Phase-field models with hysteresis

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Phase-field systems as mathematical models for phase transitions have drawn increasing attention in recent years. However, while capable of capturing many of the experimentally observed phenomena, they are only of restricted value in modelling hysteresis effects occurring during phase transition processes. To overcome this shortcoming of existing phase-field theories, the authors have recently proposed a new approach to phase-field models which is based on the mathematical theory of hysteresis operators developed in the past fifteen years. In particular, they have proved well-posedness and thermodynamic consistency for hysteretic phase field models which are related to the Caginalp and Penrose-Fife models. In this paper, these results are extended into different directions: we admit temperature-dependent relaxation coefficients and relax the growth conditions for the hysteresis operators considerably; also, a unified approach is used for a general class of systems that includes both the Caginalp and Penrose-Fife analogues.

1 Introduction and physical motivation

In this paper, we study systems of partial differential equations of the form

$$(1.1) \quad \text{(i)} \quad \mu(\theta)\, w_t + f_1[w] + \theta\, f_2[w] = 0 \ ,$$

$$\text{(ii)} \quad (\theta + F_1[w])_t - \Delta\theta = \psi(x,t,\theta)\, ,$$

which arise as *phase-field* equations from the mathematical modelling of *phase transitions*. Systems of the form (1.1) have been studied repeatedly in the literature for the case that μ, f_1, f_2, F_1, ψ are (possibly nonlinear) smooth functions of their respective variables (cf., for instance, the monographs [1] and [13]). In contrast to these works, the present contribution is devoted to the case when f_1, f_2, F_1 are no longer *real-valued functions*, but *hysteresis operators* acting between suitable function spaces.

It has already been pointed out in [7], [8] that hysteresis operators offer a natural and efficient tool for describing phase transitions. The aim of this paper is to generalize the results of the above papers and to give a new physical interpretation of hysteresis operators in the phase-field context.

Let us consider a bounded container $\Omega \subset \mathbb{R}^N$ filled by a material existing in two phases, liquid and solid, say. The state of the system is determined by the value of two state variables: the absolute temperature $\theta > 0$, and the phase fraction $\chi \in [0,1]$, both being functions of the space variable $x \in \Omega$ and the time $t \in [0,T]$, where $\chi = 1$ corresponds to the pure liquid and $\chi = 0$ to the pure solid phase. The evolution of the system is governed by the following physical laws.

$$(1.2) \qquad \mathcal{U}_t + \operatorname{div} q = \psi \qquad \text{(balance of internal energy)} \ ,$$

$$(1.3) \qquad \mu(\theta)\, \chi_t \in -\partial_\chi \mathcal{F}(\chi,\theta) \qquad \text{(melting/solidification law)} \ ,$$

where $\mathcal{U} = \mathcal{U}(\chi,\theta) \geq 0$ is the internal energy, q is the heat flux, ψ is the heat source density, $\mathcal{F}$ is the free energy, ∂_χ is a (formal) derivative w.r.t. χ, and $\mu(\theta) > 0$ is the phase relaxation coefficient. We say that the model is *thermodynamically consistent*, if

$$(1.4) \qquad \theta(x,t) > 0 \quad \text{a.e.}\, ,$$

$$(1.5) \qquad \mathcal{S}_t \geq -\operatorname{div}\left(\frac{q}{\theta}\right) + \frac{\psi}{\theta} \qquad \text{a.e.} \quad \text{(Clausius-Duhem inequality)}$$

entropy. Using the energy balance (1.2), we can formally rewrite the Clausius–Duhem inequality equivalently in the form

$$\theta \mathcal{S}_t - \mathcal{U}_t \geq \frac{1}{\theta} \langle q , \nabla\theta \rangle \quad \text{a.e.} \tag{1.6}$$

Throughout the paper we assume, for the sake of simplicity, that the Fourier law

$$q = -\kappa \nabla\theta \tag{1.7}$$

holds with a constant heat conductivity coefficient $\kappa > 0$. Then inequality (1.6) holds if and only if

$$\theta \mathcal{S}_t - \mathcal{U}_t \geq 0 \quad \text{a.e.} \tag{1.8}$$

Let us first briefly describe a model introduced by Frémond and Visintin in [3] which can be characterized as a relaxed Stefan problem with overheating and undercooling and consists in choosing the free energy $\mathcal{F}$ in the form

$$\mathcal{F}(\chi, \theta) = \mathcal{F}_0(\theta) + \tilde{\mathcal{F}}(\chi, \theta), \tag{1.9}$$

where

$$\mathcal{F}_0 (\theta) := c_V \theta \left(1 - \log \frac{\theta}{\theta_c}\right) \tag{1.10}$$

is the purely caloric component, and

$$\tilde{\mathcal{F}}(\chi, \theta) = \nu\, \theta I_{[0,1]} (\chi) + L \left(\lambda(\chi) - \frac{\theta}{\theta_c} \sigma(\chi)\right) \tag{1.11}$$

is the phase component of the free energy. Here, $c_V > 0$ (the specific heat), $L > 0$ (latent heat), $\theta_c > 0$ (a referential temperature), $\nu > 0$ (an arbitrary physical constant) are given constants, $I_{[0,1]}$ is the indicator function of the interval $[0, 1]$, and σ, λ are given smooth functions. Typical choices are

$$\sigma(\chi) = \chi, \quad \lambda(\chi) = \chi + \alpha\chi(1 - \chi), \tag{1.12}$$

where $\alpha \in [0, 1]$ can be interpreted as a dimensionless coefficient of undercooling/overheating. Figure 1 shows a diagram of $\tilde{\mathcal{F}}$ at several fixed temperatures θ. We see that it has the form of a double-obstacle potential with two local minima provided that θ is close to θ_c, that is, if $\theta_c(1 - \alpha) < \theta < \theta_c(1 + \alpha)$; for higher temperatures it has a unique local minimum at $\chi = 1$, and for lower temperatures the only minimum is $\chi = 0$.

The corresponding expressions for the internal energy $\mathcal{U}$ and the entropy $\mathcal{S}$ have the form

$$\mathcal{U} = c_V\theta + L\,\lambda(\chi), \tag{1.13}$$

$$\mathcal{S} = c_V \log \frac{\theta}{\theta_c} - \nu I_{[0,1]} (\chi) + \frac{L}{\theta_c} \sigma(\chi). \tag{1.14}$$

With these choices, the laws (1.2), (1.3) read

$$(c_V\, \theta + L\,\lambda(\chi))_t - \kappa\Delta\theta = \psi, \tag{1.15}$$

$$\mu(\theta)\chi_t + L(\lambda'(\chi) - \frac{\theta}{\theta_c} \sigma'(\chi)) \in -\partial_\chi I_{[0,1]} (\chi), \tag{1.16}$$

