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On the regularization of the ill–posed logarithmic kernel integral equation of the first kind

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ABSTRACT. The logarithmic kernel integral equation of the first kind is investigated as improperly posed problem considering its right–hand side as observed quantity in a suitable space with a weaker norm. The improperly posed problem is decomposed into a well–posed one, extensively studied in the literature (cf. e.g. [11], [13], [14]), and an ill–posed imbedding problem. For the ill–posed part a modified truncated singular value decomposition regularization method is proposed that allows an easily performable a–posteriori parameter choice. The whole problem is then solved by combining the regularization method with a numerical procedure from [13] for the well–posed part. Finally, an error estimate is given revealing the influence of the observation error on the approximation error of the numerical procedure. For a specification of the discretization parameter as a known function of the noise level only, the optimal convergence order is achieved.

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1. Introduction

The logarithmic kernel integral equation, called Symm's integral equation

$$\frac{1}{\pi}\int_\Gamma \log\frac{1}{|p-q|}v(q)ds_q = g(p)\,,\ p\in\Gamma\,,$$

and also other equations of the first kind

$$Bu = f \tag{1.1}$$

with smoothing linear operators B, have been examined extensively during the last years. In suitable spaces of smooth functions those problems are well–posed in Hadamard's sense: Existence, uniqueness and continuous dependence on the data f can be shown (cf. e.g. [14] for Symm's equation). In [4], [11], [13] and references given there, approximation methods are presented and an error analysis is carried out.

However, considering technical applications those methods are not usable immediately. In that case the data are not given exactly as a smooth function but gained by unexact observation or measurement. Interpolating a smooth function from given noisy data and inserting it as right–hand side makes little sense because nothing is known about stability or error analysis. The problem is ill–posed in Hadamard's sense.

The approach to such improperly posed problems consists in the development of regularization procedures. Examples are discretization procedures where the regularization is achieved by a proper choice of the discretization parameter relative to the observation error (cf. e.g. [12] for the general principle and [8] and papers cited there for Symm's equation).

In this paper however another approach is chosen. It consists in decomposing the problem

$$Au = g,\ A = LB\,,$$

into a (possibly nonlinear) well–posed part (1.1) and an ill–posed part

$$Lf = g\,,$$

L being a linear frequently compact mapping into the observation space. Examples are the approaches in [5] concerning the parameter determination in partial differential equations where the parameter–to–observation operator was decomposed into the parameter–to–solution mapping and the solution–to–observation mapping, and in [1] concerning a nonlinear Abel's integral equation of the first kind where the operator was decomposed into an operator of the second kind and the integration mapping.

In problems considered here this decomposition approach is advantageous as the well–posed part is very well–known and this knowledge can be used treating the problem as a whole.

In this paper that program is performed for the example of Symm's equation on a closed smooth plane curve. (The case of the open arc is completely analogous.) The well–posed part is considered on the Sobolev scale $H^p(\mathbb{T})$ where $\mathbb{T}$ is the one–dimensional torus $\mathbb{R}/\mathbb{Z}$. The papers [14] resp. [13] are taken as basic literature for existence, uniqueness, stability resp. approximation method and error analysis. The facts taken from those papers are summarized in Section 2. In Section 3 the abstract decomposition method is described. Section 4 is dedicated to the regularization of the imbedding

$$H^p(\mathbb{T}) \to L_2(\mathbb{T}).$$

A seemingly new modification of the truncated singular value decomposition method is carried out allowing an a–posteriori parameter choice by a Morozov's like principle.

Finally in Section 5 an approximation method for the solution of the ill–posed Symm's equation is given representing a combination of the collocation method in [13] and the regularization of the imbedding operator. Instead of the procedure in [13] an arbitrary other procedure could be taken, provided its error analysis is available in Sobolev spaces $H^p(\mathbb{T})$. The regularization can be interpreted as a data smoothing process. For the entire problem an error estimate is given reflecting the effect of the measurement error on the approximation error. Because of unknown constants the value of the error estimate mainly consists in its asymptotic properties. With a suitably chosen discretization parameter the optimal convergence rate is achieved.

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2. Preliminaries

In this Section Symm's integral equation is considered as a well–posed problem in the Sobolev scale $H^t(\mathbb{T})$. Here results from the literature [13], [14] are quoted. The useful formulation over $\mathbb{T} = I\!R/\mathbb{Z}$, the one–dimensional torus, is taken from [11], the definition of Sobolev spaces from [9].

Symm's integral equation

$$\frac{1}{\pi}\int_\Gamma \log\frac{1}{|p-q|}v(q) = g(p),\ p \in \Gamma,$$

Γ a plane smooth closed curve, can be transformated by a C^∞–parametrization γ of Γ with a non–vanishing Jacobian into an operator equation

$$Bu = f \text{ on } \mathbb{T}, \tag{2.1}$$

where

$$(Bu)(x) = -2\int_{\mathbb{T}} \log|\gamma(x) - \gamma(y)|u(y)dy,\ x \in \mathbb{T}$$

holds (cf. [11]).

