ON THE HOT SPOTS OF A CATALYTIC SUPER-BROWNIAN MOTION

JEAN-FRANÇOIS DELMAS AND KLAUS FLEISCHMANN

ABSTRACT. Consider the catalytic super-Brownian motion X^ϱ (reactant) in $\mathbb{R}^d$, $d \le 3$, which branching rates vary randomly in time and space and in fact are given by an ordinary super-Brownian motion ϱ (catalyst). Our main object of study is the collision local time $L = L_{[\varrho, X^\varrho]}\big(d[s,x]\big)$ of catalyst and reactant. It determines the covariance measure in the martingale problem for X^ϱ and reflects the occurence of "hot spots" of reactant which can be seen in simulations of X^ϱ. In dimension 2, spatial marginal collision densities exist and, via self-similarity, enter as factor in the long-term random ergodic limit of L (diffusiveness of the 2-dimensional model).

1. INTRODUCTION

The ordinary *super-Brownian motion* $\varrho = (\varrho_t\,, t \ge 0)$ in Euclidean space $\mathbb{R}^d$ can be obtained as a limit of branching particle systems. In this branching particle system, the particles evolve according to independent Brownian motions in $\mathbb{R}^d$, and additionally, with constant rate $\gamma > 0$, each particle splits independently into 2 or 0 particles with equal probability (this is a critical binary branching mechanism).

We now interpret ϱ as a *catalyst process*: $\varrho_t(dx)$ is the amount of catalytic "particles" at time t in the volume element dx of $\mathbb{R}^d$. We then let a super-Brownian motion $X^\varrho = (X^\varrho_t, t \ge 0)$ evolve in this *catalytic random medium* ϱ. Intuitively X^ϱ describes *reactant* "particles" which are evolving according to independent Brownian motions and which are performing a critical binary branching, but at random time-space varying rates given by ϱ. In fact, the rate of branching of an intrinsic reactant particle with Brownian path W is controlled by the *collision local time* $L_{[\varrho,W]}$ *of* ϱ *and* W, defined as the measure

$$L_{[\varrho,W]}(ds) \;:=\; \lim_{\varepsilon\downarrow 0} ds \int \varrho_s(dy)\, p(\varepsilon, y - W_s),$$

where p is the standard heat kernel $p(t,x) = [2\pi t]^{-d/2} \exp\left[-|x|^2/2t\right]$, $(t,x) \in (0,\infty) \times \mathbb{R}^d$. According to [BEP91], this collision local time $L_{[\varrho,W]}$ makes sense non-trivially in dimension $d \le 3$, and vanishes for $d \ge 4$ (where the Brownian reactant particles do not hit the catalyst ϱ). Thus we restrict our attention to $d \le 3$ (since otherwise X^ϱ degenerates to the heat flow).

Date: November 17, 1998; WIAS preprint No. 450.

1991 *Mathematics Subject Classification.* Primary 60J80; Secondary 60G57, 60K35.

Key words and phrases. Catalytic super-Brownian medium, superprocess, measure-valued process, collision local time of catalyst and reactant, two-dimensional process, catalytic medium.

The research was partially supported by the European TMR program grant ERBFMRXCT 960075 and by the NSF grant DMS-9701755 at the MSRI.

The *catalytic super-Brownian motion* X^ϱ was constructed in [DF97a]. Let ϱ and X^ϱ start at time 0 with Lebesgue measures ℓ_{c} and ℓ_{r}, respectively. From the papers [DF97a, DF97b, EF98, FK97] it is known that X^ϱ_T *converges* in law as $T \uparrow \infty$ to a limit X^ϱ_∞ with full expectation ℓ_{r} (*persistence*) [1)]. The approach of [FK97] to solve the most difficult case, namely convergence in the critical dimension $d = 2$, was to study the local structure of X^ϱ and then to use a self-similarity argument. In fact, they showed that in dimensions $d = 2, 3$, given the catalyst ϱ, the reactant X^ϱ has a *density field* ξ^ϱ :

$$X^\varrho_t(dx) \;=\; \xi^\varrho_t(x)dx, \qquad t > 0.$$

Moreover, off the time-space support of the catalyst ϱ (which is a Lebesgue zero set), ξ^ϱ can be chosen as a (time-space) C^∞-function that solves the heat equation, just as intuitively expected.

Simulations of (ϱ, X^ϱ) in dimension $d = 2$ (see the figure in [FK97]) confirm the *heuristic picture* one has. Namely, at late times T,

- the reactant X^ϱ_T is rather uniform outside of the catalyst ϱ_T,
- it is absent inside of the clumps of ϱ_T (since a huge rate of branching causes mainly killing),
- but occasionally also some *hot spots* of the reactant occur in the *interface* of ϱ_T and X^ϱ_T, that is in the boundary region of the catalytic clumps.

According to [FK97], in the two-dimensional case, the (local) long-term limit X^ϱ_∞ is in fact a random multiple of Lebesgue measure [the factor is given by $\xi^\varrho_1(0)$]. But so far the investigations on the catalytic super-Brownian motion X^ϱ do not reflect anything on the hot spots seen in the pictures. Our approach to gain some information about them is to study the *collision local time* $L := L_{[\varrho, X^\varrho]}$ *of* ϱ *and* X^ϱ defined as the limit of

$$L_\varepsilon\left(d[s,x]\right) \;:=\; ds\; \varrho_s(dx) \int X^\varrho_s(dy)\, p(\varepsilon, x-y), \tag{1}$$

as $\varepsilon \downarrow 0$.

Actually there is a further motivation to study this collision local time $L_{[\varrho, X^\varrho]}$. It occurs indeed in the description of the *martingale problem* for the process X^ϱ (see Corollary 4 below). For martingale problems of catalytic super-Brownian motions, see also [DF94, Del96, Led97].

Let us present the results. We prove that in all dimensions of non-trivial existence of X^ϱ the collision local time L of catalyst and reactant makes non-trivially sense (see Theorem 3 below). [In dimension 1, it is known that both ϱ_s and X^ϱ_s are absolutely continuous (cf. [KS88] and [DFR91], respectively); thus L_ε and hence L simplify in this case.] This non-trivial existence of L reflects the high fluctuations of X^ϱ in the interface of catalyst and reactant, seen as hot spots in simulations. Our *main result* however is that for $d = 2$ the *marginal measure* of $L = L_{[\varrho, X^\varrho]}$ concerning the space variable is *absolutely continuous* (Theorem 5). Note that this is in contrast, for instance, with the (one-dimensional) single-point catalytic model of [DF94], say X^{δ_0}, where, together with the catalyst δ_0, the space marginal of the collision local time $L_{[\delta_0, X^{\delta_0}]}$ is concentrated in the single space point 0, hence is singular (even atomic). Finally, in dimension 2, using the *self-similarity* of $L_{[\varrho, X^\varrho]}$ which follows from the

---

1) In the three-dimensional case, for simplification it was assumed in [DF97b] that the catalyst process ϱ is already in its corresponding equilibrium.

self-similarity of (ϱ, X^ϱ) , we show that $T^{-1}L\left([0,T]\times(\cdot)\right)$ has a *random ergodic limit* as $T\uparrow\infty$, which indicates *diffusive* features in the long-term behavior in $d=2$.

It remains *open* whether also in dimension 3 space-marginal collision densities exist since our L^2-approach fails in this case (see Remark 6 below).

The *outline* of the paper is as follows. In Section 2 we introduce formal definitions of the processes ϱ and X^ϱ and state the results. The following two sections are then devoted to the proofs of our two theorems. In an appendix we collect some results on ordinary and catalytic super-Brownian motions used in the proofs.

2. Statement of results

2.1. **Notation.** The lower index $+$ on a set will always refer to the collection of all its nonnegative members. Similarly, f_+ is the nonnegative part of f. The supremum norm is denoted by $\|\cdot\|_\infty$. Let c always refer to a (finite) constant whose value may vary from place to place.

We denote by $\mathcal{B}(E)$ the space of all real Borel measurable functions defined on a polish space E. We also denote by $\mathcal{B}(E)$ the Borel σ–field of E.

For a fixed constant $q>d$, introduce the reference function $\phi_q\in\mathcal{B}(\mathbb{R}^d)$:

$$\phi_q(x) \;:=\; \left[1+|x|^2\right]^{-q/2}, \qquad x\in\mathbb{R}^d. \tag{2}$$

Set $\mathcal{B}^q := \left\{f\in\mathcal{B}(\mathbb{R}^d);\|f/\phi_q\|_\infty<\infty\right\}$. Let $C_c(\mathbb{R}^d)$ denote the collection of all continuous functions on $\mathbb{R}^d$ with compact support.