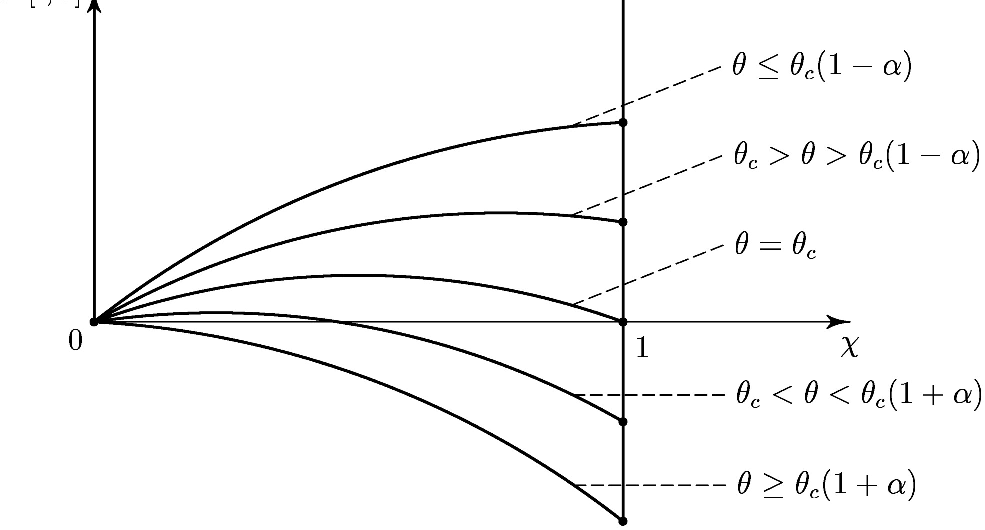

Figure 1 : *The phase component $\tilde{\mathcal{F}}$ of the free energy at different temperatures.*

where ∂_χ now denotes the subdifferential. We couple the equations (1.15), (1.16) with the initial and boundary conditions

(1.17) $$\chi(x,0) = \chi^0(x) \in [0,1], \qquad \theta(x,0) = \theta^0(x) > 0, \qquad \text{in } \Omega,$$

(1.18) $$\frac{\partial\theta}{\partial n}(x,t) = 0 \qquad \text{on } \partial\Omega \times]0,T[\,.$$

We rewrite inclusion (1.16) in a more convenient form. To this end, let us define an auxiliary function w by the formula

(1.19) $$w(x,t) := w^0(x) + \int_0^t \frac{-L}{\mu(\theta)} \left(\lambda'(\chi) - \frac{\theta}{\theta_c}\sigma'(\chi)\right)(x,\tau)\, d\tau$$

with some given initial condition w^0. Apparently, the integrand in (1.19) is (up to the factor $1/\mu(\theta)$) nothing else but the negative of the partial derivative with respect to χ of the differentiable part of the free energy $\mathcal{F}(\chi,\theta)$. Since the latter is usually seen as the *thermodynamic force* driving the phase transition, the new variable w can be interpreted as the (time-integrated) *memory* of the system during the evolution. It thus seems to be quite natural to describe the evolution in terms of w. Now, using (1.19), we obtain from (1.16) that

(1.20) $$\chi_t - w_t \in -\partial_\chi I_{[0,1]}(\chi),$$

or equivalently,

(1.21) $$\chi \in [0,1], \quad (\chi_t - w_t)(\chi - \varphi) \leq 0 \ \text{ a.e.} \qquad \forall\, \varphi \in [0,1].$$

Variational inequality (1.21) enables us to apply the theory of hysteresis operators and to simplify the problem stated above by equations (1.15) – (1.18). Recall that a mapping $f : C[0,T] \to C[0,T]$ is called a *hysteresis operator* if it is

causal, that is, the implication

(1.22) $$u(t) = v(t) \quad \forall\, t \in [0,t_0] \ \Rightarrow \ f(u)(t_0) = f(v)(t_0)$$

rate-independent, that is, for every $u \in C[0,T]$ and every continuous increasing mapping α of $[0,T]$ onto $[0,T]$ we have

$$f(u \circ \alpha)(t) = f(u)(\alpha(t)) \quad \text{for all } t \in [0,T]\,. \tag{1.23}$$

Let us note that hysteresis operators are exactly those that admit a local representation by means of superposition operators in each interval of monotonicity of the input, with a possible branching when the input changes direction.

In connection with inequality (1.21), we recall the following result (which can also be generalized to the case of vector-valued functions), see e.g. [4], [1], [5], [6].

Proposition 1.1. *For every closed interval $Z \subset \mathbb{R}$, every element $\chi^0 \in Z$ and every function $w \in W^{1,1}(0,T)$, there exists a unique $\chi \in W^{1,1}(0,T)$ such that $\chi(0) = \chi^0$ and condition* (1.21) *is satisfied. The solution operator*

$$\mathfrak{s}_Z : Z \times W^{1,1}(0,T) \to W^{1,1}(0,T) : \quad (\chi^0, w) \mapsto \chi\,, \tag{1.24}$$

is Lipschitz and admits a Lipschitz continuous extension onto $Z \times C[0,T] \to C[0,T]$.

The operator $\mathfrak{s}_Z$ is called *stop*. To simplify the notation, we write $\mathfrak{s}$ instead of $\mathfrak{s}_{[0,1]}$. The hysteretic input-output behaviour of the stop $\mathfrak{s}$ is illustrated in Fig. 2. Along the upper (lower) threshold line $\chi = 1$, ($\chi = 0$), the process is irreversible and can only move to the right (to the left, respectively), while in between, motions in both directions are admissible. This is similar to Prandtl's model of perfect elastoplasticity, where the horizontal parts of the diagram correspond to plastic yielding and the intermediate lines can be interpreted as linearly elastic trajectories.

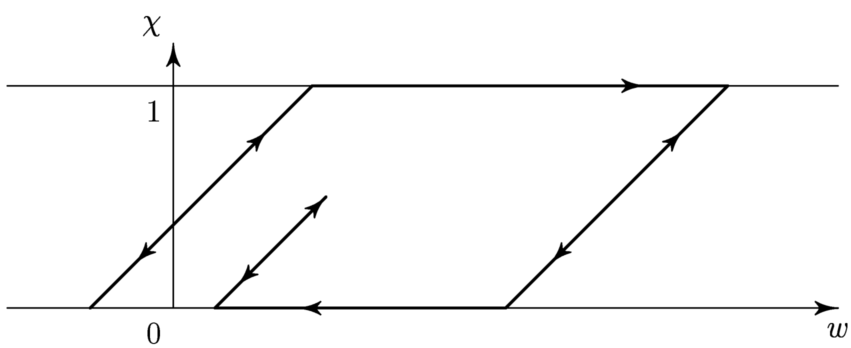

Figure 2: *A diagram of the stop* $\mathfrak{s}$.

Proposition 1.1 enables us to eliminate χ and to rewrite system (1.15), (1.16) in the form

$$(c_V\theta + L\,\lambda(\mathfrak{s}[\chi^0, w]))_t - \kappa\,\Delta\theta = \psi\,, \tag{1.25}$$

$$\mu(\theta)\,w_t + L\Big(\lambda'(\mathfrak{s}[\chi^0, w]) - \frac{\theta}{\theta_c}\,\sigma'(\mathfrak{s}[\chi^0, w])\Big) = 0\,, \tag{1.26}$$

with the initial conditions

$$w(x,0) = w^0(x), \quad \theta(x,0) = \theta^0(x)\,, \tag{1.27}$$

with hysteresis of the form

$$\mu(\theta)\, w_t \,+\, f_1[w] \,+\, \theta\, f_2[w] \,=\, 0\,, \tag{1.28}$$

$$(c_V\theta \,+\, F_1\,[w])_t \,-\, \kappa\,\Delta\theta \,=\, \psi\,, \tag{1.29}$$

with three hysteresis operators f_1, f_2, F_1.