Now, let us recall from [9] the definition of $H^p(\mathbb{T})$, $p \in I\!R$. We summarize it as follows.

Consider the space $L_2(\mathbb{T})$ with the orthonormal basis

$$\varphi_l(x) = e^{2\pi i l x},\ x \in \mathbb{T},\ l \in \mathbb{Z}$$

and let for $l \in \mathbb{Z}$

$$[l] = \begin{cases} |l|, & l \neq 0 \\ 1, & l = 0. \end{cases}$$

Then, let for $p \in I\!R$

$$H^p(\mathbb{T}) = \left\{ u, \sum_{l \in \mathbb{Z}} [l]^{2p} |(u, \varphi_l)|^2 < \infty \right\}$$

be the Hilbert space with scalar product

$$(u, v)_p = \sum_{l \in \mathbb{Z}} [l]^{2p} (u, \varphi_l) \overline{(v, \varphi_l)}$$

where for $p \geq 0$

$$u, v \in L_2(\mathbb{T}), \; (u, \varphi_l) = (u, \varphi_l)_{L_2}$$

and for $p < 0$

$$u, v \quad \text{are bounded linear functionals on} \quad H^{-p}(\mathbb{T}), \; (u, \varphi_l) = u(\varphi_l).$$

For $k \in \mathbb{N}$ let $W_2^k(\mathbb{T})$ be the Sobolev space (in the usual sense) of functions with square integrable derivatives up to the order k.

From [14] we quote the following Lemma 2.1 and Theorem 2.1.

Lemma 2.1. *(Cf. [14, Propos. 2.1].)* $H^k(\mathbb{T}) = W_2^k(\mathbb{T})$ *if* $k \in \mathbb{N}$.

Theorem 2.1. *(Cf. [14, Thm. 4.1].) Let* Γ *be a smooth, simple curve with transfinite diameter differing from* 1. *Then* B *is* $1-1$ *and "onto" as a continuous, linear mapping of* $H^s(\mathbb{T})$ *to* $H^{s+1}(\mathbb{T})$ *for any* $s \in I\!R$.

As a consequence we have the

Corollary 2.1. *Given* $f \in H^{s+1}(\mathbb{T})$, *(2.1) has exactly one solution* $u \in H^s(\mathbb{T})$ *and*

$$||u||_s \leq c(s) ||f||_{s+1} \tag{2.2}$$

holds, i.e. B *has a continuous inverse operator that is everywhere defined,* $1-1$ *and "onto".*

For proof of the corollary it suffices to mention that from the closedness of the range of B the stability (2.2) immediately follows. However, $c(s)$ is not known.

□

Now, as an example of a numerical procedure let us consider the collocation method from [13]. Let N be a natural number,

$$\mathbb{Z}_N = \left\{ l \in \mathbb{Z}, -\frac{N}{2} < l \leq \frac{N}{2} \right\}$$

and

$$\mathfrak{T}_N = \text{span}\, \{\varphi_l, \, l \in \mathbb{Z}_N\}$$

a space of trigonometric polynomials. Let $t_j, j = 1, ..., N$ be a uniform mesh of $\mathbb{T}$, $t_j = j \cdot h$, $h = 1/N$. Consider the trigonometric interpolation polynomial

$$u_h \in \mathfrak{T}_N$$

at (t_j, u_j) as approximated solution of (2.1). Here u_j, $1 \leq j \leq N$, is the solution of the linear system

$$\alpha(t_i)u_i + h\sum_{j\neq i} k(t_i, t_j)(u_j - u_i) = f(t_i) \tag{2.3}$$

where $k(x, y) = \log|\gamma(x) - \gamma(y)|$ and $\alpha(x) = \int_{\mathbb{T}} k(x, y)dy$.

Theorem 2.2. *(Cf. [13, Thm. 3.2].) Let $u \in H^s(\mathbb{T})$, $s > -1/2$ and $h > 0$ be small. Then u_h is uniquely determined and*

$$||u - u_h||_t \leq Ch^{s-t}||u||_s$$

where $-1 \leq t \leq s \leq t + 3$ and $||\cdot||_p$ is the norm in $H^p(\mathbb{T})$.

For proof take $\beta = -1$, case $a_- \equiv 0$ in Thm. 3.2 of [13].

3. Decomposition of the improperly posed problem

From now on let us consider the right–hand side of (2.1) to be observed or measured.

Imagine that in the general situation

$$V \xrightarrow{B} W \xrightarrow{L} Y,$$

where V, W, Y are normed spaces and B, L are continuous mappings, L is linear with nonclosed range, the two following problems have been settled.

