11}$$

**Proof.** Let us denote $l_1 = T^{r'} D_h^{k'-e_1}$, $l_2 = T^{r''} D_h^{k''-e_2}$. The first part of (11) follows from (10), $\|T^r v\| \le \|v\|$ and (9). Here $c_3 = 2(1 + c_2)$. The second part of this estimate immediately follows from (9). The corollary is proved.

In the two-dimensional case the following multiplicative inequality and grid embedding theorem holds:

$$\begin{aligned}\|u\|_C &\le c_4 \|u\|^{(1/2-\gamma)} \|u\|_{2,2}^{(1/2+\gamma)}, \quad 0 < \gamma < 0.5, \\ \|u\|_p &\le c_5 \|u\|_{1,2} \quad p \in [2, \infty), \quad c_5 = c_5(p).\end{aligned} \tag{12}$$

Both estimates can be found in [7].

In the sequel we also use the results of the following lemma:

**Lemma 4** *In all points of the grid $D_h^k(\bar{\omega}_x)$ the following formula for the function $D_h^k\Big(\prod_{j=1}^s v_j\Big)$ is valid:*

$$D_h^k\Big(\prod_{j=1}^s v_j\Big) = \sum_{l^{(1)}+\ldots+l^{(s)}=k} \frac{k_1! k_2!}{\prod_{j=1}^s l_1^{(j)}!\, l_2^{(j)}!} \prod_{j=1}^s T^{l^{(1)}+\ldots+l^{(j-1)}} D^{l^{(j)}} v_j. \tag{13}$$

*Here $l^{(j)} = (l_1^{(j)}, l_2^{(j)})$, $l_1^{(j)} \in \{0, 1, \ldots, k_1\}$, $l_2^{(j)} \in \{0, 1, \ldots, k_2\}$, $j = 1, \ldots, s$.*

**Proof.** Let us prove (13) for $s = 2$. We use the method of mathematical induction with respect to $k = (k_1, k_2)$. We must prove the formula

$$D_h^k(v_1 v_2) = \sum_{l^{(1)}+l^{(2)}=k} \frac{k_1! k_2!}{l_1^{(1)}!\, l_1^{(2)}!\, l_2^{(1)}!\, l_2^{(2)}!} D^{l^{(1)}} v_1\, T^{l^{(1)}} D^{l^{(2)}} v_2.$$

Note that here we have $l_j^{(2)} = k_j - l_j^{(1)}$ and, therefore, $\frac{k_j!}{l_j^{(1)}!l_j^{(2)}!} = C_{k_j}^{l_j^{(1)}}$, $j = 1, 2$.

This formula is obvious in the case when $k = 0 = (0,0)$, $k = e_1 = (1,0)$ and $k = e_2 = (0,1)$. Suppose that it is also valid for all $r$, $0 \le r_1 \le k_1$, $0 \le r_2 \le k_2$. Let us consider this formula for $D_h^{k+e_1} = D_h^{k'}$:

$$D_h^{k+e_1}(v_1v_2) = D_h^{e_1}(D_h^k(v_1v_2)) = \sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} C_{k_1}^{l_1}C_{k_2}^{l_2}D_h^{e_1}\left(D^l v_1 T^l D^{k-l}v_2\right)$$

$$= \sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} C_{k_1}^{l_1}C_{k_2}^{l_2}(D^l v_1 T^l D^{k-l+e_1}v_2 + D^{l+e_1}v_1 T^{l+e_1}D^{k-l}v_2)$$

$$= \sum_{l_2=0}^{k_2} C_{k_2}^{l_2}\Big[\sum_{l_1=1}^{k_1}(C_{k_1}^{l_1} + C_{k_1}^{l_1-1})D^l v_1 T^l D^{k'-l}v_2 + v_1 D^{k'}v_2 + D^{k'}v_1 T^{k'}v_2\Big].$$

Here the last expression is equivalent to the formula for $D_h^k(v_1v_2)$ with $k'$ instead of $k$. We can complete the proof of the formula for $D_h^{k+e_2}$ in the same way and, therefore, complete the proof of the induction step.

Now we can prove a general form of (13) using the mathematical induction with respect to $s$.

Formula (13) is obvious for $s = 1$. We have also proved it above for $s = 2$. Suppose that it is also valid for all $r$, $2 \le r \le s$. Let us introduce $w = \prod_{j=1}^{s} v_j$ and consider this formula for $D_h^k(\prod_{j=1}^{s+1} v_j) = D_h^k(wv_{s+1})$:

$$D_h^k\Big(\prod_{j=1}^{s+1} v_j\Big) = D_h^k(wv_{s+1}) = \sum_{r+l^{(s+1)}=k} \frac{k_1!k_2!}{r_1!l_1^{(s+1)}!r_2!l_2^{(s+1)}!} D^r w T^r D^{l^{(s+1)}} v_{s+1}$$

$$= \sum_{r+l^{(s+1)}=k}\ \sum_{l^{(1)}+\ldots+l^{(s)}=r} \frac{k_1!k_2!}{\prod_{j=1}^{s+1} l_1^{(j)}!l_2^{(j)}!}\prod_{j=1}^{s+1} T^{l^{(1)}+\ldots+l^{(j-1)}}D^{l^{(j)}}v_j.$$

According to this, (13) is valid for $D_h^k(\prod_{j=1}^{s+1} v_j)$. It proves the induction step and, therefore, (13) is valid. The lemma is proved.

# 4 Estimates of the nonlinear function.

We can also derive some estimates for the nonlinear part of the difference schemes. First, we show some properties which can be obtained for the nonlinear functions $f(u, u^*)$ due to requirement (2).

**Lemma 5** *Let (2) be satisfied. Then the following estimate holds:*

$$|f(v, v^*) - f(w, w^*)| \le 2\varphi(\max\{|v|, |w|)\})|v - w|.$$

**Proof.** We have the following equalities:

$$f(v, v^*) - f(w, w^*) = \frac{f(v, v^*) - f(w, v^*)}{v - w}(v - w) + \frac{f(w, v^*) - f(w, w^*)}{v^* - w^*}(v - w)^*$$

$$= \frac{\partial}{\partial u} f(\tilde{v}_1, v^*)(v-w) + \frac{\partial}{\partial u^*} f(w, \tilde{v}_2^*)(v-w)^*.$$

Here $\tilde{v}_j = \theta_j v + (1-\theta_j)w$, $\theta_j \in [0;1]$, $j = 1,2$. The supposition of this lemma follows from (2).

