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Domain separation by means of sign changing eigenfunctions of p-Laplacians

Herbert Gajewski and Klaus Gärtner

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Weierstrass Institute for
Applied Analysis and Stochastics
Mohrenstraße 39
D–10117 Berlin
Germany
E-Mail: {gajewski, gaertner}@wias-berlin.de

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Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. We are interested in algorithms for constructing surfaces Γ of possibly small measure that separate a given domain Ω into two regions of equal measure. Using the integral formula for the total gradient variation, we show that such separators can be constructed approximatively by means of sign changing eigenfunctions of the p-Laplacian, $p \to 1$, under homogeneous Neumann boundary conditions. These eigenfunctions are proven to be limits of continuous and discrete steepest descent methods applied to suitable norm quotients.

1 Introduction

Let $\Omega \subset I\!R^n$, $n \geq 1$, be an open, bounded, connected Lipschitzian domain. There is a practical interest [12] in algorithms for constructing surfaces Γ of possibly small measure $|\Gamma|$ which separate Ω into two regions of approximately equal measure, i. e. , in solving minimum problems like

$$\varphi_1(E) = 2\; P_\Omega(E) + \frac{\gamma|\;|E| - |\Omega \setminus E|\;|}{|\Omega|} \to \min, \quad E \subset \Omega, \quad \gamma > 0, \tag{1}$$

where $P_\Omega(E) = |\Gamma|$ is the perimeter of E relative to Ω and $|E|$ is the measure of E.

This paper aims to solve the geometrical problem (1) by analytical tools. Roughly speaking, we look for approximative solutions of the form $E = \{x \in \Omega,\ u(x) > 0\}$, where u minimizes

$$F_1(u) = \frac{\int_\Omega |Du| + \gamma|\bar{u}|}{||u||_1}, \quad \gamma > 0, \quad u \in BV. \tag{2}$$

Here $||\cdot||_p$ is the norm in the Lebesgue space $L^p = L^p(\Omega)$, $\bar{u}$ is the mean value of u and $\int_\Omega |Du|$ has to be interpreted in the sense of the space BV of functions of bounded variation on Ω [9], i.e.,

$$\int_\Omega |Du| = \sup_g \left(\int_\Omega u \nabla \cdot g\; dx \right),\ g \in C_0^1(\Omega, I\!R^n),\ |g(x)| \leq 1,\ x \in \Omega.$$

(We have $\int_\Omega |Du| = |||\nabla u|||_1$, provided u belongs to the Sobolev space $H^{1,1}(\Omega)$.)

The key idea for this approach is Federer's observation (comp. [4]), that the infimum of the functional

$$\varphi(E) = \frac{P_\Omega(E)}{\min(|E|^{\frac{1}{p^*}}, |\Omega \setminus E|^{\frac{1}{p^*}})} \to \min, \quad E \subset \Omega, \quad p^* = \frac{n}{n-1}, \tag{3}$$

coincides with that of

$$\phi(u) = \frac{\int_\Omega |Du|}{||u - t_0(u)||_{p*}} \to \min, \quad u \in BV, \tag{4}$$

where the functional t_0 is defined by

$$t_0(u) = \sup \{t : |E_t| \geq |\Omega \setminus E_t|\}, E_t = \{x \in \Omega, u(x) > t\}. \tag{5}$$

To specify the connection between (3) and (4) we quote some basic facts from [4], [5]:
(i) Let u be locally integrable on Ω. Then

$$\int_\Omega |Du| = \int_{-\infty}^{\infty} P_\Omega(E_t)dt.$$

(ii) Let $\Omega \subset I\!R^n$ be an open, bounded and connected Lipschitzian domain. Then Ω satisfies a relative isoperimetric inequality, i. e., there exists a constant $Q = Q(\Omega)$, such that

$$\min(|E|^{\frac{1}{p^*}}, |\Omega - E|^{\frac{1}{p^*}}) \le Q P_\Omega(E), \ \ p^* = \frac{n}{n-1}. \tag{6}$$

(iii) Let Ω, Q be as in (ii) and let u be as in (i). Then

$$||u - t_0||_{p^*} \le Q \int_\Omega |Du|. \tag{7}$$

A special case of (i) is

$$\int_\Omega |D\chi_E| = P_\Omega(E),$$

where χ is the characteristic function. Hence the map $E \to \chi_E - \chi_{\Omega \setminus E}$ directly connects (1) and (2). The inverse direction may be indicated by the map $u \to E_u$ with

$$E_u = \{x \in \Omega, \ u(x) > 0\}.$$

The functional F_1 still is unpleasant from the algorithmical point of view. Therefore we shall approximate F_1 by (apart from zero) differentiable functionals

$$F_p(u) = \frac{||\nabla u||_p^p + \gamma |\bar{u}|^p}{||u||_p^p}, \ \ u \in H^{1,p}, \ \ p \in (1,2], \ \gamma > 0. \tag{8}$$

Remark 1 *Our considerations can be generalized to the functionals*

$$F_{p,q}(u) = \frac{||\nabla u||_p^p + \gamma |\bar{u}|^p}{||u||_q^p}, \ \ q \in (1, \frac{np}{n-p}].$$

The next section contains notations and some results clarifying the connection between φ_1, F_1 and F_p, $p > 1$. In Section 3 we analyse a continuous steepest descent method for F_p. That leads to nonlinear nonlocal evolution equations which are proven to have global solutions u_p. The asymptotic behavior of u_p is studied in Section 4. It is shown that $F_p(u(t))$ tends monotonously decreasing to $F(u_p^*)$, where u_p^* are sign changing eigenfunctions of p-Laplaciens. Moreover, we show that u_p^* approximates for $p \to 1$ a function $u^* \in BV$ with

$$F_1(u^*) \le \liminf_p F_p(u_p^*), \ \ \overline{u^*} = 0.$$

A time discretization of the evolution equations is established in the last section.

