# ON THE HOT SPOTS OF A CATALYTIC SUPER-BROWNIAN MOTION

JEAN-FRANÇOIS DELMAS AND KLAUS FLEISCHMANN

ABSTRACT. Consider the catalytic super-Brownian motion $X^{\varrho}$ (reactant) in $\mathbb{R}^d$, $d \le 3$, which branching rates vary randomly in time and space and in fact are given by an ordinary super-Brownian motion $\varrho$ (catalyst). Our main object of study is the collision local time $L = L_{[\varrho, X^{\varrho}]}\big(d[s,x]\big)$ of catalyst and reactant. It determines the covariance measure in the martingale problem for $X^{\varrho}$ and reflects the occurence of "hot spots" of reactant which can be seen in simulations of $X^{\varrho}$. In dimension 2, spatial marginal collision densities exist and, via self-similarity, enter as factor in the long-term random ergodic limit of $L$ (diffusiveness of the 2-dimensional model).

## 1. INTRODUCTION

The ordinary *super-Brownian motion* $\varrho = (\varrho_t, t \ge 0)$ in Euclidean space $\mathbb{R}^d$ can be obtained as a limit of branching particle systems. In this branching particle system, the particles evolve according to independent Brownian motions in $\mathbb{R}^d$, and additionally, with constant rate $\gamma > 0$, each particle splits independently into 2 or 0 particles with equal probability (this is a critical binary branching mechanism).

We now interpret $\varrho$ as a *catalyst process*: $\varrho_t(dx)$ is the amount of catalytic "particles" at time $t$ in the volume element $dx$ of $\mathbb{R}^d$. We then let a super-Brownian motion $X^{\varrho} = (X^{\varrho}_t, t \ge 0)$ evolve in this *catalytic random medium* $\varrho$. Intuitively $X^{\varrho}$ describes *reactant* "particles" which are evolving according to independent Brownian motions and which are performing a critical binary branching, but at random time-space varying rates given by $\varrho$. In fact, the rate of branching of an intrinsic reactant particle with Brownian path $W$ is controlled by the *collision local time $L_{[\varrho,W]}$ of $\varrho$ and $W$*, defined as the measure

$$L_{[\varrho,W]}(ds) \;:=\; \lim_{\varepsilon \downarrow 0} ds \int \varrho_s(dy)\, p(\varepsilon, y - W_s),$$

where $p$ is the standard heat kernel $p(t,x) = [2\pi t]^{-d/2} \exp\left[-|x|^2/2t\right],\ (t,x) \in (0,\infty) \times \mathbb{R}^d$. According to [BEP91], this collision local time $L_{[\varrho,W]}$ makes sense non-trivially in dimension $d \le 3$, and vanishes for $d \ge 4$ (where the Brownian reactant particles do not hit the catalyst $\varrho$). Thus we restrict our attention to $d \le 3$ (since otherwise $X^{\varrho}$ degenerates to the heat flow).

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The *catalytic super-Brownian motion* $X^\varrho$ was constructed in [DF97a]. Let $\varrho$ and $X^\varrho$ start at time 0 with Lebesgue measures $\ell_{\rm c}$ and $\ell_{\rm r}$, respectively. From the papers [DF97a, DF97b, EF98, FK97] it is known that $X_T^\varrho$ *converges* in law as $T \uparrow \infty$ to a limit $X_\infty^\varrho$ with full expectation $\ell_{\rm r}$ (*persistence*) [1]. The approach of [FK97] to solve the most difficult case, namely convergence in the critical dimension $d=2$, was to study the local structure of $X^\varrho$ and then to use a self-similarity argument. In fact, they showed that in dimensions $d = 2, 3$, given the catalyst $\varrho$, the reactant $X^\varrho$ has a *density field* $\xi^\varrho$ :

$$X_t^\varrho(dx) \;=\; \xi_t^\varrho(x)dx, \qquad t>0.$$

Moreover, off the time-space support of the catalyst $\varrho$ (which is a Lebesgue zero set), $\xi^\varrho$ can be chosen as a (time-space) $C^\infty$-function that solves the heat equation, just as intuitively expected.

Simulations of $(\varrho, X^\varrho)$ in dimension $d=2$ (see the figure in [FK97]) confirm the *heuristic picture* one has. Namely, at late times $T$,

- the reactant $X_T^\varrho$ is rather uniform outside of the catalyst $\varrho_T$ ,
- it is absent inside of the clumps of $\varrho_T$ (since a huge rate of branching causes mainly killing),
- but occasionally also some *hot spots* of the reactant occur in the *interface* of $\varrho_T$ and $X_T^\varrho$ , that is in the boundary region of the catalytic clumps.

According to [FK97], in the two-dimensional case, the (local) long-term limit $X_\infty^\varrho$ is in fact a random multiple of Lebesgue measure [the factor is given by $\xi_1^\varrho(0)$]. But so far the investigations on the catalytic super-Brownian motion $X^\varrho$ do not reflect anything on the hot spots seen in the pictures. Our approach to gain some information about them is to study the *collision local time* $L := L_{[\varrho, X^\varrho]}$ *of* $\varrho$ *and* $X^\varrho$ defined as the limit of

$$L_\varepsilon\left(d[s,x]\right) \;:=\; ds\;\varrho_s(dx)\int X_s^\varrho(dy)\,p(\varepsilon, x-y), \tag{1}$$

as $\varepsilon \downarrow 0$.

Actually there is a further motivation to study this collision local time $L_{[\varrho, X^\varrho]}$. It occurs indeed in the description of the *martingale problem* for the process $X^\varrho$ (see Corollary 4 below). For martingale problems of catalytic super-Brownian motions, see also [DF94, Del96, Led97].

Let us present the results. We prove that in all dimensions of non-trivial existence of $X^\varrho$ the collision local time $L$ of catalyst and reactant makes non-trivially sense (see Theorem 3 below). [In dimension 1, it is known that both $\varrho_s$ and $X_s^\varrho$ are absolutely continuous (cf. [KS88] and [DFR91], respectively); thus $L_\varepsilon$ and hence $L$ simplify in this case.] This non-trivial existence of $L$ reflects the high fluctuations of $X^\varrho$ in the interface of catalyst and reactant, seen as hot spots in simulations. Our *main result* however is that for $d=2$ the *marginal measure* of $L = L_{[\varrho, X^\varrho]}$ concerning the space variable is *absolutely continuous* (Theorem 5). Note that this is in contrast, for instance, with the (one-dimensional) single-point catalytic model of [DF94], say $X^{\delta_0}$, where, together with the catalyst $\delta_0$ , the space marginal of the collision local time $L_{[\delta_0, X^{\delta_0}]}$ is concentrated in the single space point 0, hence is singular (even atomic). Finally, in dimension 2, using the *self-similarity* of $L_{[\varrho, X^\varrho]}$ which follows from the

[1] In the three-dimensional case, for simplification it was assumed in [DF97b] that the catalyst process $\varrho$ is already in its corresponding equilibrium.

self-similarity of $(\varrho, X^{\varrho})$ , we show that $T^{-1}L\left([0,T]\times(\cdot)\right)$ has a *random ergodic limit* as $T\uparrow\infty$, which indicates *diffusive* features in the long-term behavior in $d=2$.

It remains *open* whether also in dimension 3 space-marginal collision densities exist since our $L^2$-approach fails in this case (see Remark 6 below).

The *outline* of the paper is as follows. In Section 2 we introduce formal definitions of the processes $\varrho$ and $X^{\varrho}$ and state the results. The following two sections are then devoted to the proofs of our two theorems. In an appendix we collect some results on ordinary and catalytic super-Brownian motions used in the proofs.

## 2. Statement of results

2.1. **Notation.** The lower index $+$ on a set will always refer to the collection of all its nonnegative members. Similarly, $f_+$ is the nonnegative part of $f$. The supremum norm is denoted by $\|\cdot\|_\infty$. Let $c$ always refer to a (finite) constant whose value may vary from place to place.

We denote by $\mathcal{B}(E)$ the space of all real Borel measurable functions defined on a polish space $E$. We also denote by $\mathcal{B}(E)$ the Borel $\sigma$–field of $E$.

For a fixed constant $q>d$, introduce the reference function $\phi_q\in\mathcal{B}(\mathbb{R}^d)$ :

$$\phi_q(x) \;:=\; \left[1+|x|^2\right]^{-q/2}, \qquad x\in\mathbb{R}^d. \tag{2}$$

Set $\mathcal{B}^q := \left\{f\in\mathcal{B}(\mathbb{R}^d);\, \|f/\phi_q\|_\infty<\infty\right\}$. Let $C_c(\mathbb{R}^d)$ denote the collection of all continuous functions on $\mathbb{R}^d$ with compact support.

If $\nu$ is a Radon measure on $\mathbb{R}^d$, we write $(\nu,f)$ for $\int\nu(dx)\,f(x)$ (if the integral makes sense). Let $\mathcal{M}_q$ denote the set of all Radon measures $\nu$ on $\mathbb{R}^d$ such that $(\nu,\phi_q)<\infty$. This space of tempered measures is endowed with the coarsest topology such that the maps $\nu\mapsto(\nu,f)$ are continuous for $f\in C_c(\mathbb{R}^d)\cup\{\phi_q\}$, getting a Polish space. Since $q>d$, Lebesgue measure belongs to $\mathcal{M}_q$.

