Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

The work of Vladimir Maz'ya on integral and pseudodifferential operators

Johannes Elschner

submitted: 2nd December 1998

Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstrasse 39
10117 Berlin
Germany
E-Mail: elschner@wias-berlin.de

Preprint No. 454
Berlin 1998

1991 Mathematics Subject Classification. 31-02, 35-02, 45-02, 58-02.

Key words and phrases. Integral operators, pseudodifferntial operators, oblique derivative problem, differential operators in the half–space, characteristic Cauchy problem, multiplier theory, boundary integral equations.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail (X.400): c=de;a=d400-gw;p=WIAS-BERLIN;s=preprint
E-Mail (Internet): preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The paper presents an outline of Vladimir Maz'ya's important and influential contributions to the solvability theory of integral and pseudodifferential equations.

Since integral and pseudodifferential operators are one of the main themes of Maz'ya's vast mathematical work, it is a difficult task to describe his diverse results in this field in a short survey. In fact, this article was to have been written by Maz'ya's close friend Siegfried Prössdorf, who was my teacher at the Technical University of Chemnitz and my colleague at the Weierstrass Institute in Berlin. Siegfried's unexpected and untimely death was a tragic loss for everybody who knew him. Siegfried had followed Maz'ya's work for over thirty years. In this respect I would like to draw attention to their comprehensive joint monograph published as Volume 27 of the Encyclopaedia of Mathematical Sciences.

1 Non-elliptic operators

I learned about Vladimir's work for the first time in 1972 when two articles by Maz'ya and Plamenevskiĭ on multidimensional singular integral operators with degenerate symbol were reported in Prössdorf's seminar at the Technical University of Chemnitz. The ideas from these papers influenced the research done in Chemnitz in the same area, including some of my own work. This is why I have chosen to discuss Maz'ya's contributions to non-elliptic singular integral and pseudodifferential operators first.

I start with some definitions. An operator of the form

$$Au(x) := a(x)u(x) + \int_{\mathbb{R}^n} \frac{f(x,\theta)}{|x-y|^n} u(y)dy \,, \tag{1}$$

where $x \in \mathbb{R}^n, \theta = (x-y)|x-y|^{-1} \in S^{n-1}$, is called *singular integral operator* in $\mathbb{R}^n$. The *symbol* of A, which was first introduced by Mikhlin for $n = 2$ and somewhat later by Giraud for $n > 2$ as a series in spherical harmonics, can equivalently be defined by

$$\sigma(x,\xi) := a(x) + F_{y\Rightarrow\xi}\{|y|^{-n} f(x, y/|y|)\} \,, \quad x\,,\,\xi \in \mathbb{R}^n \,, \quad \xi \neq 0 \,,$$

where F refers to the Fourier transform. Note that σ is positively homogeneous of degree 0 in ξ. It was proved by Calderón and Zygmund that (1) can be written in the form

$$Au(x) = F^{-1}_{\xi\to x}\{\sigma(x,\xi)(Fu)(\xi)\} \,. \tag{2}$$

The operator (1) is called *elliptic* if $\sigma(x,\xi) \neq 0$ for all $x \in \mathbb{R}^n$ and $\xi \in S^{n-1}$, otherwise it is called non-elliptic or degenerate.

At the beginning of the sixties the solvability properties of elliptic multidimensional singular integral operators were well understood, due to the fundamental contributions by Tricomi, Mikhlin, Giraud, Calderón and Zygmund, and Gohberg, whereas nothing was known in the non-elliptic case. Influenced by S. Mikhlin, V. Maz'ya had already started working in this field in 1964. The short but illuminating paper [37] by Maz'ya and Plamenevskiĭ was the first dealing with non–elliptic pseudodifferential operators in higher dimensions. It was followed by another short note [38] and the longer paper [39]. Among other things, it was proved that the equation

$$Au = g\,, \quad g \in L_2(\mathbb{R}^n)\,,$$

is always solvable in an appropriate anisotropic Sobolev space provided the symbol of A does not depend on x and has zeroes of constant (finite) multiplicities on disjoint smooth submanifolds of S^{n-1}. Furthermore, a complete description of the finite dimensional kernel (null space) of A and formulations of well–posed problems for the inhomogeneous equation were given. Maz'ya and Plamenevskiĭ were also able to treat some cases of symbols depending additionally on x.

Apparently, these pioneering works on non–elliptic operators remained completely unknown outside the Iron Curtain. However, the case of degenerating symbol became rather fashionable after the theory of pseudodifferential operators had emerged in the works by Eskin and Vishik, Kohn and Nirenberg, Bokobza and Unterberger, and Hörmander. Recall that a (classical) *pseudodifferential operator* in $\mathbb{R}^n$ is defined by relation (2), where the symbol σ admits an asymptotic expansion into positively homogeneous terms in ξ,

$$\sigma(x,\xi) \sim \sum_{k=0}^{\infty} \sigma_k(x,\xi)\,, \quad \sigma_k(x,t\xi) = t^{l-k}\sigma_k(x,\xi) \quad \forall\, t > 0\,.$$

Here σ_0 is called the principal symbol, l is the order of A, and A is said to be elliptic if σ_0 is nowhere vanishing.

After 1965 solvability and regularity theory for pseudodifferential equations with various types of degeneration became a vast area of study. This theory was also applied to non–elliptic boundary value problems. In the late sixties, Maz'ya and Paneyah made an important contribution to this field. In their papers [34], [35], [36] they studied a rather general class of pseudodifferential operators on a smooth manifold Γ without boundary, with symbol vanishing on a submanifold of codimension one. Assuming that the principal symbol σ_0 satisfies the condition

$$\text{Im}\ \sigma_0(x,\xi) = 0 \quad \Longleftrightarrow \quad x \in \Gamma_0\,,$$

they introduced a classification of the types of degeneration (depending on the sign of Im σ_0 near Γ_0) and developed a complete solvability theory for each of them. Moreover, sharp a priori estimates leading to precise regularity results for weak solutions were proved. These results have direct applications to the degenerate oblique derivative problem, which will be discussed now.