**Lemma 6** *Assume that $v \in W$ and (2) is satisfied. Then the following estimates are valid:*

$$\|D_h^k f(v,v^*)\| \le d_k \|v\|_{|k|,2}, \quad |k| = 1,2, \quad d_k = d_k(\varphi(\|v\|_C), \|v\|_{|k|-1,2}). \tag{14}$$

*If, additionally, condition (2) for the function $f(u,u^*)$ is valid up to the fourth order derivatives, the following estimates are valid:*

$$\begin{aligned} \|D_h^k f(v,v^*)\| &\le d_k\Big(\|D_h^{(1,2)} v\| + \|D_h^{(2,1)} v\|\Big), \qquad k \in \{(1,2),(2,1)\}, \\ \|D_h^{(2,2)} f(v,v^*)\| &\le d_{(2,2)} \|D_h^{(2,2)} v\|. \end{aligned} \tag{15}$$

*Here we suppose that the norms in the right-hand side of the inequalities are bounded $\|D_h^k v\| \le C < \infty$ for all positive $h_1, h_2 \le h_0$ and*

$$d_k = \begin{cases} d_k(\varphi(\|v\|_C), \|v\|_{2,2}) & \text{if} \quad k \in \{(1,2),(2,1)\}; \\ d_k(\varphi(\|v\|_C), \|D_h^{(1,2)} v\|, \|D_h^{(2,1)} v\|) & \text{if} \quad k = (2,2). \end{cases}$$

**Proof.** If $|k| = 1$ the statement of this lemma follows from Lemma 5.

Assume $|k| \ge 2$. The finite difference differentiation $D_h^k v$ can be written as

$$D_h^k v\Big|_{x=\bar{x}} = \frac{1}{h_1^{k_1} h_2^{k_2}} \sum_{y \in A_k(\bar{x})} e(y) v(y) = \frac{1}{h^{|k|} d^{k_2}} \sum_{y \in A_k(\bar{x})} e(y) v(y).$$

Here $A_k(\bar{x}) = ([\bar{x}_1 - k_1 h_1; \bar{x}_1] \times [\bar{x}_2 - k_2 h_2; \bar{x}_2]) \cap \bar{\omega}_x$ is the set of neighbouring points of $\bar{x}$ on which the differentiation $D_h^k$ is defined. For $y \in A(\bar{x})$ we have $e(y) = e(\bar{x}_1 - l_1 h_1, \bar{x}_2 - l_2 h_2) = (-1)^{|l|} C_{k_1}^{l_1} C_{k_2}^{l_2}$ and $\sum_{y \in A(\bar{x})} |e(y)| = 2^{|k|}$.

Using these notation and an expansion of the function $f(v,v^*)$ into the Taylor series in the neighbourhood of $\bar{v} = v(\bar{x})$ we get the following formula:

$$\begin{aligned} D_h^k f(v,v^*)\Big|_{x=\bar{x}} &= \frac{1}{h^{|k|} d^{k_2}} \sum_{y \in A_k(\bar{x})} e(y) f(v(y), v^*(y)) \\ &= \frac{1}{h^{|k|} d^{k_2}} \sum_{y \in A_k(\bar{x})} e(y) \sum_{|s|<|k|} \frac{\partial^s f(\bar{v},\bar{v}^*)}{\partial^{s_1} v\, \partial^{s_2} v^*} \frac{(v(y)-\bar{v})^{s_1} (v(y)-\bar{v})^{*s_2}}{s_1!\ s_2!} \\ &\quad + \frac{1}{h^{|k|} d^{k_2}} \sum_{y \in A_k(\bar{x})} e(y) \sum_{|s|=|k|} \frac{\partial^s f(\tilde{v}_{1y},\tilde{v}_{2y}^*)}{\partial^{s_1} v\, \partial^{s_2} v^*} \frac{(v(y)-\bar{v})^{s_1} (v(y)-\bar{v})^{*s_2}}{s_1!\ s_2!} \\ &= \sum_{1 \le |s| < |k|} \frac{1}{s_1!\ s_2!} \frac{\partial^s f(\bar{v},\bar{v}^*)}{\partial^{s_1} v\, \partial^{s_2} v^*} D_h^k\Big((v-\bar{v})^{s_1} (v-\bar{v})^{*s_2}\Big)\Big|_{x=\bar{x}} \\ &\quad + \sum_{y \in A_k(\bar{x})} e(y) \sum_{|s|=|k|} \frac{1}{d^{k_2} s_1!\ s_2!} \frac{\partial^s f(\tilde{v}_{1y},\tilde{v}_{2y}^*)}{\partial^{s_1} v\, \partial^{s_2} v^*} \Big(\frac{v(y)-\bar{v}}{h}\Big)^{s_1} \Big(\frac{v(y)-\bar{v}}{h}\Big)^{*s_2}. \end{aligned}$$

Here we have set $\tilde{v}_{jy} = \theta_{jy} v(\bar{x}) + (1-\theta_{jy})v(y)$, $\theta_{jy} \in [0,1]$, $j = 1,2$.

Using (2), Newton's binom and the expression of $e(y)$ we have

$$\left|D_h^k f(v,v^*)\right| \le \sum_{1\le|s|<|k|} \frac{\varphi(\|v\|_C)}{s_1!\, s_2!} \left|D_h^k \Big( \sum_{r_1=0}^{s_1} \sum_{r_2=0}^{s_2} C_{s_1}^{r_1} C_{s_2}^{r_2} v^{r_1} v^{*r_2} \bar{v}^{s_1-r_1} \bar{v}^{*s_2-r_2} \Big)\right|$$
$$+\frac{\varphi(\|v\|_C) 2^{|k|}}{d^{k_2}|k|!} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} C_{k_1}^{l_1} C_{k_2}^{l_2} \left| \sum_{r_1=0}^{l_1-1} D_{h_1} T^{(r_1,l_2)} v + d \sum_{r_2=0}^{l_2-1} D_{h_2} T^{(0,r_2)} v \right|^{|k|}.$$