We consider the polish space $\mathcal{C}:=C(\mathbb{R}_+,\mathcal{M}_q)$ of all continuous functions from $\mathbb{R}_+$ to $\mathcal{M}_q$ equipped with the topology of uniform convergence on compacta.

Let $(P_t\,, t\ge 0)$ denote the semigroup of heat flow on $\mathbb{R}^d$:

$$P_t[f](x) \;:=\; \int dy\; p(t,x-y)f(y), \qquad t>0, \quad f\in\mathcal{B}_+(\mathbb{R}^d). \tag{3}$$

2.2. **Catalyst and reactant process.** We start by introducing the catalyst process.

**Definition 1** (catalyst process). Let $\gamma>0$ and $\nu\in\mathcal{M}_q$. There exists a unique probability measure $\mathbb{P}_\nu$ on $\left(\mathcal{C},\mathcal{B}(\mathcal{C})\right)$, such that the coordinate process $\varrho=(\varrho_t\,, t\ge0)$ on $\mathcal{C}$ is a super-Brownian motion with constant branching rate $\gamma$ and starting measure $\nu$. That is, $\varrho$ is a continuous time-homogeneous strong Markov process with the following properties:

- $\mathbb{P}_\nu$-almost surely, $\varrho_0=\nu$,
- for every $f\in\mathcal{B}^q_+,\ t\ge r\ge 0$, we have [2)]

$$\mathbb{E}_\nu\left[\mathrm{e}^{-(\varrho_t\,,f)}\;\Big|\;\sigma\left(\varrho_s\,, s\in[0,r]\right)\right] \;=\; \mathrm{e}^{-\left(\varrho_r\,,w(t-r)\right)}, \tag{4}$$

[2)] We use the following convention: If $\mathcal{P}$ is a probability law, then the corresponding letter $\mathcal{E}$ refers to the related expectation symbol.

where $w$ is the unique nonnegative solution on $\mathbb{R}_+ \times \mathbb{R}^d$ of the log-Laplace equation

$$w(t,x) + \gamma \int_0^t ds \; P_s[w^2(t-s)](x) \;=\; P_t\,[f]\,(x). \tag{5}$$

We write $\mathbb{P}$ for $\mathbb{P}_\nu$ in the case $\nu = i_{\mathrm{c}}\ell$, where $i_{\mathrm{c}} > 0$ and $\ell$ is the (normalized) Lebesgue measure on $\mathbb{R}^d$. $\diamond$

From now on we assume that $d \le 3$, and that $\varrho$ is distributed [3)] according to $\mathbb{P}$. Next we recall the definition of the catalytic super-Brownian motion $X^\varrho$ in the random medium $\varrho$ (see [DF97a] for details).

**Definition 2** (catalytic super-Brownian motion). Fix $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$ and $\kappa > 0$. For convenience, set $\mathcal{C}' := C\left([r,\infty), \mathcal{M}_q\right)$. There exists a (measurable) probability kernel $\varrho \mapsto \mathrm{P}^\varrho_{r,\mu}$ from $\left(\mathcal{C}, \mathcal{B}(\mathcal{C})\right)$ to $\left(\mathcal{C}', \mathcal{B}(\mathcal{C}')\right)$ such that the coordinate process $X^\varrho = (X^\varrho_t, t \ge r)$ on $\mathcal{C}'$ is a *super-Brownian motion in the catalytic medium* $\varrho$. That is, $\mathbb{P}$-a.s. under $\mathrm{P}^\varrho_{r,\mu}$, the process $X^\varrho$ is continuous time-inhomogeneous Markov with the following properties:

- $\mathrm{P}^\varrho_{r,\mu}$-almost surely, $X^\varrho_r = \mu$,
- for every $f \in \mathcal{B}^q_+$, $t \ge s \ge r$, we have

$$\mathrm{E}^\varrho_{r,\mu}\left[\mathrm{e}^{-(X^\varrho_t, f)} \,\middle|\, \sigma\left(X^\varrho_u, u \in [r,s]\right)\right] \;=\; \mathrm{e}^{-\left(X^\varrho_s,\, v_t(s)\right)}, \tag{6}$$

where $v_t$ is the unique nonnegative solution on $[r,\infty) \times \mathbb{R}^d$ of the catalytic log-Laplace equation

$$v(s,x) + \kappa \int_s^\infty du \int \varrho_u(dy)\, p(u-s, x-y)\, v^2(u,y) \;=\; J(s,x), \tag{7}$$

with $J(s) := \mathbf{1}_{t \ge s}\, P_{t-s}[f]$.

Often, we also pass from the quenched distributions $\mathrm{P}^\varrho_{r,\mu}$ to the annealed law $\mathbb{E}[\mathrm{P}^\varrho_{r,\mu}]$. $\diamond$

2.3. **Existence of collision local time of catalyst and reactant.** For our constant $q > d$, we introduce the function space $H^q := \bigcup_{T \ge 0} H^q_T$, where

$$H^q_T \;:=\; \left\{ g \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d);\ \operatorname{supp} g \subset [0,T] \times \mathbb{R}^d,\ \|g/\phi_q\|_\infty < \infty \right\}, \tag{8}$$

with $\|g/\phi_q\|_\infty = \sup_{(s,x) \in \mathbb{R}_+ \times \mathbb{R}^d} |g(s,x)|/\phi_q(x)$, and $\operatorname{supp} g$ denoting the support of $g$.

Recall the approximated collision local time $L_\varepsilon$ of $\varrho$ and $X^\varrho$ introduced already in (1). We are now ready to state our first result, the existence of the collision local time $L = L_{[\varrho, X^\varrho]}$ of $\varrho$ and $X^\varrho$. Recall that $d \le 3$ and $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

**Theorem 3** (collision local time). *There exists a random variable denoted by $L = L_{[\varrho, X^\varrho]}$ defined on $\left(\mathcal{C} \times \mathcal{C}', \mathcal{B}(\mathcal{C} \times \mathcal{C}')\right)$, taking values in the set of Radon measures on $[r,\infty) \times \mathbb{R}^d$ with the following properties:*

- **(i) (tempered measure):** *For every $T \ge r$, we have $\mathbb{E}\left[\mathrm{P}^\varrho_{r,\mu}(L, \mathbf{1}_{[r,T]}\, \phi_q)\right] < \infty$.*
- **(ii) (existence via convergence):** *For every $\varphi \in H^{2q}$,*

$$\lim_{\varepsilon \downarrow 0} (L_\varepsilon\,, \varphi) = (L, \varphi), \quad \mathbb{E}[\mathrm{P}^\varrho_{r,\mu}]\text{–}a.s.$$

[3)] In [FK97] more generally a class of so-called $\eta$-diffusive measures $\nu$ is introduced which allow that $\varrho$ under $\mathbb{P}_\nu$ may serve as the catalyst for $X^\varrho$.

**(iii) (regularity):** *For every* $\varphi \in H^{2q}$, *and* $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$*-a.s., the process* $\big((L, \mathbf{1}_{[r,t]}\varphi),\, t \geq r\big)$ *is continuous and adapted to the filtration*

$$\Big(\mathcal{F}_t := \sigma(\varrho) \vee \sigma\left(X^{\varrho}_s,\, s \in [r,t]\right),\ t \geq r\Big).$$

**(iv) (moments):** *For every* $m \geq 1$, $\varphi \in H^{2q}$, $\mathbb{P}$*-a.s.,*

$$\mathrm{E}^{\varrho}_{r,\mu}\left[\left[\int_{[r,\infty)\times\mathbb{R}^d} L\left(d[s,x]\right)\varphi(s,x)\right]^m\right] \;=\; m! \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1,\\ n_1+\cdots+n_k = m}} \prod_{i=1}^{k} \left(\mu, \chi_{n_i}(r)\right), \tag{9}$$

*where the functions* $\chi_n$, $n \geq 1$, *belong to* $H^q$ *and are recursively defined by*

$$\chi_n(s,x) \;:=\; \kappa \int_s^{\infty} du \int \varrho_u(dy)\, p(u-s, x-y) \left[\sum_{i=1}^{n-1} \chi_i(u,y)\, \chi_{n-i}(u,y)\right], \qquad n \geq 2, \tag{10}$$

*with the initial condition*

$$\chi_1(s,x) \;:=\; \int_s^{\infty} du \int \varrho_u(dy)\; p(u-s, x-y)\, \varphi(u,y), \qquad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{11}$$

Consequently, in dimensions $d \leq 3$, the collision local time $L = L_{[\varrho, X^{\varrho}]}$ of catalyst and reactant exists non-trivially, reflecting in particular the occurrence of hot spots in the mentioned 2-dimensional simulations.

The proof of this theorem is postponed to Section 3.

