TIKHONOV REGULARIZATION FOR AN INTEGRAL EQUATION OF THE FIRST KIND WITH LOGARITHMIC KERNEL

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Abstract. In this paper, we discuss stability and Tikhonov regularization for the integral equation of the first kind with logarithmic kernel. Since the kernel is analytic in our case, the problem is severely ill-posed. We prove a convergence rate for the regularized solution and describe a method for its numerical calculation.

1. Introduction

Many inverse problems from applications, such as tomography [11], geophysics [10], non-destructive detection [7], inverse contact problems [4], give rise to integral equations of the first kind with analytic kernels. Since these problems are severely ill-posed, it will be very difficult to find the numerical solution. In [3], [5], for certain integral equations of the first kind with analytic kernels, a conditional stability estimate could be proved, provided some a-priori information about the solution was known.

The purpose of our paper is to study the Tikhonov regularization for integral equations considered in [3], [5]. Applying the conditional stability estimate proved there, we can obtain a convergence rate for the regularized solution.

In this paper, in order to explain our idea, we will consider the one dimensional case only. But our method will work also for multi-dimensional problems [3], [4], [5] and also for some nonlinear ill-posed problems, which we have stability estimates from [1], [2] for. We will treat these problems in our forthcoming papers.

In the one dimensional case, we consider here the integral equation with logarithmic kernel

$$\int_0^1 \log(x-t) f(t) dt = g(x), \quad x \in [2,3]. \tag{1.1}$$

Since $[0,1] \cap [2,3] = \emptyset$, the kernel is analytic with respect to x, t. The integral equation (1.1) is severely ill–posed in Hadamard's sense.

Our main concern in this paper is conditional stability and Tikhonov regularization. To this end we need regularity assumptions for the solution. We consider two kinds of regularity assumptions: first, the solution is supposed to be H_0^1 on $[0,1]$ and second, the solution is supposed to be H^1 in a neighborhood of one point.

The paper is organized as follows: In Section 2 we formulate the problem with noisy right–hand sides in an abstract setting, and in Section 3 we discuss its conditional stability. Regularized solutions are defined in Section 4, where a logarithmic convergence rate is proved. In Section 5 a method is given for the numerical calculation of the regularized solution.

2. Formulation of the problem

We consider the following integral equation of the first kind with logarithmic kernel

$$\mathcal{A}f = g, \tag{2.1}$$

where $\mathcal{A}f = \int_0^1 \log(x-t)f(t)dt$ is an operator from $L^2(0,1)$ to $L^2(2,3)$.

Since $x \in [2,3]$ and $t \in [0,1]$, the kernel $\log(x-t)$ is an analytic function. Therefore this problem is severely ill-posed.

As to the right–hand side of the equation (2.1), let us suppose that we only know an approximation g_δ of g in the sense $\|g - g_\delta\|_{L^2(2,3)} \leq \delta$.

3. Conditional stability for the solution of (2.1)

In this section, we give some results concerning the conditional stability for the solution of (2.1).

Let H^s and H_0^s be the usual Sobolev spaces.

Theorem 3.1. *Suppose that f is the solution of* (2.1) *and* $\|f\|_{H_0^1(0,1)} \le M$. *Then we have the following estimate*

$$\|f\|_{L^2(0,1)} \le C\frac{1}{|\log\frac{1}{\varepsilon}|}, \tag{3.1}$$

where $\varepsilon = \|g\|_{H^1(2,3)}$ *and* $C > 0$ *is a constant which depends on* M

Outline of the proof:

1. Construct a new function

$$U(x,y) = \frac{1}{2}\int_0^1 \log((x-t)^2 + y^2)f(t)dt. \tag{3.2}$$

It is easy to verify that

- $U(x,y)$ is a harmonic function in $R^2 \setminus [0,1] \times \{0\}$.
- $U(x,0) = g(x)$; $\quad \frac{\partial U}{\partial y}(x,0) = 0, \quad x \in [2,3]$,
- $\frac{\partial U}{\partial y}(x,0) = cf(x), \quad x \in [0,1]$.