Taking $L_2(\omega_x)$ norms, applying Hölder's inequality and taking into account that $\|T^k D_h^l u\|_p \le \|D_h^l u\|_p$, we obtain the following estimates:

$$\|D_h^k f(v,v^*)\| \le \frac{(4|k|d)^{|k|}}{|k|\,|k|!} \varphi(\|v\|_C) \Big( k_1 \|D_{h_1} v\|_{2|k|}^{|k|} + k_2 \|D_{h_2} v\|_{2|k|}^{|k|} \Big)$$
$$+ \sum_{1\le|s|<|k|} \sum_{r_1=0}^{s_1} \sum_{r_2=0}^{s_2} \frac{\varphi(\|v\|_C)\|v\|_C^{|s-r|}}{r_1! r_2! (s_1-r_1)!(s_2-r_2)!} \|D_h^k v^{r_1} v^{*r_2}\| = \mu_1 + \mu_2.$$

Assume that $|k| = 2$. Using (11) we find that $\mu_1 \le 32 d^2 \varphi(\|v\|_C) c_3^2 \|v\|_{1,2} \|v\|_{2,2}$ and $\mu_2 = 2\varphi(\|v\|_C)\|D_h^k v\|$. This completes the proof of (14).

For $k \in \{(1,2),(2,1),(2,2)\}$ we estimate $\mu_1$ using the second inequality from (12) and estimate (9):

$$\begin{aligned}
\mu_1 &\le \frac{(4|k|d)^{|k|}}{|k|\,|k|!} \varphi(\|v\|_C) c_5^{|k|} \|v\|_{2,2}^{|k|-1} \Big( k_1 \|D_{h_1} v\|_{1,2} + k_2 \|D_{h_2} v\|_{1,2} \Big) \\
&\le \frac{(4|k|dc_5)^{|k|}}{|k|\,|k|!} \|v\|_{2,2}^{|k|-1} \varphi(\|v\|_C) c_2 \sqrt{2+c_2^2} \Big( k_1 \|D_h^{(2,1)} v\| + k_2 \|D_h^{(1,2)} v\| \Big) \\
&\le \frac{(4|k|dc_5)^{|k|}}{|k|!} \|v\|_{2,2}^{|k|-1} \varphi(\|v\|_C) c_2^2 \sqrt{2+c_2^2}\ \|D_h^{(2,2)} v\|.
\end{aligned}$$

We write $v_1, v_2, v_3$ instead of $T^r v$ or $T^r v^*$. We do not specify a value of the vector $r$ since $\|D_h^l v_j\| \le \|D_h^l v\|$ for any $l$ and $r$. Let $l' + l'' = k$ and $|l'|, |l''| \ge 1$. For the given $k$ we can suppose that $l_1' < k_1$ and $l_2'' < k_2$. Let us also set $k_j = \max\{k_1, k_2\} = 2$. Then (11) and (9) lead to the following estimate:

$$\|D_h^{l'} v_1 D_h^{l''} v_2\| \le c_2 c_3 \|D_h^{l'+e_1} v\| \|D_h^{l''+e_2} v\| \le c_2^{|k|-2} c_3 \|D_h^{k-e_j} v\| \|D_h^k v\|.$$

For $k = (2,2)$ we put $l + e_j + e_l = k$, $\bar{e}_j = (1,1) - e_j$. Using (10) we can derive the estimate

$$\|D_h^l v_1 D_h^{e_j} v_2 D_h^{e_l} v_3\| \le \Big[ \|D_h^{e_j} v_2 D_h^{e_l} v_3\| + \|D_h^{e_j} v_2 D_h^{e_l+\bar{e}_j} v_3\| + \|D_h^{(1,1)} v_2 D_h^{e_l} T^{\bar{e}_j} v_3\| \Big]^{1/2}$$
$$\times \|D_h^{e_j} v_2 D_h^{e_l} v_3\|^{1/2} \sqrt{2c_2c_3} \|D_h^{l+e_j} v\| \le \sqrt{2} c_2^4 c_3^2 \|D_h^{(1,2)} v\| \|D_h^{(2,1)} v\| \|D_h^k v\|.$$

Therefore, for $k \in \{(1,2),(2,1),(2,2)\}$ we can use (13) and obtain

$$\|D_h^k v_1 v_2\| \le (2\|v\|_C + (2^{|k|} - 2) c_2^{|k|-2} c_3 \|D_h^{k-e_j} v\|) \|D_h^k v\|, \qquad \text{here } k_j = 2;$$

$$\|D_h^{(2,2)}v_1v_2v_3\| \le \Big(3\|v\|_C^2+6\Big[7\|v\|_C+6\sqrt{2}c_2^2c_3\|D_h^{(2,1)}v\|\Big]c_2^2c_3\|D_h^{(1,2)}v\|\Big)\|D_h^{(2,2)}v\|.$$

Let us suppose that $k\in\{(1,2),(2,1)\}$. Then we get the following estimate:

$$\begin{aligned}\mu_2 &= 2\varphi(\|v\|_C)\|D_h^k v\| + \sum_{|s|=2}\sum_{r_1=0}^{s_1}\sum_{r_2=0}^{s_2}\frac{\varphi(\|v\|_C)\|v\|_C^{|s-r|}}{r_1!r_2!(s_1-r_1)!(s_2-r_2)!}\|D_h^k v^{r_1}v^{*r_2}\| \\ &\le 2\varphi(\|v\|_C)\Big(1+4\|v\|_C+6c_2c_3\|D_h^{(1,1)}v\|\Big)\|D_h^k v\|.\end{aligned}$$

Due to the first inequality of (12) we obtain the first estimate of (15) with a corresponding coefficient $d_k$ for these values of $k$.

Finally, let us consider $k=(2,2)$. Similarly as above we find that

$$\begin{aligned}\mu_2 &\le \varphi(\|v\|_C)\Big(2+4\|v\|_C+4\|v\|_C^2\Big)\|D_h^{(2,2)}v\|+\varphi(\|v\|_C)(2+4\|v\|_C)\|D_h^{(2,2)}v_1v_2\| \\ &+\frac{4}{3}\varphi(\|v\|_C)\|D_h^{(2,2)}v_1v_2v_3\| \le 2\varphi(\|v\|_C)\Big(1+4\|v\|_C+8\|v\|_C^2 \\ &\quad +c_2^2c_3\Big[14+56\|v\|_C+24\sqrt{2}c_2^2c_3\|D_h^{(2,1)}v\|\Big]\|D_h^{(1,2)}v\|\Big)\|D_h^{(2,2)}v\|.\end{aligned}$$

Due to the first inequality of (12) and (9) this leads to (15). The lemma is proved.

