FLUCTUATIONS IN THE HOPFIELD MODEL AT THE CRITICAL TEMPERATURE

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ABSTRACT. We investigate the fluctuations of the order parameter in the Hopfield model of spin glasses and neural networks at the critical temperature $1/\beta_c = 1$. The number of patterns $M(N)$ is allowed to grow with the number N of spins but the growth rate is subject to the constraint $M(N)^{15}/N \to 0$. As the system size N increases, on a set of large probability the distribution of the appropriately scaled order parameter under the Gibbs measure comes arbitrarily close (in a metric which generates the weak topology) to a non-Gaussian measure which depends on the realization of the random patterns. This random measure is given explicitly by its (random) density.

1. INTRODUCTION

In 1977, Pastur and Figotin introduced and discussed a disordered version of the Curie–Weiss model of ferromagnets (see [29], [30]). Later their model became popular under the name Hopfield model because of its impact on the theory of neural networks achieved by its rediscovery and reinterpretation by Hopfield [21]. This versatility of the Hopfield model—namely that it can be regarded as a very simple model of the brain on one hand, and as a so-called spin glass (i. e., a disordered spin system) on the other hand—has been the driving force for its popularity and the efforts which have been undertaken to obtain a better understanding of the model.

The neural network point of view has been taken in the original paper by Hopfield [21] for instance, as well as in the papers [27], [28], [23], [25], [26], and many others while in the seminal paper [29], as well as in [7], [8], [9], [3], [16], [17], [4], [5], and [31] the statistical-mechanics and thus the spin-glass aspect of the model have been in the centre of interest. Of course, it would be very difficult to give a complete list of all important papers in this area. For an overview of recent results on the Hopfield model and related models and results which deeply influenced our understanding of the model and even were able to justify some of the physicists' predictions (see [1], e. g.) we refer the reader to [31] and [11] and, in particular, [6] therein.

To be more specific, let us now define the Hopfield model. First of all we choose two numbers $N, M \in \mathbb{N}$ which will denote the number of spins or "neurons" and the number of so-called patterns, respectively. In contrast to a previous paper [20], we shall now treat the case where $M = M(N)$ may depend on N. Henceforth, we shall write M and thus drop its dependency on N whenever there is no danger of confusion and we shall refer explicitly to this dependency only when necessary. The random function

$$H_N(\sigma) = -\frac{1}{2N} \sum_{\mu=1}^{M} \sum_{i,j=1}^{N} \sigma_i \sigma_j \xi_i^\mu \xi_j^\mu, \qquad \sigma \in \{-1,+1\}^N, \tag{1.1}$$

denotes the Hamiltonian of the Hopfield model, which is a function of the spin configuration $\sigma \in \{-1,+1\}^N$. The strength of the pair interaction is random as the variables $\xi_i^\mu \in \{-1,+1\}$ with ξ_i^μ denoting the ith component of the μth pattern are random. In this paper we shall assume that the ξ_i^μ are i.i.d. unbiased random variables, i. e., that at given system size N, the family of random variables $\{\, \xi_i^\mu : i \in \{1,\dots,N\}, \mu \in \{1,\dots,M(N)\}\,\}$ is independent with

$$\mathbb{P}(\xi_i^\mu = +1) = \mathbb{P}(\xi_i^\mu = -1) = \frac{1}{2}$$

for all i and μ. Expectations with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$. Whenever convenient, we shall write ξ for the $(N \times M)$-matrix consisting of the $(\xi_i^\mu)_{i,\mu}$, while $\xi_i = (\xi_i^1, \dots, \xi_i^M)$ and $\xi^\mu = (\xi_1^\mu, \dots, \xi_N^\mu)$, respectively, stand for the ith row and the μth column of this matrix, respectively.

The spin variables are assumed to be independent with an unbiased a priori distribution P, i. e.,

$$\mathrm{P}(\sigma_i = +1) = \mathrm{P}(\sigma_i = -1) = \frac{1}{2}$$

for all $i \in \mathbb{N}$. In addition, we shall assume throughout this paper that the family $\{\, \xi_i^\mu : i \in \{1, \dots, N\}, \mu \in \{1, \dots, M\} \,\}$ is independent of the family of the spin variables $\{\, \sigma_i : i \in \{1, \dots, N\} \,\}$.

The Hopfield model at temperature $1/\beta \in (0, \infty)$ may now be identified with the Gibbs measure with respect to the Hamiltonian (1.1), i. e.,

$$\varrho_{N,\beta}(\sigma) = 2^{-N} \exp\{-\beta H_N(\sigma)\}/Z_{N,\beta}, \qquad \sigma \in \{-1, +1\}^N, \tag{1.2}$$

where the so-called partition function

$$Z_{N,\beta} = \frac{1}{2^N} \sum_{\sigma \in \{-1,+1\}^N} \exp\{-\beta H_N(\sigma)\} \tag{1.3}$$

is the normalization which makes $\varrho_{N,\beta}$ a probability measure.

In order to understand the introduction of the order parameter in the Hopfield model note that the Hamiltonian (1.1) may be rewritten in the following convenient form as a quadratic functional of the so-called overlap m_N:

$$H_N(\sigma) = -\frac{N}{2} \|m_N(\sigma)\|_2^2, \tag{1.4}$$

where

$$m_N(\sigma) = (m_N^\mu(\sigma))_{\mu=1,\dots,M} \qquad \text{with} \qquad m_N^\mu(\sigma) = \sum_{i=1}^N \xi_i^\mu \sigma_i. \tag{1.5}$$

Here and below, $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^M$. The μth component m_N^μ of the overlap m_N compares the spin configuration to the μth pattern ξ^μ in such a way that a large absolute value of $m_N^\mu(\sigma)$ means that the spin configuration σ largely agrees with ξ^μ (or its negative). These configurations are of low energy according to (1.4). Therefore, the overlap is an important quantity for the investigation of the Hopfield model, a so-called order parameter. Its distribution under $\varrho_{N,\beta}$ has been of major interest in the study of the model and also will be central in this paper.

In [7], Bovier, Gayrard, and Picco established a law of large numbers for the distribution of the overlap under the Gibbs measure $\varrho_{N,\beta}$ which holds for $\mathbb{P}$-almost all realizations of the random patterns ξ. They showed that, whenever $M(N)/N \to 0$, for $\mathbb{P}$-almost all ξ, the distribution of the overlap m_N under the Gibbs measure with external magnetic field of strength $h \neq 0$ in the direction of the first unit vector e_1 of the canonical basis in $\mathbb{R}^M$ converges weakly towards the Dirac measure $\delta_{\pm z(\beta) e_1}$ concentrated in $\pm z(\beta) e_1$ as first the system size $N \to \infty$ and then the strength $h \to 0\pm$. Here $z(\beta)$ denotes the largest root $z \in [0, 1)$ of the Curie–Weiss equation

$$z = \tanh(\beta z).$$

Note that $z(\beta) = 0$ for $\beta \le \beta_c = 1$, so that δ_0 is the unique limiting measure in the high-temperature region $\beta \le \beta_c = 1$, whereas $z(\beta) > 0$ for $\beta > \beta_c$, so that in this regime there is no unique limiting point.

Note that this result strongly resembles the law of large numbers for the mean magnetization in the Curie–Weiss model, see [14, Theorem IV.4.1(a)], for example. As already explained at the beginning this is, of course, not accidental, as the Hopfield model can be considered as a disordered version of the Curie–Weiss model and, indeed, for $M = 1$ the Hopfield model and the Curie–Weiss model agree by a simple "gauge transformation" (i. e., replacing σ_i by $\sigma_i \xi_i^1$).

On the scale of fluctuations, when analyzing the distribution of $\sqrt{N}(m_N - z(\beta)e_1)$, the character of the disorder becomes visible. Indeed, for $M/N \to 0$ and $(\beta, h) \neq (1, 0)$, the overlap satisfies $\mathbb{P}$-almost surely a central limit theorem with the covariance matrix which could be expected from the analogy with the Curie–Weiss model and a centring which differs in the cases $\beta > 1$ or $h \neq 0$ from the naively expected one by a ξ-dependent adjustment, see [16], [17], [19] and Bovier and Gayrard [4].