For simplicity we restrict ourselves to the formulation for the Laplace operator; all results hold of course for general elliptic operators of second order. Let $\Omega \subset \mathbb{R}^n, n \geq 3$, be a bounded domain with smooth boundary Γ, and denote the exterior unit normal to Γ by ν. The oblique derivative or Poincaré problem consists in determining a function u satisfying

$$\Delta u = 0 \quad \text{in } \Omega, \quad \partial u / \partial \ell = f \quad \text{on } \Gamma, \tag{3}$$

where ℓ denotes a field of unit vectors on Γ. The problem (3) can be converted into a pseudodifferential equation of first order on Γ with the principal symbol

$$\sigma_0(x,\xi) = -\cos(\nu,\ell)|\xi| + i\cos(\xi,\ell)|\xi|\,,\ x \in \Gamma\,,\ \xi \in T_x\Gamma\,,$$

where T_x stands for the tangent space at the point x. Observe that this equation is elliptic if and only if the vector field ℓ is nowhere tangent to Γ.

In the elliptic case, the Fredholm property and regularity of problem (3) follow from standard elliptic theory of pseudodifferential operators, while its unique solvability is a consequence of Giraud's theorem on the sign of the oblique derivative at the extremum point.

Until the mid-sixties almost nothing was known about the degenerate problem (3). For *transversal degeneration* where the field ℓ is tangent to Γ on some $(n-2)$–dimensional submanifold Γ_0, but is not tangential to Γ_0, this situation changed when the first results on non–elliptic pseudodifferential operators became available.

As a by–product of his subelliptic estimates for pseudodifferential equations, Hörmander [18] proved that the dimension of the kernel of this problem may be infinite or the regularity of solutions may fail. Correct formulations leading to Fredholm operators were first studied by Malyutov [26] and by Egorov, Kontrat'ev [9], using entirely different methods from the theory of elliptic second order differential operators. In the above mentioned papers, Maz'ya and Paneyah [34] – [36] presented a unified pseudodifferential approach to all cases of transversal degeneration, proving complete unique solvability results and studying regularity properties of solutions.

2 Oblique derivative problem: breakthrough in the generic case of degeneration

Geometrically, the transversal degeneration leads to the following three types of components of the set Γ_0 (where the vector field ℓ is tangent to Γ): those consisting of the so–called "entrance" points (of ℓ into Ω), "exit" points, and "status quo" points where ℓ remains on the same side of Γ; see Figs. 1 – 3. In 1969, after the Malyutov, Egorov & Kondrat'ev and Maz'ya & Paneyah studies, the following properties of transversal degeneration became clear. The status quo components do not affect the unique solvability of the problem; they only generate some loss of regularity of solutions. In order to preserve unique solvability, one should allow discontinuities of solutions on the entrance components and prescribe additional boundary conditions on the exit components.

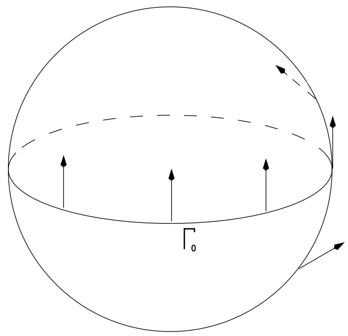

Fig. 1 Entrance points – under-determined problem

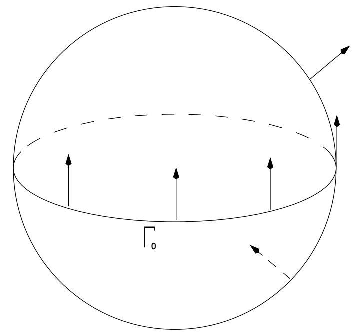

Fig. 2 Exit points – over-determined problem

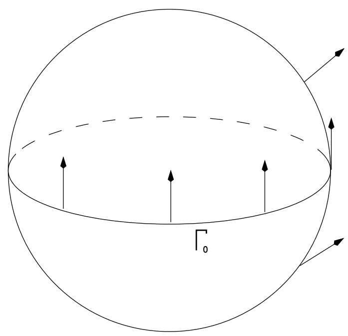

Fig. 3 Status quo – well-posed problem

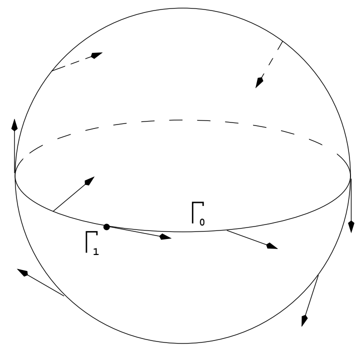

Fig. 4 Generic case of degeneration

Around 1970 V. Arnold [1] stressed the importance of the so–called *generic case of degeneration*, where the vector field ℓ is no longer transversal to Γ_0; see also his well–known book [2], p. 203. More precisely, one assumes that there are smooth manifolds (without boundary) $\Gamma_0 \supset \Gamma_1 \ldots \supset \Gamma_s$ of dimensions $n-2, n-3, \ldots, n-2-s$ such that ℓ is tangent to Γ_j exactly at the points of Γ_{j+1}, whereas ℓ is nowhere tangent to Γ_s; see Fig. 4. A local model of this situation is given by the following:

$$
\begin{aligned}
\Omega &= \{x \in \mathbb{R}^n : x_1 > 0\}\,, \quad \Gamma = \{x_1 = 0\}\,, \\
\ell &= x_2\partial_1 + x_3\partial_2 + \ldots + x_k\partial_{k-1} + \partial_k\,, \quad k \le n\,, \\
\Gamma_j &= \{x_1 = x_2 = \ldots = x_{2+j} = 0\}\,, \quad j = 0, \ldots, k-2\,.
\end{aligned}
$$

The generic case is much more difficult from the analytical point of view than the transversal one, because entrance and exit points are permitted to belong to one and the same component of Γ_0 and the usual localization technique does not apply.