2 Notations, Preliminaries

We denote by L^p, BV, $H^{1,p}$, $H^1 = H^{1,2}$, $(H^{1,p})^* = H^{-1,p'}$, $1 \le p \le 2$, $p' = \frac{p}{p-1}$ the usual spaces of functions defined on Ω and by $(\cdot,\cdot)$ the pairing between spaces and their duals. The norm in L^p is denoted by $||\cdot||_p$, $||\cdot|| = ||\cdot||_2$. For $t > 0$ and a Banach space X

$$L^2(0,T;X),\ C(0,T;X),\ C_w(0,T;X),\ \ H^1(0,T;X) = \{u \in L^2(0,T;X),\ u_t \in L^2(0,T;X)\}$$

are the usual [6], [10] spaces of functions on $[0,T]$ with values in X.
Finally, for $u \in L^1$ we define the mean value

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u\,dx.$$

To prepare the replacement of φ_1 by F_p we state some explanatory facts.

Proposition 1 *Let Q be the relative isoperimetric constant from (6). Then*

$$||u||_p \le 2^{\frac{p-1}{p}}\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q||\nabla u||_p + |\Omega|^{\frac{1}{p}}|t_0(u)|,\ u \in H^{1,p},\ \ p \in [1, \frac{n}{n-1}]. \qquad (9)$$

PROOF: By Hölder's inequality we get from (7)

$$||u - t_0||_p \le |\Omega|^{\frac{1}{p}-\frac{1}{p^*}} Q\int_\Omega |Du| = |\Omega|^{\frac{1}{p}-\frac{1}{p^*}} Q||\nabla u||_1.$$

However, an inspection of the proof of (7) (comp. the proof of Theorem 2 in [5]) shows, that herein $|\Omega|$ can be replaced by $\frac{|\Omega|}{2}$. Thus, applying Hölder's inequality once more, we get

$$||u - t_0||_p \le 2^{\frac{p-1}{p}}\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q||\nabla u||_p,$$

and by the triangle inequality (9). □

Remark 2 *The inequality (9) specifies the constant in Poincaré's inequality. For $p = 1$, (9) is sharp. Indeed, suppose equality is attained in (6) for a set E with $|E| = \frac{|\Omega|}{2}$, as for example in the case of convex domains Ω (comp. [1]). Then $u = \chi_E - \chi_{\Omega\setminus E} \in BV$ satisfies $\bar{u} = t_0(u) = 0$ and*

$$\begin{aligned} ||u||_1 &= |\Omega| = 2\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}}\left(\frac{|\Omega|}{2}\right)^{\frac{1}{p^*}} = 2\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} QP_\Omega(E) \\ &= 2^0\left(\frac{|\Omega|}{2}\right)^{\frac{1}{n}} Q\int_\Omega |Du|. \end{aligned}$$

For convex domains Ω another specification is well known [8]

$$||u||_p \leq \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1-n}{n}} d^n ||\nabla u||_p + |\Omega|^{\frac{1}{p}} |\bar{u}|,$$

where ω_n is the volume of the unit sphere in $I\!R^n$ and d is the diameter of Ω.

The following result clarifies the connection between the functionals $\varphi_1(E)$ and $F_1(u)$.

Proposition 2 *Let u_1 be minimizer for F_1. Then the set $E_1 = \{x \in \Omega,\ u_1(x) > 0\}$ is minimizer for φ_1.*

PROOF:
Let $E \subset \Omega$ be an arbitrary set with $P_\Omega(E) < \infty$. We show $\varphi_1(E) \geq \varphi_1(E_1)$ in two steps:
(i) Define $v = \chi_E - \chi_{\Omega \setminus E} \in BV$. Then

$$\frac{\varphi_1(E)}{|\Omega|} = \frac{2P_\Omega(E) + \frac{\gamma|\,|E| - |\Omega\setminus E|\,|}{|\Omega|}}{|\Omega|} = \frac{\int_\Omega |Dv| + \gamma|\bar{v}|}{||v||_1} = F_1(v) \geq F_1(u_1).$$

(ii) Let for $\varepsilon > 0$

$$w_\varepsilon(x) = \tanh\,(\frac{u_1(x)}{\varepsilon}).$$

Since u_1 is minimizer of F_1 and $w_\varepsilon \in BV$, we have

$$\frac{1}{||u_1||_1}(||w_\varepsilon||_{BV} + \gamma \mathrm{sign}\ \overline{\mathrm{u_1}}\ \overline{w_\varepsilon} - F_1(u_1)||w_\varepsilon||_1) = \frac{d}{dt} F_1(u_1 + tw_\varepsilon)|_{t=0} = 0.$$