2. Solve the Cauchy problem for the Laplace equation

$$\begin{aligned}
(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})U(x,y) &= 0, \qquad (x,y) \in R^2 \setminus [0,1] \times \{0\}, \\
U(x,0) &= g(x), \qquad x \in [2,3], \\
\frac{\partial U}{\partial y}(x,0) &= 0, \qquad x \in [2,3].
\end{aligned}$$

3. From $U(x,y)$, we can obtain f. The estimate (3.1) is just the conditional stability of the Cauchy problem for the Laplace equation.

The details can be found in [5].

In Theorem 3.1 we assumed that $f \in H_0^1$. This means that the case of a discontinuous solution is not included. Concerning a piecewise continuous solution, we give the following conditional stability estimate.

Theorem 3.2. *Suppose that f is the solution of (2.1) and $x_0 \in (0,1)$ is fixed. If*

$$\|f\|_{L^2(0,1)} \leq M$$

and there exists a neighborhood $O_r(x_0)$ of x_0 such that

$$\|f\|_{H^1(O_r(x_0))} \leq M_1.$$

Then we have the following local estimate

$$|f(x)| \leq C \frac{1}{|\log \frac{1}{\varepsilon}|^\gamma}, \qquad |x - x_0| \leq r_1 < r, \tag{3.3}$$

where $\varepsilon = \|g\|_{H^1(2,3)}$, $C > 0$ is a constant which depends on M, M_1, r and r_1 and $\gamma, 0 < \gamma < 1$, is a constant which depends on x_0, r and r_1.

Outline of the proof:

The proof is almost the same as in the previous part. The difference is that, in the third step, we will use some estimation in [6].

For details see [6].

4. Tikhonov regularization

4.1. **Concerning a continuous solution.** We consider the Tikhonov regularization for the equation (2.1).

For $\delta > 0$ fixed and $f \in H_0^1(0,1)$, we define the following functional

$$F_\alpha(f) = \|\mathcal{A}f - g_\delta\|^2_{L^2(2,3)} + \alpha \|f\|^2_{H_0^1(0,1)}, \tag{4.1}$$

where α is a positive parameter.

Since $F_\alpha(f) > 0$, there exists $\beta \geq 0$ such that

$$\beta = \inf_{f \in H_0^1(0,1)} F_\alpha(f).$$

Let $f_\alpha^\delta \in H_0^1(0,1)$ satisfy

$$F_\alpha(f_\alpha^\delta) \le \beta + \delta^2. \tag{4.2}$$

We call this function a regularized solution for (2.1).

Since δ^2 is a positive constant, such an f_α^δ exists.

Theorem 4.1. *Suppose that the exact solution of equation* (2.1) $f_0 \in H_0^1(0,1)$ *and* $\alpha = \delta^2$. *Then the regularized solution converges to* f_0 *and the following estimate holds*

$$\|f_\alpha^\delta - f_0\|_{L^2(0,1)} \le C_1 \frac{1}{|\log \frac{1}{\delta}|}, \tag{4.3}$$

where $C_1 > 0$ *is a constant which depends on* f_0.

Proof. First we estimate $\|f_\alpha^\delta\|_{H_0^1}$:

$$\begin{aligned} \alpha\|f_\alpha^\delta\|_{H_0^1} &\le F_\alpha(f_\alpha^\delta) \le \beta + \delta^2 \\ &\le F_\alpha(f_0) + \delta^2 \\ &= \|\mathcal{A}f_0 - g_\delta\|^2_{L^2(2,3)} + \alpha\|f_0\|^2_{H_0^1(0,1)} + \delta^2 \\ &\le 2\delta^2 + \alpha\|f_0\|^2_{H_0^1(0,1)}. \end{aligned}$$