In 1972 V. Maz'ya [27] published a deep result related to the generic degeneration, the only known one for the time being. He found function spaces of right–hand sides and solutions for the unique solvability of the problem. The success was achieved by an ingenious choice of weight functions in the derivation of a priori estimates for the solution. Additionally, Maz'ya proved that the inverse operator of the problem is always compact on $L_p(\Gamma), 1 < p \leq \infty$. It turned out that the manifolds Γ_j of codimension greater than one do not influence the correct statement of the problem, contrary to Arnold's expectations; see [2], p. 203. By the way, a description of the asymptotics of solutions near the points of tangency of the field ℓ to Γ_0 remains a difficult long–standing problem.

3 Estimates for differential operators in the half–space

At the beginning of the seventies Gelman and Maz'ya wrote a series of papers on the topic of this section, and the results were summarized in their monograph [13] published only in German. The fact that the book could not appear in the Soviet Union at that time sheds some light upon the antisemitic policy of the Soviet scientific administration. This is the right place to emphasize the role of Maz'ya's friends Siegfried Prössdorf and Günther Wildenhain. The first of them brought the manuscript illegally to East Germany, and the second became the editor.

The Gelman–Maz'ya book starts with the following epigraph by L. Gårding:

> "When a problem about partial differential operators has been fitted into the abstract theory, all that remains is usually to prove a suitable inequality and much of our knowledge is, in fact, essentially contained in such inequalities. But the abstract theory is not only a tool, it is also a guide to general and fruitful problems."

It contains indeed a great variety of inequalities for differential and pseudodifferential operators with constant coefficients. The authors obtain results of final character, without any restrictions on the type of the differential operators. They found necessary and sufficient conditions for the validity of the corresponding a priori estimates and presented easier verifiable either necessary or sufficient conditions.

I will now describe a few typical results from this book. Let $\mathbb{R}^n_+ = \{(x,t) : x \in \mathbb{R}^{n-1}, t \geq 0\}$ and consider pseudodifferential operators $R(D), P_j(D), Q_s(D)$ with symbols $R(\xi,\tau), P_j(\xi,\tau), Q_s(\xi,\tau)$, $\xi \in \mathbb{R}^{n-1}$, $\tau \in \mathbb{R}$, not depending on x and t, which are further assumed to be polynomials in τ with locally bounded measurable coefficients of polynomial growth in ξ. [13] presents a detailed and complete study of estimates in the half–space,

$$||R(D)u||^2_{L_2(\mathbb{R}^n_+)} \leq c \left\{ \sum_{j=1}^{m} ||P_j(D)u||^2_{L_2(\mathbb{R}^n_+)} + \sum_{s=1}^{r} ||Q_s(D)u||^2_{L_2(\partial\mathbb{R}^n_+)} \right\}, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \tag{4}$$

and of trace estimates of the form

$$||R(D)u||^2_{H^\mu(\partial\mathbb{R}^n_+)} \leq c\left\{\sum_{j=1}^{m}||P_j(D)u||^2_{L_2(\mathbb{R}^n_+)} + \sum_{s=1}^{r}||Q_s(D)u||^2_{L_2(\partial\mathbb{R}^n_+)}\right\}, \qquad u \in C_0^\infty(\mathbb{R}^n_+), \tag{5}$$

where H^μ denotes the Sobolev space of order μ. Gelman and Maz'ya found necessary and sufficient conditions stated in algebraic terms for these inequalities in full generality. Well–known results by Aronszajn, Agmon, Douglis and Nirenberg, and Schechter became part of the general theory developed in [13].

To give an idea of the results, I consider the example

$$r = 0\,, \quad m = 2\,, \quad P_1(\xi,\tau) = P(\xi,\tau)\,, \quad P_2(\xi,\tau) = 1$$

and assume that the leading coefficient of P is equal to one. Now (5) takes the form

$$||R(D)u||^2_{H^\mu(\partial\mathbb{R}^n_+)} \leq c\left\{||P(D)u||^2_{L_2(\mathbb{R}^n_+)} + ||u||^2_{L_2(\mathbb{R}^n_+)}\right\}. \tag{6}$$

Let $H(\xi,\tau)$ be a polynomial in τ with roots in the half–plane Im $\zeta > 0$, $\zeta = \tau + i\sigma$, and such that

$$|P(\xi,\tau)|^2 + 1 = |H(\xi,\tau)|^2.$$

The authors show that estimate (6) holds if and only if

$$\int_{\mathbb{R}} \frac{|T_1(\xi,\tau)|^2 + |T_2(\xi,\tau)|^2}{|P(\xi,\tau)|^2+1} d\tau \leq c(1+|\xi|^2)^{-\mu}\,,$$

where $T_1(\xi,\tau), T_2(\xi,\tau)$ denote the quotient and the remainder obtained when the polynomial (in τ) $R(\xi,\tau)\overline{H(\xi,\tau)}$ is divided by $P(\xi,\tau)$; see [13], p. 170.

It is a pity that this Gelman–Maz'ya book did not attract much attention despite the beauty and the completeness of the obtained results. Apparently they have a great potential of generalization to partial differential and pseudodifferential equations with variable coefficients, both in the half–space and on domains.

4 The characteristic Cauchy problem

If one thinks of well–posed problems for hyperbolic differential equations, the first that comes to one's mind is the Cauchy problem with initial data given on a space–like initial surface. On the other hand, it is known (see the book [8] by Courant and Hilbert) that the solution of the wave equation is already uniquely determined if its values are prescribed on the characteristic cone.

This is the simplest example of a characteristic Cauchy problem, which was the topic of an important paper by Vainberg and Maz'ya [49]. They studied general hyperbolic operators of arbitrary even order $2m$. In contrast to previous work by

Gårding, Kotake, Leray [12] and Kondrat'ev [22], they were able to avoid any assumption regarding the set where the initial surface S is characteristic. This set, which will be denoted by V, may even have positive measure. In this case derivatives of order $2m-1$ need not be prescribed on V in the formulation of the problem. The existence and uniqueness theorems and energy estimates obtained in [49] show that this formulation leads to a well–posed problem in appropriately chosen function spaces.