## 5 General properties of the difference schemes.

In order to prove convergence and stability of difference schemes (5), (6) and (5), (8) we use some auxiliary statements.

**Lemma 7** *Let $g\in W$. Then there exists a unique solution $\hat g\in W$ of (6) (or (8)) and the following estimates are valid:*

$$\|D_h^k\hat g\|\le\|D_h^k g\|,\qquad 0\le|k|=k_1+k_2\le 2$$

*If, additionally, $\|D_h^k g\|\le C<\infty$ for all positive $h_1,h_2\le h_0$ then these estimates are also valid for $k\in\{(1,2),(2,1),(2,2)\}$.*

**Proof.** Analogously as in Lemma 2, for both cases of equations (6) and (7) we define the following functions on the grid $\bar\omega_x$:

$$\begin{aligned}g(x_1,x_2) &= \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}a_{l_1,l_2}w_{l_1}(x_1)w_{l_2}(x_2),\\ g^{(j)}(x_1,x_2) &= \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}q_{l_j}^{(j)}a_{l_1,l_2}w_{l_1}(x_1)w_{l_2}(x_2),\\ \hat g(x_1,x_2) &= \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}q_{l_1,l_2}a_{l_1,l_2}w_{l_1}(x_1)w_{l_2}(x_2).\end{aligned}$$

For difference scheme (6) the constants $q_{l_1,l_2}$, $l_j = 1,\dots,N_j-1$, $j=1,2$ can be found from the equations

$$\frac{q_{l_1,l_2}-1}{2\tau} = -a(\lambda^{(1)}_{l_1}+\lambda^{(2)}_{l_2})q_{l_1,l_2}.$$

It follows from here and from the condition $\operatorname{Re} a = a_1 \geq 0$ that

$$|q_{l_1,l_2}|^2 = \left|\frac{1}{1+a\tau(\lambda^{(1)}_{l_1}+\lambda^{(2)}_{l_2})}\right|^2 = \frac{1}{(1+a_1\tau(\lambda^{(1)}_{l_1}+\lambda^{(2)}_{l_2}))^2+(a_2\tau(\lambda^{(1)}_{l_1}+\lambda^{(2)}_{l_2}))^2} \leq 1.$$

Similarly, in the case of difference scheme (7) we can find the constants

$$q_{l_1,l_2} = (q^{(1)}_{l_1}+q^{(2)}_{l_2})/2, \qquad l_j = 1,2,\dots,N_j-1, \quad j=1,2,$$

where $q^{(j)}_{l_j}$ can be derived from the equations

$$(q^{(j)}_{l_j}-1)/2\tau = -a\lambda^{(j)}_{l_j}(q^{(j)}_{l_j}+1)/2.$$

It follows that

$$|q^{(j)}_{l_j}|^2 = \left|\frac{1-a\tau\lambda^{(j)}_{l_j}}{1+a\tau\lambda^{(j)}_{l_j}}\right|^2 = 1 - \frac{4a_1\tau\lambda^{(j)}_{l_j}}{(1+a_1\tau\lambda^{(j)}_{l_j})^2+(a_2\tau\lambda^{(j)}_{l_j})^2} \leq 1$$

and

$$|q_{l_1,l_2}| \leq (|q^{(1)}_{l_1}|+|q^{(2)}_{l_2}|)/2 \leq 1.$$

Thus, similarly as in Lemma 2, for both schemes (6) and (7) and for all $0 \leq k_1, k_2 \leq 2$ we have

$$\begin{aligned} \|D^k_h\hat{g}\|^2 &= \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}(\lambda^{(1)}_{l_j})^{k_1}(\lambda^{(1)}_{l_l})^{k_2}|q_{l_1,l_2}a_{l_1,l_2}|^2 \\ &\leq \sum_{l_1=1}^{N_1-1}\sum_{l_2=1}^{N_2-1}(\lambda^{(1)}_{l_1})^{k_1}(\lambda^{(2)}_{l_2})^{k_2}|a_{l_1,l_2}|^2 = \|D^k_h g\|^2. \end{aligned}$$

Since the coefficients $q_{l_1,l_2}$ and $q_{l_1,l_2}$ exist and can be written in unique way for all $l_j = \{1,2,\dots,N_j-1\}$, $j=1,2$, the unique function $\hat{g}$ exists in the case of both schemes. The function $\hat{g}$ belongs to the space $W$ due to the estimates derived above. The lemma is proved.

**Lemma 8** *Let (2) and (3) be satisfied and $z \in \overset{\circ}{C}(\bar{\omega}_x)$. Then there is a constant $\tau_0$, such that for all positive $\tau \leq \tau_0$ there is a unique solution of (5) $\hat{z} \in \overset{\circ}{C}(\bar{\omega}_x)$ and the estimate $|\hat{z}| \leq |z|$ holds.*

**Proof.** See in [6].

**Lemma 9** *Let (2) and (3) be satisfied and $z \in W$. Then there is a constant $\tau_0$, such that for all positive $\tau \leq \tau_0$ a solution of (5) $\hat{z} \in W$ and the following estimates hold:*

$$\|\hat{z}\|_{|k|,2} \leq (1 + 2d_{|k|}\tau)\|z\|_{|k|,2}, \quad d_{|k|} = d_{|k|}(\varphi(\|z\|_C), \|z\|_{|k|-1,2}) \quad |k| = 1, 2.$$

*If, additionally, condition (2) for the function $f(u, u^*)$ is valid up to the fourth order derivatives and $\|D_h^{(2,2)}z\| \leq C < \infty$ for all positive $h_1, h_2 \leq h_0$, then the following estimates also hold:*

$$\begin{aligned}\left(\|D_h^{(1,2)}\hat{z}\|^2 + \|D_h^{(2,1)}\hat{z}\|^2\right)^{1/2} &\leq (1 + 2d_3\tau)\left(\|D_h^{(1,2)}z\|^2 + \|D_h^{(2,1)}z\|^2\right)^{1/2}; \\ \|D_h^{(2,2)}\hat{z}\| &\leq (1 + 2d_4\tau)\|D_h^{(2,2)}z\|.\end{aligned}$$

*Here $d_3 = d_3(\varphi(\|z\|_C), \|z\|_{2,2})$ and $d_4 = d_4(\varphi(\|z\|_C), \|D_h^{(1,2)}z\|, \|D_h^{(2,1)}z\|)$.*

**Proof.** Let us denote the coefficients

$$\tilde{d}_{|k|} = \sum_{1 \leq |k'| \leq |k|} d_{k'}, \ |k| = 1, 2, \quad \tilde{d}_3 = \max\{d_{(1,2)}, d_{(2,1)}\}/2, \quad \tilde{d}_4 = d_{(2,2)}.$$

Here $d_{k'} = d_{k'}(\dot{z})$ are the coefficients from Lemma 6 with the function $\dot{z}$ instead of $v$.