Therefore we have

$$\|f_\alpha^\delta\|_{H_0^1} \le 2\frac{\delta^2}{\alpha} + \|f_0\|_{H_0^1}.$$

We take $\alpha = \delta^2$ and let $M = 2 + \|f_0\|_{H_0^1}$. Then we have that

$$\|f_\alpha^\delta\|_{H_0^1} \le M$$

and

$$\|f_0\|_{H_0^1} \le M.$$

Next we will check the difference between $\mathcal{A}f_0$ and $\mathcal{A}f_\alpha^\delta$:

$$\begin{aligned}
\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{L^2(2,3)} &\leq \|\mathcal{A}f_0 - g_\delta\|_{L^2(2,3)} + \|\mathcal{A}f_\alpha^\delta - g_\delta\|_{L^2(2,3)} \\
&\leq \delta + \sqrt{F_\alpha(f_\alpha^\delta)} \\
&\leq \delta + \sqrt{F_\alpha(f_0) + \delta^2} \\
&\leq \delta + \sqrt{2\delta^2 + \alpha\|f_0\|_{H_0^1(0,1)}} = \left(1 + \sqrt{2 + \|f_0\|_{H_0^1(0,1)}}\right)\delta.
\end{aligned}$$

Here we used $\alpha = \delta^2$.

We denote $B = 1 + \sqrt{2 + \|f_0\|_{H_0^1(0,1)}}$. Then we have

$$\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{L^2(2,3)} \leq B\delta. \tag{4.4}$$

It is easy to verify that, for $x \in [2,3]$,

$$\begin{aligned}
\left|\frac{d}{dx}\int_0^1 \log(x-t)f(t)dt\right| &\leq \|f\|_{L^2(0,1)}, \\
\left|\frac{d^2}{dx^2}\int_0^1 \log(x-t)f(t)dt\right| &\leq \|f\|_{L^2(0,1)}.
\end{aligned}$$

By the Proposition in the Appendix, we have

$$\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{C(2,3)} \leq B_1\delta^{\frac{1}{2}}, \tag{4.5}$$

where B_1 is a constant which only depends on B and M.

By Lemma 5.1 in [6], we have

$$\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{C^1(2,3)} \leq 2^{\frac{3}{2}}\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{C(2,3)}^{\frac{1}{2}}\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{C^2(2,3)}^{\frac{1}{2}}. \tag{4.6}$$

Therefore we obtain

$$\|\mathcal{A}(f_0 - f_\alpha^\delta)\|_{H^1(2,3)} \leq B_2\delta^{\frac{1}{4}}, \tag{4.7}$$

where $B_2 > 0$ is a constant which depends on B and M.

Apply Theorem 3.1 to $f_0 - f_\alpha^\delta$, the solution of (2.1) with a different right–hand side. Then there exists a positive constant C which depends on M such that

$$\|f_\alpha^\delta - f_0\|_{L^2(0,1)} \le C\frac{4}{|\log(B_2\delta)|}. \tag{4.8}$$

The proof is complete.

$\square$

4.2. **Concerning a discontinuous solution.** We consider the Tikhonov regularization for the equation (2.1).

For $\delta > 0$ fixed and $f \in L^2(0,1) \cap H^1(O_r(x_0))$, we define the following functional

$$G_\alpha(f) = \|\mathcal{A}f - g_\delta\|^2_{L^2(2,3)} + \alpha(\|f\|^2_{L^2(0,1)} + \|f\|^2_{H^1(O_r(x_0))}), \tag{4.9}$$

where α is a positive parameter.

Since $G_\alpha(f) > 0$, there exists $\beta_1 \ge 0$ such that

$$\beta_1 = \inf_{f \in L^2(0,1)\cap H^1(O_r(x_0))} G_\alpha(f).$$

Let $f_\alpha^\delta \in L^2(0,1) \cap H^1(O_r(x_0))$ satisfy

$$G_\alpha(f_\alpha^\delta) \le \beta_1 + \delta^2. \tag{4.10}$$

We call this function a regularized solution for (2.1).

Since δ^2 is a positive constant, such f_α^δ exists.

Theorem 4.2. *Suppose for the exact solution of the equation* (2.1) $f_0 \in L^2(0,1) \cap H^1(O_r(x_0))$ *and* $\alpha = \delta^2$. *Then the regularized solution converges to* f_0 *in some neighborhood of* x_0, *and the following estimate holds*

$$|f_\alpha^\delta(x) - f_0(x)| \le C_1 \frac{1}{|\log\frac{1}{\delta}|^\gamma}, \qquad |x - x_0| \le r_1 < r, \tag{4.11}$$

where $C_1 > 0$ is a constant which depends on f_0, r and r_1.