If ν is a Radon measure on $\mathbb{R}^d$, we write (ν,f) for $\int\nu(dx)\,f(x)$ (if the integral makes sense). Let $\mathcal{M}_q$ denote the set of all Radon measures ν on $\mathbb{R}^d$ such that $(\nu,\phi_q)<\infty$. This space of tempered measures is endowed with the coarsest topology such that the maps $\nu\mapsto(\nu,f)$ are continuous for $f\in C_c(\mathbb{R}^d)\cup\{\phi_q\}$, getting a Polish space. Since $q>d$, Lebesgue measure belongs to $\mathcal{M}_q$.

We consider the polish space $\mathcal{C}:=C(\mathbb{R}_+,\mathcal{M}_q)$ of all continuous functions from $\mathbb{R}_+$ to $\mathcal{M}_q$ equipped with the topology of uniform convergence on compacta.

Let $(P_t\,,t\ge0)$ denote the semigroup of heat flow on $\mathbb{R}^d$:

$$P_t[f](x) \;:=\; \int dy\; p(t,x-y)f(y), \qquad t>0,\quad f\in\mathcal{B}_+(\mathbb{R}^d). \tag{3}$$

2.2. **Catalyst and reactant process.** We start by introducing the catalyst process.

Definition 1 (catalyst process)**.** Let $\gamma>0$ and $\nu\in\mathcal{M}_q$. There exists a unique probability measure $\mathbb{P}_\nu$ on $\left(\mathcal{C},\mathcal{B}(\mathcal{C})\right)$, such that the coordinate process $\varrho=(\varrho_t\,,t\ge0)$ on $\mathcal{C}$ is a super-Brownian motion with constant branching rate γ and starting measure ν. That is, ϱ is a continuous time-homogeneous strong Markov process with the following properties:

- $\mathbb{P}_\nu$-almost surely, $\varrho_0=\nu$,
- for every $f\in\mathcal{B}^q_+,\ t\ge r\ge0$, we have [2)]

$$\mathbb{E}_\nu\left[\mathrm{e}^{-(\varrho_t\,,f)}\;\middle|\;\sigma\left(\varrho_s\,,s\in[0,r]\right)\right] \;=\; \mathrm{e}^{-\left(\varrho_r\,,w(t-r)\right)}, \tag{4}$$

[2)] We use the following convention: If $\mathcal{P}$ is a probability law, then the corresponding letter $\mathcal{E}$ refers to the related expectation symbol.

where w is the unique nonnegative solution on $\mathbb{R}_+ \times \mathbb{R}^d$ of the log-Laplace equation

$$w(t,x) + \gamma \int_0^t ds\ P_s[w^2(t-s)](x) \ = \ P_t\,[f]\,(x). \tag{5}$$

We write $\mathbb{P}$ for $\mathbb{P}_\nu$ in the case $\nu = i_{\mathrm{c}}\ell$, where $i_{\mathrm{c}} > 0$ and ℓ is the (normalized) Lebesgue measure on $\mathbb{R}^d$. ◇

From now on we assume that $d \le 3$, and that ϱ is distributed[3] according to $\mathbb{P}$. Next we recall the definition of the catalytic super-Brownian motion X^ϱ in the random medium ϱ (see [DF97a] for details).

Definition 2 (catalytic super-Brownian motion)**.** Fix $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$ and $\kappa > 0$. For convenience, set $\mathcal{C}' := C\left([r,\infty), \mathcal{M}_q\right)$. There exists a (measurable) probability kernel $\varrho \mapsto \mathrm{P}^\varrho_{r,\mu}$ from $\left(\mathcal{C}, \mathcal{B}(\mathcal{C})\right)$ to $\left(\mathcal{C}', \mathcal{B}(\mathcal{C}')\right)$ such that the coordinate process $X^\varrho = (X^\varrho_t, t \ge r)$ on $\mathcal{C}'$ is a *super-Brownian motion in the catalytic medium* ϱ. That is, $\mathbb{P}$-a.s. under $\mathrm{P}^\varrho_{r,\mu}$, the process X^ϱ is continuous time-inhomogeneous Markov with the following properties:

- $\mathrm{P}^\varrho_{r,\mu}$-almost surely, $X^\varrho_r = \mu$,
- for every $f \in \mathcal{B}^q_+$, $t \ge s \ge r$, we have

$$\mathrm{E}^\varrho_{r,\mu}\left[\mathrm{e}^{-(X^\varrho_t, f)}\ \middle|\ \sigma\left(X^\varrho_u, u \in [r,s]\right)\right] \ = \ \mathrm{e}^{-\left(X^\varrho_s,\, v_t(s)\right)}, \tag{6}$$

where v_t is the unique nonnegative solution on $[r,\infty) \times \mathbb{R}^d$ of the catalytic log-Laplace equation

$$v(s,x) + \kappa \int_s^\infty du \int \varrho_u(dy)\, p(u-s, x-y)\, v^2(u,y) \ = \ J(s,x), \tag{7}$$

with $J(s) := \mathbf{1}_{t \ge s}\, P_{t-s}[f]$.

Often, we also pass from the quenched distributions $\mathrm{P}^\varrho_{r,\mu}$ to the annealed law $\mathbb{E}[\mathrm{P}^\varrho_{r,\mu}]$. ◇

2.3. **Existence of collision local time of catalyst and reactant.** For our constant $q > d$, we introduce the function space $H^q := \bigcup_{T \ge 0} H^q_T$, where

$$H^q_T \ := \ \left\{ g \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d);\ \operatorname{supp} g \subset [0,T] \times \mathbb{R}^d,\ \|g/\phi_q\|_\infty < \infty \right\}, \tag{8}$$

with $\|g/\phi_q\|_\infty = \sup_{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d} |g(s,x)|/\phi_q(x)$, and $\operatorname{supp} g$ denoting the support of g.

Recall the approximated collision local time L_ε of ϱ and X^ϱ introduced already in (1). We are now ready to state our first result, the existence of the collision local time $L = L_{[\varrho, X^\varrho]}$ of ϱ and X^ϱ. Recall that $d \le 3$ and $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

Theorem 3 (collision local time)**.** *There exists a random variable denoted by $L = L_{[\varrho, X^\varrho]}$ defined on $\left(\mathcal{C} \times \mathcal{C}', \mathcal{B}(\mathcal{C} \times \mathcal{C}')\right)$, taking values in the set of Radon measures on $[r,\infty) \times \mathbb{R}^d$ with the following properties:*

(i) (tempered measure): *For every $T \ge r$, we have* $\mathbb{E}\left[\mathrm{P}^\varrho_{r,\mu}(L, \mathbf{1}_{[r,T]}\, \phi_q)\right] < \infty$.

(ii) (existence via convergence): *For every $\varphi \in H^{2q}$,*

$$\lim_{\varepsilon \downarrow 0} (L_\varepsilon\,, \varphi) = (L, \varphi), \quad \mathbb{E}[\mathrm{P}^\varrho_{r,\mu}]\text{–}a.s.$$

[3] In [FK97] more generally a class of so-called η-diffusive measures ν is introduced which allow that ϱ under $\mathbb{P}_\nu$ may serve as the catalyst for X^ϱ.

(iii) (regularity): *For every $\varphi \in H^{2q}$, and $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$-a.s., the process $\big((L, \mathbf{1}_{[r,t]}\varphi),\ t \geq r\big)$ is continuous and adapted to the filtration*

$$\Big(\mathcal{F}_t := \sigma(\varrho) \vee \sigma\left(X^{\varrho}_s,\ s \in [r,t]\right),\ t \geq r\Big).$$

(iv) (moments): *For every $m \geq 1$, $\varphi \in H^{2q}$, $\mathbb{P}$-a.s.,*

$$\mathrm{E}^{\varrho}_{r,\mu}\left[\left[\int_{[r,\infty)\times\mathbb{R}^d} L\left(d[s,x]\right)\varphi(s,x)\right]^m\right] \;=\; m!\sum_{k=1}^{m}\frac{1}{k!}\sum_{\substack{n_1,\dots,n_k \geq 1,\\ n_1+\dots+n_k=m}}\ \prod_{i=1}^{k}\left(\mu, \chi_{n_i}(r)\right), \tag{9}$$

where the functions χ_n, $n \geq 1$, belong to H^q and are recursively defined by

$$\chi_n(s,x) \;:=\; \kappa\int_s^{\infty} du \int \varrho_u(dy)\, p(u-s, x-y)\left[\sum_{i=1}^{n-1}\chi_i(u,y)\,\chi_{n-i}(u,y)\right], \qquad n \geq 2, \tag{10}$$

with the initial condition

$$\chi_1(s,x) \;:=\; \int_s^{\infty} du \int \varrho_u(dy)\ p(u-s, x-y)\,\varphi(u,y), \qquad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{11}$$

Consequently, in dimensions $d \leq 3$, the collision local time $L = L_{[\varrho, X^{\varrho}]}$ of catalyst and reactant exists non-trivially, reflecting in particular the occurrence of hot spots in the mentioned 2-dimensional simulations.

The proof of this theorem is postponed to Section 3.