Let us apply the operator $D_h^k$ on both sides of equation (5), multiply scalarly both sides of the equation by $2\tau D_h^k \dot{z}$ and take real parts. We obtain

$$\|D_h^k\hat{z}\|^2 = \|D_h^k z\|^2 + 2\tau \operatorname{Re}(D_h^k f(\dot{z}, \dot{z}^*), D_h^k \dot{z}) \leq \|D_h^k z\|^2 + 2\tau \|D_h^k f(\dot{z}, \dot{z}^*)\| \|D_h^k \dot{z}\|.$$

Let us summate these equations for all $k'$, $|k'| \leq |k| \leq 2$ and apply the estimate $\|\hat{z}\| \leq \|z\|$ which follows from Lemma 8. Using (14) we obtain

$$\|\hat{z}\|^2_{|k|,2} \leq \|z\|^2_{|k|,2} + 2\tau\tilde{d}_{|k|}\|\dot{z}\|^2_{|k|,2} \ \Rightarrow \ \|\hat{z}\|^2_{|k|,2} \leq \left(1 + \frac{2\tau\tilde{d}_{|k|}}{1 - \tau\tilde{d}_{|k|}}\right)\|z\|^2_{|k|,2}.$$

It follows from Lemmas 6 and 8 that $\tilde{d}_1 = \tilde{d}_1(\varphi(\|\dot{z}\|_C)) \leq \tilde{d}_1(\varphi(\|z\|_C)) = d_1(\varphi(\|z\|_C))$. Taking $\tau \leq \tau_0 = 1/2d_1$, we find that $\|\hat{z}\|_{1,2} \leq (1 + 2d_1\tau)\|z\|_{1,2}$.

Due to this estimate we can introduce the coefficient $d_2 = d_2(\varphi(\|z\|_C), \|z\|_{1,2})$ such that $\tilde{d}_2 = \tilde{d}_2(\varphi(\|\dot{z}\|_C), \|\dot{z}\|_{1,2}) \leq d_2$. Now for all positive $\tau \leq \tau_0 = 1/2d_2$ we have the estimate $\|\hat{z}\|_{2,2} \leq (1 + 2d_2\tau)\|z\|_{2,2}$. This estimate completes the proof of the first part of this lemma.

Similarly, using the lower order grid derivative estimates, we can find the estimates for $\left(\|D_h^{(1,2)}\hat{z}\|^2 + \|D_h^{(2,1)}\hat{z}\|^2\right)^{1/2}$ and $\|D_h^{(2,2)}\hat{z}\|$ where $d_{|k|} = d_{|k|}(z) \geq \tilde{d}_{|k|}(\dot{z})$. The lemma is proved.

**Corollary 2** *Let (2) and (3) hold and a solution of (5), (6), or (5), (8) satisfy the estimate* $\|p(t_l)\|_C \le \alpha$, $l = 0, 1, \dots, j$. *Then for any* $u_0 \in W$ *there is a constant* $\tau_0$, *such that for all positive* $\tau \le \tau_0$ *the following estimate is valid:*

$$\|p(t_{l+1})\|_{2,2} \le \|g(t_l)\|_{2,2} \le c_6 \|u_0\|_{2,2}, \qquad l = 0, 1, \dots, j.$$

*If, additionally, condition (2) for the function* $f(u, u^*)$ *is valid up to the fourth order derivatives, and the norm* $\|D_h^{(2,2)} u_0\| \le C < \infty$ *is bounded for all* $h_1, h_2 \le h_0$, *the following estimate is also valid:*

$$\|D_h^{(2,2)} p(t_{l+1})\| \le \|D_h^{(2,2)} g(t_l)\| \le c_6' \|D_h^{(2,2)} u_0\|, \qquad l = 0, 1, \dots, j.$$

*Here* $c_6 = c_6(\varphi(\alpha), \|u_0\|_{1,2})$, $c_6' = c_6'(\varphi(\alpha), \|D_h^{(1,2)} u_0\|, \|D_h^{(2,1)} u_0\|)$. *Both these constants do not depend on the grid steps.*

**Proof.** We consider $\hat{p} = p(t_{l+1}) = \hat{g}(t_l)$, $g = g(t_l) = \hat{z}(t_l)$, $z(t_l) = p(t_l) = p$. Therefore, the results of Lemmas 7 and 9 lead to the estimates

$$\|\hat{p}\|_{1,2} \le \|g\|_{1,2} \le (1 + 2d_1\tau)\|p\|_{1,2} \le (1 + 2d_1\tau)^{l+1}\|u_0\|_{1,2} \le \exp(2d_1 T)\|u_0\|_{1,2}.$$

Here $d_1 = d_1(\varphi(\alpha))$ and $l = 0, \dots, j$. Thus, we have shown that $\|p(t_{l+1})\|_{1,2}$ is bounded with a constant, which depends only on $\|u_0\|_{1,2}$ and $\varphi(\alpha)$.

Applying the same idea for $\|p(t)\|_{2,2}$ we obtain analogous result:

$$\|p(t_{l+1})\|_{2,2} \le \|g(t_l)\|_{2,2} \le \exp(2d_2 T)\|u_0\|_{2,2}, \quad l = 0, 1, \dots, j.$$

Here $d_2$ depends on $\varphi(\alpha)$ and, due to Lemma 9 and the previous result, on

$$\max_{t \in \bar{\omega}_\tau, t \le t_j} \{\|p(t)\|_{1,2}\} \le \exp(2d_1 T)\|u_0\|_{1,2}.$$

A first statement of the corollary follows from here with $c_6 = \exp(2d_2 T)$.