Passing $\varepsilon \to 0$, the lower semicontinuity of the BV-norm [9] and Lebesgue's dominated convergence theorem imply

$$2P_\Omega(E_1) + \gamma \mathrm{sign}\ \overline{\mathrm{u_1}}\ \overline{\mathrm{sign}\ \mathrm{u_1}} \leq F_1(u_1)|\Omega|. \tag{10}$$

Now using once more that u_1 is minimizer of F_1, we find

$$\frac{1}{||u_1||_1}(\gamma \mathrm{sign}\ \overline{\mathrm{u_1}} - F(u_1)\overline{\mathrm{sign}\ \mathrm{u_1}}|\Omega|) = \frac{d}{dt} F_1(u_1 + t\chi_\Omega)|_{t=0} = 0,$$

that is

$$\mathrm{sign}\ \overline{\mathrm{u_1}} = \mathrm{sign}\ \overline{\mathrm{sign}\ \mathrm{u_1}}.$$

Thus by (10) we have

$$2P_\Omega(E_1) + \gamma|\overline{\mathrm{sign}\ \mathrm{u_1}}| \leq F_1(u_1)|\Omega|.$$

Because of (i) the assertion follows. □

Now we turn to the functional

$$F_p(u) = \frac{||\nabla u||_p^p + \gamma|\bar{u}|^p}{||u||_p^p}, \quad u \subset H^{1,p}, \quad p \in (1,2], \ \gamma > 0$$

as regularization of F_1 from (2). By Poincaré's inequality F_p is bounded from below. Minimizers of F_p satisfy necessarily the Euler Lagrange equations, i. e., the nonlinear eigenvalue problem (comp. [3])

$$A_p u = B_p u, \tag{11}$$

where the operators $A_p, B_p \in (H^{1,p} \to (H^{1,p})^*)$ are defined by

$$\begin{aligned} (A_p u, h) &= (|\nabla u|^{p-2}\nabla u, \nabla h) + \gamma \mathrm{sign}\ \bar{\mathrm{u}}|\bar{\mathrm{u}}|^{\mathrm{p}-1}\bar{\mathrm{h}}, \quad \forall \mathrm{h} \in \mathrm{H}^{1,\mathrm{p}}, \\ B_p u &= F_p(u) b_p(u), \ b_p(u) = |u|^{p-2}u. \end{aligned}$$

(12)

F_p approximates F_1 in the following sense.

Proposition 3 *Let $u_p \in H^{1,p}$, $1 < p \leq 2$, be minimizer for F_p, such that*

$$\frac{1}{||u_p||_1} + ||u_p||_p \leq c. \tag{13}$$

Then a sequence $p \to 1$ and a minimizer $u \in BV$ of F_1 exist such that

$$u_i := u_{p_i} \to u \ \ in\ L^1, \quad F_{p_i}(u_i) \to \lambda \geq F_1(u). \tag{14}$$

PROOF:
(i)Let $w \in H^1$ be fixed. Using that u_p is minimizer and (13), we find

$$|\Omega|^{1-p}||\nabla u_p||_1^p \leq ||\nabla u_p||_p^p + \gamma|\overline{u}_p|^p = F_p(u_p)||u_p||_p^p \leq F_p(w)||u||_p^p \leq c.$$

Since $H^{1,1}$ is compactly imbedded into L^1, a sequence $p_i \to 1$ and $u \in BV$ exist such that

$$u_i := u_{p_i} \to u \text{ in } L^1, \ F_{p_i}(u_i) \to \lambda.$$

(ii) Using the lower semicontinuity of the BV-norm, Hölder's and Young's inequalities, we get from (11), setting $p = p_i$ temporaryly,

$$\begin{aligned} \int_\Omega |Du| &\leq \liminf \int_\Omega |Du_i| = \liminf ||\nabla u_i||_1 \\ &\leq \liminf(|\Omega|^{\frac{p-1}{p}}||\nabla u_i||_p) \leq \liminf(\frac{p-1}{p}|\Omega| + \frac{1}{p}||\nabla u_i||_p^p) \\ &\leq \liminf ||\nabla u_i||_p^p = \liminf(F_p(u_i)||u_i||_p^p - \gamma|\bar{u}_i|^p) \\ &= \liminf(F_p(u_i)||u_i||_p^p) - \gamma|\bar{u}|^p) = \lambda \liminf ||u_i||_p^p - \gamma|\bar{u}|^p \\ &\leq \lambda \liminf(||u_i||_1^{2-p}||u_i||^{2(p-1)}) - \gamma|\bar{u}|^p \\ &\leq \lambda \liminf((2-p)||u_i||_1 + (p-1)||u_i||^2) - \gamma|\bar{u}|^p \\ &= \lambda||u||_1 - \gamma|\bar{u}|^p \end{aligned}$$

and hence

$$F_1(u) \leq \lambda. \tag{15}$$

(iii) Let $v \in BV,\ v \neq 0$. We want to show that $F_1(u) \leq F_1(v)$. To this end let $(v_j) \subset C^\infty$ be a sequence (comp. [9]) such that

$$v_j \to v \text{ in } L^1, \quad \int_\Omega |Dv_j| \to \int_\Omega |Dv|. \tag{16}$$

We have

$$\begin{aligned} F_1(v) &= F_1(v_j) + F_1(v) - F_1(v_j) = F_p(v_j) + F_1(v_j) - F_p(v_j) + F_1(v) - F_1(v_j) \\ &\geq F_p(u_p) - |F_1(v_j) - F_p(v_j)| - |F_1(v) - F_1(v_j)|. \end{aligned}$$