Problem 1: Concerning the (well–posed) equation

$$Bu = f$$

the following are known:

$(i)_1$ Existence, i.e. the range $R(B)$ of B is known.
$(ii)_1$ Uniqueness, i.e. B is on $R(B)$ uniquely invertible.
$(iii)_1$ Stability, i.e. $||u_1 - u_2||_V \leq c_S||Bu_1 - Bu_2||_W$, $u_1, u_2 \in D(B)$.
$(iv)_1$ For every $f \in R(B)$ a stable approximation procedure u_h can be constructed with given error analysis (possibly in a weaker norm):

$$||u - u_h||_{V'} \to 0 \quad (h \to 0) \quad \text{where} \quad V \subset V',\ ||u||_{V'} \leq c_E||u||_V$$

Problem 2: Concerning the (ill–posed) equation

$$Lf = g$$

$(i)_2$ a regularization $R_\alpha : y \to R(B) \subseteq W$,
$(ii)_2$ a suitable parameter choice $\alpha(\delta)$, δ the noise level, g^δ the measured data,
$(iii)_2$ an error analysis

$$||R_{\alpha(\delta)}g^\delta - f||_W \to 0 \quad (\delta \to 0)$$

are known.

Let us suppose $(i)_i$ – $(iv)_1$ and $(i)_2$, $(ii)_2$, $(iii)_2$ to be true. Then the problem

$$LBu = g \tag{3.1}$$

can be solved in the following way.

Theorem 3.1. *Let g^δ be an observed approximation of the right–hand side g with the property $||g - g^\delta|| \leq \delta$ and let u^δ be the solution of*

$$Bu^\delta = R_{\alpha(\delta)}g^\delta.$$

Then

$$||u - u_h^\delta||_{V'} \leq c_E c_S ||R_{\alpha(\delta)}g^\delta - f||_W + ||u^\delta - u_h^\delta||_{V'}. \tag{3.2}$$

Here u_h^δ is the approximated solution for $f^\delta = R_{\alpha(\delta)}g^\delta \in R(B)$ in the sense of $(iv)_1$. Consequently, if $h \to 0$, $\delta \to 0$,

$$u_h^\delta \to u \quad in \quad V'.$$

Proof. From

$$||u - u_h^\delta||_{V'} \leq ||u - u^\delta||_{V'} + ||u^\delta - u_h^\delta||_{V'},$$

inserting $(iii)_1$ and $(iv)_1$ the assertion (3.2) immediately follows. Moreover, it is clear that $(iv)_1$ and $(iii)_2$ imply $u_h^\delta \to u$ if both δ and $h \to 0$. □

Notice that in the case of the logarithmic kernel integral equation $(i)_1$ – $(iv)_1$ are fulfilled by Section 2. In that case $V = H^s(\mathbb{T})$, $W = H^{s+1}(\mathbb{T})$, $V' = H^t(\mathbb{T})$, $t \leq s$, $Y = L_2(\mathbb{T})$, $R(B) = W$, L is the (compact) imbedding $H^{s+1}(\mathbb{T}) \to L_2(\mathbb{T})$. To solve the problem (3.1) with noisy data $(i)_2$, $(ii)_2$, $(iii)_2$ have to be settled for the ill–posed imbedding problem.

4. A REGULARIZATION METHOD

In this Section let us consider a linear continuous injective compact operator

$$T : X \to Y$$

between Hilbert spaces X, Y with the property $\overline{R(T)} = Y$ and known singular value decomposition. We shall develop a modified truncated singular value decomposition method with a–posteriori parameter choice and finally apply it to the Sobolev imbedding operator in the one–dimensional case. The used literature is [10], especially chapter 3.5, besides [3] and [7].

Let $\{u_i, v_i, \sigma_i\}$ be the singular value decomposition of T, i.e. $\{u_i\}$ resp. $\{v_i\}$ are orthonormal bases of X resp. Y and

$$Tu_i = \sigma_i v_i, \quad T^*v_i = \sigma_i u_i$$

hold.

Let us recall the definition of an adjoint Hilbert scale from [3], [7] or [10], and summarize it as follows.

Consider the selfadjoint operator

$$S = (T^*T)^{1/2}$$

from X to X with eigen values $\{\sigma_i\}$ and eigen functions $\{u_i\}$. Then by definition, for $\nu \in I\!R$

$$H_\nu = S^\nu X \tag{4.1}$$

is a Hilbert space with norm defined by

$$||S^\nu x||_\nu = ||x||, \quad x \in X, \tag{4.2}$$

and scalar product defined by

$$(S^\nu x_1, S^\nu x_2)_\nu = (x_1, x_2), \quad x_1, x_2 \in X, \tag{4.3}$$

where $||\cdot||$ is the norm and $(\cdot,\cdot)$ the scalar product of X. Notice that for

$$x = \sum (x, u_i) u_i \in X$$

$S^\nu x$ can (formally) be written as

$$S^\nu x = \sum \sigma_i^\nu (x, u_i) u_i.$$

If $k > 0$

$$H_k = \left\{x \in X, \sum \sigma_i^{-2k} |(x, u_i)|^2 < \infty\right\} = \{x \in X, \ S^{-k}x \in X\}. \tag{4.4}$$

H_ν, $\nu \in I\!R$, is a scale of Hilbert spaces with

$$H_\nu \subset H_\mu \quad \text{if } \nu > \mu$$

and compact imbedding operator,

$$||x||_\mu \leq ||S||^{\nu-\mu} ||x||_\nu, \ x \in H_\nu \tag{4.5}$$

and logarithmic convexity of the norms.