Using the lower order grid derivative estimates, similarly we can find similar estimates for $\left(\|D_h^{(1,2)} p(t)\|^2 + \|D_h^{(2,1)} p(t)\|^2\right)^{1/2}$ and $\|D_h^{(2,2)} p(t)\|$ with $c_6' = \exp(2d_{(2,2)} T)$. The corollary is proved.

# 6 Convergence of the scheme (5, 6).

Let us introduce a truncation error $\Phi$ on the grid $\bar{\omega}_\tau$:

$$\Phi(t_j) = \frac{u(t_{j+1}) - u(t_j)}{\tau} - a\Delta_h u(t_{j+1}) - f\Big(\frac{u(t_{j+1}) + u(t_j)}{2}, \frac{u^*(t_{j+1}) + u^*(t_j)}{2}\Big).$$

Here and below $u(x, t)$ is the solution of (1). This error satisfies the estimate

$$\max_{t \in \bar{\omega}_\tau} \{\|\Phi(t)\|\} \le c_7(\tau + h^2)$$

if condition (4) holds.

Now we can show, that finite difference scheme (5), (6) converges to the solution of (1).

**Theorem 1** *Let (2), (3) and (4) be satisfied and $u_0 \in W$. Then there exist $h_0$ and $\tau_0$ such that for all positive $h_1, h_2 \le h_0$ and $\tau \le \tau_0$ the solution of difference scheme (5), (6) converges to the solution of differential problem (1) and the following estimates hold:*

$$\max_{t\in\bar\omega_\tau}\{\|p(t)-u(t)\|\} \le c_8(\tau+h^2),$$
$$\max_{t\in\bar\omega_\tau}\{\|p(t)-u(t)\|_C\} \le c_8'(\tau^{(1/2-\gamma)}+h^{(1-2\gamma)}), \quad \gamma\in(0,0.5) \tag{16}$$

*Here $c_8, c_8'$ depend only on $c_7$, $|a|$, $\alpha = 2\|u\|_{C(\bar Q)}$, $\varphi(\alpha)$, $T$, $\|u_0\|_{2,2}$. $c_8'$ depends also on $c_4$. Both these coefficients do not depend on the grid steps.*

**Proof.** Adding equations (5) and (6) we get the scheme

$$p_t = a\Delta_h\hat p + f(\dot z, \dot z^*).$$

We have the following difference scheme for the error of the solution $\varepsilon = u - p$:

$$\varepsilon_t = a\Delta_h\hat\varepsilon + (f(\dot u,\dot u^*) - f(\dot z,\dot z^*)) + \Phi, \quad (x,t)\in Q_{h\tau},$$
$$\varepsilon(x,t) = 0, \quad (x,t)\in\partial\omega_x\times\bar\omega_\tau, \quad \varepsilon(x,0)=0, \ x\in\omega_x. \tag{17}$$

Let as denote the constant $\alpha = 2\|u\|_{C(\bar Q)}$. By mathematical induction we can show that there exist constants $\tau_0$ and $h_0$ such that for all positive $\tau\le\tau_0$, $h_1,h_2\le h_0$ and for all $t\in\bar\omega_\tau$ the estimate $\|p(t)\|_C\le\alpha$ holds.

It is clear, that $\|p(t_0)\|_C \le \alpha$. Suppose that for all $l = 0,1,\dots,j$ the estimate $\|p(t_l)\|_C\le\alpha$ holds. Then from Corollary 2 and from (12) we know that the estimate $\|p(t_l)\|_{2,2}\le c_6\|u_0\|_{2,2}$ for all $l=0,1,\dots,j+1$ is valid. Note also that, due to Lemma 5 and to (6), we have the following estimates:

$$\|f(\dot u,\dot u^*) - f(\dot z,\dot z^*)\| \le 2\varphi(\max\{\|u\|_{C(\bar Q)},\|p\|_C\})(\|\dot\varepsilon\| + 0.5\tau\|g_t\|)$$
$$\le 2\varphi(\alpha)(\|\dot\varepsilon\| + \tau|a|c_6\|u_0\|_{2,2}).$$

Multiplying scalarly (17) by $\tau\hat\varepsilon$, taking real parts, applying Cauchy inequalities, using the property $\mathrm{Re}\,a = a_1\ge 0$, we can obtain the following inequalities:

$$\|\hat\varepsilon\|^2 \le \|\varepsilon\|\,\|\hat\varepsilon\| + \tau(\|\Phi\| + \|f(\dot u,\dot u^*) - f(\dot z,\dot z^*)\|)\|\hat\varepsilon\|.$$

From here and from the estimates for $\|\Phi\|$ and $\|f(\dot u,\dot u^*) - f(\dot z,\dot z^*)\|$ it follows:

$$\|\hat\varepsilon\| \le \|\varepsilon\| + 2\tau\varphi(\alpha)\|\dot\varepsilon\| + \tau c_7(\tau+h^2) + 2\tau^2|a|c_6\|u_0\|_{2,2}\varphi(\alpha).$$

Taking time step $\tau\le\tau_0 = 1/2\varphi(\alpha)$, we obtain

$$\|\hat\varepsilon\| \le (1+4\varphi(\alpha)\tau)\|\varepsilon\| + \tau(2c_7 + 4\varphi(\alpha)|a|c_6\|u_0\|_{2,2})(\tau+h^2).$$

Adding these estimates for time layers, using the grid Gronwall's inequality and knowing that $\|\varepsilon(t_0)\| = 0$, we can obtain

$$\|\varepsilon(t_{j+1})\| \le c_8(\tau+h^2) \quad\to\quad 0.$$

Remark 1, condition (4) and the induction's supposition lead to

$$\|\varepsilon(t_{j+1})\|_{2,2} \leq \|u(t_{j+1})\|_{2,2} + \|p(t_{j+1})\|_{2,2} \leq (c_1 + c_6)\|u_0\|_{2,2}.$$

Thus, from multiplicative inequality (12) we have

$$\|\varepsilon(t_{j+1})\|_C \leq c_8'(\tau + h^2)^{(1/2-\gamma)} \quad \rightarrow \quad 0.$$

Therefore, taking time and spatial grid steps small enough we can achieve that $\|p(t_{j+1})\|_C \leq 2\|u(t_{j+1})\|_C \leq \alpha$. Thus, a step of induction is completed. Therefore, the estimates for the $L_2$ and $C$ norms of $\varepsilon(t_{j+1})$ are valid for all time layers and (16) holds. The theorem is proved.