As an *application,* we can now describe the covariance measure of the martingale measure associated with $X^{\varrho}$. Let $C_b^{1,2}$ denote the set of bounded functions $\varphi \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ such that the partial derivatives $\frac{\partial \varphi}{\partial s}$ and $\frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ exist and are continuous and bounded. It is easy to check that under $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$ the process $(M\varphi_{r,t}\,,\, t \geq r)$ defined by

$$M\varphi_{r,t} \;:=\; \left(X^{\varrho}_t, \varphi(t)\right) - \left(X^{\varrho}_r, \varphi(r)\right) - \int_r^t ds\, \left(X^{\varrho}_s\,,\, \frac{\partial \varphi}{\partial s}(s) + \frac{1}{2}\Delta\varphi(s)\right), \tag{12}$$

is an $(\mathcal{F}_t\,, t \geq r)$*-martingale* [note that $\mathcal{F}_r = \sigma(\varrho) \vee \sigma(X_r)$]. Thanks to the Markov property of $X^{\varrho}$ (given $\varrho$), and the moment formula (A.9) for $X^{\varrho}$ stated in the appendix, we get that for $\varphi, \psi$ in $C_b^{1,2}$, $\mathbb{P}$-a.s. for all $s \geq r$ and $t \geq r$,

$$\mathrm{E}^{\varrho}_{r,\mu}\left[M\varphi_{r,s} M\psi_{r,t}\right] \;=\; 2\kappa \int \mu(dx) \int_r^{s\wedge t} du \int \varrho_u(dy)\; p(u-r, x-y)\, \varphi(u,y)\, \psi(u,y). \tag{13}$$

The functional $M : \varphi \mapsto M\varphi$ defined on $C_b^{1,2}$ can be extended to an *orthogonal martingale measure* on $H^q$. Let $\langle M \rangle$ denote its *covariance measure.* Now we show how $\langle M \rangle$ can be expressed in terms of the collision local time $L = L_{[\varrho, X^{\varrho}]}$. Recall that $d \leq 3$ and that $(r,\mu) \in \mathbb{R}_+ \times \mathcal{M}_q$.

**Corollary 4** (covariance measure). *For every* $\varphi \in H^q$, $\mathbb{E}[\mathrm{P}^{\varrho}_{r,\mu}]$*-a.s. for every* $t \geq r$, *we have*

$$\left\langle M\varphi \right\rangle_{r,t} \;=\; 2\kappa \int_{[r,t]\times\mathbb{R}^d} L\left(d[s,y]\right) \varphi^2(s,y). \tag{14}$$

*Proof.* Using the Markov property of $X^\varrho$ (given $\varrho$) and an obvious extension of the second moment formula (13), we obtain for $\varphi \in H^q$, $\mathbb{P}$-a.s. for all $t \geq s \geq r$,

$$\mathbb{E}\mathrm{E}^\varrho_{r,\mu}\left[(M\varphi_{r,t})^2 \mid \mathcal{F}_s\right] = (M\varphi_{r,s})^2 + 2\kappa \int X^\varrho_s(dx) \int_s^t du \int \varrho_u(dy)\, p(u-s, x-y)\, \varphi^2(u,y).$$

Since

$$\left(\int_{[r,t]\times\mathbb{R}^d} L\left(d[s,y]\right) \varphi^2(s,y),\ t \geq r\right)$$

is in $t$ non-decreasing and continuous, is adapted to $(\mathcal{F}_t, t \geq r)$, and zero for $t = r$, we get that

$$\left(\langle M\varphi\rangle_{r,t} - 2\kappa \int_{[r,t]\times\mathbb{R}^d} L\left(d[s,y]\right) \varphi^2(s,y),\ t \geq r\right)$$

is a continuous martingale under $\mathbb{E}[\mathrm{P}^\varrho_{r,\mu}]$ with bounded variation starting at time $t = r$ from 0. This martingale is then constant and, in fact, equal to 0, giving the claim (14). □

2.4. **Collision local time in dimension two.** We now state our results for the collision local time in the "critical" dimension $d = 2$. For convenience, we introduce the following abbreviation for an annealed law:

$$\mathbf{P} := \mathbb{E}[\mathrm{P}_{0,i_\mathrm{r}\ell}] = \mathbb{E}_{i_\mathrm{c}\ell}[\mathrm{P}_{0,i_\mathrm{r}\ell}], \quad \text{where } i_\mathrm{r} > 0.$$

(That is, we now focus on the situation $r = 0$ and $\mu = i_\mathrm{r}\ell$.)

**Theorem 5** (two-dimensional collision local time)**.** *Let $d = 2$.*

**(a) (local spatial $L^2$ collision densities):** *For every $t \geq s \geq 0$ and $z \in \mathbb{R}^2$,*

$$\left(\int_{[s,t]\times\mathbb{R}^d} L\left(d[r,y]\right) p(\varepsilon, z-y),\ \varepsilon > 0\right)$$

*converges in $L^2(\mathbf{P})$ as $\varepsilon \downarrow 0$ to a random variable denoted by $\lambda_{[s,t]}(z)$. It has expectation*

$$\mathbf{E}\left[\lambda_{[s,t]}(z)\right] = i_\mathrm{c} i_\mathrm{r}\,(t-s),$$

*and its finite variance is non-zero provided that $s < t$.*

**(b) (spatial absolute continuity):** *For $t \geq s \geq 0$, there exists a measurable version of $\lambda_{[s,t]}$ with respect to $\mathcal{B}(\mathbb{R}^2)\times\mathcal{F}_t$, and $\mathbf{P}$-a.s. the measure $L\left([s,t]\times(\cdot)\right)$ on $\mathbb{R}^2$ is absolutely continuous and can be represented as*

$$L\left([s,t]\times dx\right) = \lambda_{[s,t]}(x)dx.$$

**(c) (self-similarity):** *Under $\mathbf{P}$, the laws of the scaled collision local times*

$$K^{-2}L\left(K(\cdot)\times K^{1/2}(\cdot)\right)$$

*are independent of the scaling factor $K > 0$.*

**(d) (random ergodic limit):** *The following convergence in $\mathcal{M}_q$ holds in law with respect to $\mathbf{P}$ :*

$$\lim_{T\uparrow\infty} T^{-1}L\left([0,T]\times(\cdot)\right) = \lambda_{[0,1]}(0)\,\ell$$

(*with $\ell$ the Lebesgue measure and* $0 < \mathbf{Var}\left[\lambda_{[0,1]}(0)\right] < \infty$ ).

Consequently, in dimension 2, the spatial marginal measures $L\left([s,t]\times(\cdot)\right)$ of the collision local time $L_{[\varrho,X^\varrho]}$ of catalyst and reactant have non-degenerated densities $\lambda_{[s,t]}(z)$ (provided that $s<t$). Moreover, $\lambda_{[0,1]}(0)$ enters as random factor of Lebesgue measure in the long-term ergodic limit. Recall that this reflects the diffusive features of the hot spots.

**Remark 6** (dimension three). The $L^2(\mathbf{P})$-convergence in part (a) does *not* hold for $d=3$. In fact, in the three-dimensional case an infinite term would be involved in our calculations, see the remark following (29) in the proof below. Recall on the other hand that in dimension one, $L_{[\varrho,X^\varrho]}$ should be rather "regular". $\diamond$

**Remark 7** (regularity). It is an *open problem* whether the spatial collision density functions $\lambda_{[s,t]}$ have some regularities properties in the space variable. Note also that the exceptional set in the $\mathbf{P}$-a.s. statement in (b) depends on $[s,t]$. One would also like to know whether this situation can be improved. $\diamond$

The statement (c) follows from the self-similarity of $(\varrho, X^\varrho)$ by standard arguments (compare with [DF97b, Subsections 4.1 and 4.2]). Otherwise the proof of Theorem 5 will be provided in Section 4.

## 3. Existence of collision local time (proof of Theorem 3)

Recall that $d\le 3$. First of all we state the following lemma.

**Lemma 8** (approximated moment increments). *For every $m\ge 1$, $r\ge 0$, $\mu\in\mathcal{M}_q$, $T\ge 0$, $\xi\in(0,1/4)$, $\mathbb{P}$-a.s. there exists a finite constant $M_m$ (depending on $\varrho$) such that for every $\varphi\in H_T^{2q}$, $t'\ge t\ge 0$, $1\ge\varepsilon'\ge\varepsilon>0$,*

$$\mathrm{E}^{\varrho}_{r,\mu}\left[(L_\varepsilon\,,\varphi\,\mathbf{1}_{[t,t']})^{2m}\right]\;\le\;M_m\,\|\varphi/\phi_{2q}\|_\infty^{2m}\left[\left|t-t'\right|^\xi\left(1+\log_+\left(1/|t-t'|\right)\right)\right]^{2m}, \tag{15}$$

$$\mathrm{E}^{\varrho}_{r,\mu}\left[\left[(L_\varepsilon\,,\varphi)-(L_{\varepsilon'}\,,\varphi)\right]^{2m}\right]\;\le\;M_m\,\|\varphi/\phi_{2q}\|_\infty^{2m}\left[\left|\varepsilon-\varepsilon'\right|^\xi\left(1+\log_+\left(1/|\varepsilon-\varepsilon'|\right)\right)\right]^{2m}. \tag{16}$$

Based on this lemma, the proof of Theorem 3 (ii) and (iii) are similar to the proof of Proposition 5.1 based on Lemma 5.2 in [Del96] with the obvious changes and is left to the reader. (iv) is not stated in Proposition 5.1 there, but it is a by-product of its proof [take the limit in (32)]. Eventually, (i) is proved by using the monotone convergence theorem with the moment formula (9) and (A.2) (in the appendix) with $m=1$ and the inequality (A.1).