To illustrate that, I will present the Vainberg–Maz'ya energy estimate in the special case of the homogeneous differential equation. Let $\{S_\tau\}_{0\le\tau\le T}$ be a one–parameter family of surfaces $S_\tau = \{(t,x) : t = s(\tau,x),\, x \in \mathbb{R}^n\}$, where $S_0 = S$. On some compact sets V_τ the surfaces S_τ have characteristic directions, while at the remaining points they are space–like. Consider the weight function

$$\rho(\tau,x) = P_0(s(\tau,x), x, 1, -s'_x(\tau,x)),\ 0 \le \tau \le T\,,$$

where P_0 is the principal symbol of the hyperbolic differential operator P under consideration. The function ρ is non–negative and satisfies $\rho(\tau,x) = 0$ if and only if $(t,x) \in V_\tau$. Finally, let E^ρ denote the "weighted" energy

$$E^\rho(\tau,u) = ||\sqrt{\rho}\,\partial^{2m-1}u/\partial t^{2m-1}||^2_{L_2(S_\tau)} + \sum_{j=0}^{2m-2} ||\partial^j u/\partial t^j||^2_{H^{2m-1-j}(S_\tau)}\,,$$

whereas $E(\tau,u)$ is defined by setting $\rho = 1$ in the first term. Then, for the characteristic Cauchy problem

$$P(t,x,\partial/\partial t,\partial/\partial x)u = 0,$$
$$\partial^j u/\partial t^j|_S = \varphi_j, \quad 0 \le j \le 2m-2, \quad \partial^{2m-1}u/\partial t^{2m-1}|_{S\setminus V} = \varphi_{2m-1}\,,$$

the energy estimate

$$\max_{0\le\tau\le T} E^\rho(\tau,u) + \int_0^T E(\tau,u)d\tau \le c\left\{||\sqrt{\rho}\,\varphi_{2m-1}||^2_{L_2(S\setminus V)} + \sum_{j=0}^{2m-2} ||\varphi_j||^2_{H^{2m-1-j}(S)}\right\}$$

holds. Hörmander, who did not know about the paper by Vainberg and Maz'ya, obtained the same results in [19] for hyperbolic equations of second order. It should be noted that the proofs for higher order equations are much more complicated.

5 Applications of multiplier theory to integral operators

In 1979–1983, together with his wife Tatyana Shaposhnikova, Maz'ya developed a new theory of multipliers in spaces of differentiable functions, the results of which were summarized in their book [40]. One should not mix up these multipliers with the Fourier multipliers already studied in the thirties. Maz'ya and Shaposhnikova showed that their multipliers provide a natural language for various questions of analysis and the theory of differential and pseudodifferential operators.

Before their work, only a few separated facts concerning multipliers in Sobolev spaces were known. In [40] one can find, in particular, a complete description of the multiplier spaces $M(W_p^m \to W_p^l)$ as well as of the spaces $M^\circ(W_p^m \to W_p^l)$ of compact multipliers. Here W_p^m stands for the Sobolev–Slobodetskiĭ space on $\mathbb{R}^n$, and $M(X \Rightarrow Y)$ denotes the set of functions for which the corresponding multiplication operator maps the Banach space X into another Banach space Y. For $X = Y$, we simply write $M(X)$.

I dwell upon some of the many results in [40] which are close to the topic of the present article. As an application of their multiplier results for $p = 2$, Maz'ya and Shaposhnikova proved two–sided estimates for the essential norm and necessary and sufficient conditions for the compactness of convolution operators $u \to k * u$, acting from the weighted L_2 space $L_2(\mathbb{R}^n; (1+|x|^2)^{m/2})$ into $L_2(\mathbb{R}^n; (1+|x|^2)^{l/2})$. Note that the Fourier transform is an isomorphic map of W_2^l onto the latter weighted L_2 space.

The authors also study singular integral operators in Sobolev spaces. It turns out that the basic properties of these operators are retained under minimal smoothness assumptions on the symbol concerning the first variable. I present a typical result in this direction.

Consider singular integral operators A, B of the form (1) in $\mathbb{R}^n$, which have the symbols $a(x,\xi)$, $b(x,\xi)$. Let $A \circ B$ denote the singular integral operator with the product symbol ab. It was proved in [40], Chap. 4.5, that if a and b satisfy the smoothness conditions

$$a \in C^\infty(M(W_p^{l+1}),\, S^{n-1}), \quad \nabla_x b \in C^\infty(M(W_p^l),\, S^{n-1})$$

then the operator $C := AB - A \circ B$ is a continuous map of W_p^l into W_p^{l+1}. If, in addition, there exists $b_\infty \in C^\infty(S^{n-1})$ such that

$$b - b_\infty \in C^\infty(M^\circ(W_p^l),\, S^{n-1}),$$

then C is compact on W_p^l.

In conclusion I mention that the book [40] contains very interesting applications of multipliers to the theory of elliptic boundary value problems on domains with non–smooth boundaries.

6 Integral equations of harmonic potential theory on general non–regular surfaces

In the sixties Maz'ya and his colleagues made a major breakthrough in the theory of boundary integral equations on very general surfaces, generalizing the classical

Radon theory to higher dimensions. In order to explain this, I need some notation. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a domain with boundary Γ. The harmonic double layer potential is given by

$$Wu(x) = \int\limits_{\Gamma} u(\xi) \frac{\partial}{\partial \nu_\xi} G(x, \xi) d_\xi \Gamma, \quad x \in \mathbb{R}^n \backslash \Gamma \,,$$

where G denotes the fundamental solution of the Laplacian. Let $T = 2W_0$, with W_0 being the direct value of W on Γ. The interior and exterior Dirichlet problem for the Laplacian in Ω can be reduced to the classical second kind integral equation with the boundary integral operators $I + T$ and $I - T$, respectively, whereas the corresponding Neumann problems lead to the adjoint operators $I \mp T^*$. Here I denotes the identity operator.