As shown in a previous paper [20], the influence of the disorder is even stronger when investigating the fluctuations of the overlap at the critical temperature $1/\beta = 1/\beta_c = 1$, even when $M(N)$ remains bounded. Recall that in the Curie–Weiss model the criticality at temperature $1/\beta = 1$ can also be seen as the breakdown of the central limit theorem. As a matter of fact at the critical temperature the magnetization in the Curie–Weiss model—scaled by a factor $N^{1/4}$—converges weakly towards a random variable given by its density with respect to Lebesgue measure which is proportional to $\exp(-x^4/12)$, cf. [14, Theorem V.9.5]. In [20] we showed that in the Hopfield model with finitely many patterns (i. e., with M not depending on N) the distribution of the overlap—scaled by the same factor $N^{1/4}$—regarded as a random variable Q_N taking values in the Polish space $\mathcal{M}_1(\mathbb{R}^M)$ of probability measures on $\mathbb{R}^M$ converges weakly (with respect to $\mathbb{P}$) to a limiting random measure $\overline{Q}_M$. This limiting random measure $\overline{Q}_M$ is given by its (random) density with respect to the M-dimensional Lebesgue measure which is proportional to

$$\exp\left(-\frac{1}{12}\sum_{\mu=1}^{M} x_\mu^4 - \frac{1}{2}\sum_{1\le\mu<\nu\le M} x_\mu^2 x_\nu^2 + \sum_{1\le\mu<\nu\le M} \eta_{\mu,\nu} x_\mu x_\nu\right), \tag{1.6}$$

where η is an $M(M-1)/2$-dimensional Gaussian random variable with mean zero and the covariance matrix being the identity matrix, namely,

$$\Sigma = \left(\Sigma_{(\mu,\nu),(\mu',\nu')}\right)_{(\mu,\nu),(\mu',\nu')}$$

and

$$\Sigma_{(\mu,\nu),(\mu',\nu')} = \begin{cases} 1, & \text{if } (\mu,\nu) = (\mu',\nu'), \\ 0, & \text{otherwise}, \end{cases}$$

for $1 \le \mu < \nu \le M$ and $1 \le \mu' < \nu' \le M$.

This shows that even for finite M at the critical temperature $1/\beta = 1$, the fluctuations of the overlap depend strongly on the random disorder as even the distribution of the limiting fluctuations is random. Even to formulate the corresponding result for the case where the number of patterns $M(N)$ is actually growing with N seemed to be difficult, since, on one hand, we don't have an "infinite-dimensional Lebesgue measure" as reference measure and, on the other hand, we cannot work with finite-dimensional projections (as in the Central Limit Theorem) either, since the "mixed terms" $\sum_{1\le\mu<\nu\le M} \eta_{\mu,\nu} x_\mu x_\nu$ tend to "glue" together the coordinates.

In this paper we circumvent these difficulties by not stating a limit theorem but by showing instead that the distance between the distribution Q_N of the scaled overlap and the random measure $\overline{Q}_M$ becomes small with high probability for large N. More precisely, we shall show, under the constraint $M^{15}/N \to 0$ on the growth rate of $M(N)$, that for each large enough N there exists a set of ξ's of probability larger than $1 - \exp\{-M/L\}$ (with some constant $L > 0$) on which the distance between Q_N and $\overline{Q}_M$ is smaller than $\varepsilon_N \searrow 0$.

This paper has three more sections. Section 2 contains the explicit statement of the result concerning the non-Gaussian fluctuations of the overlap at $\beta = 1$ for the Hopfield model with a growing number of patterns. Section 3 is devoted to one of our basic tools, a multidimensional version of a strong approximation result of Komlós, Major and Tusnády [22], which allows to control the difference of a sum of i.i.d. random variables and a sum of i.i.d. Gaussian random variables with the same covariance matrix. These results go back to Zaitsev [32], [33], Einmahl [12] and Einmahl and Mason [13]. They also proved useful in [10]. Section 4 finally is devoted to the proof which is based on the Hubbard–Stratonovich transform of the measures of interest together with a Taylor expansion of the resulting density, a saddle point approximation as well as the strong Gaussian approximation mentioned before.

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2. Statement of Results

This section contains the mathematically precise statement of the result announced in the introduction. We shall state the theorem only for the case of $\beta = \beta_c = 1$ being fixed. In [20], where we considered M independent of N only, we also treated the case of variable temperature β_N converging to $\beta_c = 1$ as $N \to \infty$. It turned out that for β_N converging to β_c faster than $1/\sqrt{N}$ (recall that M was chosen as a constant), the limiting distribution is the same, while for β_N converging to β_c slower than $1/\sqrt{N}$, we have a Central-Limit-Theorem type result and at "the borderline", i. e., when $\beta_N - \beta_c$ is of the same order as $1/\sqrt{N}$, one can see the influence of both possible limiting distributions.

In the present setting, we consider such an extension of our results to variable β_N a basically technical exercise. Therefore, we shall concentrate on the most interesting case which allows us to present streamlined proofs.

In general, we shall assume that the pattern matrix ξ lives on a probability space $(\Omega, \mathcal{F}, P)$ that is rich enough to allow the strong-approximation results stated in Section 3. The pattern matrix has to be viewed as a random variable on $(\Omega, \mathcal{F}, P)$, but with slight abuse of notation, we shall formulate exceptional sets as sets of ξ-variables by writing $\{\xi : F(\xi) \in A\}$ which is to be understood in the natural way as $\{\omega \in \Omega : F(\xi(\omega)) \in A\}$.

Let

$$Q_N = \varrho_{N,1}(N^{1/4} m_N)^{-1} \tag{2.1}$$

denote the distribution of the scaled overlap under the Gibbs measure $\varrho_{N,1}$. By d we denote the metric

$$d(P_1, P_2) = \sup\left\{\left|\int f dP_1 - \int f dP_2\right| : f \in \mathcal{G}\right\} \tag{2.2}$$

with

$$\mathcal{G} = \tag{2.3}$$
$$\left\{f : \mathbb{R}^M \to \mathbb{R} : \sup_{x,y\in\mathbb{R}^M} |f(x) - f(y)| \leq 1 \text{ and } \sup_{x,y\in\mathbb{R}^M} |f(x) - f(y)| \leq ||x - y||_2\right\}$$

on the set $\mathcal{M}_1(\mathbb{R}^M)$ of all probability measures on $\mathbb{R}^M$. According to [2, Corollary 2.8] this metric generates the weak topology on $\mathcal{M}_1(\mathbb{R}^M)$. The result we are going to prove is the following.

Theorem 2.1. *Let $\beta = \beta_c = 1$. Assume that $M(N)^{15}/N \to 0$. Then there exist a constant $L > 0$, a set $\overline{\Omega}(N) \subset \Omega$ with probability*

$$\mathbb{P}(\overline{\Omega}(N)) \geq 1 - e^{-M(N)/L}, \tag{2.4}$$

an $\overline{N} \in \mathbb{N}$ and a sequence $(\varepsilon_N)_{N\in\mathbb{N}}$, satisfying $\varepsilon_N \searrow 0$ as $N \to \infty$, such that for every $N \geq \overline{N}$, there exists a set

$$(\eta_{\mu,\nu})_{1\leq\mu<\nu\leq M}$$

of $M(M-1)/2$ independent standard-Gaussian random variables such that the random measure $\overline{Q}_M$, which is given by its (random) density

$$x \mapsto \exp\{\Psi_M(x)\} \Big/ \int_{\mathbb{R}^M} \exp\{\Psi_M(x)\}\, dx \tag{2.5}$$

with

$$\Psi_M(x) = -\frac{1}{12}\sum_{\mu=1}^{M} x_\mu^4 - \frac{1}{2}\sum_{1\leq\mu<\nu\leq M} x_\mu^2 x_\nu^2 + \sum_{1\leq\mu<\nu\leq M} \eta_{\mu,\nu} x_\mu x_\nu, \tag{2.6}$$

satisfies

$$d(Q_N, \overline{Q}_M) \leq \varepsilon_N \tag{2.7}$$

for all $\xi \in \overline{\Omega}(N)$.