*Proof of Lemma* 8. Fix $\mu\in\mathcal{M}_q$, $\xi\in(0,1/4)$, and $T\ge r\ge 0$ (otherwise the moments disappear). We will verify (15); the proof of (16) is similar and is left to the reader.

Note first that for fixed $\varepsilon>0$,

$$\sup_{x\in\mathbb{R}^d,\;y\in\mathbb{R}^d}\frac{\phi_q(y)\,p(\varepsilon,x-y)}{\phi_q(x)}\;<\;\infty. \tag{17}$$

Let $\varphi\in H_T^{2q}$. Since $\varrho$ is $\mathbb{P}$-a.s. a continuous $\mathcal{M}_q$-valued path, it is then clear that the functions $(s,x)\mapsto\int\varrho_s(dy)\,p(\varepsilon,x-y)\,\varphi(s,y)$ belong to $H_T^q$. Thanks to the remarks at the beginning of Subsection A.1, we see that, for fixed $t,t',\varepsilon$, the functions

$$(s,x)\mapsto J_\varepsilon(s,x)\;:=\;\int_s^\infty du\int dz\;p(u-s,x-z)\int\varrho_u(dy)\,p(\varepsilon,z-y)\,\varphi(u,y)\,\mathbf{1}_{[t,t']}(u) \tag{18}$$

are well-defined and belong to $H_T^q$.

We will now prove that $\mathbb{P}$-a.s. there exists a finite constant $c$ such that for every $\varphi \in H_T^{2q}$, $t' \geq t \geq 0$, $1 \geq \varepsilon > 0$,

$$\big|J_\varepsilon(s,x)\big| \;\leq\; c\,\mathbf{1}_{[0,T]}(s)\,\phi_q(x)\,\|\varphi/\phi_{2q}\|_\infty \left[\big|t-t'\big|^\xi \left(1+\log_+\big(1/|t-t'|\big)\right)\right]. \tag{19}$$

Clearly $\big|J_\varepsilon(s,x)\big| \,/\, \|\varphi/\phi_{2q}\|_\infty$ is bounded from above by

$$K_1 \;:=\; \mathbf{1}_{[0,T]}(s) \int_s^T du \int \varrho_u(dy)\, p(u-s+\varepsilon, x-y)\,\phi_{2q}(y)\,\mathbf{1}_{[t,t']}(u).$$

We assume that $T \geq t$ (otherwise $K_1 = 0$). Introduce the quantity

$$K_2 \;:=\; \mathbf{1}_{[0,T\wedge t']}(s) \int_{s\vee t}^{T\wedge t'} du \int \varrho_u(dy)\, p(u-s\vee t, x-y)\,\phi_{2q}(y).$$

Thanks to (A.6), we have $K_2 \leq \mathbf{1}_{[0,T]}(s)\, C_2\, |t-t'|^\xi\, \phi_q(x)$. Now

$$|K_1 - K_2| \;\leq\; \mathbf{1}_{[0,T\wedge t']}(s) \int_{s\vee t}^{T\wedge t'} du \int \varrho_u(dy)\; \big|p(u-s+\varepsilon, x-y) - p(u-s\vee t, x-y)\big|\,\phi_{2q}(y).$$

Using the inequality

$$\big|p(v_1,z) - p(v_2,z)\big| \leq c \int_{v_1}^{v_2} dv\; v^{-1}\, p(2v,z), \tag{20}$$

where the constant $c$ is independent of $z \in \mathbb{R}^d$ and $v_2 \geq v_1 > 0$, we get that

$$\begin{aligned} |K_1 - K_2| &\leq c\,\mathbf{1}_{[0,T\wedge t']}(s) \int_{s\vee t}^{T\wedge t'} du \int \varrho_u(dy)\,\phi_{2q}(y) \int_{u-s\vee t}^{u-s+\varepsilon} dv\; v^{-1}\, p(2v, x-y) \\ &= c\,\mathbf{1}_{[0,T\wedge t']}(s) \int_0^{T\wedge t'-s+\varepsilon} dv\; v^{-1} \int_{s\vee t\vee(v+s-\varepsilon)}^{T\wedge t'\wedge(v+s\vee t)} du \int \varrho_u(dy)\,\phi_q^2(y)\, p(2v, x-y). \end{aligned}$$

In view of (A.5) and (A.1), we may continue with

$$\leq\; c\,\mathbf{1}_{[0,T\wedge t']}(s)\,\phi_q(x) \int_0^{T\wedge t'-s+\varepsilon} dv\; v^{-1}\,\big|T\wedge t'\wedge(v+s\vee t) - s\vee t\vee(v+s-\varepsilon)\big|^\xi,$$

where $c$ is independent of $t', t, \varepsilon, x$. It is easy to check that

$$\begin{aligned} &\int_0^{T\wedge t'-s+\varepsilon} dv\; v^{-1}\,\big|T\wedge t'\wedge(v+s\vee t) - s\vee t\vee(v+s-\varepsilon)\big|^\xi \\ &\qquad \leq\; c\,\big|t'-t\big|^\xi \left(1+\log_+\big(1/|t'-t|\big)\right), \end{aligned} \tag{21}$$

where $c$ is independent of $t', t$ and $\varepsilon$. As a conclusion we obtain (19).

Using the estimate (A.6), a straight forward induction shows that all the functions $\chi_n$, $n \geq 1$, of the recurrence relation (10) with initial condition $\chi_1 = J_\varepsilon$ belong to $H_T^q$ and satisfy

$$\big|\chi_n(s,x)\big| \;\leq\; c\,\mathbf{1}_{[0,T]}(s)\,\phi_q(x)\,\|\varphi/\phi_{2q}\|_\infty^n \left[\big|t-t'\big|^\xi \left(1+\log_+\big(1/|t-t'|\big)\right)\right]^n.$$

(Note that $c$ is independent of $\varphi, t, t'$ and $\varepsilon$.) Then the claim (15) is a consequence of (A.9) with $f = 0$ and

$$g(s,z) := \int \varrho_s(dy)\, p(\varepsilon, z-y)\, \varphi(s,y)\, \mathbf{1}_{[t,t']}(s), \tag{22}$$

finishing the proof. □

## 4. Two-dimensional collision local time (proof of Theorem 5)

We now assume that $d = 2$.

### 4.1. Local spatial collision densities [proof of (a)].

For the claimed $L^2$-convergence, it is enough to check that, for fixed $s, t, z$,

$$J^{\varepsilon,\varepsilon'} := \mathbf{E}\left[\int_{[s,t]\times\mathbb{R}^2} L\left(d[r,y]\right) p(\varepsilon, z-y) \int_{[s,t]\times\mathbb{R}^2} L\left(d[r',y']\right) p(\varepsilon', z-y')\right] \tag{23}$$

converges in $\mathbb{R}_+$ as $\varepsilon$ and $\varepsilon'$ decrease to 0.

For $f \in L^1_+(\mathbb{R}^2)$ with $\int dx\, f(x) = 1$, and $\varepsilon > 0$, $z \in \mathbb{R}^2$, we set

$$f_{\varepsilon,z}(x) := \varepsilon^{-1} f\left(\varepsilon^{-1/2}(x-z)\right). \tag{24}$$

Note that $f_{\varepsilon,z}(x)dx$ converges weakly to $\delta_z(dx)$, the Dirac mass at $z$, as $\varepsilon$ decreases to 0. We will prove the following stronger result.

**Lemma 9.** *For fixed $s, t, z, z'$, $f, f' \in L^1_+(\mathbb{R}^2)$ such that $\int dx\, f(x) = 1 = \int dx\, f'(x)$, the well-defined quantity*

$$J^{\varepsilon,\varepsilon'}(z,z') := \mathbf{E}\left[\int_{[s,t]\times\mathbb{R}^2} L\left(d[r,y]\right) f_{\varepsilon,z}(y) \int_{[s,t]\times\mathbb{R}^2} L\left(d[r',y']\right) f'_{\varepsilon',z'}(y')\right]$$

*converges to a finite limit independent of $f, f'$, as $\varepsilon$ and $\varepsilon'$ decrease to* 0.

Note that we need the convergence for $z = z'$ to prove (23) and then (a). Note also that although $f$ and $f'$ are not in $\mathcal{B}^q$ a priori, we show that $J^{\varepsilon,\varepsilon'}$ is well-defined.

