Deviations from typical type proportions in critical multitype Galton-Watson processes

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Abstract

Consider a critical K–type Galton-Watson process $\{\mathbf{Z}(t) : t = 0, 1, ...\}$, and a real vector $\mathbf{w} = (w_1, ..., w_K)^\top$. It is well-known that under rather general assumptions, $\langle \mathbf{Z}(t), \mathbf{w} \rangle := \sum_k Z_k(t) w_k$ conditioned on non-extinction and appropriately scaled has a limit in law as $t \uparrow \infty$ ([Vat77]). But the limit degenerates to 0 if the vector $\mathbf{w}$ deviates seriously from 'typical' type proportions, i.e. if $\mathbf{w}$ is orthogonal to the left eigenvectors related to the maximal eigenvalue of the mean value matrix. We show that in this case (under reasonable additional assumptions on the offspring laws) there exists a better normalization which leads to a non-degenerate limit. Opposed to the finite variance case, which was already resolved in Athreya and Ney [AN74] and Badalbaev and Mukhitdinov [BM89], the limit law (for instance its "index") may seriously depend on $\mathbf{w}$.

1 Introduction

1.1 Background

Let $Z_k(t)$ denote the number of particles of type $k \in \{1, ..., K\} =: \mathsf{K}$ at time t in a multitype Galton-Watson branching process $\mathbf{Z} = (Z_1, ..., Z_K)^\top$. Let the particle reproduction be specified by the vector generating function

$$\mathbf{F}(\mathbf{z}) \;=\; \big(F_1(\mathbf{z}), ..., F_K(\mathbf{z})\big)^\top \tag{1}$$

where [1)]

$$F_j(\mathbf{z}) \;:=\; E_j z_1^{Z_1(1)} \cdots z_K^{Z_K(1)}, \qquad \mathbf{z} \in \mathsf{E}, \tag{2}$$

with $\mathsf{E} := \Big\{\mathbf{z} = (z_1, ..., z_K)^\top : \; z_k \text{ complex}, \; |z_k| \le 1, \; k \in \mathsf{K}\Big\}$ and where the symbol E_j refers to the law P_j of the process started with a single particle at time $t = 0$ having type j. Denote by $\mathbf{M}$ the matrix of expectations $m_{j,k} := E_j Z_k(1)$. Assume that $\mathbf{M}$ is irreducible, aperiodic, and has maximal eigenvalue 1 (*criticality*). Let $\mathbf{u} = (u_1, ..., u_K)^\top > \mathbf{0}$ and $\mathbf{v} = (v_1, ..., v_K) > \mathbf{0}^\top$ denote the right (column) and left (row) eigenvectors of $\mathbf{M}$ corresponding to this eigenvalue normalized such that

$$\|\mathbf{u}\| := \sum_{j \in \mathsf{K}} |u_j| = \langle \mathbf{u}, \mathbf{1} \rangle = 1 \quad \text{and} \quad \mathbf{v}\mathbf{u} \;=\; \sum_{j \in \mathsf{K}} v_j u_j \;=\; 1 \tag{3}$$

(where $\mathbf{1} = (1, ..., 1)^\top$). It is well-known ([Vat77, GH78]) that in the case

$$\mathbf{v}\,\big(\mathbf{1} - \mathbf{F}\,(\mathbf{1} - x\mathbf{u})\big) \;\sim\; x - x^{1+\alpha} L(x) \quad \text{as } x \downarrow 0, \tag{4}$$

[1)] If a column vector $\mathbf{z}$ occurs as an argument in a function, by an abuse of notation it is often automatically transposed into the corresponding row vector $\mathbf{z}^\top$ if no confusion is possible.

where $\alpha \in (0,1]$ and L is a (positive) function slowly varying at $0+$. Moreover, for the *survival probability* the following asymptotics holds:

$$Q_j(t) \ := \ P_j\left(\mathbf{Z}(t) \neq \mathbf{0}\right) \ \sim \ u_j\, t^{-1/\alpha} L^*(t) \quad \text{as} \quad t \uparrow \infty. \tag{5}$$

Here L^* is an appropriate function slowly varying at infinity. Finally, for each initial type j, the conditioned random vector $\left\{q(t)\mathbf{Z}(t) \mid \mathbf{Z}(t) \neq \mathbf{0}\right\}$, where

$$q(t) \ := \sum_{j \in \mathsf{K}} v_j Q_j(t) \ \sim t^{-1/\alpha} L^*(t) \quad \text{as} \quad t \uparrow \infty, \tag{6}$$

has a well-described long-term limit in law independent of the initial state: [2)]

$$\lim_{t \uparrow \infty} E_j \left\{ \exp\left[-i\, q(t) \left\langle \mathbf{Z}(t), \mathbf{w} \right\rangle\right] \,\middle|\, \mathbf{Z}(t) \neq \mathbf{0} \right\} \ = \ 1 - \frac{i\, \mathbf{vw}}{\left(1 + (i\, \mathbf{vw})^{\alpha}\right)^{1/\alpha}}, \tag{7}$$

$j \in \mathsf{K}$, $\mathbf{w}^{\top} \in \mathsf{R}^K$. Hence, the limit law specified by its Fourier transform (7) is supported by the ray $\{\lambda \mathbf{v} : \lambda \geq 0\}$ in R^K. In this sense, the left eigenvector $\mathbf{v}$ describes *"typical" limiting type proportions.* Consequently, for a fixed $\mathbf{w}$ with $\mathbf{vw} = 0$ (i.e. fixing attention to a "deviating" type situation),

$$\left\{ q(t) \left\langle \mathbf{Z}(t), \mathbf{w} \right\rangle \,\middle|\, \mathbf{Z}(t) \neq \mathbf{0} \right\} \underset{t \uparrow \infty}{\longrightarrow} 0 \quad \text{in } P_j\text{-probability.} \tag{8}$$

Our aim is to ask for a better scaling factor $\hat{q}(t)$ in order that for such a $\mathbf{w} \neq \mathbf{0}$ a limiting distribution of the conditional random variable

$$\left\{ \hat{q}(t) \left\langle \mathbf{Z}(t), \mathbf{w} \right\rangle \,\middle|\, \mathbf{Z}(t) \neq \mathbf{0} \right\}$$

exists non-trivially. Athreya and Ney [AN74] and Badalbaev and Mukhitdinov [BM89] resolved this problem for processes with *finite covariances* [which implies that $\alpha = 1$ in (4)]. Here $\hat{q}(t) = \sqrt{q(t)}$ (which is of order $t^{-1/2}$), and the limit law is *"symmetric exponential"* (with the parameter depending on $\mathbf{w}$).

1.2 Main result

In order to find a scaling, we impose a bit stronger condition than (4). Start with introducing some additional notation. Recalling (2) and (1), we introduce vectors $\mathbf{D}(\mathbf{1}-\mathbf{z}) = \left(D_1(\mathbf{1}-\mathbf{z}), ..., D_K(\mathbf{1}-\mathbf{z})\right)^{\top}$ by

$$\mathbf{1} - \mathbf{F}(\mathbf{z}) \ =: \ \mathbf{M}(\mathbf{1}-\mathbf{z}) - \mathbf{D}(\mathbf{1}-\mathbf{z}), \qquad \mathbf{z} \in \mathsf{E}. \tag{9}$$

Recall that [3)]

$$\left|\mathbf{D}(\mathbf{1}-\mathbf{z})\right| \ \leq \ \mathbf{c}\, \left\|\mathbf{1}-\mathbf{z}\right\|. \tag{10}$$

[2)] If $\beta > 0$, in the case of the complex function $z \mapsto z^{\beta}$, we always consider the main branch, i.e. the branch for which $1^{\beta} = 1$.

[3)] If $\mathbf{z}$ is a (row) vector, we denote by $|\mathbf{z}|^{\beta}$ the *vector* with components $|z_k|^{\beta}$. Similarly, we will proceed with matrices. Furthermore, with the small letter c we always denote a positive constant, or with $\mathbf{c}$ such a vector, which may change from term to term.

Fix a vector $\mathbf{w}$ with $\mathbf{vw} = 0$, and set

$$J(\lambda, \mathbf{w}, t) \ := \ \sum_{r=t}^{\infty} \mathbf{v}\,\mathbf{D}\big(\mathbf{1} - \exp[-i\lambda \mathbf{M}^r \mathbf{w}]\big)\,, \tag{11}$$

$\lambda \in \mathsf{R}$, $t \geq 0$, where by definition $\mathrm{e}^{\mathbf{z}} = \exp[\mathbf{z}]$ is the column vector with components e^{z_1}, ..., e^{z_K}.