Remarks 2.2. 1. *Note that the scaling factor $N^{1/4}$ for the overlap vector is the same as the one for the mean magnetization in the Curie–Weiss model at the critical temperature, see* [14, Theorem V.9.5]. *Similar to that case (and, of course, similar to the Hopfield model with a finite number of patterns) the distribution of the overlap is close to a non-Gaussian distribution.*

2. *Our condition $M(N)^{15}/N \to 0$ on the growth rate of M is, of course, embarrassing. It is due to the simultaneous strong Gaussian approximation of $M(M-1)/2$ variables. Any proof using the strong Gaussian approximation as provided in* [32], *seems to produce conditions which are far off any reasonable condition on the growth rate.*

3. *In fact, we are going to show that, under the conditions of the theorem,*

$$\left| \int_{\mathbb{R}^M} f(x) Q_N(dx) - \int_{\mathbb{R}^M} f(x) \overline{Q}_M(dx) \right| \leq \varepsilon_N (K_f + \|f\|_\infty) \tag{2.8}$$

holds for all $\xi \in \overline{\Omega}(N)$ *and all* $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$*, where* $\mathrm{BL}(\mathbb{R}^M, \mathbb{R})$ *denotes the set of all bounded, Lipschitz continuous functions from* $\mathbb{R}^M$ *to* $\mathbb{R}$*,* K_f *denotes the Lipschitz constant of* f *and* $\|f\|_\infty = \sup_{x \in \mathbb{R}^M} |f(x)|$*. This implies the theorem by* (4.2) *below.*

3. Strong Gaussian Approximation

In this section we are going to collect some facts about the so-called strong Gaussian approximation and apply them to the situation of our interest. The problem of the Gaussian approximation is quickly stated. Given a sequence $(X_i)_{i \in \mathbb{N}}$ of i.i.d. random vectors in $\mathbb{R}^d$, we know that $\sum_{i=1}^n X_i$—scaled appropriately—converges in distribution to a Gaussian random vector Y. This vector can obviously be decomposed again into a sum of "small" Gaussians. The question is now, whether we can also find Gaussian vectors Y_i such that the difference

$$\Delta(X, Y, n) = \sup_{1 \leq k \leq n} \left\| \sum_{i=1}^k X_i - \sum_{i=1}^k Y_i \right\|_2 \tag{3.1}$$

becomes small in a suitable sense.

This problem was first stated and treated in a one-dimensional setting by Komlós, Major and Tusnády in [22]. The d-dimensional extension is due to Zaitsev [32] and Einmahl [12]. For a thorough treatment of the problem, we refer the reader to [33]. The form of the strong approximation we recall below proved useful in [10] and goes back to Einmahl and Mason [13].

Let P_1 and P_2 be two probability measures on $\mathbb{R}^d$ (endowed with the Borel σ-field), and for $\delta > 0$ let

$$\lambda(P_1, P_2, \delta) = \sup\{P_1(A) - P_2(A^\delta), P_2(A) - P_1(A^\delta) : A \subset \mathbb{R}^d \text{ closed}\}. \tag{3.2}$$

Here

$$A^\delta = \{x \in \mathbb{R}^d : \exists y \in A \text{ such that } \|x - y\|_2 \leq \delta\} \tag{3.3}$$

is the closed δ-neighborhood of the set A.

Furthermore, let $X_1, \ldots, X_n$ be $n \in \mathbb{N}$ independent random vectors in $\mathbb{R}^d$ with $\mathbb{E}X_1 = 0$ and finite variance which satisfy the Bernstein-type condition

$$|\mathbb{E}\langle s, X_i \rangle^2 \langle t, X_i \rangle^{m-2}| \leq \frac{1}{2} m! \tau^{m-2} \|t\|^{m-2} \mathbb{E}\langle s, X_i \rangle^2 \tag{3.4}$$

with some τ for all $m \geq 3$ and all $s, t \in \mathbb{R}^d$.

Under the condition (3.4), Zaitsev proved in [32, Theorem1.1] the following bound on $\lambda(P_{1,n}, P_{2,n}, \delta)$, where $P_{1,n}$ is the distribution of $X_1 + \ldots + X_n$ and $P_{2,n}$ is the d-dimensional normal distribution with mean zero and covariance matrix $\mathrm{cov}(X_1) + \cdots + \mathrm{cov}(X_n)$ (see also [13]).

Fact 3.1. *For all* $n \geq 1$ *and all* $\delta \geq 0$*,*

$$\lambda(P_{1,n}, P_{2,n}, \delta) \leq c_{1,d} \exp\{-\delta/(c_{2,d}\tau)\} \tag{3.5}$$

with $c_{1,d} = c_1 d^{5/2}$ *and* $c_{2,d} = c_2 d^{5/2}$ *for numerical constants* $c_1, c_2 > 0$.

As in [13], Fact 3.1, the following fact follows.

Fact 3.2. *Let $X_1, \dots, X_n$ be independent mean zero random vectors satisfying the Bernstein-type condition (3.4). If the underlying probability space is rich enough, then, for each $\delta \geq 0$, there exist independent Gaussian random vectors $Y_1, \dots, Y_n$ with mean zero and*

$$\operatorname{cov}(Y_i) = \operatorname{cov}(X_i) \qquad \text{for all } i \in \{1, \dots, n\},$$

such that

$$P\left\{ \left\| \sum_{i=1}^{n} (X_i - Y_i) \right\|_2 \geq \delta \right\} \leq c_{1,d} \exp\{-\delta/(c_{2,d}\tau)\}, \tag{3.6}$$

where the constants $c_{1,d}$, $c_{2,d}$ are the same as in Fact 3.1.

Corollary 3.3. *In the situation of Fact 3.2, for each $\delta \geq 0$, there exists a mean zero Gaussian random vector Y with covariance matrix $\operatorname{cov}(Y) = \sum_{i=1}^{n} \operatorname{cov}(X_i)$ such that*

$$P\left\{ \left\| Y - \sum_{i=1}^{n} X_i \right\|_2 \geq \delta \right\} \leq c_{1,d} \exp\{-\delta/(c_{2,d}\tau)\} \tag{3.7}$$

with the same constants $c_{1,d}$, $c_{2,d}$.

In our situation we want to apply Fact 3.2 and, in particular, Corollary 3.3 to the $M(M-1)/2$ dimensional vectors that contain the information of the mutual overlaps of the patterns in the ith component. More precisely, we will choose $d = M(M-1)/2$, $n = N$, and $X_i = (\xi_i^\mu \xi_i^\nu)_{1 \leq \mu < \nu \leq M}$ in order to replace $1/\sqrt{N} \sum_{i=1}^{N} X_i$ by a Gaussian random vector $\eta = (\eta_{\mu,\nu})_{1 \leq \mu < \nu \leq M}$. Observe that due to the independence of the ξ_i^μ, we obtain

$$\operatorname{cov}(X_i) = \mathrm{Id}$$

for each i, and hence also η will have identity covariance matrix. (By a slight abuse of notation, we denote the identity matrix by Id whatever the dimension of the underlying space $\mathbb{R}^d$ is.) In order to apply Corollary 3.3, we have to check the Bernstein-type condition (3.4). This is done in the following lemma.

Lemma 3.4. *In the above setting $X_1, \dots, X_n$ fulfill the Bernstein-type condition (3.4) with $\tau = M$.*

Proof. By Schwarz' inequality,

$$|\langle t, X_i \rangle| \leq \|t\|_2 \|X_i\|_2 \leq \|t\|_2 M.$$

Thus, for any choice of $s, t \in \mathbb{R}^M$ and all $m \geq 3$

$$|\mathbb{E}\langle s, X_i\rangle^2 \langle t, X_i\rangle^{m-2}| \leq \tau^{m-2} \|t\|_2^{m-2} E\langle s, X_i\rangle^2 \leq \frac{1}{2} m! \tau^{m-2} \|t\|_2^{m-2} \mathbb{E}\langle s, X_i\rangle^2,$$

where we have already chosen $\tau = M$. $\square$

Now we are ready to deduce the desired approximation.

Corollary 3.5. *If $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough, for each N and $\delta \geq 0$, there exist a mean zero Gaussian random variable η with covariance matrix* Id *and numerical constants $c_1, c_2 > 0$, such that*

$$P\left\{ \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i - \eta \right\|_2 \geq \delta \right\} \leq c_1 M^5 \exp\left\{ -\frac{\delta\sqrt{N}}{c_2 M^6} \right\}. \tag{3.8}$$

Proof. Apply Lemma 3.4 and Corollary 3.3 with $\tau = M$. □

Remark 3.6. *Observe that δ in (3.8) may—and will indeed in our applications—depend on N and M.*

4. Proofs

To prove Theorem 2.1, we need to show that for large system size N the distribution Q_N of the scaled overlap under the Gibbs measure $\varrho_{N,1}$ is close to the random measure $\overline{Q}_M$ with respect to the metric d on a set of large $\mathbb{P}$-measure. First we show that Q_N and its smoothed version obtained by a Hubbard–Stratonovich transform are close, so that we may investigate the Hubbard–Stratonovich transform instead of the measure itself. We recall the Hubbard–Stratonovich transform of Q_N from [20]. The core of the proof is the investigation of the density of this Hubbard–Stratonovich transform by an adaptation of Laplace's method.

Notation 4.1. *We denote by $\mu * \nu$ the convolution of two measures μ and ν.*

Lemma 4.2. *For all $M \geq 8$, all $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$ and all probability measures $\tilde{Q}$ on $\mathbb{R}^M$,*

$$\left| \int f \, d(\tilde{Q} * \mathcal{N}(0, N^{-1/2}\mathrm{Id})) - \int f \, d\tilde{Q} \right| \leq 2\sqrt{2} K_f \sqrt{\alpha} + \|f\|_\infty \mathrm{e}^{-M}, \tag{4.1}$$

where K_f denotes again the Lipschitz constant of f and $\|f\|_\infty = \sup_{x \in \mathbb{R}^M} |f(x)|$ as before.