□

Let the inverse operator of T be denoted by T^+. It is densly defined with

$$D(T^+) = R(T)$$

and given by

$$T^+ g = \sum \sigma_i^{-1} (g, v_i) u_i. \tag{4.6}$$

In order to handle noisy data $g^\delta \notin R(T)$ let us consider a regularization method

$$R_\gamma y = \sum F_\gamma(\sigma_i) \sigma_i^{-1} (y, v_i) u_i \tag{4.7}$$

that is for every fixed $\gamma \geq 0$ a continuous everywhere on Y defined operator,

$$R_\gamma : Y \to X$$

with the property $R_\gamma g \to T^+ g$ $(\gamma \to 0)$ if $g \in R(T)$. If the function F satisfies

$$F_\gamma(t) = \begin{cases} 1, & t \geq \gamma \\ 0, & t < \gamma \end{cases} \tag{4.8}$$

the method is called "truncated singular value decomposition".

Here, for the purpose of an easy a–posteriori parameter choice the following continuous modification of the "filter" (4.8) is considered. Let b_j be the positive pairwise different singular values of T arranged in the natural order

$$b_1 > b_2 > b_3 > \ldots > b_j > \ldots > 0 .$$

Then for $0 \leq \gamma < \infty$ let us define

$$F_\gamma(b_j) = \begin{cases} 1 & \text{if} \quad b_j \geq \gamma \\ 0 & \text{if} \quad b_j^+ < \gamma \\ \tau(\gamma, b_j) & \text{if} \quad b_j < \gamma \leq b_j^+ \end{cases} \tag{4.9}$$

where

$$b_j^+ = \begin{cases} b_{j-1} & \text{if} \quad j > 1 \\ \infty & \text{if} \quad j = 1 \end{cases} \tag{4.10}$$

and

$$\tau(\gamma, b_j) = \begin{cases} \frac{b_j^+ - \gamma}{b_j^+ - b_j} & \text{if} \quad \gamma \geq b_j^+ - b_j \\ b_j/\gamma & \text{if} \quad \gamma \leq b_j^+ - b_j \, . \end{cases} \tag{4.11}$$

Lemma 4.1. $\tau(\cdot, b_j)$ *is a monotone strictly decreasing continuous function,* $0 \leq \tau(\cdot, b_j) \leq 1$ *if* $b_j < \gamma \leq b_j^+$.

Lemma 4.2.

$$\sup_j |b_j^{-1} F_\gamma(b_j)| \leq \gamma^{-1} \tag{4.12}$$

$$\sup_j |1 - F_\gamma(b_j)| b_j^k \leq \gamma^k \, , \; k \geq 0 \, . \tag{4.13}$$

Proof. To prove (4.12) consider first the case $\gamma \leq b_j$. In that case $b_j^{-1} F_\gamma(b_j) = b_j^{-1} \leq \gamma^{-1}$ as $F_\gamma(b_j) = 1$. In the case $b_j < \gamma \leq b_j^+$

$$b_j^{-1} \tau(\gamma, b_j) = b_j^{-1} \frac{b_j^+ - \gamma}{b_j^+ - b_j} \leq b_j^{-1} \frac{b_j^+ - (b_j^+ - b_j)}{b_j^+ - b_j} \leq \gamma^{-1} \quad \text{if} \quad \gamma \geq b_j^+ - b_j$$

and

$$b_j^{-1} \tau(\gamma, b_j) = \gamma^{-1} \quad \text{if} \quad \gamma < b_j^+ - b_j \, .$$

(4.13) is clear as $F_\gamma(b_j) = 1$ if $b_j \geq \gamma$, and $|1 - F_\gamma(b_j)| \leq 1$ because of Lemma 4.1 in the opposite case. □

Now, let us consider the operator equation

$$Tf = g$$

where f is the searched–for solution, $g^\delta \in Y$ are noisy data with noise level δ, i.e.

$$||g - g^\sigma|| \leq \delta \, , \tag{4.14}$$

$g \in R(T)$ is unknown.

Moreover, let us regard the regularization method (4.7) with (4.9), (4.10), (4.11).

Theorem 4.1. *(A–priori parameter choice)*

(i) *If* $\gamma \to 0$, $\delta/\gamma \to 0$ *then* $R_\gamma g^\delta \to f$.
(ii) *If* $f \in H_k$, $k > 0$, *and* $\gamma(\delta) = (\delta/||f||_k)^{1/(k+1)}$ *then*

$$||R_{\gamma(\delta)}g^\delta - f|| \leq 2\delta^{\frac{k}{k+1}} \, ||f||_k^{\frac{1}{k+1}} .$$

Theorem 4.2. *(A–posteriori parameter choice).*
Let $\gamma(\delta)$ *be such that*

$$||g^\delta - TR_{\gamma(\delta)}g^\delta|| = R\delta , \tag{4.15}$$

where $R > 1$ *is fixed, then* $R_{\gamma(\delta)}g^\delta \to f$ $(\delta \to 0)$.
If in addition $f \in H_k$, $k > 0$, *then*

$$||R_{\gamma(\delta)}g^\delta - f|| \leq c_R\delta^{\frac{k}{k+1}} \, ||f||_k^{\frac{1}{k+1}} ,$$

where $c_R = (R-1)^{-\frac{1}{k+1}} + (R+1)^{\frac{k}{k+1}}$.