# 7 Convergence of the scheme (5, 7).

Now we can switch to the investigation of the other difference scheme. As it was mentioned earlier, it is enough to investigate the scheme (5), (8).

Using similar ideas as earlier we can prove the following theorem:

**Theorem 2** *Let the assumptions of Theorem 1 be satisfied. Then there exist $h_0$ and $\tau_0$ such that for all positive $h_1, h_2 \leq h_0$ and $\tau \leq \tau_0$ the solution of difference scheme (5), (8) converges to the solution of differential problem (1) and the following estimates are satisfied:*

$$\begin{aligned} \max_{t\in\bar{\omega}_\tau}\{\|p(t) - u(t)\|\} &\leq c_9(\tau^{1/2} + h^2), \\ \max_{t\in\bar{\omega}_\tau}\{\|p(t) - u(t)\|_C\} &\leq c_9'(\tau^{(1/4-\gamma)} + h^{(1-2\gamma)}), \quad \gamma \in (0, 0.25). \end{aligned} \tag{18}$$

*If, additionally, condition (2) for the function $f(u, u^*)$ is valid up to the fourth order derivatives and $\|D_h^{(2,2)}u_0\| \leq C < \infty$ for all positive $h_1, h_2 \leq h_0$, then (16) holds.*

*Coefficients $c_9, c_9', c_8, c_8'$ depend on $c_7$, $|a|$, $\alpha = 2\|u\|_{C(\bar{Q})}$, $\varphi(\alpha)$, $T$, $\|u_0\|_{2,2}$. $c_9', c_8'$ additionally depend on $c_4$; $c_8, c_8'$ depend also on $\|D_h^{(2,2)}u_0\|$. All these coefficients do not depend on the grid steps.*

**Proof.** Let us denote by $p'$, $z'$, $g'$ the solution of (5), (6) and $p$, $z$, $g$ the solution of (5), (8). Let us also denote by $\varepsilon$, $\varepsilon_z$, $\varepsilon_g$ the differences between these two solutions $p' - p$, $z' - z$ and $g' - g$ respectively.

We have the following difference scheme for $\varepsilon$, $\varepsilon_g$ and $\varepsilon_z$:

$$\begin{aligned} &(\varepsilon_z)_t = f(\dot{z}', \dot{z}'^*) - f(\dot{z}, \dot{z}^*), \quad (\varepsilon_g)_t = a\Delta_h\hat{\varepsilon}_g + 2\tau a^2(\dot{g})_{\bar{x}_1 x_1 \bar{x}_2 x_2}, \quad (x,t) \in Q_{h\tau}; \\ &\varepsilon_z = \varepsilon,\ \varepsilon_g = \hat{\varepsilon}_z,\ \hat{\varepsilon} = \hat{\varepsilon}_g;\ \ \hat{\varepsilon} = 0,\ (x,t) \in \partial\omega_x \times \omega_\tau;\ \ \varepsilon(x,0) = 0,\ x \in \bar{\omega}_x. \end{aligned} \tag{19}$$

Denote again $\alpha = 2\|u\|_{C(\bar{Q})}$. Due to Theorem 1 we have the estimates $\|p'(t)\|_C \leq \alpha$ and $\|p'(t)\|_{2,2} \leq c_6\|u_0\|_{2,2}$ for all $t \in \bar{\omega}_\tau$.

Following Theorem 1 we suppose that $\|p(t_l)\|_C \le \alpha$ holds for all $l = 0, 1, \dots, j$. Analogously as earlier, the estimate $\|p(t_l)\|_{2,2} \le c_6 \|u_0\|_{2,2}$ holds for all $l = 0, 1, \dots, j+1$.

Due to Lemmas 5 and 8, the first equation of (19) leads to the estimate

$$\|\hat{\varepsilon}_z\| \le \|\varepsilon_z\| + 2\tau\varphi(\max\{\|p\|_C, \|p'\|_C\})\|\dot{\varepsilon}_z\| \le \|\varepsilon_z\| + 2\tau\varphi(\alpha)\|\dot{\varepsilon}_z\|.$$

For all positive $\tau \le \tau_0 = 1/2\varphi(\alpha)$ we can find that

$$\|\hat{\varepsilon}_z\| \le (1 + 4\varphi(\alpha)\tau)\|\varepsilon\| \quad \Rightarrow \quad \|\hat{\varepsilon}_z\|^2 \le (1 + 16\varphi(\alpha)\tau)\|\varepsilon\|^2.$$

Note that the following estimates for grid functions $v, w$ on $\bar{\omega}_x$ with zero boundary conditions are valid:

$$|(v_{\bar{x}_1 x_1 \bar{x}_2 x_2}, w)| = |(v_{\bar{x}_1 x_1}, w_{\bar{x}_2 x_2})| \le \|v_{\bar{x}_1 x_1}\| \, \|w_{\bar{x}_2 x_2}\| \le \|v\|_{2,2} \|w\|_{2,2}.$$

Thus, multiplying scalarly the second equation of (19) by $\tau\hat{\varepsilon}_g$, taking real parts, applying Cauchy inequalities we can obtain the estimates

$$\|\hat{\varepsilon}_g\|^2 \le \|\varepsilon_g\| \, \|\hat{\varepsilon}_g\| + 2\tau^2|a|^2\|\dot{g}\|_{2,2}\|\hat{\varepsilon}_g\|_{2,2} \le \|\hat{\varepsilon}_z\|^2 + 2\tau^2|a|^2\|\hat{p} + \hat{z}\|_{2,2}\|\hat{p}' - \hat{p}\|_{2,2}.$$

Using the results of Lemma 9, Corollary 2 and the previous estimate for $\|\hat{\varepsilon}_z\|$, we have

$$\|\hat{\varepsilon}\|^2 \le (1 + 16\varphi(\alpha)\tau)\|\varepsilon\|^2 + 8c_6^2|a|^2\|u_0\|_{2,2}^2\tau^2.$$

Similarly as in Theorem 1 this leads to the estimate

$$\|\varepsilon(t_{j+1})\| \le \tilde{c}_9\tau^{1/2} \quad \to 0.$$

This estimate, (12) and boundedness of $\|\varepsilon(t_{j+1})\|_{2,2}$ lead to

$$\|\varepsilon(t_{j+1})\|_C \le \tilde{c}_9'\tau^{(1/4-\gamma)} \quad \to 0, \quad \gamma \in (0, 0.25).$$

It follows from here, that for all $\tau$ small enough $\|p(t_{j+1})\|_C \le \alpha$ and, therefore, the induction step is proved.