Now,

$$d(P_1, P_2) = \sup\left\{ \left| \int f dQ_1 - \int f dQ_2 \right| : f \in \mathcal{G}_0 \right\} \tag{4.2}$$

with

$$\mathcal{G}_0 = \mathcal{G} \cap \{f : f(0) = 0\} \tag{4.3}$$

and $\mathcal{G}_0 \subset \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$. Therefore, the following corollary is an immediate consequence of the preceding lemma.

Corollary 4.3. *For all $M \geq 8$ and all probability measures $\tilde{Q}$ on $\mathbb{R}^M$,*

$$d(\tilde{Q} * \mathcal{N}(0, N^{-1/2}\mathrm{Id}), \tilde{Q}) \leq 2\sqrt{2\alpha} + \mathrm{e}^{-M}. \tag{4.4}$$

Proof of Lemma 4.2. Let $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$ and let $\tilde{Q}$ be an arbitrary probability measure on $\mathbb{R}^M$. Then, for $\delta > 0$,

$$\begin{aligned}
&\left| \int f \, d(\tilde{Q} * \mathcal{N}(0, N^{-1/2}\mathrm{Id})) - \int f \, d\tilde{Q} \right| \\
&\quad \leq \iint 1_{B(0,\delta)}(x) |f(x+y) - f(y)| \, \tilde{Q}(dy) \, \mathcal{N}(0, N^{-1/2}\mathrm{Id})(dx) \\
&\quad\quad + 2\|f\|_\infty \left(\frac{\sqrt{N}}{2\pi} \right)^{M/2} \int 1_{B(0,\delta)^c}(x) \exp\left\{ -\frac{N}{2} \|x\|_2^2 \right\} dx \\
&\quad \leq K_f \delta + 2\|f\|_\infty \gamma_M(B(0, \delta N^{1/2})^c),
\end{aligned} \tag{4.5}$$

where γ_M denotes the M-dimensional Gaussian measure with mean zero and the covariance matrix being the identity matrix. The radius τ_M satisfying $\gamma_M(B(0,\tau_M)) = 1/2$ is bounded by $\sqrt{2M}$ for $M \geq 8$, cf. [18, Equation (4.4)]. Choosing $\delta = 2\sqrt{2\alpha}$,

$$\gamma_M(B(0,\delta N^{1/2})^{\mathrm{c}}) \leq \frac{1}{2}\exp\Big\{-\frac{1}{2}\big[N^{1/2}\delta - \tau_M\big]^2\Big\} \leq \frac{1}{2}\mathrm{e}^{-M} \tag{4.6}$$

follows by [24, Theorem 1.2]. This concludes the proof. □

The Hubbard–Stratonovich transform of the distribution of the scaled overlap is given by its density with respect to Lebesgue measure.

Lemma 4.4. *Let $0 < \beta < \infty$ and $a > 0$. Then the convolution*

$$\chi_{N,\beta,a} = Q_N * \mathcal{N}\big(0, \frac{a}{N\beta}\mathrm{Id}\big) \tag{4.7}$$

of $Q_N = \varrho_{N,\beta}(\sqrt{a}m_N)^{-1}$ with the M-dimensional Gaussian distribution with mean zero and covariance matrix $\frac{a}{N\beta}\mathrm{Id}$ is the random measure on $\mathbb{R}^M$ which is given by the (random) density

$$f_{N,\beta,a}(x) = \frac{\exp\{-N\beta\Phi_{N,\beta}(x/\sqrt{a})\}}{\int_{\mathbb{R}^M}\exp\{-N\beta\Phi_{N,\beta}(x/\sqrt{a})\}\,dx}, \qquad x \in \mathbb{R}^M, \tag{4.8}$$

with respect to the M-dimensional Lebesgue measure, where

$$\Phi_{N,\beta}(x) = \frac{1}{2}\|x\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N}\log\cosh(\beta\langle x,\xi_i\rangle), \qquad x \in \mathbb{R}^M, \tag{4.9}$$

depends on the random patterns. Here $\langle\cdot,\cdot\rangle$ stands for the inner product in $\mathbb{R}^M$.

We omit the proof as it follows by a straight-forward calculation similar to the ones given in [7, Lemma 2.2] or [15, Lemma 3.3].

Before turning to the proof of Theorem 2.1, we gather some estimates which will prove useful in the sequel. The first of these estimates is a bound on the operator norm of the random matrix arising from the patterns.

Lemma 4.5 ([6, Theorem 4.1])**.** *There exist a constant $K > 0$ and an $N_1 \in \mathbb{N}$ such that*

$$\mathbb{P}\big\{\big|\|\tfrac{1}{N}\xi^{\mathrm{T}}\xi\|_{\mathrm{Op}} - (1+\sqrt{\alpha})^2\big| \geq \sqrt{\alpha}\big\} \leq K\mathrm{e}^{-M/K} \tag{4.10}$$

for all $N \geq N_1$.

For later use, we define

$$\Omega_1(N) = \big\{\xi : \big|\|\tfrac{1}{N}\xi^{\mathrm{T}}\xi\|_{\mathrm{Op}} - (1+\sqrt{\alpha})^2\big| < \sqrt{\alpha}\big\}. \tag{4.11}$$

In particular, we know that for $N \geq N_1$, $\xi \in \Omega_1(N)$ and all $x, y \in \mathbb{R}^M$,

$$\left|\frac{1}{N}\sum_{i=1}^{N}\langle x,\xi_i\rangle\langle y,\xi_i\rangle - \langle x,y\rangle\right| \leq 4\sqrt{\alpha}\|x\|_2\|y\|_2. \tag{4.12}$$

We also need the following estimates to treat terms which involve products of components ξ_i^μ for four or six different values of μ. These are provided by the following lemma.

For $\delta > 0$ let

$$\Omega_2(N,\delta) = \Big(\bigcup_{\mu_1,\dots,\mu_4} \Big\{ \Big| \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4} \Big| > \delta\sqrt{\alpha} \Big\} \cup \bigcup_{\mu_1,\dots,\mu_6} \Big\{ \Big| \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4}\xi_i^{\mu_5}\xi_i^{\mu_6} \Big| > \delta\sqrt{\alpha} \Big\} \Big)^{\mathrm{c}}, \tag{4.13}$$

where each of the unions is taken over all sets of pairwise different indices in $\{1,\dots,M\}$.

Lemma 4.6. *For every $\delta > 0$, there exists an $N_2(\delta)$ such that for all $N \geq N_2(\delta)$*

$$\mathbb{P}\{\Omega_2(N,\delta)^{\mathrm{c}}\} \leq \exp\{-\delta^2 M/4\}. \tag{4.14}$$

Proof. Let

$$B_{N,\delta}(\mu_1,\dots,\mu_4) = \Big\{ \Big| \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4} \Big| > \delta\sqrt{\alpha} \Big\} \tag{4.15}$$

and

$$C_{N,\delta}(\mu_1,\dots,\mu_6) = \Big\{ \Big| \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4}\xi_i^{\mu_5}\xi_i^{\mu_6} \Big| > \delta\sqrt{\alpha} \Big\}. \tag{4.16}$$

For pairwise different indices $\mu_1,\dots,\mu_6 \in \{1,\dots,M\}$, Chebychev's inequality with $t = \delta\sqrt{\alpha}$ implies

$$\mathbb{P}(B_{N,\delta}(\mu_1,\dots,\mu_4)) \leq \exp\{-t\delta\sqrt{\alpha}N\}\exp\{Nt^2/2\} = \exp\{-\delta^2 M/2\}$$

and, similarly,

$$\mathbb{P}(C_{N,\delta}(\mu_1,\dots,\mu_6)) \leq \exp\{-\delta^2 M/2\}.$$

Therefore,

$$\mathbb{P}(\Omega_2(N,\delta)^{\mathrm{c}}) \leq \Big(\frac{1}{2}M(M-1) + \frac{1}{4!}M(M-1)(M-2)(M-3)\Big)\exp\{-\delta^2 M/2\}. \tag{4.17}$$

Choosing M large concludes the proof. $\square$

The next lemma provides a bound similar to (4.12) for terms involving the Gaussian η instead of $N^{-1/2}\xi^{\mathrm{T}}\xi$. Let

$$\Omega_3(N,R,\kappa) = \Big\{ \xi : \Big| \sum_{\mu<\nu} \eta_{\mu,\nu}(\xi) x_\mu x_\nu \Big| < \kappa R^2 \sqrt{M} \|x\|_2^2 \quad \forall x \in \mathbb{R}^M \Big\}. \tag{4.18}$$