The proofs of Theorems 4.1 and 4.2 are omitted here. They go along the lines of [10, chapter 3.5.], Lemmas 4.1 and 4.2 are needed to fit into the assumptions of that place.

□

Now, let us have a look at the performability of the parameter choice (4.15) in the case of our regularization method (4.9), (4.10), (4.11).

We have

$$\begin{aligned} g^\delta - TR_\gamma g^\delta &= \sum (g^\delta, v_i)v_i - T(\sum F_\gamma(\sigma_i)\sigma_i^{-1}(g^\delta, v_i)u_i) \\ &= \sum (1 - F_\gamma(\sigma_i))(g^\delta, v_i)v_i \end{aligned}$$

where the summation runs over all i with $\sigma_i \neq 0$. If $\gamma = 0$ or less or equal to the least singular value, $g^\delta - TR_\gamma g^\delta = 0$. Let $b(\gamma)$ be a suitable singular value with

$$b(\gamma) < \gamma \leq b^+(\gamma) .$$

Then

$$g^\delta - TR_\gamma g^\delta = \sum_{\sigma_i < b(\gamma)} (g^\delta, v_i)v_i + (1 - \tau(\gamma)) \sum_{\sigma_i = b(\gamma)} (g^\delta, v_i)v_i$$

where $\tau(\gamma) = \tau(\gamma, b(\gamma))$ (cf. (4.11)).

$\tau(\gamma)$ is continuous, decreasing, $0 \leq \tau(\gamma) \leq 1$, $\tau(\gamma) \to 0$ $(\gamma \to \infty)$. Hence,

$$\phi(\gamma) = ||g^\delta - TR_\gamma g^\delta|| = \left(\sum_{\sigma_i < b(\gamma)} |(g^\delta, v_i)|^2 + (1 - \tau(\gamma))^2 \sum_{\sigma_i = b(\gamma)} |(g^\delta, v_i)|^2 \right)^{1/2}$$

is a continuous, non–decreasing function with the property

$$0 \leq \phi(\gamma) \leq ||g^\delta|| .$$

Consequently, for every ϕ_o, $0 \leq \phi_o < ||g^\delta||$ the equation

$$\phi(\gamma) = \phi_o$$

is solvable.

It is quite natural to assume that the signal–to–noise ratio $||g^\delta||/\delta$ has the property

$$||g^\delta||/\delta > R > 1 \tag{4.16}$$

(cf. [7], p. 43). (R should be chosen such that (4.16) holds.) Then $R\delta < ||g^\delta||$ and the equation $\phi(\gamma) = R\delta$ is solvable.

Let us solve (4.15). Choose the singular value b such that

$$\sum_{\sigma_i < b} |(g^\delta, v_i)|^2 < (R\delta)^2 \leq \sum_{\sigma_i \leq b} |(g^\delta, v_i)|^2$$

holds. Then there is a τ, $0 \leq \tau < 1$, with

$$(R\delta)^2 = \sum_{\sigma_i < b} |(g^\delta, v_i)|^2 + (1-\tau)^2 \sum_{\sigma_i = b} |(g^\delta, v_i)|^2 ,$$

i.e.

$$\tau = 1 - \left(\frac{(R\delta)^2 - \sum_{\sigma_i < b} |(g^\delta, v_i)|^2}{\sum_{\sigma_i = b} |(g^\delta, v_i)|^2} \right)^{1/2} . \tag{4.17}$$

Then (4.11) and (4.17) imply

$$\gamma = \begin{cases} b^+ - (b^+ - b)\tau & \text{if} \quad \tau \leq b/(b^+ - b) \\ b/\tau & \text{if} \quad \tau > b/(b^+ - b) . \end{cases} \tag{4.18}$$

Notice that γ is uniquely determined calculating it in the prescribed way. The calculated γ is a solution of (4.15). We take it as our parameter choice $\gamma(\delta)$. We obtain

$$F_{\gamma(\delta)}(\sigma_j) = \begin{cases} 1 & \sigma_i > b \\ 0 & \sigma_i < b \\ \tau & \sigma_i = b \end{cases}$$

and

$$R_{\gamma(\delta)} y = \sum_{\sigma_j > b} \sigma_i^{-1} (y, v_i) u_i + \tau \sum_{\sigma_i = b} \sigma_i^{-1} (y, v_i) u_i .$$