The system is formally thermodynamically consistent provided $F_1 \geq 0$ and there exist two further operators g, F_2 such that

$$g[w]_t\, w_t \,\geq 0 \quad \text{a.e.}\,, \tag{1.30}$$

$$F_i\,[w]_t \,\leq\, g[w]_t\; f_i[w] \quad \text{a.e.}\,, \qquad i \,=\, 1,2\,, \tag{1.31}$$

for every $w \,\in\, W^{1,1}\,(0,T)$. Indeed, putting

$$\mathcal{U} \,:=\, c_V\theta \,+\, F_1[w]\;, \quad \mathcal{S} \,:=\, c_V \log\frac{\theta}{\theta_c} \,-\, F_2[w]\,,$$

we obtain

$$\begin{aligned}\mathcal{U}_t - \theta\,\mathcal{S}_t \;&=\; F_1[w]_t \,+\, \theta\,F_2[w]_t \,\leq\, g[w]_t\,(f_1[w] + \theta\,f_2[w]) \\ &=\; -\mu(\theta)\,g[w]_t\,w_t \;\leq\; 0\,,\end{aligned}$$

hence inequality (1.8) holds for every regular solution (w,θ) of the system (1.28), (1.29) satisfying $\theta > 0$. We will prove rigorously in the next sections that conditions (1.30), (1.31), together with additional technical hypothesis, also imply the positivity of temperature and enable us to justify the above formal computation.

We easily check that inequalities (1.30), (1.31) are fulfilled in the context of system (1.25), (1.26), where we put $g[w] \,=\, \mathfrak{s}\,[\chi^0, w]$, $f_1\,[w] \,=\, \lambda'\,(g[w])$, $f_2\,[w] \,=\, \sigma'\,(g[w])$, $F_1\,[w] \,=\, \lambda\,(g[w])$, $F_2\,[w] \,=\, \sigma\,(g[w])$. Our approach, however, makes it possible to model an additional hysteretic behaviour in the melting/solidification law itself. As an example, we can consider a free energy of the form (1.9) – (1.12), where the function $\lambda(\chi)$ is replaced by the operator

$$F\,[\chi] \,=\, \chi +\; \alpha\left(\mathfrak{s}_r^2\left[\chi_r^0, \chi - \frac{1}{2}\right] + \chi(1-\chi) - r^2\right)\,, \tag{1.32}$$

where $\mathfrak{s}_r \,:=\, \mathfrak{s}_{[-r,r]}$ is the stop operator corresponding to $Z \,=\, [-r,r]$ with some $r \in]0,\frac{1}{2}[$, and where $\chi_r^0 \,=\, \text{sign}\,(\chi^0 - \frac{1}{2})\,\min\,\{r, |\chi^0 - \frac{1}{2}|\} \in [-r,r]$.

In other words, the phase component $\tilde{\mathcal{F}}$ of the free energy now has the form

$$\tilde{\mathcal{F}}[\chi,\theta] \;=\, \nu\theta\, I_{[0,1]}\,(\chi) \,+\, L\left(F[\chi] - \frac{\theta}{\theta_c}\,\chi\right) \tag{1.33}$$

with F given by (1.32), see Fig. 3. Let us note that the operator F is not Gâteaux differentiable; we therefore interpret the formal condition (1.3) as an inclusion analogous to (1.16), namely

$$\mu(\theta)\,\chi_t \,+\, L\left(f[\chi] - \frac{\theta}{\theta_c}\right) \,\in\, -\,\partial_\chi\, I_{[0,1]}\,(\chi)\,, \tag{1.34}$$

where f is the operator

$$f[\chi] \,=\, 1 \,+\, \alpha\left(2\,\mathfrak{s}_r\left[\chi_r^0, \chi - \frac{1}{2}\right] \,+\, 1 - 2\chi\right). \tag{1.35}$$

(1.28), (1.29) with $F_2[w] = g[w] = \mathfrak{s}[\chi^0, w]$, $f_2[w] \equiv 1$, $f_1[w] = f[g[w]]$, $F_1[w] = F[g[w]]$. To check that f_1, F_1 satisfy inequality (1.31), we need to show that

$$F[\chi]_t \leq \chi_t \, f[\chi] \text{ a.e.} \tag{1.36}$$

for all $\chi \in W^{1,1}(0,T)$. Put $s := \mathfrak{s}_r \left[\chi_r^0, \chi - \frac{1}{2}\right]$. Then

$$F\,[\chi]_t \; - \; \chi_t\, f\,[\chi] \; = \; \frac{2\alpha}{1-2r} \; s\,(\dot{s} - \dot{\chi})\,, \tag{1.37}$$

and inequality (1.36) follows from the definition of the stop operator.

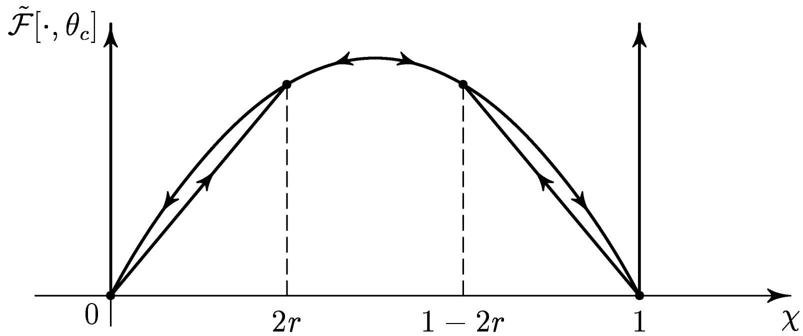

Figure 3: *Free energy (1.33) at* $\theta = \theta_c$.

Inequality (1.30) is called *piecewise* ([12], [1]) or *local* ([5]) *monotonicity*. Condition (1.36) represents the energy inequality for the hysteresis operator f with a *clockwise admissible potential* F according to the terminology of [1], see Fig. 4.

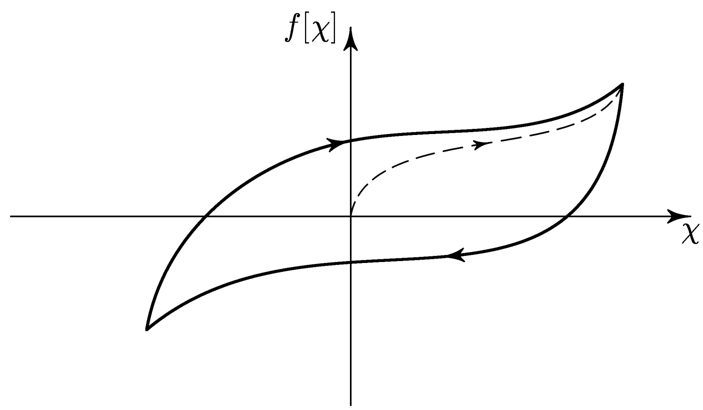

Figure 4: *Clockwise admissibility of the operator* f.

2 Statement of the problem

We consider the following system of equations:

(2.1) (i) $$\mu(\theta)\, w_t \, + \, f_1[w] \, + \, \theta\, f_2[w] \, = \, 0 \; ,$$

(ii) $$(\theta \, + \, F_1[w])_t \, - \, \Delta\theta \, = \, \psi(x,t,\theta)\,,$$

in $\Omega \times \,]0,T[$, coupled with initial and boundary conditions

$$\theta(x,0) \, = \, \theta^0(x)\,, \; w(x,0) \, = \, w^0(x) \;\; \text{in } \Omega\,, \quad \frac{\partial\theta}{\partial n} \, = \, 0 \;\; \text{on } \partial\Omega \times \,]0,T[, \tag{2.2}$$

time. We make the following hypotheses concerning the data of the system.