Lemma 4.7.

$$\mathbb{P}\{\Omega_3(N,R,\kappa)^{\mathrm{c}}\} \leq 5^{2M}\exp\{-\kappa^2 R^4 M/16\}.$$

Proof. Let $x, y \in \mathbb{R}^M$. First note that $\sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu y_\nu$ can be viewed as the scalar product of η and the vector $(x_\mu y_\nu)_{\mu<\nu}$ and that $\|(x_\mu y_\nu)_{\mu<\nu}\|_2 \leq 2^{-1/2}\|x\|_2\|y\|_2$. By Chebychev's inequality,

$$\begin{aligned} \mathbb{P}\Big\{ \sum_{\mu<\nu} \eta_{\mu,\nu} x_\mu y_\nu \geq \kappa' \Big\} &\leq \exp\{-t\kappa'\}\exp\Big\{\frac{t^2}{2}\|(x_\mu y_\nu)_{\mu<\nu}\|_2^2\Big\} \\ &\leq \exp\{-t\kappa'\}\exp\Big\{\frac{t^2}{4}\|x\|_2^2\|y\|_2^2\Big\} \end{aligned} \tag{4.19}$$

for $t > 0$. Choosing $t = 2\kappa'/(\|x\|_2^2\|y\|_2^2)$,

$$\mathbb{P}\Big\{\sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu y_\nu \geq \kappa'\Big\} \leq \exp\Big\{-\frac{\kappa'^2}{\|x\|_2^2\|y\|_2^2}\Big\} \tag{4.20}$$

follows. To obtain a uniform bound, note that

$$\begin{aligned}\mathbb{P}\Big\{\exists x \in \mathbb{R}^M : \sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu x_\nu \geq \kappa'\|x\|_2^2\Big\} &= \mathbb{P}\Big\{\exists x \in B(0,1) : \sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu x_\nu \geq \kappa'\Big\} \\ &\leq \mathbb{P}\Big\{\exists x, y \in B(0,1) : \sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu y_\nu \geq \kappa'\Big\}.\end{aligned}$$

$B(0,1)$ being a (bounded) convex, balanced set in $\mathbb{R}^M$, there exists a subset $D \subset B(0,2)$ such that $B(0,1)$ is contained in the convex hull of D and D has at most 5^M elements (see for example [31, Lemma 10.2 in the Appendix]). Now, by our previous bound and the definition of the set D,

$$\begin{aligned}&\mathbb{P}\Big\{\exists x \in \mathbb{R}^M : \sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu x_\nu \geq \kappa'\|x\|_2^2\Big\} \\ &\quad\leq \mathbb{P}\Big\{\exists x, y \in D : \sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu y_\nu \geq \kappa'\Big\} \leq 5^{2M}\sup_{x,y\in D}\mathbb{P}\Big\{\sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu y_\nu \geq \kappa'\Big\} \\ &\quad\leq 5^{2M}\sup_{x,y\in D}\exp\Big\{-\frac{\kappa'^2}{\|x\|_2^2\|y\|_2^2}\Big\} \leq 5^{2M}\exp\Big\{-\frac{\kappa'^2}{16}\Big\}.\end{aligned} \tag{4.21}$$

Choosing $\kappa' = \kappa R^2\sqrt{M}$ with $\kappa > 0$ concludes the proof. □

With these preparations we are able to prove Theorem 2.1.

Proof of Theorem 2.1. By (4.2), Theorem 2.1 follows, once we have shown that, under the conditions of the theorem,

$$\left|\int_{\mathbb{R}^M} f(x)Q_N(dx) - \int_{\mathbb{R}^M} f(x)\overline{Q}_M(dx)\right| \leq \varepsilon_N(K_f + \|f\|_\infty) \tag{4.22}$$

holds for all $\xi \in \overline{\Omega}(N)$ and all $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$. By Lemma 4.2, we may replace Q_N by its Hubbard–Stratonovich transform.

So let $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$. We need to investigate

$$\frac{\int f(x)\exp\{-N\Phi(x/N^{1/4})\}\,dx}{\int \exp\{-N\Phi(x/N^{1/4})\}\,dx}, \tag{4.23}$$

where

$$\Phi(y) = \Phi_{N,1}(y) = \frac{1}{2}\|y\|_2^2 - \frac{1}{N}\sum_{i=1}^N \log\cosh(\langle y, \xi_i\rangle), \qquad y \in \mathbb{R}^M. \tag{4.24}$$

Consider the nominator first as the denominator is a special case of the nominator. The main contribution to the integral arises from the *inner region* $B(0, RM^{1/4})$ and we shall choose a suitable $R > 0$ later on. In the inner region as well as in the *intermediate region* $B(0, rN^{1/4}) \setminus B(0, RM^{1/4})$ with $r > 0$ to be chosen later, we investigate the behaviour of the integral in the nominator with the help of a Taylor expansion of Φ. The *outer region* $B(0, rN^{1/4})^{\mathrm{c}}$ is treated separately.

Taylor expansion. Calculating the Taylor expansion of Φ around zero, we see that there exists a $\theta \in (0,1)$ such that

$$\Phi(x) = \frac{1}{2}\|x\|_2^2 - \frac{1}{N}\sum_{i=1}^N \left[\frac{1}{2}\langle x, \xi_i\rangle^2 - \frac{1}{12}\langle x, \xi_i\rangle^4\right] + R_N(x,\xi), \tag{4.25}$$

with

$$R_N(x,\xi) = -\frac{1}{N}\sum_{i=1}^N \frac{1}{15}\, h(\langle \theta x, \xi_i\rangle)\langle x, \xi_i\rangle^5, \tag{4.26}$$

where

$$h(t) = \frac{\tanh(t)}{\cosh^4(t)}[2 - \sinh^2(t)], \qquad t \in \mathbb{R}. \tag{4.27}$$

Regrouping the terms of the Taylor expansion of Φ, we find that

$$\begin{aligned} &-N\Phi(x/N^{1/4}) \\ &\quad = -\frac{1}{12}\|x\|_4^4 - \frac{1}{4}\sum\nolimits^*_{\mu_1,\mu_2} x_{\mu_1}^2 x_{\mu_2}^2 + \frac{1}{2}\sum\nolimits^*_{\mu_1,\mu_2} x_{\mu_1}x_{\mu_2}\frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2} \\ &\qquad - \frac{1}{3}\sum\nolimits^*_{\mu_1,\mu_2} x_{\mu_1}x_{\mu_2}^3 \frac{1}{N}\sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2} - \frac{1}{2}\sum\nolimits^*_{\mu_1,\mu_2,\mu_3} x_{\mu_1}x_{\mu_2}x_{\mu_3}^2\frac{1}{N}\sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2} \\ &\qquad - \frac{1}{12}\sum\nolimits^*_{\mu_1,\mu_2,\mu_3,\mu_4} x_{\mu_1}x_{\mu_2}x_{\mu_3}x_{\mu_4}\frac{1}{N}\sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4} + \mathcal{O}(N|R_N(x/N^{1/4},\xi)|), \end{aligned} \tag{4.28}$$

where $\|x\|_4^4 = \sum_{\mu=1}^M x_\mu^4$. Here and in the sequel, we use the notation $\sum^*_{\mu_1,\dots,\mu_k}$ for summation over all k-tuples $(\mu_1,\dots,\mu_k) \in \{1,\dots,M\}$ with pairwise disjoint components.

Let us consider the different ξ-dependent terms. By the strong Gaussian approximation Corollary 3.3, there exist a constant $N_0 \in \mathbb{N}$ and an $M(M-1)/2$-dimensional Gaussian vector η with mean zero and covariance matrix being the identity matrix such that

$$\Omega_0(N,\delta_N) = \left\{\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^N (\xi_i^\mu \xi_i^\nu)_{\mu<\nu} - \eta\right\|_2 < \delta_N\right\} \tag{4.29}$$

with

$$\delta_N = KM^7/\sqrt{N} \tag{4.30}$$

for some $K > 0$ satisfies

$$\mathbb{P}(\Omega_0(N,\delta_N)^{\mathrm{c}}) \le \exp\{-KM/(2c_2)\} \tag{4.31}$$

for all $N \ge N_0$ and

$$\left|\frac{1}{2}\sum\nolimits^*_{\mu_1,\mu_2} x_{\mu_1}x_{\mu_2}\frac{1}{\sqrt{N}}\sum_{i=1}^N \xi_i^{\mu_1}\xi_i^{\mu_2} - \sum_{\mu_1<\mu_2}\eta_{\mu_1,\mu_2}x_{\mu_1}x_{\mu_2}\right| \le \delta_M \|(x_{\mu_1}x_{\mu_2})_{\mu_1<\mu_2}\|_2 \le \frac{\delta_M}{\sqrt{2}}\|x\|_2^2 \tag{4.32}$$

for all $\xi \in \Omega_0(N,\delta_N)$.