To formulate Radon's classical results on these integral equations, I need some further notation. Let A be a bounded linear operator acting on a Banach space X. The Fredholm radius $R(A)$ of A, introduced by Radon [44], is the radius of the largest disk centered at the origin of the complex λ–plane such that the operators $I - \lambda A$ are Fredholm for all λ in its interior. The quantity $|A| = \inf ||A - K||_X$, where the infimum is taken over all compact operators K on X, is referred to as the essential norm of A and also appeared first in [44].

Let Γ be a planar curve of bounded rotation, i.e., Γ is rectifiable and the angle $\vartheta(s)$ between the positively oriented tangent and the abscissa is of bounded variation on, say, $0 \leq s \leq l$. In 1919 Radon [45] proved that, for the harmonic double layer potential operator T acting on the space $C(\Gamma)$ of continuous functions, the equality

$$R(T) = |T|^{-1} = \pi / \alpha$$

is satisfied with $\alpha = \sup\{|\vartheta(s+0) - \partial(s-0)| : 0 \leq s \leq l\}$.

As a corollary of this result, one obtains that if Γ has no cusps then $R(T) > 1$, so that in this case the Fredholm theory applies to the operators $I \pm T$ in $C(\Gamma)$. Moreover, Radon then obtained the basic solvability results for these operators as well, e.g., the invertibility of $I + T$.

In their famous course of functional analysis [47] F. Riesz and B. Sz.–Nagy noted that "in the case of the spatial problem an analogue of curves with bounded rotation has not yet been found." This inspired the work by Burago, Maz'ya, Sapozhnikova [4], [5] and Burago, Maz'ya [3], who not only extended Radon's theory to higher dimensions but also improved Radon's result for contours in the plane.

The basic solvability results (e.g., invertibility of $I + T$ in $C(\Gamma)$ and of $I + T^*$ in $C(\Gamma^*)$) were obtained for domains subject to the following two conditions:

(A) $\sup\{\text{var } \omega(\xi, \Gamma \backslash \xi) : \xi \in \Gamma\} < \infty,$

(B) $\lim\limits_{r \longrightarrow 0} \sup\{\text{var } \omega(\xi, \Gamma \cap B_r(\xi)) : \xi \in \Gamma\} < \sigma_n / 2.$

Here $\omega(\xi, B)$ denotes the solid angle at which the set B is seen from the point ξ, or rather its generalization to a certain set function. Furthermore, var denotes the variation of the charge, $B_r(\xi)$ is the ball with center ξ and radius r, and σ_n is the area of the unit sphere S^{n-1}.

Condition (A) is necessary to apply the potential method in the spaces under consideration, whereas (B) is solely needed to prove the Fredholm alternative. In fact, independently of each other, Král [22] and Burago and Maz'ya [3] proved that (B) is equivalent to the inequality $|T| < 1$. Condition (A) is of course valid for any curve of bounded rotation, while (B) is satisfied if it has no cusps. However, there exist plane curves satisfying (A) and (B) which are not of bounded rotation. Before the works of Král and Burago, Maz'ya and Sapozhnikova it was a common belief that Radon generalized the theory of potentials in spaces of continuous functions to its natural limit; see [47], Sect. 91.

We now discuss some deep results by Kresin and Maz'ya [23], [24] on the essential norm of the general vector–valued double layer potentials

$$Tu(x) = 2 \int\limits_{\Gamma} k(e_{xy}) u(y) \omega(x, dy) \,.$$

Here k is a continuous even $(m \times m)$–matrix–valued function on the unit sphere, which is homogeneous of degree 0 and normalized by the condition $\int_{S^{n-1}} k = I$, and e_{xy} denotes $(y-x)/|y-x|$. The authors succeeded in proving general formulas for the norm and the essential norm in the space $C(\Gamma)^m$ of continuous vector–valued functions on Γ with m components, provided the surface Γ satisfies condition (A).

The case where $k(e)$ is the unit matrix with elements

$$\sigma_n^{-1}\{(1-\kappa)\delta_{ij} + n\kappa(e, e_i)\}\,, \quad i, j = 1, \dots, n\,, \quad 0 \le \kappa \le 1\,,$$

e_j being the unit vector directed along the jth coordinate axis, is of particular interest. Note that $\kappa = 0$ yields a diagonal matrix of harmonic potentials, and putting $\kappa = 1$ and $\kappa = (\lambda + \mu)(\lambda + 3\mu)^{-1}$ with the Lamé constants λ, μ, we obtain the hydrodynamic and elastic potentials, respectively.

For $n = 2$ and Γ a polygon with interior angles $\alpha_1, \dots, \alpha_N$, Kresin and Maz'ya deduced the following beautiful formula:

$$|T| = \frac{2}{\pi}(1+\kappa) E\left(\frac{\pi - \alpha_{\min}}{2}, \frac{2\sqrt{\kappa}}{1+\kappa}\right)\,, \quad \alpha_{\min} = \min\{\alpha_i : i = 1, \dots, N\},$$

where E denotes the elliptic integral of the second kind. For the hydrodynamic potential, i.e. for $\kappa = 1$, this gives $|T| = 4\pi^{-1}\cos(\alpha_{\min}/2)$. On the other hand, Shelepov [48] found the Fredholm radius,

$$R(T) = \pi\{|\pi - \alpha_{\min}| + \kappa \sin|\pi - \alpha_{\min}|\}^{-1}\,.$$

These two formulae imply the unexpected inequality $R(T) > |T|^{-1}$ whenever $\kappa > 0$. Explicit formulae for $|T|$ were also given in [24] in the three–dimensional case where Γ has an edge or a conical point.

7 Boundary integral equations on piecewise smooth surfaces

Despite the great generality of surfaces which Burago, Maz'ya and Král dealt with in the sixties, their theory did not apply to quite natural geometries because of the rather restrictive condition (B) on the local variation of the solid angle, and of course they only studied the Dirichlet and Neumann problem for harmonic potentials. Neither specific problems of mathematical elasticity and hydrodynamics nor general elliptic boundary value problems were touched. The main obstacle was that the theories of Fredholm and singular integral equations were not sufficiently developed to deal with irregular boundaries.