Remark 1 (i) (well-defined) Note that $J(\cdot, \mathbf{w}, \cdot)$ is well-defined under our criticality assumptions and by $\mathbf{vw} = 0$, since in this case

$$(\mathbf{M}^r \mathbf{w})_k = O(\varrho^r) \quad \text{as} \quad r \uparrow \infty, \qquad k \in \mathsf{K}, \tag{12}$$

for some $\varrho \in (0,1)$ (see, for instance, [AN72, Frobenius Theorem 5.2.1]). In fact, by (10),

$$\big|D_j\left(\mathbf{1} - \exp[-i\lambda \mathbf{M}^t \mathbf{w}]\right)\big| \ \leq \ c\ \big\|\lambda \mathbf{M}^t \mathbf{w}\big\| \ \leq \ c\ |\lambda|\, \varrho^t, \tag{13}$$

$j \in \mathsf{K}$, leading to a convergent series in (11).
(ii) (conjugate) Since for the conjugate expressions,

$$\overline{\mathbf{D}(\mathbf{1}-\mathbf{z})} = \mathbf{D}(\mathbf{1} - \overline{\mathbf{z}}), \quad \text{hence} \ \ \overline{J(\lambda, \mathbf{w}, t)} = J(-\lambda, \mathbf{w}, t)\,.$$

(iii) (special J) Note also that

$$\mathbf{Mw} = \mathbf{0} \tag{14}$$

is a sufficient condition for $\mathbf{vw} = 0$ because $\mathbf{vw} = (\mathbf{vM})\,\mathbf{w} = \mathbf{v}\,(\mathbf{Mw})$. In this case even $J(\lambda, \mathbf{w}, t) = 0$, $t \geq 1$, and $J(\lambda, \mathbf{w}, 0) = \mathbf{v}\,\mathbf{D}\big(\mathbf{1} - \mathrm{e}^{-i\lambda \mathbf{w}}\big)$. ◇

For convenience we introduce the following hypothesis, a related example will be discussed in the next subsection.

Hypothesis 2 (basic assumptions) Recall that we are dealing with a critical process satisfying (4), and that we fixed a vector $\mathbf{w}$ such that $\mathbf{vw} = 0$.

(a) (index G) There exist an index $G \in [\alpha, 1)$ and a real constant φ with $|\varphi| \leq 1 - G$ such that

$$J(\lambda, \mathbf{w}, 0) \ \sim \ -\mathrm{e}^{i\frac{\pi}{2}\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G} L_1\left(|\lambda|\right) \quad \text{as} \ \lambda \to 0\pm\,, \tag{15}$$

where L_1 is a function slowly varying at $0+$.

(b) (tail behavior) There is a $\lambda_0 \in (0,1)$ such that for each $\varepsilon > 0$ there exists a $T = T(\varepsilon)$ such that for all real λ with $|\lambda| \leq \lambda_0$ and all $t \geq T$,

$$\left|\sum_{r=t}^{\infty} D_k\left(\mathbf{1} - \exp[-i\lambda \mathbf{M}^r \mathbf{w}]\right)\right| \ \leq \ \varepsilon\ \big|J(\lambda, \mathbf{w}, 0)\big|\,, \qquad k \in \mathsf{K}. \tag{16}$$

(c) (local behavior) There exists a constant $\gamma \in [0, \alpha/2)$ and an $\varepsilon > 0$ such that

$$\begin{aligned} &\left|D_k(\mathbf{1}-\mathbf{z}) - D_k(\mathbf{1}-\mathbf{z}^*)\right| \\ &\leq c \left[\|\mathbf{1}-\mathbf{z}\|^{\alpha-\gamma} + \|\mathbf{1}-\mathbf{z}^*\|^{\alpha-\gamma}\right] \|\mathbf{z}-\mathbf{z}^*\| \end{aligned} \tag{17}$$

for all $\mathbf{z}, \mathbf{z}^* \in \mathsf{E}$ with

$$\|\mathbf{1}-\mathbf{z}\| \leq \varepsilon \quad \text{and} \quad \|\mathbf{1}-\mathbf{z}^*\| \leq \varepsilon.$$

◇

Set

$$R(x) := x^{1+G} L_1(x) \tag{18}$$

[with L_1 from (a)], and let $\hat{R}$ denote the "inverse" function satisfying

$$\hat{R}\left(R(x)\right) \sim x \quad \text{and} \quad R\big(\hat{R}(x)\big) \sim x \quad \text{as } x \downarrow 0. \tag{19}$$

According to [Sen76, 5°, p.21] such a function exists, is asymptotically unique (in an obvious sense) and

$$\hat{R}(x) = x^{1/(1+G)} \hat{L}_1(x) \tag{20}$$

with $\hat{L}_1$ a function also slowly varying at $0+$. In fact, $\hat{R}$ can be selected to be monotone ([Sen76, 4°, p.19]), and, moreover, we assume throughout that $\hat{R}$ is a monotone non-decreasing function defined on all of $(0, \infty)$.

Recall the notation $q(t)$ introduced in (6). Set

$$\hat{q}(t) := \hat{R}\left(q(t)\right). \tag{21}$$

In view of (20) and (6),

$$\begin{aligned} \hat{q}(t) &= q^{1/(1+G)}(t) \hat{L}\left(q(t)\right) \\ &\sim t^{-1/\left(\alpha(1+G)\right)} \left(L^*(t)\right)^{1/(1+G)} \hat{L}\left(t^{-1/\alpha} L^*(t)\right). \end{aligned}$$

Here is our main result:

Theorem 3 (limiting deviations) *Under Hypothesis* 2, *the following convergence statements hold. For all* $j \in \mathsf{K}$ *and* $\lambda \in \mathsf{R}$,

(a) (ratio limit theorem)

$$\begin{aligned} &\lim_{t\uparrow\infty} E_j \left\{ \exp\left[-i\lambda \hat{R}\big(1/\langle \mathbf{Z}(t), \mathbf{u}\rangle\big) \langle \mathbf{Z}(t), \mathbf{w}\rangle\right] \,\Big|\, \mathbf{Z}(t) \neq \mathbf{0} \right\} \\ &= \exp\left[-\mathrm{e}^{i\frac{\pi}{2}\varphi \operatorname{sign}\lambda} |\lambda|^{1+G}\right], \end{aligned}$$

(b) (absolute scaling)

$$\lim_{t\uparrow\infty} E_j \left\{ \exp\left[-i\lambda\, \hat{q}(t) \left\langle \mathbf{Z}(t), \mathbf{w} \right\rangle\right] \;\middle|\; \mathbf{Z}(t) \neq \mathbf{0} \right\}$$
$$= 1 - \frac{\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\, |\lambda|^{1+G}}{\left(1 + \left(\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\, |\lambda|^{1+G}\right)^{\alpha}\right)^{1/\alpha}}\,.$$

Consequently, using the sample normalization such as in (a) we get ([Fel71, formula (XVII.3.18)]) a *stable* limit law of index $1+G$ (note that G might be larger than α), whereas in the 'absolute' scaling case of (b) we get a *mixture* of such laws, with the weights chosen according to the "classical" limit law (7).

Remark 4 ($G=1$) We excluded the case $G=1$, since the latter would require more delicate arguments (see, for instance (34) and (50) below) and would enlarge the exposition seriously. ◇

After some preparations in Section 2, the proof of the theorem will follow in the final section.

1.3 Example

Here we want to illustrate the assumptions to our Theorem 3 in terms of an example with $G > \alpha$.

Consider the case of $K = 3$ types. For $\theta \in (0, 1/6)$, let

$$\mathbf{F}(\mathbf{z}) \;:=\; \begin{pmatrix} \left(\frac{1}{6}-\theta\right) + \left(\frac{1}{3}+\theta\right) z_1 + \frac{1}{3} z_2 + \frac{1}{6} z_3^2 \\ \left(\frac{1}{6}+\theta\right) + \left(\frac{1}{3}-\theta\right) \left(z_1 + \frac{1}{2}\left(1-z_1\right)^{1+\frac{2}{3}}\right) + \frac{1}{3} z_2 + \frac{1}{6} z_3^2 \\ \frac{1}{3} z_1 + \frac{1}{3} z_2 + \frac{1}{3}\left(z_3 + \frac{1}{2}\left(1-z_3\right)^{1+\frac{1}{2}}\right) \end{pmatrix} \tag{22}$$

in which case

$$\mathbf{M} \;=\; \begin{pmatrix} \frac{1}{3}+\theta & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3}-\theta & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad \mathbf{D}(\mathbf{z}) \;=\; \begin{pmatrix} \frac{1}{6} z_3^2 \\ \frac{1}{2}\left(\frac{1}{3}-\theta\right) z_1^{1+\frac{2}{3}} + \frac{1}{6} z_3^2 \\ \frac{1}{6} z_3^{1+\frac{1}{2}} \end{pmatrix}.$$