As an *application,* we can now describe the covariance measure of the martingale measure associated with X^{ϱ}. Let $C_b^{1,2}$ denote the set of bounded functions $\varphi \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that the partial derivatives $\frac{\partial\varphi}{\partial s}$ and $\frac{\partial^2\varphi}{\partial x_i \partial x_j}$ exist and are continuous and bounded. It is easy to check that under $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$ the process $(M\varphi_{r,t},\ t \geq r)$ defined by

$$M\varphi_{r,t} \;:=\; \left(X^{\varrho}_t, \varphi(t)\right) - \left(X^{\varrho}_r, \varphi(r)\right) - \int_r^t ds\left(X^{\varrho}_s,\ \frac{\partial\varphi}{\partial s}(s) + \frac{1}{2}\Delta\varphi(s)\right), \tag{12}$$

is an $(\mathcal{F}_t, t \geq r)$*-martingale* [note that $\mathcal{F}_r = \sigma(\varrho) \vee \sigma(X_r)$]. Thanks to the Markov property of X^{ϱ} (given ϱ), and the moment formula (A.9) for X^{ϱ} stated in the appendix, we get that for φ, ψ in $C_b^{1,2}$, $\mathbb{P}$-a.s. for all $s \geq r$ and $t \geq r$,

$$\mathrm{E}^{\varrho}_{r,\mu}\left[M\varphi_{r,s} M\psi_{r,t}\right] \;=\; 2\kappa\int\mu(dx)\int_r^{s\wedge t} du \int \varrho_u(dy)\ p(u-r, x-y)\,\varphi(u,y)\,\psi(u,y). \tag{13}$$

The functional $M : \varphi \mapsto M\varphi$ defined on $C_b^{1,2}$ can be extended to an *orthogonal martingale measure* on H^q. Let $\langle M\rangle$ denote its *covariance measure.* Now we show how $\langle M\rangle$ can be expressed in terms of the collision local time $L = L_{[\varrho, X^{\varrho}]}$. Recall that $d \leq 3$ and that $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

Corollary 4 (covariance measure). *For every $\varphi \in H^q$, $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$-a.s. for every $t \geq r$, we have*

$$\langle M\varphi\rangle_{r,t} \;=\; 2\kappa\int_{[r,t]\times\mathbb{R}^d} L\left(d[s,y]\right)\varphi^2(s,y). \tag{14}$$

Proof. Using the Markov property of X^ϱ (given ϱ) and an obvious extension of the second moment formula (13), we obtain for $\varphi \in H^q$, $\mathbb{P}$-a.s. for all $t \geq s \geq r$,

$$\mathbb{E}\mathrm{E}^{\varrho}_{r,\mu}\left[(M\varphi_{r,t})^2 \mid \mathcal{F}_s\right] \;=\; (M\varphi_{r,s})^2 + 2\kappa \int X^{\varrho}_s(dx) \int_s^t du \int \varrho_u(dy)\, p(u-s, x-y)\, \varphi^2(u,y).$$

Since

$$\left(\int_{[r,t]\times\mathbb{R}^d} L\left(d[s,y]\right) \varphi^2(s,y),\ t \geq r\right)$$

is in t non-decreasing and continuous, is adapted to $(\mathcal{F}_t, t \geq r)$, and zero for $t = r$, we get that

$$\left(\langle M\varphi\rangle_{r,t} - 2\kappa \int_{[r,t]\times\mathbb{R}^d} L\left(d[s,y]\right) \varphi^2(s,y),\ t \geq r\right)$$

is a continuous martingale under $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$ with bounded variation starting at time $t = r$ from 0. This martingale is then constant and, in fact, equal to 0, giving the claim (14). $\square$

2.4. **Collision local time in dimension two.** We now state our results for the collision local time in the "critical" dimension $d = 2$. For convenience, we introduce the following abbreviation for an annealed law:

$$\mathbf{P} \;:=\; \mathbb{E}[\mathrm{P}_{0,i_{\mathrm{r}}\ell}] \;=\; \mathbb{E}_{i_{\mathrm{c}}\ell}[\mathrm{P}_{0,i_{\mathrm{r}}\ell}], \quad \text{where } i_{\mathrm{r}} > 0.$$

(That is, we now focus on the situation $r = 0$ and $\mu = i_{\mathrm{r}}\ell$.)

Theorem 5 (two-dimensional collision local time). *Let $d = 2$.*

(a) (local spatial L^2 collision densities): *For every $t \geq s \geq 0$ and $z \in \mathbb{R}^2$,*

$$\left(\int_{[s,t]\times\mathbb{R}^d} L\left(d[r,y]\right) p(\varepsilon, z-y),\ \varepsilon > 0\right)$$

converges in $L^2(\mathbf{P})$ as $\varepsilon \downarrow 0$ to a random variable denoted by $\lambda_{[s,t]}(z)$. It has expectation

$$\mathbf{E}\left[\lambda_{[s,t]}(z)\right] \;=\; i_{\mathrm{c}} i_{\mathrm{r}}\,(t-s),$$

and its finite variance is non-zero provided that $s < t$.

(b) (spatial absolute continuity): *For $t \geq s \geq 0$, there exists a measurable version of $\lambda_{[s,t]}$ with respect to $\mathcal{B}(\mathbb{R}^2)\times\mathcal{F}_t$, and $\mathbf{P}$-a.s. the measure $L\left([s,t]\times(\cdot)\right)$ on $\mathbb{R}^2$ is absolutely continuous and can be represented as*

$$L\left([s,t]\times dx\right) \;=\; \lambda_{[s,t]}(x)dx.$$

(c) (self-similarity): *Under $\mathbf{P}$, the laws of the scaled collision local times*

$$K^{-2}L\left(K(\cdot)\times K^{1/2}(\cdot)\right)$$

are independent of the scaling factor $K > 0$.

(d) (random ergodic limit): *The following convergence in $\mathcal{M}_q$ holds in law with respect to $\mathbf{P}$:*

$$\lim_{T\uparrow\infty} T^{-1}L\left([0,T]\times(\cdot)\right) \;=\; \lambda_{[0,1]}(0)\,\ell$$

(with ℓ the Lebesgue measure and $0 < \mathbf{Var}\left[\lambda_{[0,1]}(0)\right] < \infty$).

Consequently, in dimension 2, the spatial marginal measures $L\left([s,t]\times(\cdot)\right)$ of the collision local time $L_{[\varrho,X^\varrho]}$ of catalyst and reactant have non-degenerated densities $\lambda_{[s,t]}(z)$ (provided that $s<t$). Moreover, $\lambda_{[0,1]}(0)$ enters as random factor of Lebesgue measure in the long-term ergodic limit. Recall that this reflects the diffusive features of the hot spots.

Remark 6 (dimension three). The $L^2(\mathbf{P})$-convergence in part (a) does *not* hold for $d=3$. In fact, in the three-dimensional case an infinite term would be involved in our calculations, see the remark following (29) in the proof below. Recall on the other hand that in dimension one, $L_{[\varrho,X^\varrho]}$ should be rather "regular". $\diamond$

Remark 7 (regularity). It is an *open problem* whether the spatial collision density functions $\lambda_{[s,t]}$ have some regularities properties in the space variable. Note also that the exceptional set in the $\mathbf{P}$-a.s. statement in (b) depends on $[s,t]$. One would also like to know whether this situation can be improved. $\diamond$

The statement (c) follows from the self-similarity of (ϱ, X^ϱ) by standard arguments (compare with [DF97b, Subsections 4.1 and 4.2]). Otherwise the proof of Theorem 5 will be provided in Section 4.

3. Existence of collision local time (proof of Theorem 3)

Recall that $d\le 3$. First of all we state the following lemma.

Lemma 8 (approximated moment increments). *For every $m\ge 1$, $r\ge 0$, $\mu\in\mathcal{M}_q$, $T\ge 0$, $\xi\in(0,1/4)$, $\mathbb{P}$-a.s. there exists a finite constant M_m (depending on ϱ) such that for every $\varphi\in H_T^{2q}$, $t'\ge t\ge 0$, $1\ge\varepsilon'\ge\varepsilon>0$,*

$$\mathrm{E}^\varrho_{r,\mu}\left[(L_\varepsilon\,,\varphi\,\mathbf{1}_{[t,t']})^{2m}\right]\;\le\; M_m\,\|\varphi/\phi_{2q}\|_\infty^{2m}\left[\left|t-t'\right|^\xi\left(1+\log_+\left(1/|t-t'|\right)\right)\right]^{2m}, \tag{15}$$

$$\mathrm{E}^\varrho_{r,\mu}\left[\left[(L_\varepsilon\,,\varphi)-(L_{\varepsilon'},\varphi)\right]^{2m}\right]\;\le\; M_m\,\|\varphi/\phi_{2q}\|_\infty^{2m}\left[\left|\varepsilon-\varepsilon'\right|^\xi\left(1+\log_+\left(1/|\varepsilon-\varepsilon'|\right)\right)\right]^{2m}. \tag{16}$$

Based on this lemma, the proof of Theorem 3 (ii) and (iii) are similar to the proof of Proposition 5.1 based on Lemma 5.2 in [Del96] with the obvious changes and is left to the reader. (iv) is not stated in Proposition 5.1 there, but it is a by-product of its proof [take the limit in (32)]. Eventually, (i) is proved by using the monotone convergence theorem with the moment formula (9) and (A.2) (in the appendix) with $m=1$ and the inequality (A.1).