The proof goes along the same lines as the proof of Theorem 4.1.

The next result concerns the discontinuity points of the solution.

Theorem 4.3. *Suppose that the exact solution f_0 is a piecewise smooth function and x_0 is a discontinuity point such that $f_0 \in C^2((x_0-\epsilon, x_0)$, $f_0 \in C^2(x_0, x_0+\epsilon)$ and $f_0(x_0+0) \neq f_0(x_0-0)$. Let f_α^δ be a regularized solution of the equation* (2.1) *as defined in (4.10). Then we have*

$$\lim_{\delta\to 0} \|f_\alpha^\delta\|_{H^1(O_r(x_0))} = \infty. \tag{4.12}$$

Proof. We assume that the conclusion is not true, i.e.

$$\|f_\alpha^\delta\|_{H^1(O_r(x_0))} \leq C, \tag{4.13}$$

where $C > 0$ is a constant which is independent of δ.

Without loss of generality, we assume that there is only one discontinuity point x_0 in $O_r(x_0)$.

Let $c = f_0(x_0+0) - f_0(x_0-0) \neq 0$. For δ sufficiently small, we consider a new function

$$f_1 = \begin{cases} \frac{c}{2\delta^{\frac{1}{2}}}(x - x_0), & x \in (x_0 - \delta^{\frac{1}{2}}, x_0 + \delta^{\frac{1}{2}}) \\ f_0, & x \in (0,1) \setminus (x_0 - \delta^{\frac{1}{2}}, x_0 + \delta^{\frac{1}{2}}). \end{cases}$$

It is easy to verify that $f_1(x) \in H^1(O_r(x_0))$ and $f_1 \in L^2(0,1)$.

By the definition of f_α^δ, we have for $\alpha = \delta^2$

$$\|\mathcal{A}f_\alpha^\delta - g_\delta\|^2_{L^2(2,3)} \leq G_\alpha(f_\alpha^\delta) \leq \beta + \delta^2$$

$$\leq G_\alpha(f_1) + \delta^2 = \|\mathcal{A}f_1 - g_\delta\|^2_{L^2(2,3)} + c_1\delta^2\|f_1\|^2_{H^1(O_r(x_0))} + \delta^2.$$

We can verify directly that

$$\begin{aligned}\|\mathcal{A}f_1 - g_\delta\|_{L^2(2,3)} &\le \|\mathcal{A}f_1 - \mathcal{A}f_0\|_{L^2(2,3)} + \|\mathcal{A}f_0 - g_\delta\|_{L^2(2,3)} \\ &\le \delta + \|\mathcal{A}f_1 - \mathcal{A}f_0\|_{L^2(2,3)} \le D_1\delta^{\frac{1}{2}} + \delta \le D\delta^{\frac{1}{2}},\end{aligned}$$

where $D > 0$ is a constant which is independent of δ.

Applying the conditional stability results (Theorem 3.2) for $f_\alpha^\delta(x) - f_0(x)$, $x \in (x_0 - \epsilon, x_0)$ and $x \in (x_0, x_0 + \epsilon)$, we have that

$$\lim_{\delta\to 0} f_\alpha^\delta(x) = f_0(x), \qquad x \in (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\}. \tag{4.14}$$

This means that $f_\alpha^\delta(x)$ converges to $f_0(x)$ for almost every $x \in O_r(x_0)$. Since f_0 is not a function in $H^1(O_r(x_0))$ and $f_\alpha^\delta \in H^1(O_r(x_0))$, the assumption (4.13) is not true.

The proof is complete. $\square$

Thus from Theorems 4.2 and 4.3 we obtain the following corollary:

Theorem 4.4. *Let O_r be an open subinterval of $[0,1]$. There is a discontinuity point of the solution f_0 in O_r if and only if for the regularized solution f_α^δ defined in (4.10) holds $\|f_\alpha^\delta\|_{H^1(O_r)}$ is unbounded for $\alpha = \delta^2$ and $\delta \to 0$.*

Remark 4.5. Comparing with other results for the standard Tikhonov regularization [8], [9], if one wants to obtain a convergence rate, an assumption of the kind $g \in R(A^*)$ is necessary. Otherwise the convergence rate can be as slow as possible.