□

To finish this Section consider the Hilbert scale (4.1) and recall (4.2)–(4.5). In what follows we are going to apply our method (4.9), (4.10), (4.11) to the imbedding operator

$$E : H_p \to X , \quad p > 0 \text{ fixed.}$$

First, let us determine its singular value decomposition and the adjoint mapping E^*. For $u \in H_p$, $v \in H_o = X$ we have

$$\begin{aligned} (Eu, v) &= (u, E^* v)_p = (S^p S^{-p} u, S^p S^{-p} E^* v)_p , \\ &= (S^{-p} u, S^{-p} E^* v) = (u, S^{-2p} E^* v) , \end{aligned}$$

i.e. $v = S^{-2p}E^*v$,

$$E^* = S^{2p}.$$

Moreover,

$$\begin{aligned}(\sigma_i^p u_i, \sigma_j^p u_j)_p &= (S^p u_i, S^p u_j)_p = (u_i, u_j) = \delta_{ij}\,,\\ E(\sigma_i^p u_i) &= \sigma_i^p u_i\,,\ E^* u_i = S^{2p}u_i = \sigma_i^p(\sigma_i^p u_i)\end{aligned}$$

hence the singular value decomposition of E is

$$\{\sigma_i^p u_i, u_i, \sigma_i^p\}. \tag{4.19}$$

The adjoint Hilbert scale has the form

$$(E^*E)^{\nu/2}H_p = H_{p+p\nu} \tag{4.20}$$

as $(E^*E)^{1/2} = S^p$.

The trivial identity $||Eu||_p = ||u||_p$ says that E is "smoothing of the step p" (in the sense of Louis [10]), i.e. the image of $u \in H_p$ under E is p steps smoother than a (non–smooth) element of X.

Besides, σ_1^p being the largest singular value, the imbedding constant is σ_1^p, i.e.

$$||Eu|| \le \sigma_1^p ||u||_p.$$

We are now ready to apply our regularization method. According to (4.6) and (4.19) we have

$$E^+ g = \sum \sigma_i^{-p}(g, u_i)\sigma_i^p u_i = \sum (g, u_i)u_i = g$$

as it should be! And (4.7) reads

$$R_\gamma y = \sum F_\gamma(\sigma_i^p)(y, u_i)u_i\,,$$

u_i being the orthonormal basis of X.

To be short we restrict ourselves to the application of Theorem 4.2. Assume that (4.14) holds and R is suitable chosen according to (4.16).

Corollary 4.1. *Let $\gamma(\delta)$ be such that*

$$||g^\delta - ER_{\gamma(\delta)}g^\delta|| = R\delta$$

then $R_{\gamma(\delta)}g^\delta \to f$ $(\delta \to 0)$ in H_p.

If additionally $f \in H_{p+pk}$, $k > 0$, then

$$||R_{\gamma(\delta)}g^\delta - f||_p \le c_R \delta^{\frac{k}{k+1}} ||f||_{p+pk}^{\frac{1}{k+1}}\,,$$

where $c_R = (R-1)^{-\frac{1}{k+1}} + (R+1)^{\frac{k}{k+1}}$.

The proof of the Corollary 4.1 follows easily from Theorem 4.2 taking (4.20) into account.

It is not difficult to translate the corollary to the case of the Sobolev scale considered in Section 2, where $X = L_2(\mathbb{T})$, $H_p = H^p(\mathbb{T})$ and $\sigma_l = 1/[l]$ hold. That case will be applied in Section 5.

5. A NUMERICAL PROCEDURE

In this Section we are going to show how for the improperly posed Symm's equation one can find an approximation procedure and gain its error analysis – using a known procedure and its analysis for the well–posed equation. Let us restrict ourselves to the case of a closed smooth curve. (The open arc proceeds analogously.)

Let be given a mesh on $\mathbb{T}$ of M equidistant points

$$t_j = jd\,,\ j = 1, \ldots, M\,,\ d = 1/M\,,$$

and at every t_j a measured value g_j of $g(t_j)$ with the property

$$|g(t_j) - g_j| \le \delta\,, \tag{5.1}$$

where $\delta > 0$ is the noise level and g is the (unknown) smooth right–hand side of Symm's equation. Recalling the notations of Section 2 consider the trigonometric interpolation polynomial

$$g^\delta(x) = \sum_{l \in \mathbb{Z}_M} (g^\delta, \varphi_l)\varphi_l\,, \tag{5.2}$$

where

$$(g^\delta, \varphi_l) = \frac{1}{M} \sum_{j=1}^{M} g_j e^{-2\pi i l j/M}$$

is the discrete Fourier transform. It has the property

$$g^\delta(t_j) = g_j\,,\ j = 1, \ldots M\,.$$

Let be $g \in C(\mathbb{T})$ and

$$g_M(x) = \sum_{l \in \mathbb{Z}_M} \left(\frac{1}{M} \sum_{j=1}^{M} g(t_j) e^{-2\pi i l j/M} \right) \varphi_l$$

be the M–th interpolation polynomial of g. Then

$$|g - g_M|_C \le c(k, g)\, \frac{\log(M+2)}{M^{k+\alpha}} \tag{5.3}$$

provided $g \in C^{k,\alpha}(\mathbb{T})$ (cf. [2]).