*Proof of Lemma* 9. By a standard monotone class argument, we deduce from the quenched moment formula (9) for collision local time with $m = 2$, that for $g \in \mathcal{B}_+\left((\mathbb{R}_+)^2 \times (\mathbb{R}^2)^2\right)$,

$$\begin{aligned}
&\mathbf{E}\left[\int_{\mathbb{R}_+\times\mathbb{R}^2} L\left(d[r,y]\right) \int_{\mathbb{R}_+\times\mathbb{R}^2} L\left(d[r',y']\right) g(r,r',y,y')\right] \\
&= \mathbb{E}\Bigg[2\, i_{\mathrm{r}}\kappa \int dx \int_0^\infty ds_1 \int \varrho_{s_1}(dy_1)\, p(s_1, y_1 - x) \int_{s_1}^\infty ds_2 \int \varrho_{s_2}(dy_2)\, p(s_2 - s_1, y_2 - y_1) \\
&\qquad \int_{s_1}^\infty ds_3 \int \varrho_{s_3}(dy_3)\, p(s_3 - s_1, y_3 - y_1)\, g(s_2, s_3, y_2, y_3) \\
&\qquad + i_{\mathrm{r}}^2 \int dx_1 \int_0^\infty ds_1 \int \varrho_{s_1}(dy_1)\, p(s_1, y_1 - x_1) \\
&\qquad \int dx_2 \int_0^\infty ds_2 \int \varrho_{s_2}(dy_2)\, p(s_2, y_2 - x_2)\, g(s_1, s_2, y_1, y_2)\Bigg].
\end{aligned}$$

Thus we can write

$$J^{\varepsilon,\varepsilon'} \;=\; 2\,i_{\mathrm{r}}\kappa J_1^{\varepsilon,\varepsilon'} + i_{\mathrm{r}}^2 J_2^{\varepsilon,\varepsilon'}, \tag{25}$$

where

$$J_1^{\varepsilon,\varepsilon'}(z,z') := \int_0^t ds_1\,\mathbb{E}\Big[\int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int \varrho_{s_1}(dy_1)\int \varrho_{s_2}(dy_2)\int \varrho_{s_3}(dy_3)$$
$$p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\Big]$$

and

$$J_2^{\varepsilon,\varepsilon'}(z,z') \;:=\; \mathbb{E}\Big[\int_s^t ds_1 \int_s^t ds_2 \int \varrho_{s_1}(dy_1)\int \varrho_{s_2}(dy_2)\; f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2)\Big]$$

are third and second moment expressions of the catalyst process only, respectively. We easily compute $J_2^{\varepsilon,\varepsilon'}$ thanks to the moment formula (A.2) for ordinary super-Brownian motion (with $f=0$ and $g$ properly chosen):

$$\begin{aligned}
J_2^{\varepsilon,\varepsilon'}(z,z') \;=&\; 2\gamma\, i_{\mathrm{c}} \int dx \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2 \int dy_1 \int dy_2 \int dy_3 \\
&\quad p(s_3,y_3-x)\,p(s_1-s_3,y_1-y_3)\,p(s_2-s_3,y_2-y_3)\,f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2) \\
&+\; i_{\mathrm{c}}^2 \int dx_1 \int dx_2 \int_s^t ds_1 \int_s^t ds_2 \int dy_1 \int dy_2 \\
&\quad p(s_1,y_1-x_1)\,p(s_2,y_2-x_2)\,f_{\varepsilon,z}(y_1)\,f'_{\varepsilon',z'}(y_2) \\
=&\; 2\gamma\, i_{\mathrm{c}} \int dy_1\; f_{\varepsilon,z}(y_1) \int dy_2\; f'_{\varepsilon',z'}(y_2) \\
&\quad \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2\; p(s_1+s_2-2s_3,y_1-y_2) \\
&+\; i_{\mathrm{c}}^2\,(t-s)^2 \\
\le&\; 2\gamma\, i_{\mathrm{c}} \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2\; p(s_1+s_2-2s_3,0) +\; i_{\mathrm{c}}^2\,(t-s)^2 =: K_2 < \infty.
\end{aligned}$$

As $(\varepsilon,\varepsilon')\downarrow 0$, the quantity $J_2^{\varepsilon,\varepsilon'}(z,z')$ converges to

$$J_2^0(z,z') \;:=\; 2\gamma\, i_{\mathrm{c}} \int_0^t ds_3 \int_{s_3\vee s}^t ds_1 \int_{s_3\vee s}^t ds_2\; p(s_1+s_2-2s_3,z-z') + i_{\mathrm{c}}^2\,(t-s)^2 \;\le\; K_2. \tag{26}$$

We can also compute $J_1^{\varepsilon,\varepsilon'}$ using the Markov property of $\varrho$ at time $s_1$ and twice the moment formula (A.2):

$$\begin{aligned} J_1^{\varepsilon,\varepsilon'}(z,z') \;=\; & 2\gamma \int_0^t ds_1\, \mathbb{E}\Big[ \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_1}(dy_5) \int dy_4 \int dy_2 \int dy_3 \\ & \qquad p(s_4-s_1,y_4-y_5)\, p(s_2-s_4,y_2-y_4)\, p(s_3-s_4,y_3-y_4) \\ & \qquad p(s_2-s_1,y_2-y_1)\, p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3)\Big] \\ & + \int_0^t ds_1\, \mathbb{E}\Big[ \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int \varrho_{s_1}(dy_1) \int \varrho_{s_1}(dy_4) \int \varrho_{s_1}(dy_5) \int dy_2 \int dy_3 \\ & \qquad p(s_2-s_1,y_2-y_4)\, p(s_3-s_1,y_3-y_5)\, p(s_2-s_1,y_2-y_1) \\ & \qquad p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f'_{\varepsilon',z'}(y_3)\Big]. \end{aligned}$$

With obvious notation we write

$$J_1^{\varepsilon,\varepsilon'} \;=\; 2\gamma J_3^{\varepsilon,\varepsilon'} + J_4^{\varepsilon,\varepsilon'}. \tag{27}$$

Using again the moment formula, we get

$$J_3^{\varepsilon,\varepsilon'} \;=\; 2\gamma\, i_{\mathrm{c}}\, J_5^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2\, J_6^{\varepsilon,\varepsilon'}, \tag{28}$$

where

$$\begin{aligned} J_5^{\varepsilon,\varepsilon'}(z,z') \;:=\; & \int_0^t ds_1 \int_0^{s_1} ds_5 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \int dy_5 \int dy_6 \\ & p(s_1-s_5,y_1-y_6)\, p(s_1-s_5,y_5-y_6)\, p(s_4-s_1,y_4-y_5)\, p(s_2-s_4,y_2-y_4) \\ & p(s_3-s_4,y_3-y_4)\, p(s_2-s_1,y_2-y_1)\, p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3) \end{aligned}$$

and

$$\begin{aligned} J_6^{\varepsilon,\varepsilon'}(z,z') \;:=\; & \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \int dy_5 \\ & p(s_4-s_1,y_4-y_5)\, p(s_2-s_4,y_2-y_4)\, p(s_3-s_4,y_3-y_4) \\ & p(s_2-s_1,y_2-y_1)\, p(s_3-s_1,y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3). \end{aligned}$$

We now compute $J_6^{\varepsilon,\varepsilon'}$. Integrating over $dy_1$, $dy_5$, and $dy_4$ gives

$$\begin{aligned} J_6^{\varepsilon,\varepsilon'}(z,z') \;=\; & \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_2 \int dy_3 \\ & p(s_2+s_3-2s_4,y_2-y_3)\, p(s_2+s_3-2s_1,y_2-y_3)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3). \end{aligned}$$

The function

$$H_6(y_2,y_3) := \int_0^t ds_1 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3\; p(s_2+s_3-2s_4,y_2-y_3)\, p(s_2+s_3-2s_1,y_2-y_3)$$

is continuous in $(y_2,y_3)$ and bounded from above by $H_6(y,y)=K_6$ which is finite since $d=2$. Thus $J_6^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by $K_6$. Using that $f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3) dy_2 dy_3$ converges weakly to $\delta_z(dy_2)\delta_{z'}(dy_3)$, we deduce that $J_6^{\varepsilon,\varepsilon'}$ converges to

$$J_6^0(z,z') \;:=\; H_6(z,z') \;\le\; K_6. \tag{29}$$

Note that $H_6(z,z)=\infty$ if $d=3$, which implies that $J^{\varepsilon,\varepsilon'}(z,z)$ doesn't converge for $d=3$, however it is well-defined at least for $f(x)=f'(x)=p(1,x)$.