The mean matrix $\mathbf{M}$ has eigenvalues $1, \theta, 0,$ and we get

$$\mathbf{v} = (1,1,1) \quad \text{and} \quad \mathbf{u} = \frac{1}{3-3\theta}\,(1, 1-2\theta, 1-\theta)^{\top}$$

for the normalized eigenvectors corresponding to the eigenvalue 1. Now

$$\mathbf{v}\left(\mathbf{1} - \mathbf{F}\left(\mathbf{1} - x\mathbf{u}\right)\right) \;=\; x - \frac{1}{6}\left(\frac{1}{3}x\right)^{1+\frac{1}{2}} \left(1 + o(1)\right) \quad \text{as} \;\; x \downarrow 0,$$

hence (4) is true with $\alpha = 1/2$ and $L(x) \equiv \frac{1}{6}\left(\frac{1}{3}\right)^{3/2}$. On the other hand, the vector $\mathbf{w} = (1,-1,0)^\top$ satisfies

$$\mathbf{v}\mathbf{w} = 0, \quad \mathbf{M}^r\mathbf{w} = \theta^r\mathbf{w}, \qquad r \geq 0,$$

and, therefore, by the particular shape of $\mathbf{D}$,

$$J(\lambda, \mathbf{w}, t) \;=\; \frac{1}{2}\left(\frac{1}{3} - \theta\right)\sum_{r=t}^{\infty}\left(1 - \exp\left[-i\lambda\theta^r\right]\right)^{1+\frac{2}{3}}.$$

Clearly, as $\lambda \to 0\pm$,

$$\begin{aligned} J(\lambda, \mathbf{w}, t) &\sim \frac{1}{2}\left(\frac{1}{3} - \theta\right)(i\lambda)^{1+\frac{2}{3}}\sum_{r=t}^{\infty}\theta^{r\left(1+\frac{2}{3}\right)} \\ &= -\mathrm{e}^{i\,\frac{\pi}{2}\,\frac{1}{3}\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+\frac{2}{3}}\,\frac{1}{2}\left(\frac{1}{3} - \theta\right)\theta^{t\left(1+\frac{2}{3}\right)}\left(1 - \theta^{1+\frac{2}{3}}\right)^{-1}, \end{aligned}$$

hence (a) and (b) of Hypothesis 2 are true with $G = \frac{2}{3} > \frac{1}{2} = \alpha$, $\varphi = \frac{1}{3} = 1 - G$, and $L_1(x) \equiv c$. Moreover, (c) holds for $\gamma = 0$ and even on all of E. Finally, $q(t) \sim c\,t^{-2}$, $\hat{R}(x) \sim c\,x^{3/5}$, hence $\hat{q}(t) \sim c\,t^{-6/5}$. ◇

2 Preparations: an asymptotic expansion

2.1 The key expansion

Introduce the vector generating function $\mathbf{F}(t,\mathbf{z}) = \left(F_1(t,\mathbf{z}), ..., F_K(t,\mathbf{z})\right)^\top$ of $\mathbf{Z}(t)$, which by the branching property satisfies

$$\mathbf{F}(t+1,\mathbf{z}) = \mathbf{F}\left(\mathbf{F}(t,\mathbf{z})\right), \qquad t \geq 0, \quad \mathbf{z} \in \mathsf{E}. \tag{23}$$

We set

$$\mathbf{Q}(t,\mathbf{z}) \;:=\; \mathbf{1} - \mathbf{F}(t,\mathbf{z}), \quad \mathbf{Q}(t) \;:=\; \mathbf{Q}(t,\mathbf{0}). \tag{24}$$

From

$$|\mathbf{Q}(1,\mathbf{z})| \;\leq\; \mathbf{M}\,|\mathbf{1} - \mathbf{z}| \tag{25}$$

(see, e.g., [Sew74, p.114]), we get

$$|\mathbf{Q}(t+1,\mathbf{z})| \;=\; \left|\mathbf{Q}\left(1,\mathbf{F}(t,\mathbf{z})\right)\right| \;\leq\; \mathbf{M}\,|\mathbf{Q}(t,\mathbf{z})|. \tag{26}$$

By iteration,

$$|\mathbf{Q}(t,\mathbf{z})| \;\leq\; \mathbf{M}^t\,|\mathbf{1} - \mathbf{z}|, \qquad t \geq 0, \quad \mathbf{z} \in \mathsf{E}. \tag{27}$$

From (23) and (9), for $t \geq 1$,

$$\mathbf{Q}(t+1,\mathbf{z}) \;=\; \mathbf{1} - \mathbf{F}\left(\mathbf{1} - \mathbf{Q}(t,\mathbf{z})\right) \;=\; \mathbf{M}\mathbf{Q}(t,\mathbf{z}) - \mathbf{D}\left(\mathbf{Q}(t,\mathbf{z})\right), \tag{28}$$

and iteration gives

$$\mathbf{Q}(t+1,\mathbf{z}) \;=\; \mathbf{M}^t\mathbf{Q}(1,\mathbf{z}) - \sum_{0\le r<t-1} \mathbf{M}^r\mathbf{D}\left(\mathbf{Q}(t-r,\mathbf{z})\right), \qquad t\ge 0. \tag{29}$$

Now we introduce

$$\mathbf{D}(1,\mathbf{z}) \;:=\; \mathbf{D}(\mathbf{1}-\mathbf{z}), \tag{30}$$

set $\mathbf{z} = \mathrm{e}^{-i\lambda\mathbf{w}}$ with $\lambda\in\mathsf{R}$ and $\mathbf{w}^\top \in \mathsf{R}^K$, and define recursively

$$\mathbf{D}(t+1,\mathbf{z}) \;:=\; \mathbf{M}\mathbf{D}(t,\mathbf{z}) + \mathbf{D}\big(\mathbf{1}-\exp[-i\lambda\mathbf{M}^t\mathbf{w}]\big), \qquad t\ge 1. \tag{31}$$

This gives, for $t\ge 0$,

$$\begin{aligned} \mathbf{D}(t+1,\mathbf{z}) \;&=\; \mathbf{M}^t\mathbf{D}\big(\mathbf{1}-\mathrm{e}^{-i\lambda\mathbf{w}}\big) \\ &\quad + \sum_{0\,\le\, p\,<\, t} \mathbf{M}^p\mathbf{D}\big(\mathbf{1}-\exp[-i\lambda\mathbf{M}^{t-p}\mathbf{w}]\big). \end{aligned} \tag{32}$$

The following statement is the key in our development.

Proposition 5 (asymptotic expansion) *Impose Hypothesis* 2. *Define vectors* $\Delta(t,\lambda)$ *via*

$$\mathbf{Q}(t,\mathrm{e}^{-i\lambda\mathbf{w}}) \;=\; i\lambda\mathbf{M}^t\mathbf{w} - \mathbf{D}(t,\mathrm{e}^{-i\lambda\mathbf{w}}) + \Delta(t,\lambda), \qquad t\ge 1. \tag{33}$$

Then there are (strictly) positive constants β,δ,λ_0 *such that*

$$\big|\Delta_j(t,\lambda)\big| \;\le\; c\,t^\beta\left(|\lambda|^{(1+G+\delta)\wedge 2} + \varrho^t\right), \tag{34}$$

$t\ge 1$, $|\lambda|\le\lambda_0$, $j\in\mathsf{K}$, *with* ϱ *taken from Frobenius' Theorem* (12).

The proof requires some preparations, it will then be completed in the end of Subsection 2.4.

2.2 Preliminary estimates

As a preparation for the proof of Proposition 5, we first deal with the case $t=1$. From the definition (33) of $\Delta(1,\lambda)$ and (9) it follows that

$$\Delta(1,\lambda) \;=\; \mathbf{M}\left(\mathbf{1}-\mathrm{e}^{-i\lambda\mathbf{w}} - i\lambda\mathbf{w}\right). \tag{35}$$

But

$$\big|1-\mathrm{e}^{-ix}-ix\big| \;\le\; \frac{1}{2}\,|x|^2, \qquad x\in\mathsf{R}, \tag{36}$$

hence

$$\big|\Delta(1,\lambda)\big| \;\le\; \mathbf{M}\,\big|\mathbf{1}-\mathrm{e}^{-i\lambda\mathbf{w}}-i\lambda\mathbf{w}\big| \;\le\; \mathbf{c}\,|\lambda|^2. \tag{37}$$

This verifies (34) in the case $t=1$.

For general t, we need some preparations. Recall the definition (11) of $J(\lambda,\mathbf{w},t)$.