Around 1980 Maz'ya arrived at a simple but bold idea which brought a new light to the whole domain of boundary integral equations. He understood that such equations could be exhaustively studied without using general theories of integral equations. His talk [28] presented at the Petrovskiĭ Conference in 1981 became a breakthrough in the solvability theory of boundary integral equations on piecewise smooth surfaces.

Maz'ya's approach is based on the fact that solutions of boundary integral equations can be expressed in terms of solutions to certain auxiliary interior and exterior boundary value problems. Then the full force of the newly developed theory of boundary problems in non-smooth domains, with its theorems on solvability and Fredholm property in various function spaces as well as the results on asymptotics of solutions near boundary singularities, becomes available. In this theory Maz'ya also did major and pioneering work; see J. Rossmann's survey in the present Volume.

The new approach was first exemplified in [28] by the classical boundary value problems of potential theory, although it was clear from the very beginning that it has universal character. In [29] and [30] it was applied to the two fundamental boundary value problems of elasticity. As a result of fifteen years' work, Maz'ya and his collaborators developed an extensive solvability theory of boundary integral equations on surfaces with conical points, edges, vertices, and also cusps. A detailed account of the results obtained before 1990 can be found in Maz'ya's survey article [32], which has become a standard reference in the field.

Another important development during the last two decades was the theory of boundary equations in Lipschitz domains and with data from L_p, originating from the result of Calderón and Coifman, McIntosh, Meyer on the boundedness of the Cauchy singular integral operator over Lipschitz curves. Based on these results and on the Rellich-Nečas identities, Verchota [50], Kenig [20], Fabes [11], and others proved the solvability in L_p of various boundary integral equations on a Lipschitz surface. However, it is worth mentioning that there are simple polyhedra that are not Lipschitz (in the sense that they can in local Cartesian coordinates be described by Lipschitz functions).

To give an impression of how Maz'ya's method works, I will consider the special case of the second kind boundary integral equation

$$(I+T)u = f\,, \quad T = 2W_0 \tag{7}$$

for the Laplacian, which has already been discussed in the previous section. Let $\mathcal{D} : f \to v$ be the inverse operator of the interior Dirichlet problem

$$\Delta v = 0 \quad \text{in } \Omega\,, \quad \mathcal{T}_0 v := v|_\Gamma = f\,,$$

and let $\mathcal{N}$ be the inverse of the corresponding exterior Neumann problem. Then the inverse of $I+T$ can be represented as

$$(I+T)^{-1} = \frac{1}{2}(I - \mathcal{T}_0 \mathcal{N} \mathcal{T}_1 \mathcal{D})\,, \tag{8}$$

where $\mathcal{T}_1$ stands for the trace operator $\mathcal{T}_1 v := (\partial/\partial\nu)v|_\Gamma$. This also applies to the systems of integral equations of elasticity and hydrodynamics if the role of $\mathcal{T}_1$ is played by the stress operator. In [30] Maz'ya studied the solvability of these equations in weighted Hölder spaces in the case of a surface with conical points, edges and polyhedral angles, using the representation (8) and his results with Plamenevskiĭ on boundary value problems. Corresponding results for the mixed problem of 3D elasticity in domains with edges were obtained in his paper [31]. Moreover, Maz'ya's approach turned out to be useful to calculate the Fredholm radius of boundary integral operators. In the paper [15] with his student Grachev, an explicit formula for the harmonic double layer potential operator in weighted Hölder spaces on surfaces with edges was proved.

If one wants to obtain solvability in the space C or in weighted L_p and Sobolev spaces, then the representation (8) is not directly applicable (as long as there are no solvability results in Hardy spaces for the auxiliary boundary value problems). In 1989 Grachev and Maz'ya developed an approach which also works in this case. They established sharp estimates for the kernel of the inverse operator $(I+T)^{-1}$, implying, in particular, the solvability of the integral equation (7) in C without any assumption on the essential norm of T. For a smooth cone Γ of vertex 0 in $\mathbb{R}^n$, it was shown in their paper [16] that $(I+T)^{-1}$ decomposes into a sum $I + M_1 + M_2$ with certain integral operators M_1 and M_2 on Γ whose kernels admit the estimates

$$\begin{aligned} |M_1(x,y)|| &\leq c(1+|y|^{\mu-1}), \\ |M_2(x,y)|| &\leq \begin{cases} c|y|^{1-n}(|x|/|y|)^{\mu} & \text{for} \quad |x| < |y|/2\,, \\ c|y|^{-1}|x-y|^{2-n} & \text{for} \quad |y|/2 < |x| < 2|y|\,, \\ c|y|^{-1}|x|^{2-n}(|y|/|x|)^{\mu} & \text{for} \quad |x| > 2|y|\,, \end{cases} \end{aligned}$$

where the number $\mu, 0 < \mu \leq 1$, depends on the shape of the cone.

As a consequence of these estimates, general solvability results for equation (7) in C and weighted L_p spaces on a boundary with a finite number of conical points were obtained. Applying the same method, Maz'ya and Grachev [17] settled the long–standing classical question on solvability of (7) in the space C for an arbitrary polyhedron; see also [14]. At the same time, this problem was solved by Rathsfeld [46] using Mellin transformation and Banach algebra techniques. In addition [46] establishes the inequality $R(T) > 1$ for the Fredholm radius in C, which also holds for L_2 and certain weighted Sobolev spaces (see [10]).

As another application of Maz'ya's method, the asymptotics of solutions to boundary integral equations near singular points of the boundary can be derived, including the computation of the coefficients appearing in the asymptotic formulas. These results were set up by Zargaryan and Maz'ya [51], [52] for the integral equations of harmonic potential theory on a polygonal boundary. A different approach to solvability and asymptotics of solutions of integral equations over curves with corners was worked out by Costabel and Stephan [6], [7], who used the Mellin transform to solve the model equation on the legs of an infinite angle. The first result on the asymptotics of solutions to boundary integral equations over three–dimensional regions is due to Maz'ya and Levin [25] and refers to the case of conical points on the boundary.