For $J_5^{\varepsilon,\varepsilon'}$ we get

$$\begin{aligned} J_5^{\varepsilon,\varepsilon'}(z,z') \;=\; &\int_0^t ds_1 \int_0^{s_1} ds_5 \int_{s_1}^t ds_4 \int_{s_4\vee s}^t ds_2 \int_{s_4\vee s}^t ds_3 \int dy_1 \int dy_2 \int dy_3 \int dy_4 \\ &p(s_1+s_4-2s_5, y_1-y_4)\, p(s_2-s_4, y_2-y_4)\, p(s_3-s_4, y_3-y_4) \\ &p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1)\, f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3). \end{aligned}$$

We set

$$\begin{aligned} &h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3) \\ &:= \mathbf{1}_{0<s_5<s_1<s_4<s_2\wedge s_3} \int dy_1 \int dy_4\; p(s_1+s_4-2s_5, y_1-y_4)\, p(s_2-s_4, y_2-y_4)\, p(s_3-s_4, y_3-y_4) \\ &\qquad\qquad p(s_2-s_1, y_2-y_1)\, p(s_3-s_1, y_3-y_1), \end{aligned}$$

and

$$H_5(y_2,y_3) \;:=\; \int ds_1 \int ds_2 \int ds_3 \int ds_4 \int ds_5\; \mathbf{1}_{s<s_2,s_3<t}\, h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3),$$

so that

$$J_5^{\varepsilon,\varepsilon'}(z,z') = \int dy_2 \int dy_3\; f_{\varepsilon,z}(y_2)\, f_{\varepsilon',z'}(y_3) H_5(y_2,y_3).$$

Let us now prove that $H_5$ is bounded and continuous. Note first that $p(s_1+s_4-2s_5, y_1-y_4) \le p(s_1+s_4-2s_5, 0)$. Thus, we easily get

$$\begin{aligned} &h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3) \\ &\qquad \le\; \mathbf{1}_{0<s_5<s_1<s_4<s_2\wedge s_3}\, p(s_1+s_4-2s_5,0)\, p(s_2+s_3-2s_4,0)\, p(s_2+s_3-2s_1,0). \end{aligned}$$

Now it is easy to check that

$$\begin{aligned} H_5(y_2,y_3) &= \int ds_1 \cdots \int ds_5\; \mathbf{1}_{s\,<\,s_2,s_3\,<\,t}\, h_5(s_1,s_2,s_3,s_4,s_5,y_2,y_3) \\ &\le \int_s^t ds_2 \int_s^t ds_3 \int_0^{s_2\wedge s_3} ds_4 \int_0^{s_4} ds_1 \int_0^{s_1} ds_5 \\ &\qquad p(s_1+s_4-2s_5,0)\, p(s_2+s_3-2s_4,0)\, p(s_2+s_3-2s_1,0) \;=\; K_5 < \infty. \end{aligned}$$

The function $h_5$ is continuous and bounded in $(y_2,y_3)$. From dominated convergence we deduce that $H_5$ is continuous and bounded. Using that $f_{\varepsilon,z}(y_2)f'_{\varepsilon',z'}(y_3)dy_2dy_3$ converges weakly to $\delta_z(dy_2)\delta_{z'}(dy_3)$, we see that $J_5^{\varepsilon,\varepsilon'}$ tends to

$$J_5^0(z,z') \;:=\; H_5(z,z') \le K_5 \tag{30}$$

when $\varepsilon$ and $\varepsilon'$ decrease to 0. Note that $J_5^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by $K_5$.

Finally, we study $J_4^{\varepsilon,\varepsilon'}$. Let $g \in \mathcal{B}_+\big((\mathbb{R}^2)^3\big)$ and $\overline{g}(x_1,x_2,x_3) := \sum_\sigma g(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$, where the sum is over all the permutations $\sigma$ of $\{1,2,3\}$. By a standard monotone class argument we deduce from the moment formula (A.2) for $\varrho$ that

$$\begin{aligned}
&\mathbb{E}\Big[\int \varrho_v(dy_1)\int \varrho_v(dy_4)\int \varrho_v(dy_5)\; g(y_1,y_4,y_5)\Big]\\
&= 2\,i_{\mathrm{c}}\gamma^2 \int_0^v ds_4 \int_{s_4}^v ds_5 \int dy_1 \int dy_4 \int dy_5 \int dy_6\; p(v+s_5-2s_4, y_1-y_6)\\
&\qquad\qquad p(v-s_5,y_4-y_6)\,p(v-s_5,y_5-y_6)\,\overline{g}(y_1,y_4,y_5)\\
&\quad +\; i_{\mathrm{c}}^2\,\gamma \int_0^v ds_4 \int dy_1\int dy_4\int dy_5\; p(2v-2s_4,y_1-y_4)\,\overline{g}(y_1,y_4,y_5)\\
&\quad +\; \frac{1}{3!}\,i_{\mathrm{c}}^3 \int dy_1\int dy_4\int dy_5\; \overline{g}(y_1,y_4,y_5).
\end{aligned}$$

This implies

$$J_4^{\varepsilon,\varepsilon'} \;=\; 2\,i_{\mathrm{c}}\gamma^2 J_7^{\varepsilon,\varepsilon'} + i_{\mathrm{c}}^2\,\gamma J_8^{\varepsilon,\varepsilon'} + \frac{1}{3!}\,i_{\mathrm{c}}^3\,J_9^{\varepsilon,\varepsilon'}, \tag{31}$$

where

$$\begin{aligned}
J_7^{\varepsilon,\varepsilon'}(z,z') \;:=\; 2\int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \int dy_1\int dy_4\int dy_5\int dy_2\int dy_3\int dy_6\\
p(s_2-s_1,y_2-y_4)\,p(s_3-s_1,y_3-y_5)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\\
f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\\
\Big[p(s_1+s_5-2s_4,y_1-y_6)\,p(s_1-s_5,y_4-y_6)\,p(s_1-s_5,y_5-y_6)\\
+\; p(s_1+s_5-2s_4,y_4-y_6)\,p(s_1-s_5,y_1-y_6)\,p(s_1-s_5,y_5-y_6)\\
+\; p(s_1+s_5-2s_4,y_5-y_6)\,p(s_1-s_5,y_1-y_6)\,p(s_1-s_5,y_4-y_6)\Big]
\end{aligned}$$

and

$$\begin{aligned}
J_8^{\varepsilon,\varepsilon'}(z,z') :=2\int_0^t ds_1\int_0^{s_1} ds_4 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int dy_1\int dy_4\int dy_5\int dy_2\int dy_3\; p(s_2-s_1,y_2-y_4)\\
p(s_3-s_1,y_3-y_5)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\\
\Big[p(2s_1-2s_4,y_1-y_4)+p(2s_1-2s_4,y_1-y_5)+p(2s_1-2s_4,y_4-y_5)\Big]
\end{aligned}$$

as well as

$$\begin{aligned}
J_9^{\varepsilon,\varepsilon'}(z,z') := 3!\int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int dy_1\int dy_4\int dy_5\int dy_2\int dy_3\; p(s_2-s_1,y_2-y_4)\\
p(s_3-s_1,y_3-y_5)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\,f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3).
\end{aligned}$$

We are left to study the convergence of $J_9^{\varepsilon,\varepsilon'}$, $J_8^{\varepsilon,\varepsilon'}$ and $J_7^{\varepsilon,\varepsilon'}$.

First of all, we have

$$\begin{aligned}J_9^{\varepsilon,\varepsilon'}(z,z') &= 3!\int_0^t ds_1\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\int dy_2\int dy_3\; p(s_2+s_3-2s_1,y_2-y_3)f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)\\ &\le 6\int_0^t ds_1\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\; p(s_2+s_3-2s_1,0) =: K_9<\infty.\end{aligned}$$

As $\varepsilon$ and $\varepsilon'$ decrease to zero, $J_9^{\varepsilon,\varepsilon'}$ converges to

$$J_9^0(z,z') := 6\int_0^t ds_1\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\; p(s_2+s_3-2s_1,z-z') \le K_9. \tag{32}$$

Next, we have

$$J_8^{\varepsilon,\varepsilon'}(z,z') = \int dy_2\int dy_3\; f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)H_8(y_2,y_3),$$

where

$$\begin{aligned}H_8(y_2,y_3) &= 2\int_0^t ds_1\int_0^{s_1} ds_4\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\\ &\quad\Big[\int dy_1\; p(s_2+s_1-2s_4,y_1-y_2)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\\ &\quad+\int dy_1\; p(s_3+s_1-2s_4,y_3-y_1)\,p(s_2-s_1,y_2-y_1)\,p(s_3-s_1,y_3-y_1)\\ &\quad+\; p(s_2+s_3-2s_1,y_2-y_3)\,p(s_2+s_3-2s_4,y_3-y_2)\Big].\end{aligned}$$

Since $p(s_2+s_1-2s_4,y_1-y_2)\le p(s_2+s_1-2s_4,0)$ and $p(s_3+s_1-2s_4,y_3-y_1)\le p(s_3+s_1-2s_4,0)$, we deduce that

$$\begin{aligned}H_8(y_2,y_3) &\le 6\int_0^t ds_1\int_0^{s_1} ds_4\int_{s_1\vee s}^t ds_2\int_{s_1\vee s}^t ds_3\, p(s_2+s_1-2s_4,0)\,p(s_2+s_3-2s_1,0)\\ &= K_8<\infty.\end{aligned}$$