The other ξ-dependent terms become small due to the law of large numbers. For $N \geq N_1$ and $\xi \in \Omega_1(N)$, the bound (4.12) on the random matrix yields

$$\left| \frac{1}{3} \sum_{\mu_1,\mu_2}^{*} x_{\mu_1} x_{\mu_2}^3 \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu_1} \xi_i^{\mu_2} \right| \leq \frac{4}{3} \sqrt{\alpha} \|x\|_2^4 \tag{4.33}$$

as well as

$$\begin{aligned} &\left| \frac{1}{2} \sum_{\mu_1,\mu_2,\mu_3}^{*} x_{\mu_1} x_{\mu_2} x_{\mu_3}^2 \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu_1} \xi_i^{\mu_2} \right| \\ &\quad = \left| \frac{1}{2} \sum_{\mu_1,\mu_2}^{*} x_{\mu_1} x_{\mu_2} \|x\|_2^2 \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu_1} \xi_i^{\mu_2} - \sum_{\mu_1,\mu_2}^{*} x_{\mu_1} x_{\mu_2}^3 \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu_1} \xi_i^{\mu_2} \right| \leq 6\sqrt{\alpha} \|x\|_2^4. \end{aligned} \tag{4.34}$$

Furthermore, for $N \geq N_2(\delta)$ and $\xi \in \Omega_2(N,\delta)$, by the definition of $\Omega_2(N,\delta)$,

$$\begin{aligned} &\left| \frac{1}{12} \sum_{\mu_1,\mu_2,\mu_3,\mu_4}^{*} x_{\mu_1} x_{\mu_2} x_{\mu_3} x_{\mu_4} \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu_1} \xi_i^{\mu_2} \xi_i^{\mu_3} \xi_i^{\mu_4} \right| \\ &\quad \leq \frac{\delta\sqrt{\alpha}}{12} \sum_{\mu_1,\mu_2,\mu_3,\mu_4} |x_{\mu_1} x_{\mu_2} x_{\mu_3} x_{\mu_4}| \leq \frac{\delta\sqrt{\alpha} M^2}{12} \|x\|_2^4 = \frac{\delta}{12} \left(\frac{M^5}{N} \right)^{1/2} \|x\|_2^4. \end{aligned} \tag{4.35}$$

It remains to consider the remainder of the Taylor expansion. Now, $|h(t)| \leq 2|t|$ and $0 < \theta < 1$ together with Schwarz' inequality imply that

$$|R_N(y,\xi)| \leq \frac{2}{15N} \sum_{i=1}^{N} \langle y, \xi_i \rangle^6 \leq \frac{2}{15} \sum_{\mu_1,\dots,\mu_6} |y_{\mu_1} \dots y_{\mu_6}| \left| \frac{1}{N} \sum_{i=1}^{N} \xi_i^{\mu_1} \dots \xi_i^{\mu_6} \right|. \tag{4.36}$$

The right-hand side is bounded above by a combinatorial factor times the sum of terms similar to the ones treated above (with two, four or six different ξ_i^{μ}) plus the term arising from $\mu_1 = \dots = \mu_6$. This yields

$$|R_N(y,\xi)| \leq C \left[\sqrt{\alpha} \|y\|_2^6 + \delta \left(\frac{M^5}{N} \right)^{1/2} \|y\|_2^6 + \delta \left(\frac{M^7}{N} \right)^{1/2} \|y\|_2^6 + \|y\|_6^6 \right] \tag{4.37}$$

for $N \geq \max\{N_1, N_2(\delta)\}$ and $\xi \in \Omega_1(N) \cap \Omega_2(N,\delta)$, so that

$$N |R_N(x/N^{1/4}, \xi)| \leq \frac{C}{\sqrt{N}} \left[\sqrt{\alpha} \|x\|_2^6 + 2\delta \left(\frac{M^7}{N} \right)^{1/2} \|x\|_2^6 + \|x\|_6^6 \right]. \tag{4.38}$$

From now on, we shall always assume that $N \geq \max\{N_0, N_1, N_2(\delta)\}$ and that $\xi \in \Omega_0(N, \delta_N) \cap \Omega_1(N) \cap \Omega_2(N,\delta)$. We have already seen that this implies that $-N\Phi(x/N^{1/4})$ differs from

$$\Psi(x) = -\frac{1}{12} \|x\|_4^4 - \frac{1}{2} \sum_{\mu<\nu} x_\mu^2 x_\nu^2 + \sum_{\mu<\nu} \eta_{\mu,\nu} x_\mu x_\nu \tag{4.39}$$

by at most a constant times

$$g_N(x) = \delta_M \|x\|_2^2 + \left[\sqrt{\alpha} + \delta \left(\frac{M^5}{N} \right)^{1/2} \right] \|x\|_2^4 + \frac{1}{\sqrt{N}} \left[\sqrt{\alpha} + \delta \left(\frac{M^7}{N} \right)^{1/2} \right] \|x\|_2^6 + \frac{\|x\|_6^6}{\sqrt{N}}. \tag{4.40}$$

The inner region. For $\|x\|_2 \le RM^{1/4}$, the main contribution to $g_N(x)$ arises from the first summand. Therefore, we shall use the estimate

$$g_N(x) \le h_N(\delta, R) = \Big(\frac{M^{15}}{N}\Big)^{1/2}(K+\delta)R^6 \to 0, \tag{4.41}$$

provided $M^{15}/N \to 0$. (Recall that $\delta_M = KM^7/\sqrt{N}$.) Therefore, the estimate for the inner region is immediate: For $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$,

$$\begin{aligned}\int_{B(0,RM^{1/4})} & f(x)\exp\{-N\Phi(x/N^{1/4})\}\,dx \\ &= \exp\{\mathcal{O}(h_N(\delta,R)\}\int_{B(0,RM^{1/4})} f(x)\exp\{\Psi(x)\}\,dx. \end{aligned}\tag{4.42}$$

The intermediate region. For $RM^{1/4} \le \|x\|_2 \le rN^{1/4}$,

$$g_N(x) \le \delta_M\|x\|_2^2 + \left[(1+r^2)\sqrt{\alpha} + (1+r^2)\delta\Big(\frac{M^7}{N}\Big)^{1/2} + r^2\right]\|x\|_2^4, \tag{4.43}$$

which implies, that there exists an $N_3(\delta, r) \in \mathbb{N}$ such that

$$g_N(x) \le \delta_M\|x\|_2^2 + 2r^2\|x\|_2^4 \tag{4.44}$$

for all $N \ge N_3(\delta, r)$, provided provided $M^7/N \to 0$.

Assuming $N \ge \max\{N_0, N_1, N_2(\delta), N_3(\delta\}$ and $\xi \in \Omega_0(N,\delta_N)\cap\Omega_1(N)\cap\Omega_2(N,\delta)\cap\Omega_3(N,R,\kappa)$ from now on, our previous estimates together with the definition of $\Omega_3(N,R,\kappa)$ yield

$$\begin{aligned} -N&\Phi(x/N^{1/4}) \\ &\le \Psi(x) + \mathcal{O}(g_N(x)) \\ &\le -\frac{1}{12}\|x\|_4^4 - \frac{1}{2}\sum_{\mu<\nu} x_\mu^2x_\nu^2 + \sum_{\mu<\nu}\eta_{\mu,\nu}x_\mu x_\nu + \mathcal{O}(\delta_M\|x\|_2^2 + 2r^2\|x\|_2^4) \\ &\le -\frac{1}{12}\|x\|_4^4 - \frac{1}{12}\big[\|x\|_2^4 - \|x\|_4^4\big] + \kappa R^2\sqrt{M}\|x\|_2^2 + \mathcal{O}(\delta_M\|x\|_2^2 + 2r^2\|x\|_2^4). \end{aligned}\tag{4.45}$$

For $\|x\|_2 \ge RM^{1/4}$, $\|x\|_2^4 \ge R^2\sqrt{M}\|x\|_2^2$ is trivial. By choosing r and $0 < \kappa \le 1/48$ small enough, we see that there exists an $N_4(R,K) \in \mathbb{N}$ such that δ_M becomes so small that

$$-N\Phi(x/N^{1/4}) \le -\frac{R^2}{24}\sqrt{M}\|x\|_2^2 \tag{4.46}$$

holds for all $N \ge N_4(R,K)$ and all x from the intermediate region. Therefore, for all $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$ and N and ξ chosen as before,

$$\begin{aligned}\Big|\int_{\{RM^{1/4}\le\|x\|_2\le rN^{1/4}\}} & f(x)\exp\{-N\Phi(x/N^{1/4})\}\,dx\Big| \\ &\le \|f\|_\infty \int_{\{\|x\|_2\ge RM^{1/4}\}} \exp\Big\{-\frac{R^2}{24}\sqrt{M}\|x\|_2^2\Big\}\,dx \\ &\le \|f\|_\infty \exp\{-R^4M/48\}\int_{\mathbb{R}^M}\exp\Big\{-\frac{R^2}{48}\sqrt{M}\|x\|_2^2\Big\}\,dx \\ &= \|f\|_\infty\exp\{-R^4M/48\}\Big(\frac{48\pi}{R^2\sqrt{M}}\Big)^{M/2}. \end{aligned}\tag{4.47}$$

This bound will allow us to deduce that the integral over the intermediate region is negligible.