H1. The initial data are given in such a way that

(2.3) (i) $$w^0 \in L^\infty(\Omega)\,,\ \theta^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)\,,$$

(ii) $$\exists \delta > 0\ :\ \theta^0(x) \ge \delta \quad \text{for} \quad \text{a.e. } x \in \Omega.$$

H2. The function $\mu :]0,\infty[\to]0,\infty[$ is Lipschitz continuous on compact subsets of $]0,\infty[$, and either

(2.4) $$\exists \mu_0 > 0 : \quad \mu(\theta) \ge \mu_0 \min\{\theta, 1\} \quad \forall \theta > 0\,,$$

or

(2.4)* $$\exists \mu_0 > 0 : \quad \mu(\theta) \ge \mu_0 \quad \forall \theta > 0\,.$$

H3. The operators $f_1, f_2 : C[0,T] \to C[0,T]$ are causal, and there exists some $K_1 > 0$ such that

(2.5) $$w_1, w_2 \in C[0,T] \Rightarrow |\, f_i\,[w_1]\,(t) - f_i\,[w_2](t)| \le K_1 |w_1 - w_2|_{[0,t]} \quad \forall t \in [0,T]\,,\ i = 1,2\,,$$

where for $z \in C[0,T]$ and $t \in [0,T]$ we denote

(2.6) $$|z|_{[0,t]} := \max\{|z(\tau)|\,;\, \tau \in [0,t]\}.$$

We moreover assume that either

(2.7) $$\exists \lambda :]0,\infty[\to]0,\infty[\quad \text{nondecreasing, with } \limsup_{s\to\infty} \lambda(s)/s = 0\,, \text{ such that} \quad |f_2[w](t)| \le \lambda(|w|_{[0,t]}) \quad \forall w \in C[0,T]\,,\ \forall t \in [0,T]\,,$$

or

(2.7)* $$\exists K_2 > 0 : \quad |f_i[w](t)| \le K_2 \quad \forall w \in C[0,T]\,,\ \forall t \in [0,T]\,,\ i = 1,2\,.$$

H4. The operator $F_1 : W^{1,1}(0,T) \to W^{1,1}(0,T)$ is causal, and it holds:

(2.8) $$\exists K_3 > 0 : \quad |F_1[w]_t\,(t)| \le K_3 |w_t(t)| \ \text{ a.e.} \quad \forall w \in W^{1,1}(0,T),$$

(2.9) $$\forall R > 0\ \exists \Phi_R > 0 : w_1, w_2 \in W^{1,1}(0,T)\,,\ |w_i|_{W^{1,1}(0,T)} \le R,\ i = 1,2, \Rightarrow |F_1[w_1](t) - F_1[w_2](t)| \le \Phi_R\,|w_1 - w_2|_{W^{1,1}(0,t)} \quad \forall t \in [0,T]\,,$$

where for $z \in W^{1,1}(0,T)$ and $t \in [0,T]$ we denote

(2.10) $$|z|_{W^{1,1}(0,t)} := |z(0)| + \int_0^t |\dot z(\tau)|\, d\tau.$$

assume that $\psi : \Omega \times]0,T[\times \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$(2.11) \qquad \exists \psi_0 \in L^q(\Omega\times]0,T[) : \quad \theta \le 0 \Rightarrow \psi(x,t,\theta) = \psi_0(x,t),$$

$$(2.12) \qquad \exists K_4 > 0 : \quad \left| \frac{\partial \psi}{\partial \theta} \right| \le K_4 \quad \text{a.e.},$$

$$(2.13) \qquad \psi_0(x,t) \ge 0 \quad \text{a.e.}$$

H6. There exist causal operators $F_2, g : W^{1,1}(0,T) \to W^{1,1}(0,T)$ and a constant $K_5 > 0$ such that for all $w \in W^{1,1}(0,T)$ we have

$$(2.14) \qquad 0 \le g[w]_t \, w_t \le K_5 \, w_t^2 \quad \text{a.e.},$$

$$(2.15) \qquad F_i[w]_t \le g[w]_t \, f_i[w] \quad \text{a.e.}, \quad i = 1,2,$$

$$(2.16) \qquad F_1[w](t) \ge 0 \quad \forall t \in [0,T].$$

Remark 2.1. Assumption (2.4) is for instance satisfied if $\mu(\theta) = \mu_0\,\theta$, $\mu_0 > 0$ fixed. Then system (2.1) constitutes a hysteretic analogue of the *Penrose-Fife model* for phase transitions with zero interfacial energy (cf. [10]); on the other hand, (2.4)* is the hysteretic analogue of the *Caginalp model* with zero interfacial energy (see [2]). Note that also the intermediate models $\mu(\theta) = \mu_0\,\theta^\alpha$, $0 < \alpha < 1$, $\mu_0 > 0$ are included in (2.4).

The main result of this paper reads as follows.

Theorem 2.2. *Let hypotheses* **H1** – **H6** *hold with either (2.4) and (2.7)* * *or (2.4)* * *and (2.7). Then there exists a unique solution* $(w,\theta) \in L^\infty(\Omega\times]0,T[) \times L^\infty(\Omega\times]0,T[)$ *to problem (2.1), (2.2) such that* $w_t \in L^\infty(\Omega\times]0,T[)$, $\theta_t, \Delta\theta \in L^2(\Omega\times]0,T[)$, *equations (2.1) are satisfied almost everywhere, and there exists a constant* $\beta > 0$ *such that* $\theta(x,t) \ge \delta\, e^{-\beta t}$ *a.e. in* $\Omega\times]0,T[$.

3 An auxiliary problem

We first solve the system

$$(3.1) \qquad \text{(i)} \quad w_t = \gamma[w,\theta],$$
$$\qquad \text{(ii)} \quad (\theta + F_1[w])_t - \Delta\theta = \psi(x,t,\theta),$$

with the initial and boundary conditions (2.2), where $\gamma : C[0,T] \times L^\infty(0,T) \to L^\infty(0,T)$ is a causal operator satisfying the following hypotheses.

$$(3.2) \quad \exists K_6 > 0 : \ |\gamma[w,\theta](t)| \le K_6(1 + |\theta(t)|) \text{ a.e. } \quad \forall (w,\theta) \in C[0,T] \times L^\infty(0,T);$$

$$(3.3) \quad \forall R > 0 \, \exists \Gamma_R > 0 : \ \theta_1, \theta_2 \in L^\infty(0,T), w_1, w_2 \in C[0,T], \ |\theta_1|_\infty, |\theta_2|_\infty \le R$$
$$\Rightarrow |\gamma[w_1,\theta_1](t) - \gamma[w_2,\theta_2](t)| \le \Gamma_R(|w_1 - w_2|_{[0,t]} + |\theta_1(t) - \theta_2(t)|),$$
for a.e. $t \in (0,T)$.

hold. Then problem (3.1), (2.2) admits a unique solution $(w,\theta) \in L^\infty(\Omega\times]0,T[) \times L^\infty(\Omega\times]0,T[)$ *such that* $w_t \in L^\infty(\Omega\times]0,T[)\,, \theta_t, \Delta\theta \in L^2(\Omega\times]0,T[)$, *and such that the equations (3.1) are satisfied almost everywhere.*

Let us first consider equation (3.1) (i) independently of the space variable.