(17)

By (16) we can choose j such that for given $\varepsilon > 0$

$$|F_1(v) - F_1(v_j)| < \varepsilon.$$

Further we have

$$\begin{aligned} ||\nabla v_j||_p^p &\leq ||\nabla v_j||_1 ||\nabla v_j||_\infty^{p-1} \leq \frac{1}{p}||\nabla v_j||_1^p + \frac{p-1}{p}||\nabla v_j||_\infty^p \\ &\leq ||\nabla v_j||_1 (1 + |\frac{1}{p}||\nabla v_j||_1^{p-1} - 1|) + \frac{p-1}{p}||\nabla v_j||_\infty^p \end{aligned}$$

and

$$||v_j||_p^p \leq \frac{1}{p}||v_j||_1^p + \frac{p-1}{p}||v_j||_\infty^p \leq ||v_j||_1 (1 + |\frac{1}{p}||v_j||_1^{p-1} - 1|) + \frac{p-1}{p}||v_j||_\infty^p.$$

Consequently, we can choose $p_i = p_i(j)$ such that

$$|F_1(v_j) - F_{p_i}(v_j)| < \varepsilon.$$

Thus, using (14) and (15), we get from (17)

$$F_1(u) \leq F_{p_i}(u_i) \leq F_1(v) + 2\varepsilon.$$

Passing to $\varepsilon \to 0$, we finish the proof. □

3 Continuous steepest descent method

Due to the Propositions 2,3 the original minimum problem (2) is approximatively reduced to the construction of minimizers for the functionals $F_p,\ p \in (1, 2]$ from

(8). Because of (11), these minimizers are steady states of the nonlinear, nonlocal evolution problem

$$u_t + Au = Bu, \ \ u(0) = u_0 \in H^1, \tag{18}$$

where the operators $A = A_p,\ B_p$ are defined by (12).
In this section we fix p and drop the index p in order to simplify the notation.
The initial value problem (18) can be understood as continuous steepest descent method applied to the functional $F = F_p$ from (8). Accordingly, F turns out to be Lyapunov functional of (18). This will be essential as well for proving existence of global solutions u to (18) in the present section, as for showing that $u(t_i)$ tends to steady states u^*, i. e., solutions to the nonlinear eigenvalue problem (11), for suitable sequences $t_i \to \infty$, in the forthcoming section.

The function b and consequently the operator B are not Lipschitz continuous. The inequalities (comp. [2])

$$0 \le (|y|^{p-1} - |z|^{p-1})(|y| - |z|) \le (|y|^{p-2}y - |z|^{p-2}z, y - z) \le c(p)|y - z|^p, \ y, z \in I\!R^n, \tag{19}$$

imply only continuity and monotonicity of the operators A and B. We introduce

$$b_\varepsilon(u) = (u^2 + \varepsilon)^{p/2-1}u, \ \varepsilon > 0,$$

as Lipschitz continuous approximation of b. Accordingly we define

$$B_\varepsilon u = F_\varepsilon(u) b_\varepsilon(u),$$

$$F_\varepsilon(u) = \frac{||\nabla u||_p^p + \gamma|\bar{u}|^p}{||(u^2 + \varepsilon)^{\frac{p}{2}}||_1}, \tag{20}$$

and consider auxiliary problems

$$(u - \varepsilon\Delta u)_t + Au = B_\varepsilon u, \ u(0) = u_0 \in H^1. \tag{21}$$

Lemma 1 *Let $\varepsilon > 0$. Then the initial value problem (21) has a unique solution $u_\varepsilon \in C^1(0, T; H^1)$. Moreover, for $t \in [0, T]$ u_ε satisfies*

$$||u_\varepsilon(t)||_\varepsilon^2 \le ||u_0||_\varepsilon^2, \tag{22}$$

$$\int_0^t \frac{||u_{\varepsilon t}||_\varepsilon^2}{||(u_\varepsilon^2 + \varepsilon)^{\frac{p}{2}}||_1} ds + \frac{1}{p}(F_\varepsilon(u_\varepsilon(t)) - F_\varepsilon(u_0)) = 0, \tag{23}$$

where

$$||v||_\varepsilon^2 := ||v||^2 + \varepsilon||\nabla v||^2.$$

PROOF:
(i) By (19) the operator $A \in (H^1 \to H^{-1})$ is continuous and monotone [6], [13]. Set for $K > 0$

$$B_\varepsilon^K u = F_\varepsilon^K(u) b_\varepsilon(u), \ F_\varepsilon^K = \frac{\min(K, ||\nabla u||_p^p) + \gamma|\bar{u}|^p}{||(u^2 + \varepsilon)^{\frac{p}{2}}||_1}.$$