Proof of Lemma 8. Fix $\mu\in\mathcal{M}_q$, $\xi\in(0,1/4)$, and $T\ge r\ge 0$ (otherwise the moments disappear). We will verify (15); the proof of (16) is similar and is left to the reader.

Note first that for fixed $\varepsilon>0$,

$$\sup_{x\in\mathbb{R}^d,\ y\in\mathbb{R}^d}\frac{\phi_q(y)\,p(\varepsilon,x-y)}{\phi_q(x)}\;<\;\infty. \tag{17}$$

Let $\varphi\in H_T^{2q}$. Since ϱ is $\mathbb{P}$-a.s. a continuous $\mathcal{M}_q$-valued path, it is then clear that the functions $(s,x)\mapsto\int\varrho_s(dy)\,p(\varepsilon,x-y)\,\varphi(s,y)$ belong to H_T^q. Thanks to the remarks at the beginning of Subsection A.1, we see that, for fixed t,t',ε, the functions

$$(s,x)\mapsto J_\varepsilon(s,x)\;:=\;\int_s^\infty du\int dz\;p(u-s,x-z)\int\varrho_u(dy)\,p(\varepsilon,z-y)\,\varphi(u,y)\,\mathbf{1}_{[t,t']}(u) \tag{18}$$

are well-defined and belong to H_T^q.

We will now prove that $\mathbb{P}$-a.s. there exists a finite constant c such that for every $\varphi \in H_T^{2q}$, $t' \geq t \geq 0$, $1 \geq \varepsilon > 0$,

$$\left|J_\varepsilon(s,x)\right| \;\leq\; c\,\mathbf{1}_{[0,T]}(s)\,\phi_q(x)\,\|\varphi/\phi_{2q}\|_\infty \left[\left|t-t'\right|^\xi \left(1+\log_+\left(1/|t-t'|\right)\right)\right]. \tag{19}$$

Clearly $\left|J_\varepsilon(s,x)\right| / \left\|\varphi/\phi_{2q}\right\|_\infty$ is bounded from above by

$$K_1 \;:=\; \mathbf{1}_{[0,T]}(s) \int_s^T du \int \varrho_u(dy)\, p(u-s+\varepsilon, x-y)\,\phi_{2q}(y)\,\mathbf{1}_{[t,t']}(u).$$

We assume that $T \geq t$ (otherwise $K_1 = 0$). Introduce the quantity

$$K_2 \;:=\; \mathbf{1}_{[0,T\wedge t']}(s) \int_{s\vee t}^{T\wedge t'} du \int \varrho_u(dy)\, p(u-s\vee t, x-y)\,\phi_{2q}(y).$$

Thanks to (A.6), we have $K_2 \leq \mathbf{1}_{[0,T]}(s)\, C_2\, |t-t'|^\xi\, \phi_q(x)$. Now

$$|K_1 - K_2| \;\leq\; \mathbf{1}_{[0,T\wedge t']}(s) \int_{s\vee t}^{T\wedge t'} du \int \varrho_u(dy)\, \left|p(u-s+\varepsilon, x-y) - p(u-s\vee t, x-y)\right| \phi_{2q}(y).$$

Using the inequality

$$\left|p(v_1,z) - p(v_2,z)\right| \leq c \int_{v_1}^{v_2} dv\; v^{-1}\, p(2v,z), \tag{20}$$

where the constant c is independent of $z \in \mathbb{R}^d$ and $v_2 \geq v_1 > 0$, we get that

$$\begin{aligned}
|K_1 - K_2| &\leq c\,\mathbf{1}_{[0,T\wedge t']}(s) \int_{s\vee t}^{T\wedge t'} du \int \varrho_u(dy)\,\phi_{2q}(y) \int_{u-s\vee t}^{u-s+\varepsilon} dv\; v^{-1}\, p(2v, x-y) \\
&= c\,\mathbf{1}_{[0,T\wedge t']}(s) \int_0^{T\wedge t'-s+\varepsilon} dv\; v^{-1} \int_{s\vee t\vee(v+s-\varepsilon)}^{T\wedge t'\wedge(v+s\vee t)} du \int \varrho_u(dy)\,\phi_q^2(y)\, p(2v,x-y).
\end{aligned}$$

In view of (A.5) and (A.1), we may continue with

$$\leq\; c\,\mathbf{1}_{[0,T\wedge t']}(s)\,\phi_q(x) \int_0^{T\wedge t'-s+\varepsilon} dv\; v^{-1} \left|T\wedge t'\wedge(v+s\vee t) - s\vee t\vee(v+s-\varepsilon)\right|^\xi,$$

where c is independent of t', t, ε, x. It is easy to check that

$$\begin{aligned}
&\int_0^{T\wedge t'-s+\varepsilon} dv\; v^{-1} \left|T\wedge t'\wedge(v+s\vee t) - s\vee t\vee(v+s-\varepsilon)\right|^\xi \\
&\qquad \leq\; c\left|t'-t\right|^\xi \left(1+\log_+\left(1/|t'-t|\right)\right),
\end{aligned} \tag{21}$$

where c is independent of t', t and ε. As a conclusion we obtain (19).

Using the estimate (A.6), a straight forward induction shows that all the functions χ_n, $n \geq 1$, of the recurrence relation (10) with initial condition $\chi_1 = J_\varepsilon$ belong to H_T^q and satisfy

$$\left|\chi_n(s,x)\right| \;\leq\; c\,\mathbf{1}_{[0,T]}(s)\,\phi_q(x)\,\|\varphi/\phi_{2q}\|_\infty^n \left[\left|t-t'\right|^\xi \left(1+\log_+\left(1/|t-t'|\right)\right)\right]^n.$$

(Note that c is independent of φ, t, t' and ε.) Then the claim (15) is a consequence of (A.9) with $f = 0$ and

$$g(s,z) \;:=\; \int \varrho_s(dy)\, p(\varepsilon, z-y)\, \varphi(s,y)\, \mathbf{1}_{[t,t']}(s), \tag{22}$$

finishing the proof. □

4. Two-dimensional collision local time (proof of Theorem 5)

We now assume that $d = 2$.

4.1. Local spatial collision densities [proof of (a)]. For the claimed L^2-convergence, it is enough to check that, for fixed s, t, z,

$$J^{\varepsilon,\varepsilon'} \;:=\; \mathbf{E}\left[\int_{[s,t]\times\mathbb{R}^2} L\left(d[r,y]\right) p(\varepsilon, z-y) \int_{[s,t]\times\mathbb{R}^2} L\left(d[r',y']\right) p(\varepsilon', z-y')\right] \tag{23}$$

converges in $\mathbb{R}_+$ as ε and ε' decrease to 0.

For $f \in L^1_+(\mathbb{R}^2)$ with $\int dx\, f(x) = 1$, and $\varepsilon > 0$, $z \in \mathbb{R}^2$, we set

$$f_{\varepsilon,z}(x) := \varepsilon^{-1} f\left(\varepsilon^{-1/2}(x-z)\right). \tag{24}$$

Note that $f_{\varepsilon,z}(x)dx$ converges weakly to $\delta_z(dx)$, the Dirac mass at z, as ε decreases to 0. We will prove the following stronger result.

Lemma 9. *For fixed s, t, z, z', $f, f' \in L^1_+(\mathbb{R}^2)$ such that $\int dx\, f(x) = 1 = \int dx\, f'(x)$, the well-defined quantity*

$$J^{\varepsilon,\varepsilon'}(z,z') \;:=\; \mathbf{E}\left[\int_{[s,t]\times\mathbb{R}^2} L\left(d[r,y]\right) f_{\varepsilon,z}(y) \int_{[s,t]\times\mathbb{R}^2} L\left(d[r',y']\right) f'_{\varepsilon',z'}(y')\right]$$

converges to a finite limit independent of f, f', as ε and ε' decrease to 0.

Note that we need the convergence for $z = z'$ to prove (23) and then (a). Note also that although f and f' are not in $\mathcal{B}^q$ a priori, we show that $J^{\varepsilon,\varepsilon'}$ is well-defined.