Lemma 6 *Under Hypothesis* 2, *there exists a constant* $\lambda_0 \in (0,1)$, *and for all* $\varepsilon > 0$ *there is a* $T = T(\varepsilon) \geq 1$, *such that*

$$\left|D_k(t, \mathrm{e}^{-i\lambda\mathbf{w}}) - u_k J(\lambda, \mathbf{w}, 0)\right| \;\leq\; \varepsilon \left|J(\lambda, \mathbf{w}, 0)\right| + t^2 \varrho^t, \tag{38}$$

$|\lambda| \leq \lambda_0$, $t \geq T$, $k \in \mathsf{K}$, *with* ϱ *from* (12).

Proof By the representation formula (32) and definition (11),

$$\begin{aligned}\left|\mathbf{D}(t, \mathrm{e}^{-i\lambda\mathbf{w}}) - \mathbf{u}J(\lambda, \mathbf{w}, 0)\right| \;&\leq\; \mathbf{u}\left(\mathbf{v}\left|\sum_{p=t}^{\infty} \mathbf{D}\left(\mathbf{1} - \exp\left[-i\lambda\mathbf{M}^p\mathbf{w}\right]\right)\right|\right) \\ &+ \sum_{p=0}^{t-1} \left|\mathbf{M}^p - \mathbf{u}\mathbf{v}\right| \left|\mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^{t-p-1}\mathbf{w}]\right)\right|.\end{aligned} \tag{39}$$

According to the Frobenius Theorem (see, for example, [AN72, Theorem 5.2.1])

$$\left|\mathbf{M}^p - \mathbf{u}\mathbf{v}\right| \;\leq\; c\,\varrho^p\,\mathbf{I} \tag{40}$$

where ϱ is the same as in (12), and $\mathbf{I}$ is the unit matrix. From condition (c) in Hypothesis 2 (with $\mathbf{z}^* = 1$) it follows that there exist constants $\gamma \in [0, \alpha/2)$ and $\delta > 0$ such that

$$\left|\mathbf{D}(\mathbf{1} - \mathbf{z})\right| \;\leq\; \mathbf{c}\left\|\mathbf{1} - \mathbf{z}\right\|^{1+\alpha-\gamma} \tag{41}$$

for all $\mathbf{z} \in \mathsf{E}$ with $\|\mathbf{1} - \mathbf{z}\| \leq \delta$. By (12),

$$\left\|\mathbf{1} - \exp[-i\lambda\mathbf{M}^{t-p-1}\mathbf{w}]\right\| \;\leq\; \left\|\lambda\mathbf{M}^{t-p-1}\mathbf{w}\right\| \;\leq\; |\lambda|\,c\,\varrho^{t-p-1}. \tag{42}$$

Choose $\lambda_0 = \lambda_0(\delta) \in (0,1)$ such that the latter expression is smaller than δ whenever $|\lambda| \leq \lambda_0$. For these λ, from (41) and (42),

$$\left|\mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^{t-p-1}\mathbf{w}]\right)\right| \;\leq\; \mathbf{c}\left(|\lambda|\,\varrho^{t-p-1}\right)^{1+\alpha-\gamma} \;\leq\; \mathbf{c}\,|\lambda|^{1+\alpha-\gamma}\varrho^{t-p-1}. \tag{43}$$

Combining with (40),

$$\sum_{p=0}^{t-1} \left|\mathbf{M}^p - \mathbf{u}\mathbf{v}\right| \left|\mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^{t-p-1}\mathbf{w}]\right)\right| \;\leq\; t\,\mathbf{c}\,\varrho^{t-1}\,\mathbf{1} \;\leq\; t^2\varrho^t\,\mathbf{1}$$

for all sufficiently large t. Inserting into (39), and using part (b) of Hypothesis 2 for the other term in (39), the proof is complete. □

Corollary 7 *Under Hypothesis* 2, *there exist constants* $\gamma \in [0, \alpha/2)$ *and* $\lambda_0 \in (0,1)$ *such that for all* $|\lambda| \leq \lambda_0$ *and* $t \geq 1$,

$$\left|D_k(t, \mathrm{e}^{-i\lambda\mathbf{w}})\right| \;\leq\; c\left(|\lambda|^{1+G-\gamma} + t^2\varrho^t\right), \qquad k \in \mathsf{K}.$$

Proof This follows from the preceding lemma and condition (a) in Hypothesis 2. □

For convenience, we expose the following elementary observation.

Lemma 8 *Fix a constant* $B > 0$. *For all complex numbers* $x, y \neq 0$,

$$\left|(x+y)^{B+1} - x^{B+1}\right| \leq c\left|yx^B\right|\left(1 + |y/x|^B\right) = c\left(\left|yx^B\right| + \left|y^{B+1}\right|\right).$$

Proof Indeed,

$$\left|(x+y)^{B+1} - x^{B+1}\right| = \left|yx^B\right|\left|\frac{(1+z)^{B+1} - 1}{z}\right| \tag{44}$$

with $z = y/x$. Clearly,

$$\left|\frac{(1+z)^{B+1} - 1}{z}\right| \leq c\left(1 + |z|^B\right). \tag{45}$$

(To check this, built the ratio, and consider the cases $|z| \downarrow 0$ and $|z| \uparrow \infty$.) Hence the needed inequality follows. □

2.3 First estimates of $\bar{\mathbf{Q}}$ and Δ

First observe that by (27),

$$\left|\mathbf{Q}(t, \mathrm{e}^{-i\lambda\mathbf{w}})\right| \leq \mathbf{M}^t\left|\mathbf{1} - \mathrm{e}^{-i\lambda\mathbf{w}}\right| \leq \mathbf{M}^t|\mathbf{w}|\,|\lambda| \leq \mathbf{c}\,|\lambda|, \tag{46}$$

$t \geq 1$, $\lambda \in \mathsf{R}$, where in the last step we used the criticality. Hence, as in (41), there are constants $\gamma \in [0, \alpha/2)$ and $\lambda_0 \in (0,1)$ such that for all $|\lambda| \leq \lambda_0$,

$$\left|\mathbf{D}\left(\mathbf{Q}(t, \mathrm{e}^{-i\lambda\mathbf{w}})\right)\right| \leq \mathbf{c}\left\|\mathbf{Q}(t, \mathrm{e}^{-i\lambda\mathbf{w}})\right\|^{1+\alpha-\gamma} \leq \mathbf{c}\,|\lambda|^{1+\alpha-\gamma}. \tag{47}$$

Set

$$\bar{\mathbf{Q}}(t, \mathrm{e}^{-i\lambda\mathbf{w}}) := \mathbf{Q}(t, \mathrm{e}^{-i\lambda\mathbf{w}}) - i\lambda\mathbf{M}^t\mathbf{w}, \qquad t \geq 1. \tag{48}$$

Our next aim is to prove the following estimate.

Lemma 9 (first estimates) *There are constants* $\gamma \in [0, \alpha/2)$ *and* $\lambda_0 \in (0,1)$ *such that for all* $k \in \mathsf{K}$, $t \geq 1$ *and* $|\lambda| \leq \lambda_0$,

$$\left|\bar{Q}_k(t, \mathrm{e}^{-i\lambda\mathbf{w}})\right| \leq c\,t\,|\lambda|^{1+\alpha-\gamma} \tag{49}$$

and

$$\left|\Delta_k(t,\lambda)\right| \leq c\,t^2\,|\lambda|^{(1+2(\alpha-\gamma))\wedge 2}. \tag{50}$$

Proof Using (28), the definition (33) of $\Delta(t,\lambda)$, and adding and subtracting a $\mathbf{D}$–term, for $t \geq 1$ we obtain

$$\begin{aligned}&\mathbf{Q}(t+1,\mathrm{e}^{-i\lambda\mathbf{w}})\\&= \mathbf{M}\left[i\lambda\mathbf{M}^t\mathbf{w} - \mathbf{D}(t,\mathrm{e}^{-i\lambda\mathbf{w}}) + \Delta(t,\lambda)\right] - \mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^t\mathbf{w}]\right)\\&\quad - \left[\mathbf{D}\left(\mathbf{Q}(t,\mathrm{e}^{-i\lambda\mathbf{w}})\right) - \mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^t\mathbf{w}]\right)\right].\end{aligned}$$

On the other hand, again by (33),

$$\mathbf{Q}(t+1,\mathrm{e}^{-i\lambda\mathbf{w}}) \;=\; i\lambda\mathbf{M}^{t+1}\mathbf{w} - \mathbf{D}(t+1,\mathrm{e}^{-i\lambda\mathbf{w}}) + \Delta\left(t+1,\lambda\right). \tag{51}$$

Using the recursive relation (31), we conclude for the following recursion formula

$$\Delta\left(t+1,\lambda\right) \;=\; \mathbf{M}\Delta(t,\lambda) - \left[\mathbf{D}\left(\mathbf{Q}(t,\mathrm{e}^{-i\lambda\mathbf{w}})\right) - \mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^t\mathbf{w}]\right)\right],$$