Lemma 3.2. *Let conditions* (3.2), (3.3) *hold and let* $\theta \in L^\infty(0,T)$ *be given. Then the equation*

$$\dot{w}(t) = \gamma[w,\theta](t)\,, \quad w(0) = w^0\,, \tag{3.4}$$

admits a unique solution $w \in W^{1,\infty}(0,T)$ *for each* $w^0 \in \mathbb{R}$. *Moreover, two solutions* w_1, w_2 *corresponding to two different input functions* θ_1, θ_2 *satisfy for every* $R > 0$ *and* $t \in [0,T]$ *the following implication.*

$$|\theta_1|_\infty\,, |\theta_2|_\infty \le R \quad \Rightarrow \quad |w_1(t) - w_2(t)| \le \Gamma_R\, e^{\Gamma_R t} \int_0^t |\theta_1 - \theta_2|(\tau)\, d\tau\,. \tag{3.5}$$

Proof of Lemma 3.2. For $w \in C[0,T]$ put

$$G[w](t) := w^0 + \int_0^t \gamma[w,\theta](\tau)\, d\tau. \tag{3.6}$$

Condition (3.3) yields for $R := |\theta|_\infty$,

$$\Big|G[w_1](t) - G[w_2](t)\Big| \le \Gamma_R \int_0^t |w_1 - w_2|_{[0,\tau]}\, d\tau\,, \tag{3.7}$$

for all $w_1, w_2 \in C[0,T]$. By induction we easily check that the n-th iteration G^n of G fulfills the inequality

$$\Big|G^n[w_1](t) - G^n[w_2](t)\Big| \le \frac{\Gamma_R^n}{(n-1)!} \int_0^t (t-\tau)^{n-1}\, |w_1 - w_2|_{[0,\tau]}\, d\tau\,, \tag{3.8}$$

that is, G^n is a contraction on $C[0,T]$ for sufficiently large n. Therefore, there exists a unique fixed point $w \in C[0,T]$ of G which satisfies equation (3.4) almost everywhere. To derive inequality (3.5), we just notice that for $|\theta_1|_\infty$, $|\theta_2|_\infty \le R$, the hypothesis (3.3) entails that

$$\Big|w_1(t) - w_2(t)\Big| \le \Gamma_R \int_0^t \Big(|w_1 - w_2|_{[0,\tau]} + |\theta_1 - \theta_2|(\tau)\Big)\, d\tau\,, \tag{3.9}$$

and the assertion follows from Gronwall's inequality. □

The proof of Theorem 3.1 is based on the following classical properties of the linear heat equation, see e.g. [9].

Lemma 3.3 *Consider the problem*

$$u_t - \Delta u + u = g \quad in \ \ \Omega \times]0,T[\,, \tag{3.10}$$

$$u(x,0) = u^0(x) \quad in \ \ \Omega\,, \qquad \frac{\partial u}{\partial n} = 0 \quad on \ \ \partial\Omega \times]0,T[\,, \tag{3.11}$$

where $\Omega \subset \mathbb{R}^N$ *is a bounded domain with a lipschitzian boundary and* g, u^0 *are given functions. Then the following statements hold.*

(3.10) – (3.11) *satisfies for every* $t \in [0,T]$ *the estimate*

$$\left|u(\cdot\,,t)\right|_p^p \le \left|u^0\right|_p^p + \int_0^t \left|g(\cdot\,,\tau)\right|_p^p d\tau\,, \tag{3.12}$$

where $|\cdot|_p$ *denotes the norm in* $L^p(\Omega)$.

(ii) *Let* r_N *and* q *be as in Hypothesis* **H5**. *Then there exists a constant* $K_\infty > 0$ *such that for every* $u^0 \in L^\infty(\Omega)$ *and* $g \in L^q(\Omega\times\,]0,T[)$ *the solution* u *of* (3.10)–(3.11) *satisfies the estimate*

$$\|u\|_\infty \le K_\infty \max\left\{|u^0|_\infty\,,\ \|g\|_q\right\}, \tag{3.13}$$

where $\|\cdot\|_q$ *denotes the norm of* $L^q(\Omega\times\,]0,T[)$.

Proof of Theorem 3.1. We construct the solution of system (3.1), (2.2) by successive approximation. We put $\theta^0(x,t) := \theta^0(x)$, and for $k \ge 1$ we define recursively the sequences $\{w^k,\theta^k\}_{k=1}^\infty$ as solution to the system

$$\text{(3.14)}\quad \begin{aligned} &\text{(i)} && w_t^k(x,t) = \gamma\left[w^k(x,\cdot),\theta^{k-1}(x,\cdot)\right](t)\,,\\ &\text{(ii)} && \theta_t^k - \Delta\theta^k + \theta^k = \theta^{k-1} + \psi(x,t,\theta^{k-1}) - F_1[w^k]_t\,, \end{aligned}$$

together with the initial and boundary conditions (2.2). Inequalities (3.2) and (2.8) yield that

$$\text{(3.15)}\quad \begin{aligned} &\text{(i)} && |w_t^k(x,t)| \le K_6\Big(1 + |\theta^{k-1}(x,t)|\Big) \quad \text{a.e.}\,,\\ &\text{(ii)} && |F_1[w^k]_t(x,t)| \le K_3\,K_6\Big(1 + |\theta^{k-1}(x,t)|\Big) \quad \text{a.e.}\,, \end{aligned}$$

and from Lemma 3.3 (i) we infer that

$$\int_\Omega |\theta^k(x,t)|^q\,dx \le C_1\left(1 + \int_0^t\int_\Omega |\theta^{k-1}(x,\tau)|^q\,dx\,d\tau\right), \tag{3.16}$$

for all $k \ge 1$ and $t \in [0,T]$, with some constant $C_1 \ge |\theta^0|_\infty^q$ that is independent of k. By induction, we obtain from (3.16)

$$\int_\Omega |\theta^k(x,t)|^q\,dx \le C_1\,e^{C_1 t} \quad \forall k \in \mathbb{N},\ \forall t \in [0,T]\,. \tag{3.17}$$

Applying Lemma 3.3 (ii) to equation (3.14) (ii) and using inequalities (3.15), (3.17) and Hypothesis **H5**, we can find a constant $C_2 > 0$, independent of k, such that

$$\|\,\theta^k\,\|_\infty \le C_2 \quad \forall k \ge 0\,. \tag{3.18}$$

Taking a bigger C_2, if necessary, we also have

$$\|\,\theta_t^k\|_2\,,\ \|\Delta\,\theta^k\,\|_2 \le C_2 \quad \forall k \in \mathbb{N}\,. \tag{3.19}$$

According to Lemma 3.2 and hypothesis (3.3), there exists some constant $C_3 > 0$, independent of k, such that

$$\begin{aligned} \left|w_t^{k+1}(x,t) - w_t^k(x,t)\right| \le C_3\Big(&\left|\theta^k(x,t) - \theta^{k-1}(x,t)\right| \\ &+ \int_0^t \left|\theta^k(x,\tau) - \theta^{k-1}(x,\tau)\right| d\tau\Big)\,, \end{aligned} \tag{3.20}$$