Proof of Lemma 9. By a standard monotone class argument, we deduce from the quenched moment formula (9) for collision local time with $m = 2$, that for $g \in \mathcal{B}_+\left((\mathbb{R}_+)^2 \times (\mathbb{R}^2)^2\right)$,

$$\begin{aligned}
&\mathbf{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^2} L\left(d[r,y]\right) \int_{\mathbb{R}_+\times\mathbb{R}^2} L\left(d[r',y']\right) g(r,r',y,y')\right] \\
&= \mathbb{E}\Bigg[2\, i_{\mathrm{r}}\kappa \int dx \int_0^\infty ds_1 \int \varrho_{s_1}(dy_1)\, p(s_1, y_1 - x) \int_{s_1}^\infty ds_2 \int \varrho_{s_2}(dy_2)\, p(s_2 - s_1, y_2 - y_1) \\
&\qquad\qquad \int_{s_1}^\infty ds_3 \int \varrho_{s_3}(dy_3)\, p(s_3 - s_1, y_3 - y_1)\, g(s_2, s_3, y_2, y_3) \\
&\qquad + i_{\mathrm{r}}^2 \int dx_1 \int_0^\infty ds_1 \int \varrho_{s_1}(dy_1)\, p(s_1, y_1 - x_1) \\
&\qquad\qquad \int dx_2 \int_0^\infty ds_2 \int \varrho_{s_2}(dy_2)\, p(s_2, y_2 - x_2)\, g(s_1, s_2, y_1, y_2)\Bigg].
\end{aligned}$$

Thus we can write

$$J^{\varepsilon,\varepsilon'} \;=\; 2\,i_{\mathrm{r}}\kappa J_1^{\varepsilon,\varepsilon'} + i_{\mathrm{r}}^2 J_2^{\varepsilon,\varepsilon'}, \tag{25}$$

where

$$\begin{aligned} J_1^{\varepsilon,\varepsilon'}(z,z') := \int_0^t ds_1\,\mathbb{E}\Big[\int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int \varrho_{s_1}(dy_1)\int \varrho_{s_2}(dy_2)\int \varrho_{s_3}(dy_3) \\ p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\Big] \end{aligned}$$

and

$$J_2^{\varepsilon,\varepsilon'}(z,z') \;:=\; \mathbb{E}\Big[\int_s^t ds_1 \int_s^t ds_2 \int \varrho_{s_1}(dy_1)\int \varrho_{s_2}(dy_2)\; f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2)\Big]$$

are third and second moment expressions of the catalyst process only, respectively. We easily compute $J_2^{\varepsilon,\varepsilon'}$ thanks to the moment formula (A.2) for ordinary super-Brownian motion (with $f=0$ and g properly chosen):

$$\begin{aligned} J_2^{\varepsilon,\varepsilon'}(z,z') \;&=\; 2\gamma\, i_{\mathrm{c}} \int dx \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2 \int dy_1 \int dy_2 \int dy_3 \\ &\qquad p(s_3,y_3-x)\,p(s_1-s_3,y_1-y_3)\,p(s_2-s_3,y_2-y_3)\,f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2) \\ &\quad+\; i_{\mathrm{c}}^2 \int dx_1 \int dx_2 \int_s^t ds_1 \int_s^t ds_2 \int dy_1 \int dy_2 \\ &\qquad p(s_1,y_1-x_1)\,p(s_2,y_2-x_2)\,f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2) \\ &=\; 2\gamma\, i_{\mathrm{c}} \int dy_1\; f_{\varepsilon,z}(y_1) \int dy_2\; f'_{\varepsilon',z'}(y_2) \\ &\qquad \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2\; p(s_1+s_2-2s_3,y_1-y_2) \\ &\quad+\; i_{\mathrm{c}}^2\,(t-s)^2 \\ &\le\; 2\gamma\, i_{\mathrm{c}} \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2\; p(s_1+s_2-2s_3,0) +\; i_{\mathrm{c}}^2\,(t-s)^2 =: K_2 < \infty. \end{aligned}$$

As $(\varepsilon,\varepsilon')\downarrow 0$, the quantity $J_2^{\varepsilon,\varepsilon'}(z,z')$ converges to

$$J_2^0(z,z') \;:=\; 2\gamma\, i_{\mathrm{c}} \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2\; p(s_1+s_2-2s_3,z-z') + i_{\mathrm{c}}^2\,(t-s)^2 \;\le\; K_2. \tag{26}$$

We can also compute $J_1^{\varepsilon,\varepsilon'}$ using the Markov property of ϱ at time s_1 and twice the moment formula (A.2):

$$\begin{aligned} J_1^{\varepsilon,\varepsilon'}(z,z') \;=\; & 2\gamma \int_0^t ds_1\, \mathbb{E}\Big[\int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_1}(dy_5) \int dy_4 \int dy_2 \int dy_3 \\ & \qquad p(s_4-s_1,y_4-y_5)\, p(s_2-s_4,y_2-y_4)\, p(s_3-s_4,y_3-y_4) \\ & \qquad p(s_2-s_1,y_2-y_1)\, p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3)\Big] \\ & + \int_0^t ds_1\, \mathbb{E}\Big[\int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_1}(dy_4) \int \varrho_{s_1}(dy_5) \int dy_2 \int dy_3 \\ & \qquad p(s_2-s_1,y_2-y_4)\, p(s_3-s_1,y_3-y_5)\, p(s_2-s_1,y_2-y_1) \\ & \qquad p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3)\Big]. \end{aligned}$$

With obvious notation we write

$$J_1^{\varepsilon,\varepsilon'} \;=\; 2\gamma J_3^{\varepsilon,\varepsilon'} + J_4^{\varepsilon,\varepsilon'}. \tag{27}$$

Using again the moment formula, we get

$$J_3^{\varepsilon,\varepsilon'} \;=\; 2\gamma\, i_{\mathrm{c}}\, J_5^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2\, J_6^{\varepsilon,\varepsilon'}, \tag{28}$$

where

$$\begin{aligned} J_5^{\varepsilon,\varepsilon'}(z,z') \;:=\; & \int_0^t ds_1 \int_0^{s_1} ds_5 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \int dy_5 \int dy_6 \\ & p(s_1-s_5,y_1-y_6)\, p(s_1-s_5,y_5-y_6)\, p(s_4-s_1,y_4-y_5)\, p(s_2-s_4,y_2-y_4) \\ & p(s_3-s_4,y_3-y_4)\, p(s_2-s_1,y_2-y_1)\, p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3) \end{aligned}$$

and

$$\begin{aligned} J_6^{\varepsilon,\varepsilon'}(z,z') \;:=\; & \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \int dy_5 \\ & p(s_4-s_1,y_4-y_5)\, p(s_2-s_4,y_2-y_4)\, p(s_3-s_4,y_3-y_4) \\ & p(s_2-s_1,y_2-y_1)\, p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3). \end{aligned}$$

We now compute $J_6^{\varepsilon,\varepsilon'}$. Integrating over dy_1, dy_5, and dy_4 gives

$$\begin{aligned} J_6^{\varepsilon,\varepsilon'}(z,z') \;=\; & \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_2 \int dy_3 \\ & p(s_2+s_3-2s_4,y_2-y_3)\, p(s_2+s_3-2s_1,y_2-y_3)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3). \end{aligned}$$

The function

$$H_6(y_2,y_3) := \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3\; p(s_2+s_3-2s_4,y_2-y_3)\, p(s_2+s_3-2s_1,y_2-y_3)$$

is continuous in (y_2,y_3) and bounded from above by $H_6(y,y)=K_6$ which is finite since $d=2$. Thus $J_6^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by K_6. Using that $f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3)dy_2dy_3$ converges weakly to $\delta_z(dy_2)\delta_{z'}(dy_3)$, we deduce that $J_6^{\varepsilon,\varepsilon'}$ converges to

$$J_6^0(z,z') := H_6(z,z') \le K_6. \tag{29}$$

Note that $H_6(z,z)=\infty$ if $d=3$, which implies that $J^{\varepsilon,\varepsilon'}(z,z)$ doesn't converge for $d=3$, however it is well-defined at least for $f(x)=f'(x)=p(1,x)$.