In our case, we know that, for any $v \in L^2(2,3)$, A^*v is an analytic function in $(0,1)$. This means that only in the case where the solution is an analytic function, the convergence rate can be obtained. But this is not reasonable in applications.

Our result does not need this strong assumption. But the convergence rate is weak. Only a log type convergence rate can be obtained. Even if the solution is not a continuous function, we can only prove a local convergence rate in a neighborhood of x_0. Outside this neighborhood we have no information.

5. Numerical Analysis

We assume $f_0 \in H_0^1$ throughout this section.

Here we fix the positive numbers α and δ and give a method for the computation of a regularized solution f_α^δ defined in (4.2).

To this end, let n be a natural number and consider in the interval [0,1] the equidistant discretization

$$t_i = i/n,\ i = 1, \cdots, n-1.$$

Define the finite-dimensional subspace X_n of H_0^1 as

$$X_n = span\{d_i, i = 1, \cdots, n-1\},$$

where d_i is linear and continuous with $d_i(t_j) = 1$ for $j = i$ and $= 0$ for $j \neq i$, $i = 1, \cdots, n-1$. (I.e., we consider the so–called hat-functions.)

It is well–known that, given a function $\phi \in H_0^1$, its approximation $\phi_n = \sum_{i=1}^{n-1} \phi(t_i) d_i$ will converge to ϕ for $n \to \infty$. Moreover, if $\phi \in H^{1+\nu}$ we have

$$\|\phi - \phi_n\|_{H_0^1} \leq c \cdot n^{-\nu} \|\phi\|_{H^{1+\nu}}. \tag{5.1}$$

Now, consider the functional on H_0^1

$$F(f) = F_\alpha(f) = \|\mathcal{A}f - g_\delta\|_{L^2}^2 + \alpha \|f\|_{H_0^1}^2.$$

>From the identity

$$F(cf+(1-c)g) = cF(f) + (1-c)F(g) - c(1-c)\{\|\mathcal{A}f - \mathcal{A}g\|_{L^2}^2 + \alpha\|f-g\|_{H_0^1}^2\},$$

where $0 \le c \le 1$, we see that F is strongly convex. Besides, F is locally Lipschitz continuous and weakly lower semicontinuous. Hence there is a unique $f^* \in H_0^1$ with the property

$$F(f^*) = \inf_{f \in H_0^1} F(f).$$

>From the same reason, there is a unique $f_n^* \in X_n$ with

$$F(f_n^*) = \inf_{f_n \in X_n} F(f_n).$$

Let $f^\delta \in H_0^1$ have the property $F(f^\delta) \le \inf_{f \in H_0^1} F(f) + \delta^2$. (I.e., f^δ is a regularized solution in the sense of (4.2).)

Theorem 5.1. *For $n > n(\delta)$ the choice*

$$f^\delta = f_n^*$$

is possible.

Proof. It suffices to show that

$$F(f_n^*) \longrightarrow F(f^*) \ (n \to \infty).$$

To this end take $f_n \in X_n$ such that $f_n \to f^*$ in H_0^1 in the strong sense. Then the continuity of F implies $F(f_n) \to F(f^*)$. Going to the limit in the obvious inequality $F(f^*) \le F(f_n^*) \le F(f_n)$, we get the assertion. $\square$

Remark 5.2. Using $|F(f^*) - F(f_n^*)| \le |F(f^*) - F(f_n)|$, the local Lipschitz continuity of F and (5.1), we can obtain an estimate for $n(\delta)$.