$t \geq 1$. Hence, by iteration, for $t \geq 0$,

$$\begin{aligned}\Delta\left(t+1,\lambda\right) \;&=\; \mathbf{M}^t\Delta(1,\lambda)\\&- \sum_{0\,\leq\, p\,<\,t} \mathbf{M}^p\left[\mathbf{D}\left(\mathbf{Q}(t-p,\mathrm{e}^{-i\lambda\mathbf{w}})\right) - \mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^{t-p}\mathbf{w}]\right)\right].\end{aligned} \tag{52}$$

In view of the recursive relation (28), for $t \geq 1$,

$$\left|\bar{\mathbf{Q}}(t+1,\mathbf{z})\right| \;\leq\; \mathbf{M}\left|\bar{\mathbf{Q}}(t,\mathbf{z})\right| + \left|\mathbf{D}\left(\mathbf{Q}(t,\mathbf{z})\right)\right|, \tag{53}$$

and by iteration we conclude for

$$\left|\bar{\mathbf{Q}}(t+1,\mathbf{z})\right| \;\leq\; \mathbf{M}^t\left|\bar{\mathbf{Q}}(1,\mathbf{z})\right| + \sum_{0\,\leq\, r\,<t} \mathbf{M}^r\left|\mathbf{D}\left(\mathbf{Q}(t-r,\mathbf{z})\right)\right|, \qquad t \geq 0.$$

With $\mathbf{z} = \mathrm{e}^{-i\lambda\mathbf{w}}$, for the second term we use (47), whereas the first term is rewritten by means of (33), and the two new expressions $\left|\mathbf{D}(1,\mathrm{e}^{-i\lambda\mathbf{w}})\right| = \left|\mathbf{D}(1-\mathrm{e}^{-i\lambda\mathbf{w}})\right|$ and $\left|\Delta(1,\lambda)\right|$ are estimated by (41) and (37), respectively. This gives (49).

Clearly, by (52) and criticality, for $t \geq 1$,

$$\begin{aligned}\left|\Delta(t,\lambda)\right| \;\leq\; c\Bigg[&\left|\Delta(1,\lambda)\right|\\&+ \sum_{0\,<\, r\,\leq t-1}\left|\mathbf{D}\left(\mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right) - \mathbf{D}\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^r\mathbf{w}]\right)\right|\Bigg].\end{aligned} \tag{54}$$

Recalling (46), by assumption (c) in Hypothesis 2, for $k \in \mathsf{K}$ and $r \geq 1$,

$$\begin{aligned}&\left|D_k\left(\mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right) - D_k\left(\mathbf{1} - \exp[-i\lambda\mathbf{M}^r\mathbf{w}]\right)\right|\\&\quad\leq\; c\left[\left\|\mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right\|^{\alpha-\gamma} + \left\|\mathbf{1} - \exp[-i\lambda\mathbf{M}^r\mathbf{w}]\right\|^{\alpha-\gamma}\right] \times\\&\qquad\qquad \times \left\|\mathbf{1} - \exp[-i\lambda\mathbf{M}^r\mathbf{w}] - \mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right\|.\end{aligned}$$

Applying estimates as in (46) to the first line at the right hand side of this inequality and notation (48) to the second one, the inequality can be continued with

$$\leq\ c\ |\lambda|^{\alpha-\gamma}\Big(\big\|\mathbf{1}-\exp[-i\lambda\mathbf{M}^r\mathbf{w}]-i\lambda\mathbf{M}^r\mathbf{w}\big\|+\big\|\bar{\mathbf{Q}}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\big\|\Big).$$

Thus, by (36) and criticality,

$$\begin{aligned}&\Big|D_k\left(\mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right)-D_k\left(\mathbf{1}-\exp[-i\lambda\mathbf{M}^r\mathbf{w}]\right)\Big|\\&\qquad\leq\ c\ |\lambda|^{\alpha-\gamma}\left(\lambda^2+\big\|\bar{\mathbf{Q}}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\big\|\right).\end{aligned}\tag{55}$$

Inserting (49) gives,

$$\Big|D_k\left(\mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right)-D_k\left(\mathbf{1}-\exp[-i\lambda\mathbf{M}^r\mathbf{w}]\right)\Big|\ \leq\ c\,r\ |\lambda|^{1+2(\alpha-\gamma)}.\tag{56}$$

Using this bound for (54), combined with the estimate (37) for $\big|\Delta(1,\lambda)\big|$, also the second inequality follows. This completes the proof. □

2.4 Generalization by induction

Lemma 9 can be generalized as follows.

Lemma 10 (higher order estimates) *Impose Hypothesis* 2. *There is a constant* $\gamma\in[0,\alpha/2)$, *and to each* $N\geq 1$ *there are positive constants* $C_{1,N}$, $C_{2,N}$, $\beta_{1,N}$, $\beta_{2,N}$, δ_N, *and* $\lambda_N\in(0,1)$ *such that*

$$\big|\bar{Q}_k(t,\mathrm{e}^{-i\lambda\mathbf{w}})\big|\ \leq\ C_{1,N}\,t^{\beta_{1,N}}\Big[|\lambda|^{1+G-\gamma}+|\lambda|^{1+N(\alpha-\gamma)}+\varrho^t\Big]\tag{57}$$

and

$$\big|\Delta_k(t,\lambda)\big|\ \leq\ C_{2,N}\,t^{\beta_{2,N}}\Big(|\lambda|^{(1+G+\alpha-2\gamma)\wedge 2}+|\lambda|^{1+N(\alpha-\gamma)}+\varrho^t\Big),\tag{58}$$

$t\geq 1$, $|\lambda|\leq\lambda_N$, $k\in\mathsf{K}$, *with* $0<\varrho<1$ *taken from* (12).

Proof We proceed by induction on N. First of all, the case $N=1$ follows from Lemma 9. Assume the statements (57) and (58) hold for some $N\geq 1$. Then from (55) we get, for $1\leq r\leq t$,

$$\begin{aligned}&\Big|D_k\left(\mathbf{Q}(r,\mathrm{e}^{-i\lambda\mathbf{w}})\right)-D_k\left(\mathbf{1}-\exp[-i\lambda\mathbf{M}^r\mathbf{w}]\right)\Big|\\&\qquad\leq\ c\ |\lambda|^{\alpha-\gamma}\ t^{\beta_{1,N}}\Big(|\lambda|^{1+G-\gamma}+|\lambda|^{1+N(\alpha-\gamma)}+\varrho^t\Big)\\&\qquad\leq c\,t^{\beta_{1,N}}\Big(|\lambda|^{1+G+\alpha-2\gamma}+|\lambda|^{1+(N+1)(\alpha-\gamma)}+\varrho^t\Big).\end{aligned}$$

Inserting into (54), and combined with the estimate (37) for $\left|\Delta(1,\lambda)\right|$, gives, for $t \geq 1$,

$$\begin{aligned} |\Delta_k(t,\lambda)| \;&\leq\; c\left\{|\lambda|^2 + t^{1+\beta_{1,N}}\left(|\lambda|^{1+G+\alpha-2\gamma} + |\lambda|^{1+(N+1)(\alpha-\gamma)} + \varrho^t\right)\right\} \\ &\leq\; c\, t^{1+\beta_{1,N}}\left(|\lambda|^{(1+G+\alpha-2\gamma)\wedge 2} + |\lambda|^{1+(N+1)(\alpha-\gamma)} + \varrho^t\right). \end{aligned}$$

This implies (58) with N replaced by $N+1$.

By the definition (33) of $\Delta_k(t,\lambda)$,

$$\left|\bar{Q}_k(t,\mathrm{e}^{-i\lambda\mathbf{w}})\right| \;\leq\; \left|D_k(t,\mathrm{e}^{-i\lambda\mathbf{w}})\right| + |\Delta_k(t,\lambda)|\,, \qquad t \geq 1. \tag{59}$$

Apply Corollary 7 to the first term at the right hand side, and the already proved estimate (58) with N replaced by $N+1$ to the second one to get

$$\begin{aligned} &\left|\bar{Q}_k(t,\mathrm{e}^{-i\lambda\mathbf{w}})\right| \\ &\leq\; c\left\{|\lambda|^{1+G-\gamma} + t^2\varrho^t + t^{\beta_{2,N+1}}\left(|\lambda|^{(1+G+\alpha-2\gamma)\wedge 2} + |\lambda|^{1+(N+1)(\alpha-\gamma)} + \varrho^t\right)\right\}. \end{aligned}$$

This yields (57) in the case $N+1$, finishing the proof by induction. □

Completion of Proof of Proposition 5 Actually the expansion in Proposition 5 is now a simple consequence of (58) in Lemma 10. In fact, choose an N such that $1 + N(\alpha-\gamma) > 1+G$. □

3 Proof of Theorem 3

The proof of Theorem 3 still needs some additional arguments involving in particular ideas from [AN74]. Once for all, fix the type $j \in \mathsf{K}$ of the initial particle. We start from the representation [4)]

$$\mathbf{Z}(t+s) \;=\; \sum_{k\in\mathsf{K}} \sum_{n=1}^{Z_k(t)} \mathbf{Z}^{(k,n)}(t,s), \qquad t,s \geq 0, \tag{60}$$

where $\mathbf{Z}^{(k,n)}(t,s)$ denotes the descendents vector at time $t+s$ coming from the n^{th} particle of type k at time t. Set

$$X(t) \;:=\; \langle \mathbf{Z}(t), \mathbf{u}\rangle \quad\text{ and }\quad Y(t) \;:=\; \langle \mathbf{Z}(t), \mathbf{w}\rangle\,. \tag{61}$$

Then

$$Y(t+s) \;=\; \sum_{k\in\mathsf{K}} \sum_{n=1}^{Z_k(t)} \langle \mathbf{Z}^{(k,n)}(t,s), \mathbf{w}\rangle. \tag{62}$$

[4)] Of course, such representation requires a finer model description than we started off in Subsection 1.1. But we skip such details and rely at this stage only on the readers knowledge about such family tree constructions.