For $J_5^{\varepsilon,\varepsilon'}$ we get

$$\begin{aligned} J_5^{\varepsilon,\varepsilon'}(z,z') \;=\; & \int_0^t ds_1 \int_0^{s_1} ds_5 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \\ & p(s_1+s_4-2s_5, y_1-y_4)\, p(s_2-s_4, y_2-y_4)\, p(s_3-s_4, y_3-y_4) \\ & p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3). \end{aligned}$$

We set

$$\begin{aligned} & h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3) \\ & := \mathbf{1}_{0<s_5<s_1<s_4<s_2\wedge s_3} \int dy_1 \int dy_4\; p(s_1+s_4-2s_5, y_1-y_4)\, p(s_2-s_4, y_2-y_4)\, p(s_3-s_4, y_3-y_4) \\ & \qquad\qquad p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1), \end{aligned}$$

and

$$H_5(y_2,y_3) \;:=\; \int ds_1 \int ds_2 \int ds_3 \int ds_4 \int ds_5\; \mathbf{1}_{s<s_2,s_3<t}\, h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3),$$

so that

$$J_5^{\varepsilon,\varepsilon'}(z,z') = \int dy_2 \int dy_3\; f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3) H_5(y_2,y_3).$$

Let us now prove that H_5 is bounded and continuous. Note first that $p(s_1+s_4-2s_5, y_1-y_4) \le p(s_1+s_4-2s_5, 0)$. Thus, we easily get

$$\begin{aligned} & h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3) \\ & \qquad \le\; \mathbf{1}_{0<s_5<s_1<s_4<s_2\wedge s_3}\, p(s_1+s_4-2s_5,0)\, p(s_2+s_3-2s_4,0)\, p(s_2+s_3-2s_1,0). \end{aligned}$$

Now it is easy to check that

$$\begin{aligned} H_5(y_2,y_3) &= \int ds_1 \cdots \int ds_5\; \mathbf{1}_{s\,<\,s_2,s_3\,<\,t}\, h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3) \\ &\le \int_s^t ds_2 \int_s^t ds_3 \int_0^{s_2\wedge s_3} ds_4 \int_0^{s_4} ds_1 \int_0^{s_1} ds_5 \\ &\qquad p(s_1+s_4-2s_5,0)\, p(s_2+s_3-2s_4,0)\, p(s_2+s_3-2s_1,0) \;=\; K_5 < \infty. \end{aligned}$$

The function h_5 is continuous and bounded in (y_2,y_3). From dominated convergence we deduce that H_5 is continuous and bounded. Using that $f_{\varepsilon,z}(y_2)f'_{\varepsilon',z'}(y_3)dy_2dy_3$ converges weakly to $\delta_z(dy_2)\delta_{z'}(dy_3)$, we see that $J_5^{\varepsilon,\varepsilon'}$ tends to

$$J_5^0(z,z') \;:=\; H_5(z,z') \le K_5 \tag{30}$$

when ε and ε' decrease to 0. Note that $J_5^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by K_5.

Finally, we study $J_4^{\varepsilon,\varepsilon'}$. Let $g \in \mathcal{B}_+\left((\mathbb{R}^2)^3\right)$ and $\overline{g}(x_1,x_2,x_3) := \sum_\sigma g(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$, where the sum is over all the permutations σ of $\{1,2,3\}$. By a standard monotone class argument we deduce from the moment formula (A.2) for ϱ that

$$\begin{aligned}
&\mathbb{E}\left[\int\varrho_v(dy_1)\int\varrho_v(dy_4)\int\varrho_v(dy_5)\,g(y_1,y_4,y_5)\right]\\
&= 2\,i_{\mathrm{c}}\gamma^2\int_0^v ds_4\int_{s_4}^v ds_5\int dy_1\int dy_4\int dy_5\int dy_6\,p(v+s_5-2s_4,y_1-y_6)\\
&\qquad\qquad p(v-s_5,y_4-y_6)\,p(v-s_5,y_5-y_6)\,\overline{g}(y_1,y_4,y_5)\\
&\quad+\ i_{\mathrm{c}}^2\,\gamma\int_0^v ds_4\int dy_1\int dy_4\int dy_5\,p(2v-2s_4,y_1-y_4)\,\overline{g}(y_1,y_4,y_5)\\
&\quad+\ \frac{1}{3!}\,i_{\mathrm{c}}^3\int dy_1\int dy_4\int dy_5\,\overline{g}(y_1,y_4,y_5).
\end{aligned}$$

This implies

$$J_4^{\varepsilon,\varepsilon'} \ = \ 2\,i_{\mathrm{c}}\gamma^2 J_7^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2\,\gamma J_8^{\varepsilon,\varepsilon'} + \frac{1}{3!}\,i_{\mathrm{c}}^3\,J_9^{\varepsilon,\varepsilon'}, \tag{31}$$

where

$$\begin{aligned}
J_7^{\varepsilon,\varepsilon'}(z,z') \ := \ 2&\int_0^t ds_1\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\int_0^{s_1} ds_4\int_{s_4}^{s_1} ds_5\int dy_1\int dy_4\int dy_5\int dy_2\int dy_3\int dy_6\\
&p(s_2-s_1,y_2-y_4)\,p(s_3-s_1,y_3-y_5)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\\
&\qquad f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\\
&\Big[p(s_1+s_5-2s_4,y_1-y_6)\,p(s_1-s_5,y_4-y_6)\,p(s_1-s_5,y_5-y_6)\\
&\ +\ p(s_1+s_5-2s_4,y_4-y_6)\,p(s_1-s_5,y_1-y_6)\,p(s_1-s_5,y_5-y_6)\\
&\ +\ p(s_1+s_5-2s_4,y_5-y_6)\,p(s_1-s_5,y_1-y_6)\,p(s_1-s_5,y_4-y_6)\Big]
\end{aligned}$$

and

$$\begin{aligned}
J_8^{\varepsilon,\varepsilon'}(z,z') :=2&\int_0^t ds_1\int_0^{s_1} ds_4\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\int dy_1\int dy_4\int dy_5\int dy_2\int dy_3\,p(s_2-s_1,y_2-y_4)\\
&p(s_3-s_1,y_3-y_5)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\\
&\Big[p(2s_1-2s_4,y_1-y_4)+p(2s_1-2s_4,y_1-y_5)+p(2s_1-2s_4,y_4-y_5)\Big]
\end{aligned}$$

as well as

$$\begin{aligned}
J_9^{\varepsilon,\varepsilon'}(z,z') := 3!&\int_0^t ds_1\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\int dy_1\int dy_4\int dy_5\int dy_2\int dy_3\,p(s_2-s_1,y_2-y_4)\\
&p(s_3-s_1,y_3-y_5)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3).
\end{aligned}$$

We are left to study the convergence of $J_9^{\varepsilon,\varepsilon'}$, $J_8^{\varepsilon,\varepsilon'}$ and $J_7^{\varepsilon,\varepsilon'}$.

First of all, we have

$$\begin{aligned} J_9^{\varepsilon,\varepsilon'}(z,z') \;&=\; 3! \int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int dy_2 \int dy_3 \; p(s_2+s_3-2s_1, y_2-y_3) f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3) \\ &\le\; 6 \int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \; p(s_2+s_3-2s_1, 0) =: K_9 < \infty. \end{aligned}$$

As ε and ε' decrease to zero, $J_9^{\varepsilon,\varepsilon'}$ converges to

$$J_9^0(z,z') \;:=\; 6 \int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \; p(s_2+s_3-2s_1, z-z') \;\le K_9. \tag{32}$$

Next, we have

$$J_8^{\varepsilon,\varepsilon'}(z,z') \;=\; \int dy_2 \int dy_3 \; f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3) H_8(y_2,y_3),$$

where

$$\begin{aligned} H_8(y_2,y_3) \;=\; 2 \int_0^t ds_1 \int_0^{s_1} ds_4 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \\ \Big[\int dy_1 \; p(s_2+s_1-2s_4, y_1-y_2)\, p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1) \\ + \int dy_1 \; p(s_3+s_1-2s_4, y_3-y_1)\, p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1) \\ + \; p(s_2+s_3-2s_1, y_2-y_3)\, p(s_2+s_3-2s_4, y_3-y_2) \Big]. \end{aligned}$$

Since $p(s_2+s_1-2s_4, y_1-y_2) \le p(s_2+s_1-2s_4, 0)$ and $p(s_3+s_1-2s_4, y_3-y_1) \le p(s_3+s_1-2s_4, 0)$, we deduce that

$$\begin{aligned} H_8(y_2,y_3) &\le 6 \int_0^t ds_1 \int_0^{s_1} ds_4 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \, p(s_2+s_1-2s_4, 0)\, p(s_2+s_3-2s_1, 0) \\ &= K_8 < \infty. \end{aligned}$$

Arguments similar to those used for the convergence of $J_5^{\varepsilon,\varepsilon'}$ show that H_8 is continuous and bounded. Thus $J_8^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by K_8 and converges to

$$J_8^0(z,z') \;:=\; H_8(z,z') \;\le\; K_8. \tag{33}$$

Finally, we have

$$J_7^{\varepsilon,\varepsilon'}(z,z') = \int dy_2 \int dy_3 \; f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3) H_7(y_2,y_3),$$

where

$$
\begin{aligned}
H_7(y_2,y_3) \;:=\; & 2\int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \int dy_1 \int dy_6 \\
& \Big[p(s_2-s_5,y_2-y_6)\,p(s_3-s_5,y_3-y_6)\,p(s_2-s_1,y_2-y_1) \\
& \qquad p(s_3-s_1,y_3-y_1)\,p(s_1+s_5-2s_4,y_1-y_6) \\
& + \; p(s_2+s_5-2s_4,y_2-y_6)\,p(s_3-s_5,y_3-y_6)\,p(s_2-s_1,y_2-y_1) \\
& \qquad p(s_3-s_1,y_3-y_1)\,p(s_1-s_5,y_1-y_6) \\
& + \; p(s_2-s_5,y_2-y_6)\,p(s_3+s_5-2s_4,y_3-y_6)\,p(s_2-s_1,y_2-y_1) \\
& \qquad p(s_3-s_1,y_3-y_1)\,p(s_1-s_5,y_1-y_6) \Big].
\end{aligned}
$$