The outer region. The investigation of the outer region consists of two parts. First, we show that there exists an $r_0 > 0$ such that the integral over $B(0, r_0 N^{1/4})^{\mathrm{c}}$ is negligible and then, in a second step, we show that this r_0 can be replaced by an arbitrarily small $r > 0$.

For convenience, we denote by $f_{\mathrm{CW}}(\beta)$ the free energy in the Curie–Weiss model at temperature $1/\beta$, i. e.,

$$f_{\mathrm{CW}}(\beta) = -\frac{\beta}{2} z(\beta)^2 + \log\cosh(\beta z(\beta)). \tag{4.48}$$

Then,

$$\log\cosh x \le \frac{1}{4\beta} x^2 + \max_{t\in\mathbb{R}} \left\{ -\frac{1}{4\beta} t^2 + \log\cosh t \right\} = \frac{1}{4\beta} x^2 + f_{\mathrm{CW}}(2\beta), \tag{4.49}$$

which implies in particular that

$$\begin{aligned} -N\Phi(x/N^{1/4}) &= -\frac{\sqrt{N}}{2} \|x\|_2^2 + \sum_{i=1}^{N} \log\cosh\langle x/N^{1/4}, \xi_i\rangle \\ &\le -\frac{\sqrt{N}}{2} \|x\|_2^2 + \frac{1}{4\sqrt{N}} \sum_{i=1}^{N} \langle x, \xi_i\rangle^2 + N f_{\mathrm{CW}}(2). \end{aligned} \tag{4.50}$$

Estimating the sum with the help of the bound (4.12) on the random matrix $\frac{1}{N}\xi^{\mathrm{T}}\xi$, we see that there exist $r_0 > 0$ and $N_5 \ge N_1$ such that

$$-N\Phi(x/N^{1/4}) \le -\frac{\sqrt{N}}{6} \|x\|_2^2 \tag{4.51}$$

holds for all x satisfying $\|x\|_2 \ge r_0 N^{1/4}$, all $N \ge N_5$ and all $\xi \in \Omega_1(N)$.

Let now $rN^{1/4} \le \|x\|_2 \le r_0 N^{1/4}$ with an arbitrary $r \in (0, r_0)$. First note that

$$\begin{aligned} \Phi(x/N^{1/4}) \;\; &\ge \;\; \mathbb{E}\left\{ \frac{1}{2} \langle x/N^{1/4}, \xi_1\rangle^2 - \log\cosh\langle x/N^{1/4}, \xi_1\rangle \right\} \\ &\quad - \sup_{\|y\|_2 \le r_0} \left| \frac{1}{N} \sum_{i=1}^{N} \log\cosh\langle x/N^{1/4}, \xi_i\rangle - \mathbb{E}\log\cosh\langle x/N^{1/4}, \xi_1\rangle \right|. \end{aligned} \tag{4.52}$$

The first summand on the right-hand side is bounded below by

$$c_{r,r_0} = \inf_{y\,:\, r \le \|y\|_2 \le r_0} \mathbb{E}\,\phi(\langle y, \xi_1\rangle), \tag{4.53}$$

where

$$\phi(t) = t^2/2 - \log\cosh t, \qquad t \in \mathbb{R}, \tag{4.54}$$

attains its unique minimum at $t = 0$. The fact that $\langle y, \xi_1\rangle$ is a (finite) Rademacher average (see [24, Chapter I.4], for instance), implies that

$$\mathbb{P}(|\langle y, \xi_1\rangle| \ge \tfrac{1}{8}\|y\|_2) > 1/3 \tag{4.55}$$

(cf. [17, Lemma 4.3]), so that

$$c_{r,r_0} = \inf_{y\,:\, r \le \|y\|_2 \le r_0} \mathbb{E}\,\phi(\langle y, \xi_1\rangle) > 0, \tag{4.56}$$

because there is a set of positive $\mathbb{P}$-measure, on which ϕ is bounded away from its unique minimum at zero.

The second summand on the right-hand side of (4.52) becomes small due to so-called self-averaging. Inspection of the proof of [17, Lemma 4.2] shows that not only

$$\lim_{N\to\infty} \sup_{\|x\|_2 \le r_0} \left| \frac{1}{N} \sum_{i=1}^N f(\langle x, \xi_i \rangle) - \mathbb{E} f(\langle x, \xi_1 \rangle) \right| = 0 \tag{4.57}$$

holds $\mathbb{P}$-almost surely for Lipschitz continuous f, but we obtained also bounds valid for large but fixed N:

Lemma 4.8 ([17, Lemma 4.2]). *There exist a constant $c > 0$ and an $N_6 \ge N_1$ such that for all $\varepsilon > 0$ and all $N \ge \max\{N_6, 2/\varepsilon^2\}$*

$$\begin{aligned} \mathbb{P}\bigg\{ \sup_{\|y\|_2 \le r_0} \left| \frac{1}{N} \sum_{i=1}^N \log\cosh\langle y, \xi_i \rangle - \mathbb{E} \log\cosh\langle y, \xi_1 \rangle \right| \ge (3 + 2r_0)\varepsilon \bigg\} \\ \le 2\exp\{M(\log(r_0/\varepsilon) + c)\} \exp\{-N\varepsilon^2/8\} + \mathbb{P}(\Omega_1(N)^{\mathrm{c}}). \end{aligned}$$

With

$$\varepsilon = \frac{c_{r,r_0}}{2(3 + 2r_0)}$$

and

$$\Omega_4(N, r, r_0) = \left\{ \xi : \sup_{\|y\|_2 \le r_0} \left| \frac{1}{N} \sum_{i=1}^N \log\cosh\langle y, \xi_i \rangle - \mathbb{E} \log\cosh\langle y, \xi_1 \rangle \right| \le \frac{c_{r,r_0}}{2} \right\} \tag{4.58}$$

we obtain the following corollary.

Corollary 4.9. *There exist a constant $K(r, r_0) > 0$ and an $N_7(r, r_0) \in \mathbb{N}$ such that for all $N \ge N_7(r, r_0)$*

$$\mathbb{P}(\Omega_4(N, r, r_0)^{\mathrm{c}}) \le \exp\{-K(r, r_0)N\} + \mathbb{P}(\Omega_1(N)^{\mathrm{c}}).$$

Now, by our estimates on the two summands on the right-hand side of (4.52), we find

$$-N\Phi(x/N^{1/4}) \le -Nc_{r,r_0}/2 \tag{4.59}$$

for all x such that $rN^{1/4} \le \|x\|_2 \le r_0 N^{1/4}$, all $N \ge N_7(r, r_0)$ and all $\xi \in \Omega_4(N, r, r_0)$.

Gathering our estimates on the outer region yields

$$\begin{aligned} &\left| \int_{\{\|x\|_2 \ge rN^{1/4}\}} f(x) \exp\{-N\Phi(x/N^{1/4})\}\, dx \right| \\ &\quad \le \int_{\{\|x\|_2 \ge r_0 N^{1/4}\}} \|f\|_\infty \exp\left\{ -\frac{\sqrt{N}}{6} \|x\|_2^2 \right\} dx \\ &\qquad + \int_{\{rN^{1/4} \le \|x\|_2 \le r_0 N^{1/4}\}} \|f\|_\infty \exp\{-Nc_{r,r_0}/2\}\, dx \\ &\quad \le \|f\|_\infty \big[\exp\{-Nr_0^2/12\} + \exp\{-Nc_{r,r_0}/4\} \big] \end{aligned} \tag{4.60}$$

for all $N \ge N_8(r, r_0)$ for some $N_8(r, r_0) \in \mathbb{N}$.