Since $|u - p| \le |u - p'| + |p' - p| \to 0$, (18) is valid. The convergence rates for the $C$ or $L_2$ norms of the first term at the right-side of the estimate were found in Theorem (1), and the convergence rates for the second term were obtained above in this theorem.

Suppose that condition (2) is satisfied for the functions $f(u, u^*)$ up to the fourth order derivatives and $\|D_h^{(2,2)}u_0\| \le C < \infty$ for all $h_1, h_2 \le h_0$. Similarly we can prove the second part of the theorem and obtain (16).

Assume that $\|p(t_l)\|_C \le \alpha$, $l = 0, 1, \dots, j$. Then $\|p(t_l)\|_{2,2} \le c_6\|u_0\|_{2,2}$ and, due to Corollary 2, $\|D_h^{(2,2)}p(t_l)\| \le c_6'\|D_h^{(2,2)}u_0\|$ for all $l = 0, 1, \dots, j+1$.

From the first and the second equations of (19) we have the estimates

$$\|\hat{\varepsilon}_z\| \le (1 + 4\varphi(\alpha)\tau)\|\varepsilon\|, \qquad \|\hat{\varepsilon}\| \le \|\hat{\varepsilon}_z\| + 2c_6'|a|^2\|D_h^{(2,2)}u_0\|\tau^2.$$

From these two estimates, the grid Gronwall's inequality and multiplicative inequalities (12) we find

$$\|\varepsilon(t_{j+1})\| \leq \tilde{c}_8\tau \quad \to 0, \qquad \|\varepsilon(t_{j+1})\|_C \leq \tilde{c}_8'\tau^{(1/2-\gamma)} \quad \to 0, \quad \gamma \in (0, 0.5).$$

Analogously as above, these estimates lead to (16), where coefficients $c_8$ and $c_8'$ satisfy the supposition of our theorem. The theorem is proved.

# 8 Stability of the difference schemes.

Suppose that $p_1$ and $p_2$ are the two solutions of the same difference scheme with different initial conditions $u_{0,1}$ and $u_{0,2}$ respectively. We shall say that the scheme is stable in the norm of the space $B$, if the inequality

$$\max_{t\in\bar{\omega}_\tau}\{\|p_1(t) - p_2(t)\|_B\} \leq c\|u_{0,1} - u_{0,2}\|_B^\nu, \quad \nu \in (0, 1].$$

holds for all positive $\tau \leq \tau_0$, $h_1, h_2 \leq h_0$. Here we suppose that constant $c$ is not dependent on the grid steps.

**Theorem 3** *Let the conditions of Theorem 1 be satisfied. Then difference schemes (5), (6) and (5), (7) (or (8)) are stable in the norm of the spaces $L_2$ and $C$ and the following estimates hold:*

$$\begin{aligned} \max_{t\in\bar{\omega}_\tau}\{\|p_1(t) - p_2(t)\|\} &\leq c_{10}\|u_{0,1} - u_{0,2}\|, \\ \max_{t\in\bar{\omega}_\tau}\{\|p_1(t) - p_2(t)\|_C\} &\leq c_{10}'\|u_{0,1} - u_{0,2}\|_C^{(1/2-\gamma)}, \quad \gamma \in (0, 0.5). \end{aligned} \tag{20}$$

*here $c_{10}, c_{10}'$ depend on $T$, $\alpha = 2\max\{\|u_1\|_{C(\bar{Q})}, \|u_2\|_{C(\bar{Q})}\}$, $\varphi(\alpha)$. $c_{10}'$ additionally depends on $c_4$, $\|u_{0,1}\|_{2,2} + \|u_{0,2}\|_{2,2}$. Both these coefficients do not depend on the grid steps.*

**Proof.** We shall investigate both schemes together. Analogously as in Theorem 2 we denote $\varepsilon = p_1 - p_2$, $\varepsilon_z = z_1 - z_2$, $\varepsilon_g = g_1 - g_2$. The difference schemes for $\varepsilon$, $\varepsilon_z$ and $\varepsilon_g$ can be written as follows:

$$\begin{aligned} &(\varepsilon_z)_t = f(\dot{z}_1, \dot{z}_1^*) - f(\dot{z}_2, \dot{z}_2^*), \ (\varepsilon_g)_t = a\Delta_h\hat{\varepsilon}_g - 2\kappa\tau a^2(\dot{\varepsilon}_g)_{\bar{x}_1x_1\bar{x}_2x_2}, \ (x,t) \in Q_{h\tau}; \\ &\varepsilon_z = \varepsilon, \ \varepsilon_g = \hat{\varepsilon}_z, \ \hat{\varepsilon} = \hat{\varepsilon}_g; \ \ \hat{\varepsilon} = 0, \ x \in \partial\omega_x \times \omega_\tau, \ \ \varepsilon(x, 0) = u_{0,1} - u_{0,2}, \ x \in \bar{\omega}_x. \end{aligned}$$

Here $\kappa = 0$ in a case of difference scheme (5), (6) and $\kappa = 1$ in a case of (5), (8).

Analogously as in Theorem 2, from the first equation we can obtain the estimate $\|\hat{\varepsilon}_z\| \leq (1 + 4\tau\varphi(\alpha))\|\varepsilon\|$ for $\tau \leq \tau_0 = 1/2\varphi(\alpha)$.

Due to Lemma 7, the second equation for both difference schemes leads to the estimate $\|\hat{\varepsilon}\| \leq \|\varepsilon_g\|$.

Combining both these estimates, adding them for all time layers and applying Gronwall's inequality we came to the first inequality of (20) indicating stability in $L_2$ norm with the parameter $\nu = 1$ and $c_{10} = \exp(4\varphi(\alpha)T)$.

Due to multiplicative inequality (12) and the boundedness of $\|p_1(t)\|_{2,2} + \|p_2(t)\|_{2,2}$, the stability in $L_2$ norm leads also to the stability in $C$ norm with the coefficient $c'_{10} = c_{10}^{1/2-\gamma}\Big(c_6(\|u_{0,1}\|_{2,2} + \|u_{0,2}\|_{2,2})\Big)^{1/2+\gamma}$ and the parameter $\nu = 1/2 - \gamma$. The theorem is proved.

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