To conclude this section, let us calculate f_n^*, the solution of the uniquely solvable optimization problem

$$\min_{f \in X_n} \{\|\mathcal{A}f - g_\delta\|_{L^2}^2 + \alpha(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2)\}.$$

Set

$$f = \sum_{i=1}^{n-1} x_i d_i, \quad \mathcal{A}d_i = \phi_i,$$

and let $(\cdot,\cdot)$ denote the scalar product in L^2. Then we have equivalently to solve

$$\min_{\mathbf{x} \in R^{n-1}} \Phi(\mathbf{x}), \tag{5.2}$$

where

$$\begin{aligned} \Phi(\mathbf{x}) &= \sum_{i,j} x_i x_j (\phi_i, \phi_j) - 2 \sum_i x_i (\phi_i, g_\delta) \\ &\quad + (g_\delta, g_\delta) + \alpha \left(\sum_{i,j} x_i x_j ((d_i, d_j) + (d_i', d_j')) \right) \\ &= < \mathbf{W}\mathbf{x} + \mathbf{u}, \mathbf{x} > + b. \end{aligned}$$

Here $\mathbf{x}$ is the vector with entries x_i, $\mathbf{W}$ is the matrix with entries

$$W_{i,j} = (\phi_i, \phi_j) + \alpha((d_i, d_j) + (d_i', d_j')),$$

$\mathbf{u}$ is the vector with entries $u_i = -2(\phi_i, g_\delta)$, $b = (g_\delta, g_\delta)$, and $< \cdot, \cdot >$ is the scalar product in R^{n-1}. After differentiating

$$\lim_{t \to 0} \frac{\Phi(\mathbf{x} + t\mathbf{h}) - \Phi(\mathbf{x})}{t} = < (\mathbf{W} + \mathbf{W}^*)\mathbf{x} + \mathbf{u}, \mathbf{h} >,$$

and taking into account the symmetry of $\mathbf{W}$, we obtain as a necessary (and sufficient) condition for a minimum

$$2\mathbf{W}\mathbf{x} + \mathbf{u} = 0.$$

But that means that the solution $\mathbf{x}_0$ of (5.2) can be calculated from a linear system. We obtain

$$\mathbf{x}_0 = -\mathbf{W}^{-1}\mathbf{u}/2.$$

A regularized solution in the discontinuous case (Section 4.2) can be calculated analogously.

6. Appendix

In this appendix, we will prove

Proposition 6.1. *Let be $h \in C^1[2,3]$ and ϵ a small positive constant. If $|h\|_{L^1[2,3]} \leq \epsilon$ and $\|h\|_{C^1[2,3]} \leq M$ (M is a constant), then there exists a constant B which depends on M such that*

$$\|h\|_{C[0,1]} \leq C\epsilon^{\frac{1}{2}}. \tag{6.1}$$

Proof. Since $|h|$ is a continuous function on $[2,3]$, there exists a point $x_0 \in [2,3]$ such that $|h(x)|$ attends the maximum at $x = x_0$.

Let us consider two lines which cross $(x_0, |h(x_0)|)$:

$$\begin{aligned} l_1 : \qquad y - |h(x_0)| &= 2M(x - x_0), \\ l_2 : \qquad y - |h(x_0)| &= -2M(x - x_0). \end{aligned}$$

Then $y = |h(x)|$ and l_1 have no intersection point for $x \in [0, x_0]$, and $y = |h(x)|$ and l_2 have no intersection point for $x \in [x_0, 1]$.

If l_1 and l_2 do not intersect with $[0,1] \times \{y = 0\}$, then we have

$$\frac{1}{2}|h(x_0)| \leq \int_0^1 |h(t)|dt \leq \epsilon,$$

i.e.

$$|h(x_0)| \leq 2\epsilon. \tag{6.2}$$

If l_1 or l_2 intersect with $[0,1] \times \{y=0\}$, without loss of generality we assume that l_1 intersects with $[0,1] \times \{y=0\}$. Then we have

$$\frac{1}{2}\frac{|h(x_0)|}{2M}|h(x_0)| \leq \int_0^1 |h(t)|dt \leq \epsilon,$$

i.e.

$$|h(x_0)| \leq 2\sqrt{M}\sqrt{\epsilon}. \tag{6.3}$$

Combining ((6.2)) and ((6.3)), we have the conclusion. The proof is complete.

□

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