>From (2.9), (3.15), (3.18), and (3.20), it follows that

$$(3.21) \quad \left|F_1[w^{k+1}](x,t) - F_1[w^k](x,t)\right| \le C_4 \int_0^t |\theta^k - \theta^{k-1}|(x,\tau)\,d\tau \quad \text{a.e.}$$

for all $k \in \mathbb{N}$, where $C_4 > 0$ is a constant independent of k. This enables us to estimate the difference $\theta^{k+1} - \theta^k$. Indeed, for almost all (x,t) we have

$$(3.22) \quad \theta^{k+1}(x,t) - \theta^k(x,t) - \int_0^t \left(\Delta\left(\theta^{k+1} - \theta^k\right) - \left(\theta^{k+1} - \theta^k\right)\right)(x,\tau)\,d\tau$$
$$= \int_0^t \left(\left(\theta^k - \theta^{k-1}\right)(x,\tau) + \psi\left(x,\tau,\theta^k(x,\tau)\right) - \psi\left(x,\tau,\theta^{k-1}(x,\tau)\right)\right)d\tau$$
$$- F_1\left[w^{k+1}\right](x,t) + F_1\left[w^k\right](x,t).$$

Multiplying the above identity by $\left(\theta^{k+1} - \theta^k\right)(x,t)$, integrating over Ω and using inequalities (3.21), (2.12), we conclude that there exists a constant $C_5 > 0$, independent of k, such that

$$(3.23) \quad \int_\Omega \left|\theta^{k+1} - \theta^k\right|^2(x,t)\,dx + \frac{d}{dt}\int_\Omega \left(\left|\int_0^t \nabla\left(\theta^{k+1} - \theta^k\right)(x,\tau)\,d\tau\right|^2\right.$$
$$\left. + \left|\int_0^t \left(\theta^{k+1} - \theta^k\right)(x,\tau)\,d\tau\right|^2\right)dx \le C_5 \int_0^t \int_\Omega \left|\theta^k - \theta^{k-1}\right|^2(x,\tau)\,dx\,d\tau,$$

for all $k \in \mathbb{N}$ and $t \in [0,\tau]$. By induction, this implies that

$$(3.24) \quad \int_0^t \int_\Omega \left|\theta^{k+1} - \theta^k\right|^2(x,\tau)\,dx\,d\tau \le \frac{C_5^k\, t^k}{k!} \int_0^t \int_\Omega \left|\theta^1 - \theta^0\right|^2(x,\tau)\,dx\,d\tau,$$

independently of k and t. Hence, $\{\theta^k\}$ is a Cauchy sequence in $L^2(\Omega\times\,]0,T[\,)$. Let $\theta \in L^2(\Omega\times\,]0,T[\,)$ be its limit. >From inequalities (3.18), (3.19) it follows that $\theta \in L^\infty(\Omega\times\,]0,T[\,), \theta_t, \Delta\theta \in L^2(\Omega\times\,]0,T[\,)$, $\theta^k \to \theta$ in $L^\infty(\Omega\times\,]0,T[\,)$ weakly-star, and $\theta_t^k \to \theta_t$, $\Delta\theta^k \to \Delta\theta$, in $L^2(\Omega\times\,]0,T[\,)$ weakly.

From inequalities (3.5), (3.15) (i), and (3.20), it follows that $\{w^k\}, \{w_t^k\}$ are Cauchy sequences in $L^2(\Omega\times]0,T[)$ which are bounded in $L^\infty(\Omega\times]0,T[)$. Consequently, there exists some $w \in L^2(\Omega; C[0,T]) \cap L^\infty(\Omega\times]0,T[)$ with $w_t \in L^\infty(\Omega\times]0,T[)$ such that $w^k \to w$ strongly in $L^2(\Omega; C[0,T])$, and weakly-star in $L^\infty(\Omega\times]0,T[)$, $w_t^k \to w_t$ strongly in $L^2(\Omega\times]0,T[)$, and weakly-star in $L^\infty(\Omega\times]0,T[)$.

Hypothesis (2.9), and inequality (3.15) (ii), yield that $F[w^k] \to F[w]$ strongly in $L^2(\Omega; C[0,T])$, as well as $F[w^k]_t \to F[w]_t$ weakly in $L^2(\Omega\times\,]0,T[\,)$. Passing to the limit in the system (3.14) as $k \to \infty$, and using hypothesis (3.3), we see that w, θ satisfy system (3.1) almost everywhere.

The above convergences immediately yield that $w(x,0) = w^0(x)$ for a.e. $x \in \Omega$, as well as

$$\int_0^T \int_\Omega \eta(t)\left(\Delta\theta \cdot \varphi(x) + \langle\nabla\theta, \nabla\varphi(x)\rangle\right)dx\,dt = 0 \quad \forall\varphi \in W^{1,2}(\Omega), \quad \forall\eta \in L^2(0,T),$$

so that $\dfrac{\partial\theta}{\partial n} = 0$ a.e. on $\partial\Omega\times]0,T[$. We further have, for all k and t,

$$(3.25) \int_\Omega \left|\theta(x,0) - \theta^0(x)\right|^2 dx \le 3\left(\int_\Omega |\theta(x,0) - \theta(x,t)|^2\,dx\right.$$
$$\left. + \int_\Omega \left|\theta(x,t) - \theta^k(x,t)\right|^2 dx + \int_\Omega \left|\theta^k(x,t) - \theta^0(x)\right|^2 dx\right).$$

$$\int_\Omega |\theta(x,0) - \theta^0(x)|^2\,dx \le \frac{3}{2}\hat t \int_0^{\hat t}\int_\Omega \left(|\theta_t|^2 + |\theta_t^k|^2\right)(x,t)\,dx\,dt + \frac{3}{\hat t}\int_0^{\hat t}\int_\Omega |\theta - \theta^k|^2\,(x,t)\,dx\,dt\,. \tag{3.26}$$

Taking $\hat t$ sufficiently small, and then k sufficiently large, we conclude that $\theta(x,0) = \theta^0(x)$ a.e.; hence, the initial and boundary conditions (2.2) are fulfilled. We thus have proved the existence of a solution in Theorem 3.1. To prove uniqueness, we consider two solutions $(w_1,\theta_1),(w_2,\theta_2)$. Analogously as in inequality (3.23), we have, for all $t \in [0,T]$,

$$\int_\Omega |\theta_1 - \theta_2|^2\,(x,t)\,dx + \frac{d}{dt}\int_\Omega \left|\int_0^t \nabla(\theta_1-\theta_2)\,d\tau\right|^2 dx \le C_5 \int_0^t\int_\Omega |\theta_1 - \theta_2|^2(x,\tau)\,dx\,d\tau\,. \tag{3.27}$$

Gronwall's inequality yields $\theta_1 = \theta_2$, hence $w_1 = w_2$. Theorem 3.1 is proved. □

4 Proof of Theorem 2.2. Case I.

First, we prove Theorem 2.2 in the case when hypotheses (2.4) and (2.7)* hold. We fix some $\varepsilon > 0$ (to be specified later) and define auxiliary functions $T_\varepsilon, \mu_\varepsilon : \mathbb{R} \to \mathbb{R}^+$ by the formulae

(4.1) (i) $$T_\varepsilon(s) := \max\{\varepsilon, |s|\}\,,$$

(ii) $$\mu_\varepsilon(s) := \mu(T_\varepsilon(s))\,,$$

for $s \in \mathbb{R}$. Let γ_ε be the operator

$$\gamma_\varepsilon[w,\theta] := -\frac{1}{\mu_\varepsilon(\theta)}\Big(f_1[w] + T_\varepsilon(\theta)\,f_2[w]\Big)\,. \tag{4.2}$$

Using hypotheses **H1**, **H2**, we easily check that the conditions (3.2), (3.3) are fulfilled. By Theorem 3.1, the system

(4.3) (i) $$\mu_\varepsilon(\theta)\,w_t + f_1[w] + T_\varepsilon(\theta)\,f_2[w] = 0\,,$$

(ii) $$(\theta + F_1[w])_t - \Delta\theta = \psi(x,t,\theta)\,,$$

has a unique solution $(w,\theta) =: (w^\varepsilon,\theta^\varepsilon)$ satisfying the initial and boundary conditions (2.2) such that θ^ε, w^ε, $w_t^\varepsilon \in L^\infty(\Omega\times]0,T[)$, $\theta_t^\varepsilon, \Delta\theta^\varepsilon \in L^2(\Omega\times]0,T[)$.