Arguments similar to those used for the convergence of $J_5^{\varepsilon,\varepsilon'}$ show that $H_8$ is continuous and bounded. Thus $J_8^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by $K_8$ and converges to

$$J_8^0(z,z') := H_8(z,z') \le K_8. \tag{33}$$

Finally, we have

$$J_7^{\varepsilon,\varepsilon'}(z,z') = \int dy_2\int dy_3\; f_{\varepsilon,z}(y_2)\,f'_{\varepsilon',z'}(y_3)H_7(y_2,y_3),$$

where

$$
\begin{aligned}
H_7(y_2,y_3) \ := 2\int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \int dy_1 \int dy_6 \\
\Big[ p(s_2-s_5,y_2-y_6)\, p(s_3-s_5,y_3-y_6)\, p(s_2-s_1,y_2-y_1) \\
p(s_3-s_1,y_3-y_1)\, p(s_1+s_5-2s_4,y_1-y_6) \\
+\ p(s_2+s_5-2s_4,y_2-y_6)\, p(s_3-s_5,y_3-y_6)\, p(s_2-s_1,y_2-y_1) \\
p(s_3-s_1,y_3-y_1)\, p(s_1-s_5,y_1-y_6) \\
+\ p(s_2-s_5,y_2-y_6)\, p(s_3+s_5-2s_4,y_3-y_6)\, p(s_2-s_1,y_2-y_1) \\
p(s_3-s_1,y_3-y_1)\, p(s_1-s_5,y_1-y_6) \Big].
\end{aligned}
$$

Check now that the following upper bound is finite:

$$
\begin{aligned}
K_7 \ := 2\int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3 \int_0^{s_1} ds_4 \int_{s_4}^{s_1} ds_5 \\
\Big[ p(s_2+s_3-2s_5,0)\, p(s_2+s_3-2s_1,0)\, p(s_1+s_5-2s_4,0) \\
+\ p(s_2+s_5-2s_4,0)\, p(s_1+s_3-2s_5,0)\, p(s_2+s_3-2s_1,0) \\
+\ p(s_3+s_5-2s_4,0)\, p(s_1+s_2-2s_5,0)\, p(s_2+s_3-2s_1,0) \Big].
\end{aligned}
$$

Arguments similar to those used for the convergence of $J_5^{\varepsilon,\varepsilon'}$ show that $H_7$ is continuous and bounded by $K_7 < \infty$. Thus $J_7^{\varepsilon,\varepsilon'}(z,z')$ is uniformly bounded by $K_7$ and converges to

$$
J_7^0(z,z') \ := \ H_7(z,z') \ \le \ K_7. \tag{34}
$$

Altogether, for each $i \in \{1,\dots,9\}$, $J_i^{\varepsilon,\varepsilon'}$ exists, is uniformly bounded and has a finite limit as $(\varepsilon,\varepsilon') \downarrow 0$. Thus, $J^{\varepsilon,\varepsilon'}(z,z')$ is well-defined and converges in $\mathbb{R}_+$ as $\varepsilon$ and $\varepsilon'$ decrease to 0. □

*Completion of the proof of* (a). The claimed expectation expression for $\lambda_{[s,t]}(z)$ easily follows from the moment formula (9) for $L$ in the case $m = 1$.

The second moment of $\lambda_{[s,t]}(z)$ is given by the limit $J^0$, say, of $J^{\varepsilon,\varepsilon}(z,z)$ from Lemma 9 as $\varepsilon \downarrow 0$. By the formulas (25), (27), (28), and (31),

$$
J^0 \ = \ 2\,i_{\mathrm{r}}\kappa\left[2\gamma\left(2\gamma\, i_{\mathrm{c}}\, J_5^0 + i_{\mathrm{c}}^2\, J_6^0\right) + \left\{2\, i_{\mathrm{c}}\gamma^2 J_7^0 + i_{\mathrm{c}}^2\,\gamma J_8^0 + \frac{1}{3!}\, i_{\mathrm{c}}^3\, J_9^0\right\}\right] + i_{\mathrm{r}}^2 J_2^0 \ < \ \infty \tag{35}
$$

which, in the case $s < t$, is strictly larger than $\left(\mathbf{E}[\lambda_{[s,t]}(z)]\right)^2$, occurring from the $J_2^0$-term [see (26)]. This completes the proof of (a). □

**Remark 10** (variance formula). For $t \ge s \ge 0$ and $z \in \mathbb{R}^d$, from the representation (35) combined with (30), (29), (34), (33), (32), and (26), as well as the expectation formula in (a), we obtain the following formula for the *variance of* $\lambda_{[s,t]}(z)$ :

$$
\begin{aligned}
&2 i_{\mathrm{c}}\, i_{\mathrm{r}} \left(i_{\mathrm{c}}^2\, \kappa + i_{\mathrm{r}}\gamma\right) \int_0^t ds_1 \int_{s_1\vee s}^t ds_2 \int_{s_1\vee s}^t ds_3\; p(s_2+s_3-2s_1,0)\\
&+\; 8\, i_{\mathrm{c}}^2\, i_{\mathrm{r}} \gamma\, \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2\vee s}^t ds_3 \int_{s_2\vee s}^t ds_4\; p(s_3+s_4-2s_2,0)\, p(s_3+s_4-2s_1,0)\\
&+\; 8\, i_{\mathrm{c}}^2\, i_{\mathrm{r}} \gamma\, \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2\vee s}^t ds_3 \int_{s_2\vee s}^t ds_4 \int dy\; p(s_2+s_3-2s_1,y)\, p(s_3-s_2,y)\\
&\qquad\qquad p(s_4-s_2,y)\\
&+\; 16\, i_{\mathrm{c}}\, i_{\mathrm{r}} \gamma^2\, \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2}^t ds_3 \int_{s_3\vee s}^t ds_4 \int_{s_3\vee s}^t ds_5 \int dy_1 \int dy_2\\
&\qquad\qquad p(s_2+s_3-2s_1,y_1-y_2)\, p(s_4-s_2,y_1)\, p(s_4-s_3,y_2)\\
&\qquad\qquad p(s_5-s_2,y_1)\, p(s_5-s_3,y_2)\\
&+\; 16\, i_{\mathrm{c}}\, i_{\mathrm{r}} \gamma^2\, \kappa \int_0^t ds_1 \int_{s_1}^t ds_2 \int_{s_2}^t ds_3 \int_{s_3\vee s}^t ds_4 \int_{s_3\vee s}^t ds_5 \int dy_1 \int dy_2\\
&\qquad\qquad p(s_2+s_4-2s_1,y_2-y_1)\, p(s_4-s_3,y_2)\, p(s_3-s_2,y_1)\\
&\qquad\qquad p(s_5-s_2,y_2-y_1)\, p(s_5-s_3,y_2).
\end{aligned}
$$

◇

4.2. **Spatial absolute continuity [proof of (b)].** We first prove that,

$$
\phi_q(x) \int L\left(d[r,y]\right) \mathbf{1}_{[s,t]}(r)\, p(\varepsilon, x-y), \tag{36}
$$

converges in $L^1(\ell\otimes \mathbf{P})$ as $\varepsilon$ decreases to 0, to $\phi_q(x)\xi(x)$, where for almost every $x$, $\mathbf{P}$-a.s. $\xi=\lambda$. Thanks to the statement (a), it is enough to check that the function

$$
(x,\varepsilon) \mapsto \mathbf{E}\left[\int L\left(d[r,y]\right) \mathbf{1}_{[s,t]}(r)\, p(\varepsilon, x-y)\right], \tag{37}
$$

is uniformly bounded on $\mathbb{R}^2\times(0,1]$. But this is clear since

$$
\begin{aligned}
\mathbf{E}\left[\int L\left(d[r,y]\right) \mathbf{1}_{[s,t]}(r)\, p(\varepsilon,x-y)\right] &= \mathbf{E}\left[\int_s^t dr\; i_{\mathrm{r}} \int dz \int \varrho_r(dy)\, p(r,z-y)\, p(\varepsilon,x-y)\right]\\
&= i_{\mathrm{r}} i_{\mathrm{c}}\, (t-s).
\end{aligned}
$$

Statement (b) is then a straight forward consequence of the following criterion with $\nu(dy) = L\left([s,t],dy\right)$ [recall Theorem 3 (i)].