Then $B_\varepsilon^K \in (H^1 \to H^{-1})$ is Lipschitz continuous. Hence

$$C := A - B_\varepsilon^K$$

is continuous and satisfies,

$$(Cu - Cv, v - w) \geq -c(K, \varepsilon, p)||u - v||_\varepsilon^2.$$

Consequently [7], the pseudo-parabolic initial value problem

$$(u - \varepsilon\Delta u)_t + Cu = 0,\ u(0) = u_0 \in H^1, \tag{24}$$

has a unique solution $u \in C^1(0, T; H^1)$.
(ii) Testing (24) with u gives

$$\frac{1}{2}(||u(t)||_\varepsilon^2)_t + ||\nabla u||_p^p + \gamma|\bar{u}|^p = (B_\varepsilon^K u, u) \leq ||\nabla u||_p^p + \gamma|\bar{u}|^p$$

and, after integrating with respect to t, (22). From this we get

$$||\nabla u(t)||_p^p \leq |\Omega|^{1-p/2}||\nabla u(t)||^p \leq |\Omega|^{1-p/2}\varepsilon^{-\frac{p}{2}}||u_0||_\varepsilon^p =: K_0(\varepsilon).$$

Thus, choosing $K \geq K_0$, we see that actually

$$B_\varepsilon^K(u(t)) = B_\varepsilon(u(t)).$$

Hence, $u_\varepsilon := u_\varepsilon^K$ is the unique solution to (21).
(iii) For proving (23), we test (21) with $u_t/||(u^2 + \varepsilon)^{p/2}||_1$ to get

$$||u_t(t)||_\varepsilon^2/||(u^2 + \varepsilon)^{p/2}||_1 + \frac{1}{p}\frac{d}{dt}F_\varepsilon(u) = 0.$$

Integration over t yields (23). □
Now we will let $\varepsilon \to 0$ in order to obtain existence for (18).

Theorem 1 *Let u_ε be the solution to (21). Then a sequence $\varepsilon_i \to 0$ and a solution $u \in C(0, T; H^{1,p}) \cap H^1(0, T; L^2)$ to (18) exists such that*

$$u_i := u_{\varepsilon_i} \to u \text{ in } C(0, T; L^2 \cap H^{1,p}) \tag{25}$$

and

$$\|u(t)\| = ||u_0||,\ \ t \in [0, T], \tag{26}$$

$$\int_0^t ||u_t||^2/||u||_p^p ds + \frac{1}{p}(F(u(t)) - F(u_0)) \leq 0. \tag{27}$$

Moreover, the function $t \to F(u(t))$ is decreasing.

PROOF:
(i) (22), (23) along with the compactness of the imbedding ([10])

$$W = L^2(0,T;H^{1,p}) \cap H^1(0,T;L^2) \subset L^2(0,T;L^p)$$

and the continuity of the imbedding of $H^1(0,T;L^p)$ into $C(0,T;L^p)$ guarantee existence of a sequence $\varepsilon_i \to 0$ and of a function $u \in W$ such that

$$u_i := u_{\varepsilon_i} \to u \ \text{ in } \ C(0,T;L^p), \tag{28}$$

$$u_i \rightharpoonup u \ \text{ in } \ L^2(0,T;L^2) \ \textit{ and } \ L^2(0,T;H^{1,p}), \ \ \|u(t)\| \le \|u_0\|, \ \ t \in [0,T], \tag{29}$$

$$u_{it} \rightharpoonup u_t \ \text{ in } \ L^2(0,T;L^2). \tag{30}$$

Further the relation (23) shows, that the function family

$$\lambda_\varepsilon(t) = F_\varepsilon(u_\varepsilon(t))$$

is uniformly bounded and decreasing with respect to t. Thus we can suppose [11], that

$$\lambda_i(t) := \lambda_{\varepsilon_i}(t) \to \lambda(t) \ \ \forall\, t \in [0,T], \tag{31}$$

where the limit function λ is also bounded and decreasing. Because the weak lower semicontinuity of the L^p norm and (29), we have in addition

$$F(u(t)) \le \lambda(t). \tag{32}$$

(ii) For passing $\varepsilon \to 0$ we apply the usual monotonicity arguments. Define

$$g = \lambda b(u)$$

Now, (28) and (31) imply

$$\begin{aligned} \limsup_{i\to\infty} \int_0^t (u_{it} + Au_i, u_i)\, ds &= \limsup_{i\to\infty} \int_0^t (||\nabla u_i||_p^p + \gamma |\bar{u}_i|^p)\, ds + \varepsilon_i(\|\nabla u_0\|^2 - \|\nabla u_{\varepsilon_i}\|^2) \\ &\le \lim_{i\to\infty} \int_0^t \lambda_i ||u_i||_p^p ds = \int_0^t \lambda ||u||_p^p\, ds = \int_0^t (g,u)\, ds \end{aligned}$$

(33)

and, due to the continuity of b (comp. (19)), (22)), (30) and (31),

$$\lim_{i\to\infty} \int_0^t (u_{it} + Au_i, h)\, ds = \lim_{i\to\infty} \int_0^t (\varepsilon_i \Delta u_{it} + g, h)\, ds = \int_0^t (g,h)\, ds, \ \ h \in L^2(0,t;H^1).$$

Since H^1 lies densely in $H^{1,p}$ and $u_{it} + Au_i$ is bounded in $L^2(0,T;(H^{1,p})^*)$, that means

$$u_{it} + Au_i \rightharpoonup g \ \text{ in } \ L^2(0,T;(H^{1,p})^*). \tag{34}$$