Check now that the following upper bound is finite:

$$
\begin{aligned}
K_7 \;:=\; & 2\int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \\
& \Big[p(s_2+s_3-2s_5,0)\,p(s_2+s_3-2s_1,0)\,p(s_1+s_5-2s_4,0) \\
& + \; p(s_2+s_5-2s_4,0)\,p(s_1+s_3-2s_5,0)\,p(s_2+s_3-2s_1,0) \\
& + \; p(s_3+s_5-2s_4,0)\,p(s_1+s_2-2s_5,0)\,p(s_2+s_3-2s_1,0) \Big].
\end{aligned}
$$

Arguments similar to those used for the convergence of $J_5^{\varepsilon,\varepsilon'}$ show that H_7 is continuous and bounded by $K_7<\infty$. Thus $J_7^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by K_7 and converges to

$$
J_7^0(z,z') \;:=\; H_7(z,z') \;\le\; K_7. \tag{34}
$$

Altogether, for each $i\in\{1,\dots,9\}$, $J_i^{\varepsilon,\varepsilon'}$ exists, is uniformly bounded and has a finite limit as $(\varepsilon,\varepsilon')\downarrow 0$. Thus, $J^{\varepsilon,\varepsilon'}(z,z')$ is well-defined and converges in $\mathbb{R}_+$ as ε and ε' decrease to 0. □

Completion of the proof of (a). The claimed expectation expression for $\lambda_{[s,t]}(z)$ easily follows from the moment formula (9) for L in the case $m=1$.

The second moment of $\lambda_{[s,t]}(z)$ is given by the limit J^0, say, of $J^{\varepsilon,\varepsilon}(z,z)$ from Lemma 9 as $\varepsilon\downarrow 0$. By the formulas (25), (27), (28), and (31),

$$
J^0 \;=\; 2\,i_{\mathrm{r}}\kappa\left[2\gamma\left(2\gamma\,i_{\mathrm{c}}\,J_5^0+i_{\mathrm{c}}^2\,J_6^0\right)+\left\{2\,i_{\mathrm{c}}\gamma^2J_7^0+i_{\mathrm{c}}^2\,\gamma J_8^0+\frac{1}{3!}\,i_{\mathrm{c}}^3\,J_9^0\right\}\right]+i_{\mathrm{r}}^2J_2^0 \;<\;\infty \tag{35}
$$

which, in the case $s<t$, is strictly larger than $\big(\mathbf{E}[\lambda_{[s,t]}(z)]\big)^2$, occurring from the J_2^0-term [see (26)]. This completes the proof of (a). □

Remark 10 (variance formula)**.** For $t\ge s\ge 0$ and $z\in\mathbb{R}^d$, from the representation (35) combined with (30), (29), (34), (33), (32), and (26), as well as the expectation formula in (a), we obtain the following formula for the *variance of* $\lambda_{[s,t]}(z)$:

$$
\begin{aligned}
&2i_{\mathrm{c}}\, i_{\mathrm{r}}\left(i_{\mathrm{c}}^2\,\kappa + i_{\mathrm{r}}\gamma\right) \int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3\; p(s_2+s_3-2s_1,0)\\
&+\ 8\, i_{\mathrm{c}}^2\, i_{\mathrm{r}}\gamma\,\kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2\vee s}^t ds_3 \int_{s_2\vee s}^t ds_4\; p(s_3+s_4-2s_2,0)\, p(s_3+s_4-2s_1,0)\\
&+\ 8\, i_{\mathrm{c}}^2\, i_{\mathrm{r}}\gamma\,\kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2\vee s}^t ds_3 \int_{s_2\vee s}^t ds_4 \int dy\; p(s_2+s_3-2s_1,y)\, p(s_3-s_2,y)\\
&\hspace{28em} p(s_4-s_2,y)\\
&+\ 16\, i_{\mathrm{c}}\, i_{\mathrm{r}}\gamma^2\,\kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2}^t ds_3 \int_{s_3\vee s}^t ds_4 \int_{s_3\vee s}^t ds_5 \int dy_1 \int dy_2\\
&\qquad\qquad p(s_2+s_3-2s_1,y_1-y_2)\, p(s_4-s_2,y_1)\, p(s_4-s_3,y_2)\\
&\qquad\qquad p(s_5-s_2,y_1)\, p(s_5-s_3,y_2)\\
&+\ 16\, i_{\mathrm{c}}\, i_{\mathrm{r}}\gamma^2\,\kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2}^t ds_3 \int_{s_3\vee s}^t ds_4 \int_{s_3\vee s}^t ds_5 \int dy_1 \int dy_2\\
&\qquad\qquad p(s_2+s_4-2s_1,y_2-y_1)\, p(s_4-s_3,y_2)\, p(s_3-s_2,y_1)\\
&\qquad\qquad p(s_5-s_2,y_2-y_1)\, p(s_5-s_3,y_2).
\end{aligned}
$$

$\diamond$

4.2. **Spatial absolute continuity [proof of (b)].** We first prove that,

$$
\phi_q(x) \int L\left(d[r,y]\right) \mathbf{1}_{[s,t]}(r)\, p(\varepsilon, x-y), \tag{36}
$$

converges in $L^1(\ell \otimes \mathbf{P})$ as ε decreases to 0, to $\phi_q(x)\xi(x)$, where for almost every x, $\mathbf{P}$-a.s. $\xi = \lambda$. Thanks to the statement (a), it is enough to check that the function

$$
(x,\varepsilon) \mapsto \mathbf{E}\left[\int L\left(d[r,y]\right) \mathbf{1}_{[s,t]}(r)\, p(\varepsilon, x-y)\right], \tag{37}
$$

is uniformly bounded on $\mathbb{R}^2 \times (0,1]$. But this is clear since

$$
\begin{aligned}
\mathbf{E}\left[\int L\left(d[r,y]\right) \mathbf{1}_{[s,t]}(r)\, p(\varepsilon, x-y)\right] &= \mathbf{E}\left[\int_s^t dr\; i_{\mathrm{r}} \int dz \int \varrho_r(dy)\, p(r,z-y)\, p(\varepsilon,x-y)\right]\\
&= i_{\mathrm{r}} i_{\mathrm{c}}\,(t-s).
\end{aligned}
$$

Statement (b) is then a straight forward consequence of the following criterion with $\nu(dy) = L\left([s,t],dy\right)$ [recall Theorem 3 (i)].

Proposition 11 (sufficient criterion for absolute continuity). *Let $\nu \in \mathcal{M}_q$ be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that $\mathcal{E}\left[(\nu, \phi_q)\right] < \infty$ and that*

$$
\left((x,\omega)\ \mapsto\ \phi_q(x) \int \nu(dy)\, p(\varepsilon, x-y),\ \ \varepsilon > 0\right) \tag{38}
$$

converges in $L^1(\ell\otimes\mathcal{P})$ *to some* $\phi_q\xi$ *as* $\varepsilon\downarrow 0$. *Then* $\mathcal{P}$*-a.s. the measure* ν *is absolutely continuous (with respect to the Lebesgue measure) and has the density function* ξ :

$$\nu(dy) \;=\; \xi(y)dy. \tag{39}$$

Proof. Let β be any bounded random variable on $(\Omega,\mathcal{F},\mathcal{P})$, and $f\in\mathcal{B}^q$ continuous. Because of the assumed convergence in $L^1(\ell\otimes\mathcal{P})$, we get that

$$J_\varepsilon \;:=\; \int\!dx\, f(x)\,\mathcal{E}\left[\beta\int\!\nu(dy)\; p(\varepsilon,x-y)\right]$$

converges to $\int dx\, f(x)\,\mathcal{E}\left[\beta\,\xi(x)\right]$ as $\varepsilon\downarrow 0$. On the other hand, the function

$$(y,\varepsilon)\mapsto \int\!dx\, f(x)\, p(\varepsilon,x-y)$$

is bounded by $\phi_q(y)$ [thanks to (A.1)], continuous and converges to f as $\varepsilon\downarrow 0$. By dominated convergence, we get that J_ε converges to $\mathcal{E}\left[\beta\,(\nu,f)\right]$. Since β and f are arbitrary, the equality

$$\int\!dx\, f(x)\,\mathcal{E}\left[\beta\,\xi(x)\right] \;=\; \mathcal{E}\left[\beta\,(\nu,f)\right]$$

implies that ν is $\mathcal{P}$-a.s. absolutely continuous with respect to the Lebesgue measure, and that $\nu(dy)=\xi(y)dy$, $\mathcal{P}$-a.s. □