During the last decade, Maz'ya and Solov'ev developed a theory of boundary integral equations on plane contours with cusps, including the Dirichlet and Neumann problem for the Laplace and the Lamé operator. Their extensive research on this topic started in 1988 with [41] and is presently still going on; see e.g. the recent papers [42], [43]. They proved theorems on the unique solvability in appropriate weighted L_p, Sobolev and Hölder spaces and on asymptotic representations for solutions near peaks, which are the only known results in the domain. I refer the interested reader to Maz'ya's survey [33] for further information. It is a tempting perspective to generalize the Maz'ya–Solov'ev results to multidimensional domains with cusps as well as to other boundary value problems.

I hope what I have said in this section shows convincingly that Maz'ya's approach opened new horizons in the theory of boundary integral equations on non–regular surfaces, and it will definitely inspire fruitful research in this field in the future.

References

[1] V. I. Arnold, *On local problems of mathematical analysis (Russian)*, Vestnik Moscov. Univ. **2** (1970), 52–56.

[2] V. I. Arnold, *Additional chapters of the theory of ordinary differential equations (Russian)*, Nauka, Moscow, 1978.

[3] Yu. D. Burago and V. G. Maz'ya, *Potential theory and function theory for irregular regions*, Sem. Math. V. A. Steklov Inst. Leningr. **3** (1969), 1–68, translation from Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova **3** (1967), 1–152.

[4] Yu. D. Burago, V. G. Maz'ya, and V. D. Sapozhnikova, *On the double layer potential for nonregular regions*, Sov. Math. Dokl. **3** (1962), 1640–1642, translation from Dokl. Akad. Nauk SSSR **147** (1962), 523–525.

[5] ———, *On the theory of simple and double–layer potentials for domains with irregular boundaries*, Probl. Mat. Anal. **1** (1968), 1–30, translation from Probl. Mat. Analiza, Kraevye Zadachi, Integral. Uravn. Leningrad (1966), 3–34.

[6] M. Costabel and E. Stephan, *Curvature terms in the asymptotic expansions for solutions of boundary integral equations on curved polygons*, J. Integral Equations **5** (1983), 353–371.

[7] ———, *Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation*, Mathematical Models in Mechanics, Banach Center Publication, vol. 15, PWN–Polish Scientific Publishers, Warsaw, 1985, pp. 175–251.

[8] R. Courant and D. Hilbert, *Methods of mathematical physics*, vol. 2, Interscience Publishers, New York, 1962.

[9] Yu. V. Egorov and V. A. Kondrat'ev, *The oblique derivative problem*, Math. USSR-Sb. **7** (1969), 139–169, translation from Mat. Sb. **78** (1969), 148–176.

[10] J. Elschner, *The double layer potential operator over polyhedral domains. I: Solvability in weighted Sobolev spaces*, Appl. Anal. **45** (1992), 117–134.

[11] E. B. Fabes, *Boundary value problems of linear elastostatics and hydrostatics on Lipschitz domains*, Proc. Centre Math. Anal. Austral. Nat. Univ. **9** (1985), 27–45.

[12] L. Gårding, T. Kotake, and J. Leray, *Uniformisation et development asymptotique de la solution du problème de Cauchy lineaire*, Bull. Soc. Math. France **92** (1964), 263–361.

[13] I. W. Gelman and V. G. Maz'ya, *Abschätzungen für Differentialoperatoren im Halbraum*, Akademie–Verlag, Berlin, 1981.

[14] N. V. Grachev, *Representations and estimates for inverse operators of the potential theory integral equations in a polyhedron*, Potential Theory (M. Kishi, ed.), Walter de Gruyter, Berlin, 1991, pp. 201–206.

[15] N. V. Grachev and V. G. Maz'ya, *On the Fredholm radius of operators of the double layer potential type on piecewise smooth surfaces*, Vestnik Leningrad Univ. Math. **19** (1986), no. 4, 20–25, translation from Vestnik Leningr. Univ., Ser. I, no. 4 (1986), 60–64.

[16] ———, *Representations and estimates of inverse operators of integral equations of potential theory for surfaces with conic points (Russian)*, Soobshch. Akad. Nauk Gruzin. SSR **132**, no. 1 (1988), 21–24.

[17] ———, *Solvability of a boundary integral equation on a polyhedron*, Preprint LITH-MAT-R91-50, Linköping University, Department of Mathematics, 1992.

[18] L. Hörmander, *Pseudo–differential operators and non–elliptic boundary value problems*, Ann. Math. **83** (1966), 129–209.

[19] ———, *A remark on the characteristic Cauchy problem*, J. Funct. Anal. **93** (1990), 270–277.

[20] C. E. Kenig, *Boundary value problems of linear elastostatics and hydrostatics on Lipschitz domains*, Seminar Goulaouic–Meyer–Schwartz, Equat. Deriv. Partielles 1983–1984, Exp. no. 21, 1984, pp. 1–12.

[21] V. A. Kondrat'ev, *The Cauchy problem with characteristic points on the initial surface*, Moscow Univ. Math. Bull. **29** (1974), no. 1/2, 68–74, translation from Vestnik Moskov. Univ., Ser. I, **29**, no. 1 (1974), 84–92.

[22] J. Král, *The Fredholm radius of an operator in potential theory*, Czechoslovak Math. J. **15** (1965), 454—473 and 565–588.

[23] G. I. Kresin and V. G. Maz'ya, *On the essential norm of an operator of double layer potential type in the space C_m*, Sov. Math. Dokl. **20** (1979), 459–462, translation from Dokl. Akad. Nauk SSSR **246** (1979), 272–275.

[24] ———, *The norm and the essential norm of the double layer elastic and hydrodynamic potentials in the space of continuous functions*, Math. Methods Appl. Sci. **18** (1995), 1095–1131.

[25] A. V. Levin and V. G. Maz'ya, *On the asymptotics of the density of harmonic potentials close to the vertex of a cone (Russian)*, Z. Anal. Anwendungen **8** (1989), 501–514.

[26] M. B. Malyutov, *On the boundary value problem of Poincaré (Russian)*, Trudy Moscov. Mat. Obshch. **20** (1969), 173–204.