Since $A \in (L(0,T;H^{1,p}) \to L^2(0,T;(H^{1,p})^*)$ is continuous and monotone, (29), (33) and (34) imply (comp. [6], [10])

$$u_t + Au = g. \tag{35}$$

Testing (35) with u and using (32) yield

$$\frac{d}{2dt}\|u\|^2+\int_0^t (Au,u)\,ds = \int_0^t (g,u)\,ds = \int_0^t \lambda||u||_p^p\,ds \geq \int_0^t F(u)||u||_p^p\,ds = \int_0^t (Au,u)\,ds.$$

This implies $||u(t)|| \geq ||u_0||$ and by (29)

$$||u(t)|| = ||u_0||, \text{ i.e., } \frac{d}{dt}||u|| = 0,$$

and

$$\lambda = F(u). \tag{36}$$

From this and (35) the equation (18) follows.
Finally (28), (31) and (36) show that

$$||\nabla u_i(t)||_p \to ||\nabla u(t)||_p.$$

Since $H^{1,p}$ is uniformly convex, this along with (30) prove (25) and (27). □

4 Global behavior

In this section we shall show that the trajectories $u_p(t)$ of the initial value problem (18) for $t \to \infty$ tend to solutions u_p^* of the nonlinear eigenvalue problem (11). Further the behavior of u_p^* for $p \to 1$ is studied.

Theorem 2 *Let u be a solution to (18) as guaranteed by Theorem 1 and let be*

$$\lambda^* = \lim_{t\to\infty} F(u(t)).$$

Then a sequence $(t_i) \to \infty$ and a solution $u^* \in L^2 \cap H^{1,p}$ to (11) exist such that

$$\begin{aligned} u_i := u(t_i) &\to u^* \text{ in } L^2 \cap H^{1,p}, \\ ||u^*|| &= ||u_0||, \quad F(u^*) = \lambda^*, \\ u_{ti} &\to 0 \text{ in } L^2. \end{aligned} \tag{37}$$

PROOF:
(i) By (26), (27) we have

$$||u(t)||^2 + \int_0^t ||u_t||^2 ds \leq c, \quad ||u(t)|| = ||u_0||, \quad t \geq 0.$$

Testing (18) with $|u|^{2-p}u$ gives

$$\begin{aligned} \frac{1}{4-p}(||u||_{4-p}^{4-p})_t + (\frac{p}{2})^p(3-p)||\nabla|u|^{\frac{2}{p}}||_p^p &= F(u)||u||^2 - \gamma \mathrm{sign}\ \bar{\mathrm{u}}|\bar{\mathrm{u}}|^{\mathrm{p}-1}\overline{|\mathrm{u}|^{2-\mathrm{p}}\mathrm{u}} \\ &\leq F(u_0)||u_0||^2 + \frac{\gamma}{|\Omega|^p}||u||_1^{p-1}||u||_{3-p}^{3-p} \\ &\leq (F(u_0) + \frac{\gamma}{|\Omega|})||u_0||^2. \end{aligned}$$

By integrating over t we get

$$\int_0^t ||\nabla|u|^{\frac{2}{p}}||_p^p\, ds \leq c(1+t).$$

(ii) Since $H^{1,p}$ is compactly imbedded into L^p, these a priori estimates ensure the existence of a sequence $(t_i) \to \infty$ and a function $u^* \in L^2 \cap H^{1,p}$, such that

$$u_i := u(t_i) \to u^* \text{ in } L^p \text{ and a. e. in } \Omega, \tag{38}$$

$$|u_i|^{\frac{2}{p}} \to |u^*|^{\frac{2}{p}} \text{ in } L^p,\ \ u_i \rightharpoonup u^* \text{ in } L^2, \tag{39}$$

$$u_{ti} \to 0 \text{ in } L^2,\ \ u_i \rightharpoonup u^* \text{ in } H^{1,p}. \tag{40}$$

Further, (39) implies

$$u_i \to u^* \text{ in } L^2 \text{ and } ||u^*|| = ||u_0||.$$

Moreover, since $F(u_i)$ is decreasing, we have

$$F(u_i) \downarrow \lambda^*. \tag{41}$$

(iii) In order to show that u^* is solution to (11), we repeat the monotonicity arguments of step (ii) in the proof of Theorem 1:
Define

$$g = \lambda^* b(u^*),$$

then (20) and (38) yield

$$\begin{aligned} \lim_{i\to\infty}(Au_i, u_i) &= \lim_{i\to\infty}(||\nabla u_i||_p^p + \gamma|\bar{u}_i|^p) = \lim_{i\to\infty}(F(u_i)||u_i||_p^p) \\ &= \lambda^*||u^*||_p^p = (g, u^*). \end{aligned} \tag{42}$$

Further, using the continuity of b, (38), Lebesgue's dominated convergence theorem (40) and (41), we get

$$Au_i \rightharpoonup g. \tag{43}$$

Since $A \in (H^{1,p} \to (H^{1,p})^*)$ is continuous and monotone, (40), (42) and (43) imply

$$Au^* = g.$$

Testing this equation with u^* yields

$$||\nabla u^*||_p^p + \gamma|\bar{u^*}|^p = (Au^*, u^*) = (g, u^*) = \lambda^*||u^*||_p^p,$$

that means

$$\lambda^* = F(u^*), \tag{44}$$

and that u^* satisfies (11):

$$Au^* = F(u^*)b(u^*).$$

Finally, (38), (41), and (44) imply

$$||\nabla u_i||_p \to ||\nabla u^*||_p,$$

thus, in view of (40) and the uniform convexity of $H^{1,p}$, (37) follows. □

The next result gives a condition ensuring that u^* changes the sign in Ω (comp. Proposition 2).