Lemma 5.1. *Let be $g \in C^1(\mathbb{T})$ and $M > |g - g_M|_C/\delta$. Then*

$$||g - g^\delta||_{L_2} \le c_1 \delta$$

where c_1 does not depend on δ, M, g, g^δ.

Proof. Using the trapezoidal rule and its error analysis (cf. e.g. [6]) we obtain

$$\begin{aligned} ||g - g^\delta||_{L_2}^2 &= \int_{\mathbb{T}} |g(t) - g^\delta(t)|^2 dt \le \frac{1}{M} \sum_{j=1}^{M} |g(t_j) - g_j|^2 + \varepsilon_M\,, \\ \varepsilon_M &= d^2 \int_{\mathbb{T}} K_2(t, d) |g^\delta(t) - g(t)|^2 dt \\ &\le 2d^2 \int_{\mathbb{T}} K_2(t, d)[|g^\delta(t) - g_M(t)|^2 + |g_M(t) - g(t)|^2] dt\,. \end{aligned}$$

Moreover, from the estimate

$$\begin{aligned} |g^\delta(x) - g_M(x)| &= \left| \sum_{l\in\mathbb{Z}_M} \Big(\frac{1}{M}\sum_{j=1}^{M}(g_j - g(t_j))e^{-2\pi i l j/M}\Big) e^{2\pi i l x}\right| \\ &\le \sum_{l\in\mathbb{Z}_M} \Big(\frac{1}{M}\sum_{j=1}^{M}|g_j - g(t_j)|\Big) \le M\delta \end{aligned}$$

we get

$$\varepsilon_M \le d^2 \cdot 2\int_{\mathbb{T}} K_2(t,d)dt(M^2\delta^2 + |g - g_M|_C^2),$$

and by the assumption,

$$\varepsilon_M \le 4c\delta^2,$$

where $c = \int_{\mathbb{T}} K_2(t,d)dt$ is bounded in d.

Finally, by (5.1) we get the asserted estimate. □

Now, recall Theorems 2.1 and 2.2 of Section 2 and consider Symm's integral operator as a mapping from $H^s(\mathbb{T})$ to $H^{s+1}(\mathbb{T})$, $s \ge 0$. Let be

$$p = s + 1$$

and consider the regularization method of Section 4 for the imbedding

$$E : H^p(\mathbb{T}) \to L_2(\mathbb{T}).$$

After replacing δ by

$$\delta_1 = c_1\delta,$$

σ_l by $1/[l]^p$, u_l by $\varphi_l = e^{2\pi i l x}$ and after having performed the parameter choice $\gamma(\delta_1)$ along the lines (4.16), (4.17), (4.18) we calculate

$$R_{\gamma(\delta_1)}g^\delta = \sum_{l\in\mathbb{Z}} F_{\gamma(\delta_1)}(1/[l]^p)(g^\delta, \varphi_l)\varphi_l \tag{5.4}$$

where F_γ is defined by (4.9), (4.10), (4.11). Applying Corollary 4.1 we get the

Lemma 5.2. *Let g, g^δ be defined by (5.1), (5.2) and let $\gamma(\delta_1)$ be such that*

$$||g^\delta - R_{\gamma(\delta_1)}g^\delta||_{L_2} = R\delta_1,$$

$R > 1$, fixed, then

$$R_{\gamma(\delta_1)}g^\delta \to g \quad (\delta \to 0) \quad \text{in } H^p(\mathbb{T}).$$

If in addition $g \in H^{p+pk}(\mathbb{T})$, $k > 0$, then

$$||R_{\gamma(\delta_1)}g^\delta - g||_p \le c_R\delta_1^{\frac{k}{k+1}}\,||g||_{p+pk}^{\frac{1}{k+1}}, \tag{5.5}$$

where $c_R = (R-1)^{-\frac{1}{k+1}} + (R+1)^{\frac{k}{k+1}}$.

The regularization (5.4) can be interpreted as a data smoothing process as higher oszillations are cut off.

Finally we come to the announced numerical procedure and its error analysis for the solution of the improperly posed Symm's integral equation. Let u^δ be the solution and u_h^δ the approximate solution concerning the procedure (2.3) for the equation (2.1) with the right–hand side $R_{\gamma(\delta_1)}g^\delta$. Then using Theorems 3.1, 2.2 and Lemmas 5.1, 5.2 we obtain the

Theorem 5.1. *Let the assumptions of Theorems* 2.1, 2.2 *and* 3.1 *and Lemmas* 5.1, 5.2 *be fulfilled. Let*

$$\begin{aligned} & g \in C^1(\mathbb{T}) \cap H^{\alpha+1}(\mathbb{T}),\ \alpha = s + (s+1)k,\ k > 0, \\ & 0 \le t \le s \le r \le 3, \\ & \text{if } s < r,\ g^\delta \in \mathfrak{T}_M,\ d = 1/M;\ \text{if } s = r,\ d = 1. \end{aligned}$$

Then

$$||u - u_h^\delta||_t \le c_E c_S c_R \delta_1^{\frac{k}{k+1}} ||g||_{\alpha+1}^{\frac{1}{k+1}} + c h^{r-t} d^{s-r}, \tag{5.6}$$

where $c_E = 1$, $c_S =$ *stability constant (dependening on* s*),* $c_R = (R-1)^{-\frac{1}{k+1}} + (R+1)^{\frac{k}{k+1}}$, c *a constant depending on* r, s, t.