Therefore, on the events $\mathcal{N}_t := \{\mathbf{Z}(t) \neq \mathbf{0}\}$, and recalling our assumtions on $\hat{R}$ imposed after (20), we have

$$\hat{R}\left(1/X(t)\right) Y(t+s) \;=\; \hat{R}\left(1/X(t)\right) \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle + \sum_{k \in \mathsf{K}} \frac{\hat{R}\left(1/X(t)\right)}{\hat{R}\left(1/Z_k(t)\right)}\, \hat{R}\left(1/Z_k(t)\right) \sum_{n=1}^{Z_k(t)} \eta_k^{(n)}(t,s), \tag{63}$$

where,

$$\eta_k^{(n)}(t,s) \;:=\; \left\langle \mathbf{Z}^{(k,n)}(t,s), \mathbf{w} \right\rangle - \left(\mathbf{M}^s\right)_k \mathbf{w}, \qquad k \in \mathsf{K}, \quad n \geq 1, \quad s \geq 0,$$

with $\left(\mathbf{M}^s\right)_k$ being the k^{th} row of the matrix $\mathbf{M}^s$. We start with estimating the first summand in the right-hand side of (63).

Lemma 11 *Consider the case* $s = b \log t$ *where the constant* b *is chosen so large such that* $b \log \varrho + 1/2\alpha < 0$. *Then*

$$\left\{ \hat{R}\left(1/X(t)\right) \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \;\middle|\; \mathcal{N}_t \right\} \to 0 \quad \textit{in } P_j\textit{-probability as } t \uparrow \infty.$$

Proof Since we selected the function $\hat{R}$ to be monotone non-decreasing on $(0,\infty)$, we have, for $\varepsilon > 0$ and $\theta \in (0, 1/\alpha)$,

$$P_j \left\{ \left| \hat{R}\left(1/X(t)\right) \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \right| > \varepsilon \;\middle|\; \mathcal{N}_t \right\} \leq\; P_j \left\{ X(t) < t^{-\theta + 1/\alpha} \;\middle|\; \mathcal{N}_t \right\} \;+\; P_j \left\{ \left| \hat{R}\left(t^{\theta - 1/\alpha}\right) \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \right| > \varepsilon \;\middle|\; \mathcal{N}_t \right\}.$$

Note that by the limit law (7) with $\mathbf{w} = \mathbf{u}$ and the asymptotics (5) of the scaling factor $q(t)$,

$$P_j \left\{ X(t) < t^{-\theta + 1/\alpha} \;\middle|\; \mathcal{N}_t \right\} \;<\; \varepsilon \quad \text{for } t \text{ sufficiently large.} \tag{64}$$

On the other hand, by Markov's inequality,

$$P_j \left\{ \left| \hat{R}(t^{\theta - 1/\alpha}) \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \right| > \varepsilon \;\middle|\; \mathcal{N}_t \right\} \;\leq\; \frac{\hat{R}\left(t^{\theta - 1/\alpha}\right)}{\varepsilon\, P_j(\mathcal{N}_t)}\, E_j \left| \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \right|.$$

In view of (20) and (5), for $\delta \in (0, 1 - G)$,

$$\frac{\hat{R}\left(t^{\theta - 1/\alpha}\right)}{P_j(\mathcal{N}_t)} \;\leq\; c\, t^{1/\alpha + \delta} \left(t^{\theta - 1/\alpha}\right)^{1/(1+G+\delta)}.$$

Neglecting the factor corresponding to θ, the exponent of t can be estimated from above by $\delta + \theta + \frac{1}{\alpha}\left(1 - \frac{1}{1+G+\delta}\right)$. But the term in brackets is smaller than $1/2$, so the whole expression is not larger than $1/2\alpha$ if we choose δ and θ sufficiently small. Combining this with (64), we get

$$P_j \left\{ \left| \hat{R}\left(1/X(t)\right) \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \right| > \varepsilon \;\middle|\; \mathcal{N}_t \right\} \;\leq\; \varepsilon + c\, \varepsilon^{-1}\, t^{1/2\alpha}\, E_j \left| \left\langle \mathbf{Z}(t), \mathbf{M}^s \mathbf{w} \right\rangle \right|$$

for t sufficiently large. In view of Frobenius (12) and criticality, this inequality can be continued with

$$\leq \varepsilon + c\,\varepsilon^{-1}\,t^{1/2\alpha}\,\varrho^{s} \;\leq\; \varepsilon + c\,\varepsilon^{-1} t^{1/2\alpha + b\log\varrho},$$

where in the last step we took $s = b\log t$ with b as in the lemma. Letting $t \uparrow \infty$, the proof is finished since ε is arbitrary. □

Proof of claim (a) By construction, $\eta_k^{(n)}(t,s)$ are zero mean random variables, and the characteristic functions

$$\phi_k(\lambda, s) \;:=\; E_k \exp\left[-i\lambda\eta_k^{(n)}(t,s)\right], \qquad \lambda \in \mathsf{R}, \tag{65}$$

do not depend on t and n. We can write them as

$$\begin{aligned}\phi_k(\lambda, s) \;=\; 1 + &\Big(1 - \exp\left[i\lambda\left(\mathbf{M}^s\right)_k \mathbf{w}\right]\Big)\left(1 - E_k \exp\left[-i\lambda\mathbf{w}\langle\mathbf{Z}(s), \mathbf{w}\rangle\right]\right)\\ &- \Big(1 - \exp\left[i\lambda\left(\mathbf{M}^s\right)_k \mathbf{w}\right]\Big) - \left(1 - E_k \exp\left[-i\lambda\langle\mathbf{Z}(s), \mathbf{w}\rangle\right]\right).\end{aligned}$$

Observe now that as $\lambda \to 0$,

$$1 - \exp\left[i\lambda\left(\mathbf{M}^s\right)_k \mathbf{w}\right] \;=\; -i\lambda\left(\mathbf{M}^s\right)_k \mathbf{w} + O\left(\lambda^2\right), \quad \text{uniformly in } s,$$

and

$$\begin{aligned}&\Big|\Big(1 - \exp\left[i\lambda\left(\mathbf{M}^s\right)_k \mathbf{w}\right]\Big)\left(1 - E_k \exp\left[-i\lambda\langle\mathbf{Z}(s), \mathbf{w}\rangle\right]\right)\Big|\\ &\leq\; c\,|\lambda|^2\, E_k\left|\langle\mathbf{Z}(s), \mathbf{w}\rangle\right| \;\leq\; c\,|\lambda|^2.\end{aligned}$$

On the other hand, by the notations (24) and (33),

$$1 - E_k \exp\left[-i\lambda\langle\mathbf{Z}(s), \mathbf{w}\rangle\right] \;=\; i\lambda\left(\mathbf{M}^s\right)_k \mathbf{w} - D_k\left(s, \mathrm{e}^{-i\lambda\mathbf{w}}\right) + \Delta_k\left(s, \lambda\right).$$

Putting these considerations together gives

$$\phi_k(\lambda, s) \;=\; 1 + D_k\left(s, \mathrm{e}^{-i\lambda\mathbf{w}}\right) - \Delta_k\left(s, \lambda\right) + O\left(\lambda^2\right) \quad \text{as } \lambda \to 0, \tag{66}$$

uniformly in s (and k).