Theorem 3 *Let* $F(u_0) < \frac{\gamma}{|\Omega|}$. *Let*

$$E_{u^*} = \{x \in \Omega,\ u^*(x) > 0\}.$$

Then $0 < |E_{u^*}| < |\Omega|$.

PROOF: Suppose $|E_{u^*}| = 0$ or $|E_{u^*}| = |\Omega|$. Then, testing (11) with 1, we get

$$\gamma|\Omega|^{1-p}||u^*||_1^{p-1} = F(u^*)\int_\Omega |u^*|^{p-1}dx \le F(u_0)||u^*||^{p-1}|\Omega|^{2-p},$$

but this contradicts our assumption. □

Finally we study the behavior of solutions u_p^* of (11) for $p \to 1$.

Theorem 4 *Let*

$$F_p(u_0) < \frac{\gamma}{|\Omega|}. \tag{45}$$

Let u_p^*, $1 < p \le 2$, *be a solution to (11) as guaranteed by Theorem 2. Then there exists a sequence* $(p_i) \to 1$ *and a function* $u^* \in BV$ *such that*

$$u_i^* := u_{p_i}^* \quad \to \quad u^* \ \text{ in } \ L^1, \tag{46}$$

$$|\overline{u_i^*}| \quad \to \quad |\overline{u^*}| = 0, \tag{47}$$

$$F_i(u_i^*) \quad \to \quad \lambda^* \ge F_1(u^*). \tag{48}$$

PROOF:
(i) By Theorem 2 we have

$$||u_p^*|| = ||u_0||, \ \ F_p(u_p^*) \le F_p(u_0) \le c(u_0)$$

and hence

$$\int_\Omega |Du_p^*| \le c.$$

Since BV is compactly imbedded into L^1 [9], there exist a sequence $(p_i) \to 1$ and a function $u^* \in BV$ such that

$$u_i^* := u_{p_i}^* \to u^* \ \text{ in } \ L^1 \ \text{ and } \ \text{ a.e. in } \Omega,$$

$$F_{p_i}(u_i^*) \to \lambda^*.$$

(ii) Testing the equation

$$Au_i^* = F_{p_i}(u_i^*)|u_i^*|^{p_i-2}u_i^*$$

with sign $\overline{u_i^*}$, we get

$$\begin{aligned}\gamma|\bar{u}_i^*|^{p_i-1} &= F_{p_i}(u_i^*)|\int_\Omega |u_i^*|^{p_i-2}u_i^* dx| \\ &\leq F_{p_i}(u_i^*)||u_i^*||^{p_i-1}|\Omega|^{3/2-p_i/2} \\ &\leq F_{p_i}(u_0^*)||u_0^*||^{p_i-1}|\Omega|^{3/2-p_i/2}\end{aligned}$$

and hence

$$|\bar{u}_i^*| \leq \left(\frac{F_{p_i}(u_0)|\Omega|}{\gamma}\right)^{\frac{1}{p_i-1}} ||u_0|||\Omega|^{\frac{-1}{2}}.$$

Letting $p_i \to 1$ and taking into account (45) and (46) we get (47).
(iii) For proving that $F_1(u^*) \leq \lambda^*$, we can proceed as in step (ii) of the proof of Proposition 3. □

5 Time discretization

In this section we establish a (discrete) steepest descent method for solving (11). To this end we consider the following time discrete version of (18):

$$\frac{u_i - u_{i-1}}{\tau} + Au_i = Bu_{i-1}, \;\; i = 1, 2, ..., \;\; u_{i=0} = u_0, \;\; \tau > 0. \tag{49}$$

Theorem 5 *Problem (49) has a unique solution* $u_i \in L^2 \cap H^{1,p}$. *The sequence* $(F(u_i))$ *is decreasing. Let* $\lambda^* = \lim_{i\to\infty} F(u_i)$. *Let* $\tau F(u_0) < 1$. *Then a subsequence* (u_j) *and a solution* $u^* \in L^2 \cap H^{1,p}$ *to (11) exist such that*

$$u_j \to u^* \;\text{ in } L^2 \cap H^{1,p}, \;\; F(u^*) = \lambda^*. \tag{50}$$

PROOF: (i) The operator $A \in (L^2 \cap H^{1,p} \to (L^2 \cap (H^{1,p})^*)$ is continuous, strictly monotone and coercitiv. The operator B maps $H^{1,p}$ into $(L^2 \cap H^{1,p})^*$. Thus the Browder-Minty theorem ensures existence of a unique solution $u_i \in L^2 \cap H^{1,p}$ for given $u_{i-1} \in H^{1,p}$.
(ii) Testing (49) with $u_i - u_{i-1}$, applying Hölder's and Young's inequalities, we get