4.3. **Random ergodic limit [proof of (d)].** Let $f\in L^1_+(\mathbb{R}^2)$. Thanks to Lemma 9, we know that $T^{-1}\int_{[0,T]\times\mathbb{R}^2} L\left(d[r,y]\right)\, f(y)$ is well-defined and even belongs to $L^2(\mathbf{P}\,)$. By self-similarity this has the same law as

$$I_T \;=\; T\int_{[0,1]\times\mathbb{R}^2} L\left(d[r,y]\right)\, f(y\sqrt{T}).$$

Thanks to Lemma 9 and (a), we see that I_T converges in $L^2(\mathbf{P}\,)$ to $\lambda_{[0,1]}(0)\int dx\, f(x)$ as $T\uparrow\infty$. Thus we deduce that for any $f\in L^1_+(\mathbb{R}^2)$, the following convergence in law holds with respect to $\mathbb{P}$:

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_{[0,T]\times\mathbb{R}^2} L\left(d[r,y]\right)\, f(y) \;=\; \lambda_{[0,1]}(0)\int\!dx\; f(x).$$

This ends the proof of (d). □

A. Appendix: Some basic properties of catalyst and reactant

A.1. **Moment formulas for the catalyst.** Let $d\geq 1$ and fix $\nu\in\mathcal{M}_q$. It is easy to check that for every $T>0$, there exists a constant $c>0$ such that for every $x\in\mathbb{R}^d$ and $\varepsilon\in(0,T]$,

$$\int\!dy\; p(\varepsilon,x-y)\,\phi_q(y) \;\leq\; c\,\phi_q(x). \tag{A.1}$$

Therefore we get that if $g\in H^q_T$, then the function $(r,x)\mapsto\int_r^\infty ds\; P_{s-r}[g(s)](x)$ is well-defined and belongs to H^q_T. If $f\in\mathcal{B}^q$, then the function $(r,x)\mapsto \mathbf{1}_{t\geq r}\; P_{t-r}[f](x)$ is also well-defined and belongs to H^q_t.

It is well-known that for every $t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$\mathbb{E}_\nu \left[\left[(\varrho_t, f) + \int_0^\infty ds \ (\varrho_s, g(s)) \right]^m \right] \ = \ m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1, \\ n_1+\dots+n_k = m}} \prod_{i=1}^k (\nu, \chi_{n_i}(0)) \,, \tag{A.2}$$

where the sequence $(\chi_n, n \geq 1)$ is defined by the recurrence formula

$$\chi_n(r,x) \ := \ \gamma \int_r^\infty ds \int dy \ p(s-r, x-y) \left[\sum_{i=1}^{n-1} \chi_i(s,y)\, \chi_{n-i}(s,y) \right], \tag{A.3}$$

$(r,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $n \geq 2$, with initial condition

$$\chi_1(r,x) \ := \ \mathbf{1}_{t \geq r} \ P_{t-r}[f](x) + \int_r^\infty ds \ P_{s-r}[g(s)](x), \qquad (r,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{A.4}$$

Thanks to the remark at the beginning of this subsection, we see that the functions $\chi_n, n \geq 1$, are well-defined and belong to H^q.

A.2. **Regularity properties of the catalyst.** We now assume that $d \leq 3$. Recall that we write $\mathbb{P}$ for $\mathbb{P}_{i_c\ell}$. It is clear from the Hölder continuity Theorem 3 of [DF97a] (p254) that for every $\xi \in (0, 1/4)$, $T \geq 0$, $\mathbb{P}$-a.s. there exists a constant $C_1 := C(T, \varrho, \xi)$ such that for every $T \geq t \geq r \geq 0$, $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\int_r^t ds \int \varrho_s(dz)\, \phi_q(z) f(z) \ \leq \ C_1 \, |t-r|^\xi \int f(z) dz. \tag{A.5}$$

We have also [cf. Definition 2 b) and Theorem 4 of [DF97a], pp 224 and 259, respectively] that for every $T \geq 0$, $\xi \in (0, 1/4)$, $\mathbb{P}$-a.s. there exists $C_2 := C(T, \varrho, \xi)$ such that for every $x \in \mathbb{R}^d$, $T \geq t \geq r \geq 0$,

$$\int_r^t ds \int \varrho_s(dz)\, p(s-r, x-z)\, \phi_q^2(z) \ \leq \ C_2 \, |t-r|^\xi \, \phi_q(x). \tag{A.6}$$

A.3. **Moment formulas for the reactant.** Recall that $d \leq 3$. Using the Markov property of X^ϱ (given ϱ), it is easy to get that $\mathbb{P}$-a.s. for every $n \geq 1$, $t_n \geq \dots \geq t_1 \geq 0$, and $f_n, \dots, f_1 \in \mathcal{B}_+^q$,

$$\mathrm{E}_{r,\mu}^\varrho \left[\mathrm{e}^{-\sum_{t_i \geq r} (X_{t_i}^\varrho, f_i)} \right] \ = \ \mathrm{e}^{-(\mu, v(r))}, \tag{A.7}$$

where v is the unique nonnegative solution of the catalytic log-Laplace equation (7) with $J(s) := \sum_{t_i \geq s} P_{t_i - s}[f_i]$. Using the continuity of X^ϱ, it can be shown that $\mathbb{P}$-a.s. for every nonnegative $g \in H^q$,

$$\mathrm{E}_{r,\mu}^\varrho \left[\mathrm{e}^{-\int_r^\infty ds \, (X_s^\varrho, g(s))} \right] \ = \ \mathrm{e}^{-(\mu, v(r))}, \tag{A.8}$$

where v is the unique nonnegative solution of (7) with $J(s) := \int_s^\infty du \ P_{u-s}[g(u)]$.

We deduce the next result on the *moments of the reactant process* X^ϱ from Theorem 4, Lemma 4 and Remark 2 of [DF97a] (pp 259 and 232, respectively). We have $\mathbb{P}$-a.s. for every

$t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$\text{(A.9)} \qquad \mathrm{E}^{\varrho}_{r,\mu}\left[\Big[(X^{\varrho}_t, f) + \int_r^{\infty} ds\ \big(X^{\varrho}_s, g(s)\big)\Big]^m\right] \;=\; m! \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1,\\ n_1+\cdots+n_k = m}} \prod_{i=1}^{k} (\mu, \chi_{n_i}(r)),$$

where $(\chi_n\,, n \geq 1)$ is defined by the recurrence formula (10) with initial condition

$$\text{(A.10)} \qquad \chi_1(s,x) \;:=\; \mathbf{1}_{t \geq s}\, P_{t-s}[f](x) + \int_s^{\infty} du\ P_{u-s}[g(u)](x) \qquad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Since $\chi_1 \in H^q$, inequality (A.6) implies that all the functions χ_n belong to H^q.

References

[BEP91] M. T. BARLOW, S. N. EVANS, and E. A. PERKINS. Collision local times and measure-valued process. *Canad. J. Math.*, 43(5):897–938, 1991.

[DF94] D. A. DAWSON and K. FLEISCHMANN. A super-Brownian motion with a single point catalyst. *Stoch. Proc. Appl.*, 49:3–40, 1994.

[DF97a] D. A. DAWSON and K. FLEISCHMANN. A continuous super-Brownian motion in a super-Brownian medium. *J. Theoretical Probab.*, 10(1):213–276, 1997.

[DF97b] D. A. DAWSON and K. FLEISCHMANN. Longtime behavior of a branching process controlled by branching catalysts. *Stoch. Proc. Appl.*, 71(2):241–257, 1997.

[DFR91] D. A. DAWSON, K. FLEISCHMANN, and S. ROELLY. Absolute continuity of the measure states in a branching model with catalysts. In *Seminar Stoch. Proc. 1990*, volume 24 of *Progr. Probab.*, pages 117–160. Birkhäuser, Boston, 1991.

[Del96] J.-F. DELMAS. Super-mouvement brownien avec catalyse. *Stochastics Stoch. Rep.*, 58:303–347, 1996.

[EF98] A. M. ETHERIDGE and K. FLEISCHMANN. Persistence of a two-dimensional super-Brownian motion in a catalytic medium. *Probab. Th. Rel. Fields*, 110(1):1–12, 1998.

[FK97] K. FLEISCHMANN and A. KLENKE. Smooth density field of catalytic super-Brownian motion. WIAS Berlin, Preprint No 331, 1997; *Ann. Appl. Probab.* (to appear).

[KS88] N. KONNO and T. SHIGA. Stochastic partial differential equations for some measure-valued diffusions. *Probab. Th. Rel. Fields*, 1988.

[Led97] G. LEDUC. Martingale problem for (ξ, ϕ, k)-superprocesses. WIAS Berlin, Preprint No 319, 1997.

hotspot.tex typeset by LaTeX printed November 18, 1998

MSRI, 1000 Centennial Drive, Berkeley, CA 94720, U.S.A., and ENPC-CERMICS, 6 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France.

E-mail address: delmas@msri.org

URL: http://cermics.enpc.fr/~delmas/home.html

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany

E-mail address: fleischmann@wias-berlin.de

URL: http://www.wias-berlin.de/Klaus.Fleischmann