Let us test equation (4.3) (ii) with an arbitrary function $p \in W^{1,2}(\Omega\times]0,T[)$ such that $p \le 0$ almost everywhere. Assumptions (2.11) – (2.15) yield, for a.e. $t \in]0,T[$,

$$\begin{aligned}
&\int_\Omega \Big(p\,\theta_t^\varepsilon + \langle\nabla p, \nabla\theta^\varepsilon\rangle\Big)(x,t)\,dx \\
&= \int_\Omega p\Big(\psi_0(x,t) + \psi(x,t,\theta^\varepsilon) - \psi(x,t,0)\Big)\,dx + \int_\Omega \Big(|p|\,F_1[w^\varepsilon]_t\Big)(x,t)\,dx \\
&\le K_4\int_\Omega \Big(|p||\theta^\varepsilon|\Big)(x,t)\,dx + \int_\Omega \Big(|p|\,g[w^\varepsilon]_t\,f_1[w^\varepsilon]\Big)(x,t)\,dx\,,
\end{aligned} \tag{4.4}$$

$$(4.5)\qquad \int_\Omega \Big(|p|\, g[w^\varepsilon]_t\, f_1[w^\varepsilon]\Big)(x,t)\,dx$$
$$= \; - \int_\Omega \left(|p|\, \frac{g[w^\varepsilon]_t}{w^\varepsilon_t}\, \frac{f_1[w^\varepsilon]}{\mu_\varepsilon(\theta^\varepsilon)}\right) \left(f_1[w^\varepsilon] \,+\, T_\varepsilon(\theta^\varepsilon)\, f_2[w^\varepsilon]\right)(x,t)\,dx\,.$$

To estimate the last integral, we first notice that for every $a, b, r \in \mathbb{R}$ we have

$$(4.6)\qquad -\,a^2 \,-\, rab \;\le\; \frac{1}{2}\left(\sqrt{1\,+\,r^2} \,-\, 1\right)\left(a^2 \,+\, b^2\right)$$
$$\le\; \frac{|r|}{2}\min\Big\{1, |r|\Big\}\left(a^2 \,+\, b^2\right)\,.$$

Hence, by assumptions (2.14) and (2.4),

$$(4.7)\quad -\,\frac{g[w^\varepsilon]_t}{w^\varepsilon_t}\,\frac{f_1[w^\varepsilon]}{\mu_\varepsilon(\theta^\varepsilon)}\Big(f_1[w^\varepsilon] \,+\, T_\varepsilon(\theta^\varepsilon)\, f_2[w^\varepsilon]\Big) \;\le\; \frac{K_5}{2\mu_0}\left((f_1[w^\varepsilon])^2 + (f_2[w^\varepsilon])^2\right) T_\varepsilon(\theta^\varepsilon)\,.$$

Combining inequalities (4.4), (4.5) and (4.7) with assumption (2.7)*, we obtain that

$$(4.8)\quad \int_\Omega \Big(p\,\theta^\varepsilon_t \,+\, \langle\nabla p, \nabla\theta^\varepsilon\rangle\Big)(x,t)\,dx \;\le\; \left(K_4 \,+\, \frac{K_5 K_2^2}{\mu_0}\right)\int_\Omega \Big(|p|\, T_\varepsilon(\theta^\varepsilon)\Big)(x,t)\,dx\,.$$

Put $\beta := K_4 + K_5\, K_2^2/\mu_0\,,\ \ \varepsilon := \delta e^{-\beta T}$, and

$$(4.9)\qquad p(x,t) \;:=\; -\,\Big(\delta e^{-\beta t} \,-\, \theta^\varepsilon(x,t)\Big)^+ \qquad \text{for } (x,t) \,\in\, \Omega \times\,]0,T[\,.$$

Then it follows from inequality (4.8) that

$$(4.10)\quad \int_\Omega \Big(p\big(p \,+\, \delta e^{-\beta t}\big)_t \,+\, |\nabla p|^2\Big)(x,t)\,dx \;\le\; \beta\int_\Omega |p|\,\Big(|p| \,+\, \delta e^{-\beta t}\Big)(x,t)\,dx\,.$$

This yields, in particular,

$$(4.11)\qquad \frac{1}{2}\,\frac{d}{dt}\int_\Omega p^2(x,t)\,dx \,+\, \int_\Omega |\nabla p|^2(x,t)\,dx \;\le\; \beta\int_\Omega p^2(x,t)\,dx\,,$$

hence, by Gronwall's inequality, $p \equiv 0$. We therefore have $\theta^\varepsilon(x,t) \ge \delta e^{-\beta t} > \varepsilon$ a.e., and, in particular, $T_\varepsilon(\theta^\varepsilon) \,=\, \theta^\varepsilon, \mu_\varepsilon(\theta^\varepsilon) \,=\, \mu(\theta^\varepsilon)$. We thus have proved that $(w,\theta) = (w^\varepsilon, \theta^\varepsilon)$ is a solution satisfying the conditions of Theorem 2.2. Uniqueness follows from Theorem 3.1. □

5 Proof of Theorem 2.2. Case II.

Assume that hypotheses (2.4)* and (2.7) hold. We introduce a parameter $R > 0$ and define the auxiliary operators

$$(5.1)\qquad f_i^R[w] \;:=\; f_i\Big[\mathfrak{s}_R[w]\Big]\,, \quad i \,=\, 1,2\,,$$

$$(5.2)\qquad F_i^R[w] \;:=\; F_i\Big[\mathfrak{s}_R[w]\Big]\,, \quad i \,=\, 1,2\,,$$

$$(5.3)\qquad g^R[w] \;:=\; g\Big[\mathfrak{s}_R[w]\Big]\,,$$

$-R,\, R$. Since $\mathfrak{s}_R$ is causal, and Lipschitz continuous with respect to both the norms of $C[0,T]$ and $W^{1,1}(0,T)$, and since $\mathfrak{s}_R[w]_t\, w_t = (\, s_R[w]_t)^2$ a.e. for all $w \in W^{1,1}(0,T)$, all the hypotheses **H1** to **H6** are satisfied if we replace f_i, F_i, g by f_i^R, F_i^R, g^R. Moreover, there exists a function $K_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(5.4) \qquad |f_i^R[w](t)| \,\le\, K_2(R) \qquad \forall\, w \in C[0,T]\,,\; R > 0\,,\; t \in [0,T]\,,\; i = 1,2\,.$$

Indeed, inequality (5.4) is obvious for $i = 2$. Since $|\mathfrak{s}_R[w]| \le R$ by definition of the stop operator, it suffices to choose any function

$$(5.5) \qquad K_2(R) \,\ge\, \lambda(R)\,,$$

where λ is the function introduced in (2.7).