Proof. Adapting (3.2) to our situation we know from Theorem 3.1

$$||u - u_h^\delta||_t \le c_E c_S ||R_{\gamma(\delta_1)} g^\delta - g||_p + ||u^\delta - u_h^\delta||_t.$$

After having inserted (5.5) in what follows there is only need to consider the expression $||u^\delta - u_h^\delta||_t$.

To this end let us first realize that $u^\delta \in H^\lambda(\mathbb{T})$ for any $\lambda \in I\!R$. Indeed, since $Bu^\delta = f^\delta$ where $f^\delta = R_{\gamma(\delta_1)} g^\delta$ is a trigonometric polynomial (5.4) being element of an arbitrary Sobolev space $H^{\lambda+1}(\mathbb{T})$ we find from Theorem 2.1 that $u^\delta \in H^\lambda(\mathbb{T})$.

Now, applying Theorem 2.2 to the situation

$$u^\delta \in H^r,\ t \le s \le r \le 3$$

we obtain

$$||u^\delta - u_h^\delta||_t \le c' h^{r-t} ||u^\delta||_r. \tag{5.7}$$

Moreover, by Corollary 2.1

$$||u^\delta||_r \le c(r) ||f^\delta||_{r+1}. \tag{5.8}$$

Since $g^\delta \in \mathfrak{T}_M$ we have also $f^\delta \in \mathfrak{T}_M$. Then the inverse property of the trigonometric polynomials gives

$$||f^\delta||_{r+1} \le c_{r,s} d^{s-r} ||f^\delta||_p. \tag{5.9}$$

(Recall $p = s + 1$. (5.9) can easily be verified by direct calculation of the Sobolev norms using (4.1), (4.4).) By (5.5)

$$||f^\delta||_p \le ||f^\delta - g||_p + ||g||_p \le c_R \delta_1^{\frac{k}{k+1}} ||g||_{\alpha+1}^{\frac{1}{k+1}} + ||g||_p \le c_o$$

Now, from (5.9), (5.8), (5.7) the assertion (5.6) follows. □

In order to apply Theorem 5.1 to concrete cases we have to specify the parameters d, h, r, s, t. The quantity $d = 1/M$ characterizes the number of measurement points necessary at the noise level δ. Taking

$$d = \delta^{\varrho} \quad \text{for some } \varrho > 0$$

and $g \in C^1(\mathbb{T})$, to fulfil the assumption of Lemma 5.1 we obtain from (5.3) (since $C^1(\mathbb{T}) \subset C^{0,1}(\mathbb{T})$)

$$|g - g_M|_C \le c\frac{\log(M+2)}{M} \le c\delta^{\varrho(1-\varepsilon)}$$

for $\varepsilon > 0$ and small δ. Moreover

$$c\delta^{\varrho(1-\varepsilon)} \le \delta^{1-\varrho} = \delta M$$

provided $1 - \varrho < \varrho(1-\varepsilon)$, i.e.

$$\varrho > \frac{1}{2-\varepsilon}. \tag{5.10}$$

Furthermore, let us specify the discretization parameter h as

$$h = \delta^{\kappa} \quad \text{for some } \kappa > 0 \tag{5.11}$$

to be given later.

According to (5.10)

$$k = \frac{\alpha - s}{s+1}, \; \frac{k}{k+1} = \frac{\alpha - s}{\alpha + 1}.$$

Then the asymptotics of (5.6) for $\delta \to 0$ read as

$$||u - u_h^{\delta}||_t = O\left(\delta^{Min\{\frac{\alpha-s}{\alpha+1},\, \kappa(r-t)+\varrho(s-r)\}}\right).$$

Choosing

$$t = s\,,\; r = 3\,,\; 0 \le s < 3 \tag{5.12}$$

the velocity $O\left(\delta^{\frac{\alpha-s}{\alpha+1}}\right)$ is achieved if

$$\kappa \ge \varrho + \frac{1}{3-s}\,\frac{\alpha-s}{\alpha+1}. \tag{5.13}$$

With the choice (5.11), (5.12) and (5.13) we get

$$||u - u_h^{\delta}||_s = O\left(\delta^{\frac{\alpha-s}{\alpha+1}}\right).$$

That is the optimal order of regularization procedures in the sense of [10] where the operator maps H^s to L_2 and the solution lies in H^{α}. Especially in the usually considered case $s = 0$ one has the optimal order $O\left(\delta^{\frac{\alpha}{\alpha+1}}\right)$ for mappings from L_2 to L_2.

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