[27] V. G. Maz'ya, *On the degenerate oblique derivative problem*, Math. USSR-Sb. **16** (1972), 429–469, translation from Mat. Sb. **87** (1972), 417–454.

[28] ———, *The integral equations of potential theory in domains with piecewise smooth boundary (Russian)*, Uspekhi Mat. Nauk **36**, no. 4 (1981), 229–230.

[29] ———, *On the solvability of the integral equations of classical elasticity theory in domains with piecewise smooth boundary*, Proceedings Conference on General Mechanics and Elasticity Theory, Telavi, Metsniereba, Tbilisi, 1981, pp. 55–56.

[30] ———, *Boundary integral equations of elasticity in domains with piecewise smooth boundaries*, Differential equations and their applications, Equadiff 6, Proceedings 6th Int. Conf., Brno/Czech., 1985, Lecture Notes Math. **1192**, Springer, Berlin, 1986, pp. 235–242.

[31] ———, *Potential theory for the Lamé equations in domains with piecewise smooth boundary*, Proceedings All–Union Symposium, Tbilisi, April 21–23, 1982, Metsniereba, Tbilisi, 1986, pp. 123–129.

[32] ———, *Boundary integral equations*, Analysis IV (V. G. Maz'ya and S. M. Nikol'skiĭ, eds.), Encyclopaedia of Mathematical Sciences, vol. 27, Springer–Verlag, Berlin, 1991, pp. 127–222.

[33] ———, *Boundary integral equations on a contour with peaks*, Boundary integral methods for nonlinear problems (L. Morino et al., ed.), Proceedings of the IABEM symposium, Pontignano, Italy, May 28–June 3, 1995, Kluwer Academic Publishers, Dortrecht, 1997, pp. 145–153.

[34] V. G. Maz'ya and B. P. Paneyah, *Degenerate elliptic pseudo–differential operators on a smooth manifold without boundary*, Functional Anal. Appl. **3** (1969), 159–160, translation from Funktional. Anal. Prilozhen. **3**, no. 2 (1969), 91–92.

[35] ———, *Coercive estimates and regularity of the solution of degenerate elliptic pseudodifferential equations*, Functional Anal. Appl. **4** (1970), 299–311, translation from Funktional. Anal. Prilozhen. **4**, no. 4 (1970), 41 –56.

[36] ———, *Degenerate elliptic pseudodifferential operators and the oblique derivative problem*, Trans. Moscow Math. Soc. **31** (1974), 247–305, translation from Trudy Moscov. Mat. Obshch. **31** (1974), 237–295.

[37] V. G. Maz'ya and B. A. Plamenevskiĭ, *Singular equations with a vanishing symbol*, Sov. Math. Dokl. **6** (1965), 294–297, translation from Dokl. Akad. Nauk SSSR **160** (1965), 1250–1253.

[38] ———, *The Cauchy problem for hyperbolic singular integral equations of convolution type (Russian)*, Vestnik Leningr. Univ. **20**, no. 19 (1965), 161–163.

[39] V. G. Maz'ya, B. A. Plamenevskiĭ, and Yu. E. Khaikin, *A correctly posed problem for singular integral equations with vanishing symbols*, Differential Equations **13** (1977), 1028–1033, translation from Differencial'nye Uravneniya **13** (1977), 1479–1486.

[40] V. G. Maz'ya and T. O. Shaposhnikova, *Theory of multipliers in spaces of differentiable functions*, Pitman, Boston, 1985.

[41] V. G. Maz'ya and A. A. Solov'ev, *Solvability of an integral equation for the Dirichlet problem in a plane domain with cusps on the boundary*, Sov. Math.

Dokl. **37** (1988), 255–258, translation from Dokl. Akad. Nauk SSSR **298**, no. 6 (1988), 1312–1315.

[42] ———, *L_p–theory of a boundary integral equation on a cuspidal contour*, Appl. Anal. **65** (1997), 289–305.

[43] ———, *L_p–theory of a boundary integral equation on a contour with outward peak*, Integral Equation Operator Theory **32** (1998), 75–100.

[44] J. Radon, *Über lineare Funktionaltransformationen und Funktionalgleichungen*, Sitzungsberichte Akad. Wiss., Abt. 2a, Wien **128** (1919), 1083–1121.

[45] ———, *Über die Randwertaufgaben beim logarithmischen Potential*, Sitzungsberichte Akad. Wiss., Abt. 2a, Wien **128** (1919), 1123–1167.

[46] A. Rathsfeld, *The invertibility of the double layer potential operator in the space of continuous functions defined on a polyhedron. The panel method*, Appl. Anal. **45** (1992), 135–177.

[47] F. Riesz and B. Sz.-Nagy, *Functional analysis*, Ungar Publishing Co., New York, 1955, French original: Akad. Kiado, Budapest, 1952.

[48] V. Yu. Shelepov, *On investigations by Ya. B. Lopatinskiĭ's method of matrix integral equations in the space of continuous functions (Russian)*, General theory of boundary value problems, Collect. Sci. Works, Naukova Dumka, Kiev, 1983, pp. 220–226.

[49] B. R. Vainberg and V. G. Maz'ya, *Characteristic Cauchy problem for a hyperbolic equation*, J. Soviet Math. **31** (1985), 3135–3147, translation from Trudy Sem. Petrovsk. **7** (1981), 101–117.

[50] G. Verchota, *Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains*, J. Funct. Anal. **59** (1984), 572–611.

[51] S. S. Zargaryan and V. G. Maz'ya, *On singularities of solutions of a system of integral equations in potential theory for the Zaremba problem*, Vestnik Leningrad Univ. Math. **16** (1984), 49–55, translation from Vestnik Leningr. Univ., Ser. I, no. 1 (1983), 43–48.

[52] ———, *The asymptotic form of solutions of integral equations of potential theory in the neighbourhood of the corner points of a contour*, J. Appl. Math. Mech. **48** (1985), 120–124, translation from Prikl. Mat. Mekh. **48** (1984), 169–174.