For $i = 1$, we use the Lipschitz continuity of f_1 for proving inequality (5.4). Let $\varphi := f_1[0] \in C[0,T]$ be the image of the null function under f_1. By (2.5), we have

$$(5.6) \qquad \left|f_1^R[w](t)\right| \,\le |\varphi(t)| \,+\, K_1 \left|\mathfrak{s}_R[w]\right|_{[0,t]}\,,$$

so that (5.4) holds with

$$(5.7) \qquad K_2(R) \,:=\, \max\left\{\lambda(R)\,,\, |\varphi|_\infty \,+\, K_1\, R\right\}.$$

The results of Section 4 imply that the system

$$(5.8) \quad \text{(i)} \qquad \mu(\theta)\, w_t \,+\, f_1^R[w] \,+\, \theta\, f_2^R[w] \,=\, 0\,,$$

$$\text{(ii)} \qquad \left(\theta \,+\, F_1^R[w]\right)_t \,-\, \Delta\theta \,=\, \psi(x,t,\theta)\,,$$

together with the initial and boundary conditions (2.2), has for each $R > 0$ a unique solution $(w,\theta) = (w^R, \theta^R)$ satisfying the conditions of Theorem 2.2.

Integrating equation (5.8) (ii) with respect to t, and using the fact that $F_1^R[w] \ge 0$ a.e. by hypothesis (2.16), we obtain that

$$(5.9) \quad \theta^R(x,t) \,-\, \Delta \int_0^t \theta^R(x,\tau)\, d\tau$$
$$\le\, \theta^0(x) \,+\, F_1^R[w^R(x,\,\cdot)](0) + \int_0^t \psi_0(x,\tau)\, d\tau \,+\, K_4 \int_0^t \theta^R(x,\tau)\, d\tau\,.$$

The operator F_1 is causal; hence, for any arbitrary input $\tilde{w} \in W^{1,1}(0,T)$, the output value $F_1[\tilde{w}](0)$ depends only on the value of $\tilde{w}(0)$. From hypotheses (2.8), (2.9) it follows that there exists a locally Lipschitz continuous function $\varphi : [0,\infty[\to [0,\infty[$ such that

$$(5.10) \qquad F_1[\tilde{w}](0) \,=\, \varphi(\tilde{w}(0)) \qquad \forall\, \tilde{w} \in W^{1,1}(0,T)\,.$$

We fix R in such a way that

$$(5.11) \qquad |w^0(x)| \,\le\, R \quad \text{a.e.}$$

Then $\mathfrak{s}_R[w^R(x,\,\cdot)](0) = w^0(x)$ a.e., hence $F_1^R[w^R(x,\,\cdot)](0) = \varphi(w^0(x))$ a.e.

Next, observe that inequality (5.9) has the form

$$(5.12) \qquad u_t \,-\, \Delta u \,+\, u \,\le\, \rho(x,t) \quad \text{a.e.}\,,$$

with $u(x,t) := e^{\ldots}\int_0^t \theta^R(x,\tau)\,d\tau$ and

$$\rho(x,t) := \Big(\theta^0(x) + \varphi(w^0(x)) + \int_0^t \psi_0(x,\tau)\,d\tau\Big)\, e^{-(K_4+1)\,t}\,.$$

Thus, $\rho \in L^q(\Omega\times]0,T[)$ is independent of R, $u(x,0) = 0$ in Ω, $\partial u/\partial n = 0$ on $\partial\Omega\times]0,T[$.

Let v be the solution of the equation

$$\text{(5.13)}\quad v_t - \Delta v + v = \rho(x,t)\,,\quad v(x,0) = 0 \text{ in } \Omega\,,\ \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\times]0,T[\,.$$

By Lemma 3.3, there exists a constant $\tilde C > 0$, independent of R, such that

$$\text{(5.14)}\qquad \|v\|_\infty \le \tilde C\,.$$

On the other hand, testing the inequality

$$\text{(5.15)}\qquad (u - v)_t - \Delta(u - v) + (u - v) \le 0$$

with $(u-v)^+$, we find that $(u-v)^+ \equiv 0$, whence

$$\text{(5.16)}\qquad 0 \le u(x,t) \le v(x,t) \quad \text{a.e. in } \Omega\times]0,T[\,.$$

Consequently,

$$\text{(5.17)}\qquad \int_0^t \theta^R(x,\tau)\,d\tau \le \tilde C\, e^{(K_4+1)\,t} \quad \text{a.e.}$$

By the definition of the stop operator, we have

$$\text{(5.18)}\qquad \Big|\mathfrak{s}_R[\tilde w]\Big|_{[0,t]} \le |\tilde w|_{[0,t]} \qquad \forall\, \tilde w \in C[0,T]\,,$$

independently of R. Integrating equation (5.8) (i), and using inequalities (2.4)*, (2.7), (5.4) and (5.17), we obtain that

$$\begin{aligned}
\text{(5.19)}\ |w^R(x,t)| &\le |w^0(x)| + \frac{1}{\mu_0}\int_0^t \Big|f_1^R[w^R] + \theta^R f_2^R[w^R]\Big|(x,\tau)\,d\tau \\
&\le |w^0(x)| + \frac{1}{\mu_0}\int_0^t \Big(|\varphi(\tau)| + K_1|w^R(x,\cdot)|_{[0,\tau]} \\
&\qquad + \theta^R(x,\tau)\,\lambda\big(|w^R(x,\cdot)|_{[0,\tau]}\big)\Big)\,d\tau \\
&\le C_6\Big(1 + \int_0^t |w^R(x,\cdot)|_{[0,\tau]}\,d\tau + \lambda\big(|w^R(x,\cdot)|_{[0,t]}\big)\int_0^t \theta^R(x,\tau)\,d\tau\Big) \\
&\le C_7\Big(1 + \int_0^t |w^R(x,\cdot)|_{[0,\tau]}\,d\tau + \lambda\big(|w^R(x,\cdot)|_{[0,t]}\big)\Big)\,,
\end{aligned}$$

with some constants C_6, C_7 which is independent of R. Note that hypothesis (2.4)* was substantial in the above computation.

Next, we choose a constant $C_8 > 0$ such that

$$\text{(5.20)}\qquad \lambda(s) \le \frac{1}{2\,C_7}\,s + C_8 \qquad \forall\, s > 0\,.$$

Then inequality (5.19) implies that

$$\text{(5.21)}\qquad \Big|w^R(x,\cdot)\Big|_{[0,t]} \le C_9\,\Big(1 + \int_0^t \Big|w^R(x,\cdot)\Big|_{[0,\tau]}\,d\tau\Big)$$

yields that $|w^R(x,t)| \le C_9 e^{C_9 t}$ a.e., and, choosing

$$R > C_9 \, e^{C_9 T} \tag{5.22}$$

in addition to (5.11), we obtain that

$$|w^R(x,t)| < R \quad \text{a.e.} \ , \tag{5.23}$$

whence $\mathfrak{s}_R[w^R] = w^R$ a.e. The functions w^R, θ^R therefore satisfy (2.1), (2.2), and Theorem 2.2 is proved. □

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