**Proposition 11** (sufficient criterion for absolute continuity). *Let $\nu\in\mathcal{M}_q$ be a random variable defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$. We assume that $\mathcal{E}\left[(\nu,\phi_q)\right]<\infty$ and that*

$$
\left((x,\omega)\;\mapsto\;\phi_q(x)\int \nu(dy)\; p(\varepsilon,x-y),\quad \varepsilon>0\right) \tag{38}
$$

*converges in* $L^1(\ell\otimes\mathcal{P})$ *to some* $\phi_q\xi$ *as* $\varepsilon\downarrow 0$. *Then* $\mathcal{P}$*-a.s. the measure* $\nu$ *is absolutely continuous (with respect to the Lebesgue measure) and has the density function* $\xi$ :

$$\nu(dy) \;=\; \xi(y)dy. \tag{39}$$

*Proof.* Let $\beta$ be any bounded random variable on $(\Omega,\mathcal{F},\mathcal{P})$, and $f\in\mathcal{B}^q$ continuous. Because of the assumed convergence in $L^1(\ell\otimes\mathcal{P})$, we get that

$$J_\varepsilon \;:=\; \int\! dx\, f(x)\,\mathcal{E}\left[\beta\int\!\nu(dy)\; p(\varepsilon,x-y)\right]$$

converges to $\int dx\, f(x)\,\mathcal{E}\,[\beta\,\xi(x)]$ as $\varepsilon\downarrow 0$. On the other hand, the function

$$(y,\varepsilon)\mapsto\int\! dx\, f(x)\,p(\varepsilon,x-y)$$

is bounded by $\phi_q(y)$ [thanks to (A.1)], continuous and converges to $f$ as $\varepsilon\downarrow 0$. By dominated convergence, we get that $J_\varepsilon$ converges to $\mathcal{E}\,[\beta\,(\nu,f)]$. Since $\beta$ and $f$ are arbitrary, the equality

$$\int\! dx\, f(x)\,\mathcal{E}\,[\beta\,\xi(x)] \;=\; \mathcal{E}\,[\beta\,(\nu,f)]$$

implies that $\nu$ is $\mathcal{P}$-a.s. absolutely continuous with respect to the Lebesgue measure, and that $\nu(dy)=\xi(y)dy$, $\mathcal{P}$-a.s. □

4.3. **Random ergodic limit [proof of (d)].** Let $f\in L^1_+(\mathbb{R}^2)$. Thanks to Lemma 9, we know that $T^{-1}\int_{[0,T]\times\mathbb{R}^2} L\left(d[r,y]\right)f(y)$ is well-defined and even belongs to $L^2(\mathbf{P}\,)$. By self-similarity this has the same law as

$$I_T \;=\; T\int_{[0,1]\times\mathbb{R}^2} L\left(d[r,y]\right)f(y\sqrt{T}).$$

Thanks to Lemma 9 and (a), we see that $I_T$ converges in $L^2(\mathbf{P}\,)$ to $\lambda_{[0,1]}(0)\int dx\,f(x)$ as $T\uparrow\infty$. Thus we deduce that for any $f\in L^1_+(\mathbb{R}^2)$, the following convergence in law holds with respect to $\mathbb{P}$ :

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_{[0,T]\times\mathbb{R}^2} L\left(d[r,y]\right)f(y) \;=\; \lambda_{[0,1]}(0)\int\! dx\; f(x).$$

This ends the proof of (d). □

## A. Appendix: Some basic properties of catalyst and reactant

A.1. **Moment formulas for the catalyst.** Let $d\geq 1$ and fix $\nu\in\mathcal{M}_q$. It is easy to check that for every $T>0$, there exists a constant $c>0$ such that for every $x\in\mathbb{R}^d$ and $\varepsilon\in(0,T]$,

$$\int\! dy\; p(\varepsilon,x-y)\,\phi_q(y) \;\leq\; c\,\phi_q(x). \tag{A.1}$$

Therefore we get that if $g\in H^q_T$, then the function $(r,x)\mapsto\int_r^\infty ds\; P_{s-r}[g(s)](x)$ is well-defined and belongs to $H^q_T$. If $f\in\mathcal{B}^q$, then the function $(r,x)\mapsto\mathbf{1}_{t\geq r}\;P_{t-r}[f](x)$ is also well-defined and belongs to $H^q_t$.

It is well-known that for every $t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$\mathbb{E}_\nu \left[ \left[ (\varrho_t\,, f) + \int_0^\infty ds\ \left( \varrho_s\,, g(s) \right) \right]^m \right] \;=\; m! \sum_{k=1}^m \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1, \\ n_1+\cdots+n_k = m}} \prod_{i=1}^k \left( \nu, \chi_{n_i}(0) \right), \tag{A.2}$$

where the sequence $(\chi_n\,, n \geq 1)$ is defined by the recurrence formula

$$\chi_n(r,x) \;:=\; \gamma \int_r^\infty ds \int dy\ p(s-r, x-y) \left[ \sum_{i=1}^{n-1} \chi_i(s,y)\, \chi_{n-i}(s,y) \right], \tag{A.3}$$

$(r,x) \in \mathbb{R}_+ \times \mathbb{R}^d,\ n \geq 2$, with initial condition

$$\chi_1(r,x) \;:=\; \mathbf{1}_{t \geq r}\ P_{t-r}[f](x) + \int_r^\infty ds\ P_{s-r}[g(s)](x), \qquad (r,x) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{A.4}$$

Thanks to the remark at the beginning of this subsection, we see that the functions $\chi_n\,, n \geq 1$, are well-defined and belong to $H^q$.

A.2. **Regularity properties of the catalyst.** We now assume that $d \leq 3$. Recall that we write $\mathbb{P}$ for $\mathbb{P}_{i_c\ell}$. It is clear from the Hölder continuity Theorem 3 of [DF97a] (p254) that for every $\xi \in (0, 1/4)$, $T \geq 0$, $\mathbb{P}$-a.s. there exists a constant $C_1 := C(T, \varrho, \xi)$ such that for every $T \geq t \geq r \geq 0$, $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\int_r^t ds \int \varrho_s(dz)\, \phi_q(z) f(z) \;\leq\; C_1\, |t-r|^\xi \int f(z) dz. \tag{A.5}$$

We have also [cf. Definition 2 b) and Theorem 4 of [DF97a], pp 224 and 259, respectively] that for every $T \geq 0$, $\xi \in (0, 1/4)$, $\mathbb{P}$-a.s. there exists $C_2 := C(T, \varrho, \xi)$ such that for every $x \in \mathbb{R}^d$, $T \geq t \geq r \geq 0$,

$$\int_r^t ds \int \varrho_s(dz)\, p(s-r, x-z)\, \phi_q^2(z) \;\leq\; C_2\, |t-r|^\xi\, \phi_q(x). \tag{A.6}$$

A.3. **Moment formulas for the reactant.** Recall that $d \leq 3$. Using the Markov property of $X^\varrho$ (given $\varrho$), it is easy to get that $\mathbb{P}$-a.s. for every $n \geq 1$, $t_n \geq \cdots \geq t_1 \geq 0$, and $f_n\,, \cdots, f_1 \in \mathcal{B}_+^q$,

$$\mathrm{E}_{r,\mu}^\varrho \left[ \mathrm{e}^{-\sum_{t_i \geq r} (X_{t_i}^\varrho, f_i)} \right] \;=\; \mathrm{e}^{-\left(\mu, v(r)\right)}, \tag{A.7}$$

where $v$ is the unique nonnegative solution of the catalytic log-Laplace equation (7) with $J(s) := \sum_{t_i \geq s} P_{t_i - s}\,[f_i]$. Using the continuity of $X^\varrho$, it can be shown that $\mathbb{P}$-a.s. for every nonnegative $g \in H^q$,

$$\mathrm{E}_{r,\mu}^\varrho \left[ \mathrm{e}^{-\int_r^\infty ds\ \left( X_s^\varrho, g(s) \right)} \right] \;=\; \mathrm{e}^{-\left(\mu, v(r)\right)}, \tag{A.8}$$

where $v$ is the unique nonnegative solution of (7) with $J(s) := \int_s^\infty du\ P_{u-s}\,[g(u)]$.

We deduce the next result on the *moments of the reactant process* $X^\varrho$ from Theorem 4, Lemma 4 and Remark 2 of [DF97a] (pp 259 and 232, respectively). We have $\mathbb{P}$-a.s. for every

$t \geq 0$, $g \in H^q$, $f \in \mathcal{B}^q$, and $m \geq 1$,

$$\text{(A.9)} \qquad \mathrm{E}^{\varrho}_{r,\mu}\left[\left[(X^{\varrho}_t, f) + \int_r^{\infty} ds \ \left(X^{\varrho}_s, g(s)\right)\right]^m\right] = m! \sum_{k=1}^{m} \frac{1}{k!} \sum_{\substack{n_1,\dots,n_k \geq 1, \\ n_1+\cdots+n_k = m}} \prod_{i=1}^{k} (\mu, \chi_{n_i}(r)),$$

where $(\chi_n, n \geq 1)$ is defined by the recurrence formula (10) with initial condition

$$\text{(A.10)} \qquad \chi_1(s,x) := \mathbf{1}_{t \geq s} P_{t-s}[f](x) + \int_s^{\infty} du \ P_{u-s}[g(u)](x) \qquad (s,x) \in \mathbb{R}_+ \times \mathbb{R}^d.$$

Since $\chi_1 \in H^q$, inequality (A.6) implies that all the functions $\chi_n$ belong to $H^q$.

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MSRI, 1000 Centennial Drive, Berkeley, CA 94720, U.S.A., and ENPC-CERMICS, 6 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France.

*E-mail address*: `delmas@msri.org`

*URL*: `http://cermics.enpc.fr/~delmas/home.html`

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D-10117 Berlin, Germany

*E-mail address*: `fleischmann@wias-berlin.de`

*URL*: `http://www.wias-berlin.de/Klaus.Fleischmann`