We know that under the conditions of the theorem, as $t \uparrow \infty$ and in P_j–law,

$$\left\{Z_k(t) \;\middle|\; \mathcal{N}_t\right\} \;\to\; \infty \tag{67}$$

([Sew74, p.114]) and

$$\left\{\frac{Z_k(t)}{X(t)} \;\middle|\; \mathcal{N}_t\right\} \;\to\; v_k \tag{68}$$

([Vat78]). Moreover,

$$\left\{Z_k(t)\, t^{-1/\alpha + \varepsilon} \;\middle|\; \mathcal{N}_t\right\} \;\to\; \infty \tag{69}$$

and

$$\left\{ Z_k(t)\, t^{-1/\alpha-\varepsilon} \;\middle|\; \mathcal{N}_t \right\} \;\to\; 0, \tag{70}$$

for each fixed $\varepsilon > 0$.

By (68) and since $\hat{R}$ is regularly varying of order $1/(1+G)$,

$$\left\{ \frac{\hat{R}\left(1/X(t)\right)}{\hat{R}\left(1/Z_k(t)\right)} \;\middle|\; \mathcal{N}_t \right\} \to v_k^{1/(1+G)}, \qquad t \uparrow \infty, \tag{71}$$

in P_j–law.

From now on, we fix a $\lambda \in \mathsf{R}$. Using the Markov and branching properties,

$$\begin{aligned} &E_j \left\{ \exp\left[-\, i\lambda \hat{R}\left(1/Z_k(t)\right) \sum_{n=1}^{Z_k(t)} \eta_k^{(n)}(t,s) \right] \;\middle|\; \mathcal{N}_t \right\} \\ &= E_j \left\{ \phi_k\left(\lambda \hat{R}\left(1/Z_k(t)\right), s\right)^{Z_k(t)} \;\middle|\; \mathcal{N}_t \right\}. \end{aligned} \tag{72}$$

By the asymptotics (66), "explosion" (67), and regular variation (20), for the fixed λ, (conditioned on $\mathcal{N}_t$ and in P_j–law)

$$\begin{aligned} \phi_k\left(\lambda \hat{R}\left(1/Z_k(t)\right), s\right) = 1 + D_k\left(s, \exp\left[-i\lambda \hat{R}\left(1/Z_k(t)\right) \mathbf{w}\right]\right) \\ -\; \Delta_k\left(s, \lambda \hat{R}\left(1/Z_k(t)\right)\right) + O\left(\left(\hat{R}\left(1/Z_k(t)\right)\right)^2\right) \end{aligned} \tag{73}$$

as $t \uparrow \infty$. Moreover, by Lemma 6,

$$\begin{aligned} &D_k\left(s, \exp\left[-\, i\lambda \hat{R}\left(1/Z_k(t)\right) \mathbf{w}\right]\right) \\ &=\; u_k J\left(\lambda \hat{R}\left(1/Z_k(t)\right), \mathbf{w}, 0\right)\left(1 + o(1)\right) + s^2 \varrho^s \end{aligned} \tag{74}$$

as $s \uparrow \infty$ and for all sufficiently large t. In addition, by Hypothesis 2 (a), and with R from (18),

$$J\left(\lambda \hat{R}\left(1/Z_k(t)\right), \mathbf{w}, 0\right) \;\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\, \mathrm{sign}\,\lambda}\, R\left(|\lambda|\, \hat{R}\left(1/Z_k(t)\right)\right)$$

as $t \uparrow \infty$. By regular variation, we may continue with

$$\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\, \mathrm{sign}\,\lambda}\, |\lambda|^{1+G}\, R\left(\hat{R}\left(1/Z_k(t)\right)\right),$$

and (19) gives (conditioned on $\mathcal{N}_t$ and in P_j–law)

$$J\left(\lambda \hat{R}\left(1/Z_k(t)\right), \mathbf{w}, 0\right) \;\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\, \mathrm{sign}\,\lambda}\, |\lambda|^{1+G}/Z_k(t) \quad \text{as } t \uparrow \infty. \tag{75}$$

On the other hand, if we take $s = b \log t$ with $b > 0$ to be chosen later, according to Proposition 5 there exist constants $\beta > 1$ and $\delta > 0$ satisfying $1 + G + 2\delta < 2$ such that

$$\left|\Delta_k\left(s, \lambda \hat{R}\left(1/Z_k(t)\right)\right)\right| \;\le\; c\left(\left|\hat{R}\left(1/Z_k(t)\right)\right|^{1+G+2\delta} + t^{b \log \varrho}\right) \log^\beta t$$

for all sufficiently large t. Since $\hat{R}$ is a regularly varying function of order $1/(1+G)$ at $x = 0+$,

$$0 < \hat{R}(x) \;\le\; x^{1/(1+G+\delta)}, \qquad 0 < x < x_0\,, \tag{76}$$

for some x_0 and the chosen δ ([Sen76, 1°, p.18]). From this estimate,

$$\begin{aligned}\left|\Delta_k\left(s, \lambda \hat{R}\left(1/Z_k(t)\right)\right)\right| \;&\le\; c\left(\left(1/Z_k(t)\right)^{\frac{1+G+2\delta}{1+G+\delta}} + t^{b \log \varrho}\right) \log^\beta t \\ &=\; o\left(1/Z_k(t)\right) + t^{b \log \varrho} \log^\beta t\end{aligned}$$

(conditioned on $\mathcal{N}_t$ and in P_j–law) as $t \uparrow \infty$ [recall (69)]. If we chose b so large that $b \log \varrho + 1/\alpha < 0$,

$$t^{b \log \varrho} \log^\beta t \;=\; o\left(1/Z_k(t)\right),$$

hence

$$\left|\Delta_k\left(s, \lambda \hat{R}\left(1/Z_k(t)\right)\right)\right| \;=\; o\left(1/Z_k(t)\right). \tag{77}$$

Similarly,

$$s^2 \varrho^s \;=\; o\left(s^\beta \varrho^s\right) \;=\; o\left(t^{b \log \varrho} \log^\beta t\right) \;=\; o\left(1/Z_k(t)\right) \tag{78}$$

as $t \uparrow \infty$. Finally, since $1 + G + 2\delta < 2$,

$$O\left(\left(\hat{R}\left(1/Z_k(t)\right)\right)^2\right) \;=\; o\left(1/Z_k(t)\right) \quad \text{as } t \uparrow \infty. \tag{79}$$

Combining (73) – (79), we see that

$$\phi_k\left(\lambda \hat{R}\left(1/Z_k(t)\right), s\right) \;=\; 1 - u_k\, \mathrm{e}^{i \frac{\pi}{2} \varphi \operatorname{sign} \lambda} |\lambda|^{1+G}/Z_k(t) + o\left(1/Z_k(t)\right)$$

as $t \uparrow \infty$. Inserting into the right hand side of (72) gives

$$E_j\left\{\left\{1 - u_k\, \mathrm{e}^{i \frac{\pi}{2} \varphi \operatorname{sign} \lambda} |\lambda|^{1+G}/Z_k(t) + o\left(1/Z_k(t)\right)\right\}^{Z_k(t)} \,\middle|\, \mathcal{N}_t\right\}.$$

Hence,

$$\begin{aligned}&\lim_{t \uparrow \infty} E_j\left\{\exp\left[-i\lambda \hat{R}\left(1/Z_k(t)\right) \sum_{n=1}^{Z_k(t)} \eta_k^{(n)}(t, s)\right] \,\middle|\, \mathcal{N}_t\right\} \\ &= \exp\left[-u_k\, \mathrm{e}^{i \frac{\pi}{2} \varphi \operatorname{sign} \lambda} |\lambda|^{1+G}\right].\end{aligned} \tag{80}$$

Since this limiting expression is continuos at $\lambda = 0$, it must be a characteristic function and in fact of a random variable Θ_k over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a stable distribution with parameter $1+G$. By the way, suppose for the moment, we did not restrict the range of φ as done in Hypothesis 2 (a), here we would get the restriction $|\varphi| \leq 1-G$ automatically, compare with [Fel71, formula (XVII.6.2)]. In view of (71), this means that conditioned on $\mathcal{N}_t$ and in P_j–law,

$$\sum_{k \in \mathsf{K}} \frac{\hat{R}\left(1/X(t)\right)}{\hat{R}\left(1/Z_k(t)\right)} \hat{R}\left(1/Z_k(t)\right) \sum_{n=1}^{Z_k(t)} \eta_k^{(n)}(t,s) \ \to \ \sum_{k \in \mathsf{K}} v_k^{1/(1+G)} \Theta_k \tag{81}$$

as $t \uparrow \infty$, where Θ_k are independent random variables over $(\Omega, \mathcal{F}, \mathcal{P})$. Turning back to (63), along with Lemma 11 it shows that in P_j–law,

$$\left\{ \hat{R}\left(1/X(t)\right) Y(t+s) \;\middle|\; \mathcal{N}_t \right\} \to \sum_{k \in \mathsf{K}} v_k^{1/(1+G)} \Theta_k \tag{82}$$

as $t \uparrow \infty$.