Completing the proof. From now on we shall always assume that

$$\begin{aligned} \xi \in \overline{\Omega}(N) &= \overline{\Omega}(N, R, r, r_0, \delta, \kappa) \\ &= \Omega_0(N, \delta_N) \cap \Omega_1(N) \cap \Omega_2(N, \delta) \cap \Omega_3(N, R, \kappa) \cap \Omega_4(N, r, r_0) \end{aligned} \tag{4.61}$$

and that

$$N \geq \max\bigl\{N_0, N_1, N_2(\delta), N_3(\delta, r), N_4(R, K), N_5, N_6, N_7(r, r_0), N_8(r, r_0)\bigr\}. \tag{4.62}$$

Note that there exists a constant $L > 0$ such that

$$\mathbb{P}(\overline{\Omega}(N)^{\mathrm{c}}) \leq \exp\{-M/L\}, \tag{4.63}$$

provided R is chosen large compared to κ and M is large enough, cf. Lemma 4.7. Naturally, L depends on our choice of R, r, r_0, δ and κ.

Let $f \in \mathrm{BL}(\mathbb{R}^M, \mathbb{R})$ be arbitrary. We have already shown that

$$\begin{aligned} &\int f(x) \exp\{-N\Phi(x/N^{1/4})\}\, dx \\ &\quad = \exp\{\mathcal{O}(h_N(\delta, R)\} \int_{B(0,RM^{1/4})} f(x) \exp\{\Psi(x)\}\, dx \\ &\qquad + \mathcal{O}\left(\|f\|_\infty \exp\{-R^4 M/48\} \left(\frac{48\pi}{R^2\sqrt{M}} \right)^{M/2} \right) \\ &\qquad + \mathcal{O}\Bigl(\|f\|_\infty \bigl[\exp\{-Nr_0^2/12\} + \exp\{-Nc_{r,r_0}/4\}\bigr] \Bigr) \end{aligned} \tag{4.64}$$

with $h_N(\delta, R) \to 0$. Next, we want to replace the integral

$$\int_{B(0,RM^{1/4})} f(x) \exp\{\Psi(x)\}\, dx \tag{4.65}$$

by the integral over $\mathbb{R}^M$. First note, that (4.45) already provides an upper bound on $\Psi(x)$, valid for all x satisfying $\|x\|_2 \geq RM^{1/4}$:

$$\Psi(x) \leq -\frac{1}{12}\|x\|_4^4 - \frac{1}{12}\bigl[\|x\|_2^4 - \|x\|_4^4\bigr] + \kappa R^2\sqrt{M}\|x\|_2^2 \leq -\frac{R^2}{24}\sqrt{M}\|x\|_2^2. \tag{4.66}$$

As an immediate consequence,

$$\begin{aligned} &\left| \int_{\{\|x\|_2 \geq RM^{1/4}\}} f(x) \exp\{\Psi(x)\}\, dx \right| \\ &\qquad \leq \|f\|_\infty \int_{\{\|x\|_2 \geq RM^{1/4}\}} \exp\Bigl\{-\frac{R^2}{24}\sqrt{M}\|x\|_2^2\Bigr\}\, dx \\ &\qquad \leq \|f\|_\infty \exp\{-R^4 M/48\} \left(\frac{48\pi}{R^2\sqrt{M}} \right)^{M/2}, \end{aligned} \tag{4.67}$$

which implies by (4.64) that

$$\begin{aligned}\int f(x)\exp\{-N\Phi(x/N^{1/4})\}\,dx \\ &= \exp\{\mathcal{O}(h_N(\delta,R)\}\int_{\mathbb{R}^M} f(x)\exp\{\Psi(x)\}\,dx \\ &\quad + \mathcal{O}\biggl(\|f\|_\infty \exp\{-R^4M/48\}\Bigl(\frac{48\pi}{R^2\sqrt{M}}\Bigr)^{M/2}\biggr) \\ &\quad + \mathcal{O}\Bigl(\|f\|_\infty\bigl[\exp\{-Nr_0^2/12\}+\exp\{-Nc_{r,r_0}/4\}\bigr]\Bigr). \qquad (4.68)\end{aligned}$$

In order to compare

$$\frac{\int_{\mathbb{R}^M} f(x)\exp\{-N\Phi(x/N^{1/4})\}}{\int_{\mathbb{R}^M}\exp\{-N\Phi(x/N^{1/4})\}\,dx} \qquad \text{to} \qquad \frac{\int_{\mathbb{R}^M} f(x)\exp\{\Psi(x)\}}{\int_{\mathbb{R}^M}\exp\{\Psi(x)\}\,dx},$$

we need a lower bound on $\int_{\mathbb{R}^M}\exp\{\Psi(x)\}\,dx$. To obtain a lower bound on Ψ first, we proceed as in (4.45):

$$\Psi(x) \ge -\frac{1}{12}\|x\|_4^4 - \frac{1}{4}\bigl[\|x\|_2^4 - \|x\|_4^4\bigr] - \kappa R^2\sqrt{M}\|x\|_2^2 \ge -\frac{1}{4}\|x\|_2^4 - \kappa R^2\sqrt{M}\|x\|_2^2. \qquad (4.69)$$

For $\|x\|_2 \le RM^{1/4}$,

$$\Psi(x) \ge -\frac{R^2}{3}\sqrt{M}\|x\|_2^2 \qquad (4.70)$$

follows. (Recall, that $\kappa \le 1/48$.) Now,

$$\int_{\mathbb{R}^M}\exp\{\Psi(x)\}\,dx \ge \int_{B(0,RM^{1/4})}\exp\Bigl\{-\frac{R^2}{3}\sqrt{M}\|x\|_2^2\Bigr\}\,dx \ge \frac{1}{2}\Bigl(\frac{3\pi}{R^2\sqrt{M}}\Bigr)^{M/2} \qquad (4.71)$$

for M large enough, i. e., $N \ge N_9(R)$ for some $N_9(R) \in \mathbb{N}$.

With these preparations, it is easy to see that

$$\begin{aligned}&\left|\frac{\int_{\mathbb{R}^M} f(x)\exp\{-N\Phi(x/N^{1/4})\}}{\int_{\mathbb{R}^M}\exp\{-N\Phi(x/N^{1/4})\}\,dx} - \frac{\int_{\mathbb{R}^M} f(x)\exp\{\Psi(x)\}}{\int_{\mathbb{R}^M}\exp\{\Psi(x)\}\,dx}\right| \\ &\qquad \le \|f\|_\infty \frac{\mathcal{O}}{\int_{\mathbb{R}^M}\exp\{\Psi(x)\}\,dx + \mathcal{O}}, \qquad (4.72)\end{aligned}$$

where we use $\mathcal{O}$ as an abbreviation for

$$\begin{aligned}&\mathcal{O}\Bigl(h_N(\delta,R)\int_{\mathbb{R}^M}\exp\{\Psi(x)\}\,dx\Bigr) + \mathcal{O}\biggl(\exp\{-R^4M/48\}\Bigl(\frac{48\pi}{R^2\sqrt{M}}\Bigr)^{M/2}\biggr) \\ &\quad + \mathcal{O}\Bigl(\exp\{-Nr_0^2/12\} + \exp\{-Nc_{r,r_0}/4\}\Bigr).\end{aligned}$$

By our lower bound on $\int_{\mathbb{R}^M} \exp\{\Psi(x)\}\,dx$, we see that R can be chosen so large that there exist a constant $\overline{K} > 0$ and an $N_{10}(R, r, r_0, \delta, \kappa) \in \mathbb{N}$ such that

$$\begin{aligned} &\left| \frac{\int_{\mathbb{R}^M} f(x) \exp\{-N\Phi(x/N^{1/4})\}}{\int_{\mathbb{R}^M} \exp\{-N\Phi(x/N^{1/4})\}\,dx} - \frac{\int_{\mathbb{R}^M} f(x) \exp\{\Psi(x)\}}{\int_{\mathbb{R}^M} \exp\{\Psi(x)\}\,dx} \right| \\ &\quad \leq \|f\|_\infty \big[\mathcal{O}(h_N(\delta, R)) + \mathcal{O}(\exp\{-R^4 M/\overline{K}\})\big] \end{aligned} \tag{4.73}$$

for all $N \geq N_{10}(R, r, r_0, \delta, \kappa)$. Now the theorem follows from Lemma 4.2 and Lemma 4.4 with $\overline{\Omega}(N)$ as defined in the beginning of this subsection and

$$\begin{aligned} N \geq \overline{N} &= \overline{N}(R, r, r_0, \delta, \kappa) \\ &= \max\big\{N_0, N_1, N_2(\delta), N_3(\delta, r), N_4(R, K), N_5, N_6, N_7(r, r_0), N_8(r, r_0), \\ &\qquad N_9(R), N_{10}(R, r, r_0, \delta, \kappa)\big\}. \end{aligned} \tag{4.74}$$

$\square$

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