$$\begin{aligned}\frac{||u_i - u_{i-1}||^2}{\tau} &+ (Au_i, u_i) = (Au_i, u_{i-1}) + F(u_{i-1})\int_\Omega |u_{i-1}|^{p-2}u_{i-1}(u_i - u_{i-1})\, dx \\ &\leq ||\nabla u_i||_p^{p-1}||\nabla u_{i-1}||_p + \gamma|\overline{u_i}|^{p-1}|\overline{u_{i-1}}| + F(u_{i-1})(||u_{i-1}||_p^{p-1}||u_i||_p - ||u_{i-1}||_p^p) \\ &\leq \frac{1}{p}[(p-1)||\nabla u_i||_p^p + ||\nabla u_{i-1}||_p^p + \gamma((p-1)|\overline{u_i}|^p + |\overline{u_{i-1}}|^p) \\ &+ F(u_{i-1})(||u_i||_p^p - ||u_{i-1}||_p^p)] = \frac{||u_i||_p^p}{p}((p-1)F(u_i) + F(u_{i-1}))\end{aligned}$$

and hence

$$\frac{||u_i - u_{i-1}||^2}{\tau ||u_i||_p^p} + \frac{1}{p}(F(u_i) - F(u_{i-1})) \leq 0. \tag{51}$$

Testing (49) with u_i gives

$$\begin{aligned}
\frac{||u_i||^2}{\tau} \quad &+ \quad (Au_i, u_i) = \frac{(u_{i-1}, u_i)}{\tau} + F(u_{i-1}) \int_\Omega |u_{i-1}|^{p-2} u_{i-1} u_i \, dx \\
&\leq \quad \frac{1}{2\tau}(||u_i||^2 + ||u_{i-1}||^2) + \frac{F(u_{i-1})}{p}((p-1)||u_{i-1}||_p^p + ||u_i||_p^p) \\
&= \quad \frac{1}{2\tau}(||u_i||^2 + ||u_{i-1}||^2) + (Au_{i-1}, u_{i-1}) \\
&+ \quad \frac{1}{p}\left(||u_i||_p^p (F(u_{i-1}) - F(u_i)) + ||u_i||_p^p F(u_i) - ||u_{i-1}||_p^p F(u_{i-1})\right),
\end{aligned}$$

that is

$$\frac{1}{2\tau}(||u_i||^2 - ||u_{i-1}||^2) + \frac{p-1}{p}((Au_i, u_i) - (Au_{i-1}, u_{i-1})) \leq \frac{||u_i||_p^p}{p}(F(u_{i-1}) - F(u_i)).$$

Summing over $i = 1, j$ and taking into account that $(F(u_i))$ is decreasing by (51), we get

$$\begin{aligned}
\frac{||u_j||^2}{2\tau} \quad &+ \quad \frac{p-1}{p}(Au_j, u_j) \leq \frac{||u_0||^2}{2\tau} + \frac{p-1}{p}(Au_0, u_0) + \frac{1}{p} \max_i \{||u_i||_p^p\}(F(u_0) - F(u_j)) \\
&\leq \quad \frac{||u_0||^2}{2\tau} + \frac{p-1}{p}(Au_0, u_0) + \frac{1}{p} \max_i \{\frac{p}{2}||u_i||^2 + \frac{(2-p)}{2}|\Omega|\}(F(u_0) - F(u_j)).
\end{aligned}$$

Since this holds for all j, we conclude

$$\max_i \{||u_i||^2\} \leq \frac{1}{1 - \tau F(u_0)} \left(||u_0||^2 + \frac{2\tau}{p}[(p-1)(Au_0, u_0) + (2-p)|\Omega| F(u_0)] \right)$$

and thus

$$||u_j||^2 + \frac{2\tau(p-1)}{p}(Au_j, u_j) \leq ||u_0||^2 + c\tau. \tag{52}$$

(iv) Testing (49) with $|u_i|^{2-p} u_i$ yields

$$\begin{aligned}
\frac{1}{\tau}(||u_i||_{4-p}^{4-p}) \quad &+ \quad (\frac{p}{2})^p (3-p) ||\nabla |u_i|^{\frac{2}{p}}||_p^p = \frac{1}{\tau}(u_{i-1}, |u_i|^{2-p} u_i) \\
&- \quad \gamma \mathrm{sign}\, \overline{\mathrm{u_i}} |\overline{\mathrm{u_i}}|^{\mathrm{p}-1} \overline{|\mathrm{u_i}|^{2-\mathrm{p}} \mathrm{u_i}} + \mathrm{F}(\mathrm{u_{i-1}}) \int_\Omega |\mathrm{u_{i-1}}|^{\mathrm{p}-2} \mathrm{u_{i-1}} |\mathrm{u_i}|^{2-\mathrm{p}} \mathrm{u_i} \, \mathrm{dx} \\
&\leq \quad \frac{1}{\tau(4-p)}((3-p)||u_i||_{4-p}^{4-p} + ||u_{i-1}||_{4-p}^{4-p}) + \frac{\gamma}{|\Omega|^p} ||u_i||_1^{p-1} ||u_i||_{3-p}^{3-p} \\
&+ \quad F(u_0) ||u_{i-1}||^{p-1} ||u_i||^{3-p}
\end{aligned}$$

and by (52)

$$\frac{1}{\tau(4-p)}(||u_i||_{4-p}^{4-p} - ||u_{i-1}||_{4-p}^{4-p}) + (\frac{p}{2})^p (3-p) ||\nabla |u_i|^{\frac{2}{p}}||_p^p \leq c. \tag{53}$$

(v) Using (51)-(53), we can proceed as in the steps (ii) and (iii) of the proof of Theorem 2, in order to prove (50). □

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