Recalling $s = b \log t$, next we use that

$$\left\{ \frac{X(t+s)}{X(t)} \;\middle|\; \mathcal{N}_{t+s} \right\} \to 1 \tag{83}$$

as $t \uparrow \infty$ in P_j–law (compare with [AN74, p. 342]; note that the arguments there remain true under our more general assumptions). Hence, since $\hat{R}$ is regularly varying,

$$\left\{ \frac{\hat{R}\left(1/X(t)\right)}{\hat{R}\left(1/X(t+s)\right)} \;\middle|\; \mathcal{N}_{t+s} \right\} \to 1. \tag{84}$$

Thus,

$$\begin{aligned} &\lim_{t \uparrow \infty} E_j \left\{ \exp\left[-i\lambda \hat{R}\left(X(t+s)\right) Y(t+s) \right] \;\middle|\; \mathcal{N}_{t+s} \right\} \\ &= \lim_{t \uparrow \infty} E_j \left\{ \exp\left[-i\lambda \hat{R}\left(X(t)\right) Y(t+s) \right] \;\middle|\; \mathcal{N}_{t+s} \right\} \end{aligned} \tag{85}$$

(provided the limit exist).

For convenience, set now

$$V \ := \ \exp\left[-i\lambda \hat{R}\left(1/X(t)\right) Y(t+s) \right]. \tag{86}$$

Then

$$E_j \left\{ V; \mathcal{N}_{t+s} \right\} \ = \ E_j \left\{ V \mid \mathcal{N}_t \right\} P_j(\mathcal{N}_t) - P_j\left(\mathcal{N}_t \cap \mathcal{N}_{t+s}^{\mathrm{c}} \right).$$

Thus

$$E_j \left\{ V \mid \mathcal{N}_{t+s} \right\} \ = \ \left[P_j\left(\mathcal{N}_t\right) / P_j(\mathcal{N}_{t+s}) \right] \left[E_j \left\{ V \mid \mathcal{N}_t \right\} - P_j \left\{ \mathcal{N}_{t+s}^{\mathrm{c}} \mid \mathcal{N}_t \right\} \right].$$

Recalling $s = b \log t$, by the survival probability asymptotics (5),

$$P_j(\mathcal{N}_t)/P_j(\mathcal{N}_{t+s}) \to 1 \quad \text{as } t \uparrow \infty. \tag{87}$$

Therefore, $P_j\left\{\mathcal{N}_{t+s}^{\mathrm{c}} \mid \mathcal{N}_t\right\} \to 0$, and

$$\lim_{t\uparrow\infty} E_j\left\{V \mid \mathcal{N}_{t+s}\right\} = \lim_{t\uparrow\infty} E_j\left\{V \mid \mathcal{N}_t\right\} \tag{88}$$

(provided the limit exist). Combining with (85) and (82),

$$\begin{aligned}&\lim_{t\uparrow\infty} E_j\left\{\exp\left[-i\lambda \hat{R}\left(X(t+s)\right)Y(t+s)\right] \,\Big|\, \mathcal{N}_{t+s}\right\}\\ &= \prod_{k\in\mathsf{K}} \mathcal{E}\exp\left[-i\lambda v_k^{1/(1+G)}\Theta_k\right]\end{aligned}$$

(with $\mathcal{E}$ referring to expectation corresponding to the underlying law $\mathcal{P}$). But the latter expression equals

$$\prod_{k\in\mathsf{K}} \exp\left[-u_k\,\mathrm{e}^{i\frac{\pi}{2}\varphi\,\mathrm{sign}\,\lambda}\left|\lambda v_k^{1/(1+G)}\right|^{1+G}\right] = \exp\left[-\mathrm{e}^{i\frac{\pi}{2}\varphi\,\mathrm{sign}\,\lambda}\left|\lambda\right|^{1+G}\right].$$

This completes the proof of (a). □

Proof of claim (b) To prove part (b), we use similar ideas and therefore only sketch the argument. Recalling the representation (62) and notation (65), by the Markov and branching properties,

$$\begin{aligned}&E_j\left\{\exp\left[-i\lambda\,\hat{q}(t)\,Y(t+s)\right] \,\Big|\, \mathcal{N}_t\right\} \\ &= E_j\left\{\prod_{k\in\mathsf{K}}\left(\phi_k\left(\lambda\,\hat{q}(t),s\right)\right)^{Z_k(t)} \,\Big|\, \mathcal{N}_t\right\}.\end{aligned} \tag{89}$$

Similarly to (73),

$$\begin{aligned}\phi_k\left(\lambda\,\hat{q}(t),s\right) &= 1 + D_k\left(s,\exp\left[-i\lambda\,\hat{q}(t)\mathbf{w}\right]\right)\\ &\quad - \Delta_k\left(s,\lambda\,\hat{q}(t)\right) + O\left(\left(\hat{q}(t)\right)^2\right).\end{aligned} \tag{90}$$

As in (74),

$$D_k\left(s,\exp\left[-i\lambda\,\hat{q}(t)\mathbf{w}\right]\right) = u_k J\left(\lambda\,\hat{q}(t),\mathbf{w},0\right)\left(1+o(1)\right) + s^2\varrho^s. \tag{91}$$

Now, as $t \uparrow \infty$,

$$R\left(\hat{q}(t)\right) = R\left(\hat{R}\left(q(t)\right)\right) \sim q(t) \tag{92}$$

[recall (21), (6), and (19)], and by Hypothesis 2 (a),

$$J\left(\lambda \hat{R}\left(q(t)\right), \mathbf{w}, 0\right) \;\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\, R\left(|\lambda|\,\hat{R}\left(q(t)\right)\right).$$

By regular variation and (92), we may continue with

$$\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G} R\left(\hat{R}\left(q(t)\right)\right) \;\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\;|\lambda|^{1+G}\; q(t).$$

Thus,

$$J\left(\lambda\,\hat{q}(t), \mathbf{w}, 0\right) \;\sim\; -\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G} q(t) \tag{93}$$

[compare with (75)]. As in (77) – (79), with $s = b\log t$,

$$\Delta_k\left(s, \lambda\,\hat{q}(t)\right) + O\left(\left(\hat{q}(t)\right)^2\right) + s^2\varrho^s \;=\; o\left(q(t)\right). \tag{94}$$

Inserting (91), (93), and (94) into (90) gives

$$\phi_k\left(\lambda\,\hat{q}(t), s\right) \;=\; 1 - u_k\,\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G}\,q(t) + o\left(q(t)\right).$$

Hence, the right hand side of (89) can be written as

$$E_j\left\{\prod_{k\in\mathsf{K}}\left(1 - u_k\,\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G}\,q(t) \;+ o\left(q(t)\right)\right)^{Z_k(t)}\;\middle|\;\mathcal{N}_t\right\}.$$

Its limt as $t\uparrow\infty$ equals

$$\lim_{t\uparrow\infty} E_j\left\{\prod_{k\in\mathsf{K}}\exp\left[-\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G}\,q(t)\,Z_k(t)\,u_k\right]\;\middle|\;\mathcal{N}_t\right\}$$

$$= \lim_{t\uparrow\infty} E_j\left[\exp\left[-\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G} q(t)\,\langle\mathbf{Z}(t), \mathbf{u}\rangle\right]\;\middle|\;\mathcal{N}_t\right]. \tag{95}$$

Recalling notation (61), under our conditions (see [Vat77]),

$$\left\{q(t)\,X(t)\,\middle|\,\mathcal{N}_t\right\} \to \zeta \tag{96}$$

in P_j–law as $t\uparrow\infty$, where ζ is a (non-negative) random variable with characteristic function

$$\mathcal{E}\mathrm{e}^{-i\zeta\lambda} \;=\; 1 - \frac{i\lambda}{\left(1 + (i\lambda)^\alpha\right)^{1/\alpha}}, \qquad \lambda\in\mathsf{R}, \tag{97}$$

[recall (7)]. Since $\zeta\geq 0$, by analytic continuation we even have

$$\mathcal{E}\mathrm{e}^{y\zeta} \;=\; 1 - \frac{-y}{\left(1 + (-y)^\alpha\right)^{1/\alpha}}, \qquad y \text{ complex with } \Re\mathrm{e}\,y \leq 0.$$

But $\Re\mathrm{e}\left(-\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\right) = -\cos\left(\frac{\pi}{2}\,\varphi\right) \leq 0$ under our restriction on φ, hence the right hand side of (95) coincides with

$$\mathcal{E}\exp\left[-\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G}\zeta\right] \;=\; 1 - \frac{\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G}}{\left(1 + \left(\mathrm{e}^{i\,\frac{\pi}{2}\,\varphi\,\mathrm{sign}\,\lambda}\,|\lambda|^{1+G}\right)^\alpha\right)^{1/\alpha}},$$

and the proof is complete. □

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