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A RIGOROUS RENORMALIZATION GROUP METHOD FOR INTERFACES IN RANDOM MEDIA

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Abstract: We prove the existence Gibbs states describing rigid interfaces in a disordered solid-on-solid (SOS) for low temperatures and for weak disorder in dimension $D \geq 4$. This extends earlier results for hierarchical models to the more realistic models and proves a long-standing conjecture. The proof is based on the renormalization group method of Bricmont and Kupiainen originally developed for the analysis of low-temperature phases of the random field Ising model. In a broader context, we generalize this method to a class of systems with non-compact single-site state space.

Key Words: Disordered systems, interfaces, SOS-model, renormalization group, contour models

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I. Introduction

In 1988 a remarkable article by Bricmont and Kupiainen [BK1] settled the long-standing dispute on the lower critical dimension of the random field Ising model through a rigorous mathematical proof of the existence of at least two phases at low temperatures in dimension three and above (the less disputed absence of a phase transition in dimension two was later proven by Aizenman and Wehr [AW]). Their proof was based on a renormalization group (RG) analysis that clearly should provide a valuable tool for the investigation of the low temperature phase of disordered systems in general. Unfortunately, the technical complexity of this approach has so far prevented more wide-spread applications, a notable exception being the proof by the same authors [BK2] of the diffusive behaviour of random walks in asymmetric random environments in dimensions greater than two.

Another problem of considerable interest that invites an application of this method is that of the stability of interfaces in random media; one may think in particular of domain walls in random bond or random field Ising models. In a series of articles [BoP,BoK1,BoK2] a hierarchical approximation of such interface models has been investigated; the purpose of the present paper is to go beyond this hierarchical approximation and to analyse the physically more realistic solid-on-solid (SOS) model. We emphasize that the analysis of the hierarchical models shed considerable light on some aspects of this problem, in particular the more probabilistic ones, and has helped us in finding our way through the full model. We recommend reading of in particular Ref. [BoK1] as a warm-up before entering the technical parts of the present work. This reference also contains a fairly detailed introduction into the physical background and heuristic arguments which we prefer not to repeat here again in order to keep down the size of the present paper. For even more physical background on interfaces in random systems, we recommend the review by Forgacs et al. in Domb and Lebowitz Vol. 14 [FLN].

As is to be expected, the analysis of the interface model is in several respects considerable more complicated than that of the random field model; however, sometimes it is the case that added complications entail more clarity: it is our hope to convince the reader of the enormous virtues of this approach and of its conceptual clarity – and even simplicity – and in particular of its wide applicability and flexibility. From this point of view, we would like see the present work in a broader context as a generalization of the RG method for the analysis of the low-temperature phase of disordered systems to models with possibly non-compact single site state space. With this in mind, we have tried to give a fairly detailed and, hopefully, somewhat pedagogical exposition, emphasizing the conceptual ideas and presenting the method in more detail than has been done in [BK].

In presenting our approach we have chosen to stick to a concrete model and show how the RG method can be used to solve it rather than to aim directly at more generally valid results. Overall, we have tried to stress the physically relevant ideas and keep the level of mathematical abstraction as low as compatible with rigour. This is clearly to some extent a matter of taste, but we hope that this choice will make our work more accessible to a wider audience.

We would like to mention that another approach to the low temperature phase of disordered systems has recently been announced by Zahradnik [Za1]. This approach is based on the Pirogov-Sinai theory and aims at dealing with systems with finite spin space but possibly asymmetric ground states (like q-state Potts models). Although full details of this method have not yet been published, it is our believe that the two techniques are not incompatible and that an 'ultimate theory' of the low temperature disordered systems may be obtained by melting together these methods.

Let us now describe the model we want to analyse. A SOS-surface is described by a family of heights, $\{h_x\}_{x\in\mathbb{Z}^d}$, where h_x takes values in $\mathbb{Z}$. The Hamiltonian, that describes the energy difference between the 'flat' surface ($h_x \equiv 0$) and an arbitrary one is formally given by

$$H(h) = \sum_{x,y\in\mathbb{Z}^d:|x-y|=1} |h_x - h_y| + \sum_{x\in\mathbb{Z}^d} J_x(h_x) \tag{1.1}$$

where $J_x(h)$ are random variables that describe the disorder in the system. We will generally assume that for $x \neq x'$, $\{J_x(h)\}_{h\in\mathbb{Z}}$ and $\{J_{x'}(h)\}_{h\in\mathbb{Z}}$ are independent stochastic sequences with identical distributions The properties of the stochastic sequences $\{J_x(h)\}_{h\in\mathbb{Z}}$ themselves depend on the particular physical system under consideration. Two particular examples were highlighted in our previous work [BoP,BoK1,BoK2]:

(i) (Random bond model) The distribution of the sequence $\{J_x(h)\}_{h\in\mathbb{Z}}$ is stationary with respect to translations $h \to h+k$, $k \in \mathbb{Z}$. The marginal distributions satisfy gaussian bounds of the form

$$\mathbb{P}\left(|J_x(h)| > \epsilon\right) \leq e^{-\frac{\epsilon^2}{2\sigma^2}} \tag{1.2}$$

and the $J_x(h)$ are centered, i.e.

$$\mathbb{E} J_x(h) = 0 \tag{1.3}$$

In fact, one may think of the $J_x(h)$ as sequences of i.i.d. random variables. However, it turns out in the proofs that independence is unessential and impossible to maintain in the course of renormalization, while stationarity is an important invariant property.

(ii) (Random field model) Here, a priori the $J_x(h)$ should be thought of as sums of i.i.d. random variables. But again, this is not a property that is maintained under renormalization and is replaced by a weaker condition: Let $D_x(h,h') \equiv J_x(h) - J_x(h')$. Then the distribution of the stochastic array $\{D_x(h,h')\}_{h,h'\in\mathbb{Z}}$ is invariant under the diagonal translations $(h,h') \to$

$(h+k, h'+k)$, $k \in \mathbb{Z}$,

$$\mathbb{E} D_x(h,h') = 0 \tag{1.4}$$

and the marginals satisfy gaussian bounds of the form

$$\mathbb{P}\left(D_x(h,h') > \epsilon\right) \leq e^{-\frac{\epsilon^2}{2\sigma^2|h-h'|}} \tag{1.5}$$

For further physical motivations of these choices we refer to our previous articles. Let us remark that the hamiltonian (1.1) differs from the one of the d-dimensional random field Ising model essentially only in that the variables h take values in $\mathbb{Z}$ rather than $\{-1,1\}$ (this has been observed in [BFG]) which fact suggests the application of the techniques of [BK1]. In the present paper we will actually consider only the case (i); the details in the case (ii) may be found in [K].

Our aim is to prove that, for $d \geq 3$, at low temperature and for small σ, there exist infinite volume Gibbs states corresponding to the Hamiltonian (1.1) describing surfaces with everywhere finite heights, for almost all realizations of the disorder. To be more precise, let us denote by $\Omega \equiv \mathbb{Z}^{\mathbb{Z}^d}$ the configuration space and Σ the Borel sigma-algebra of Ω. For any finite subset $\Lambda \subset \mathbb{Z}^d$, we set $\Omega_\Lambda \equiv \mathbb{Z}^\Lambda$ and denote by Σ_Λ the sigma algebra generated by the functions $h_x, x \in \Lambda$. For any configuration $h \in \Omega$ we write h_Λ, h_{Λ^c} for the restrictions of the function h to Λ and Λ^c, respectively. For two configurations h and η we write $(h_\Lambda, \eta_{\Lambda^c})$ for the element of Ω for which

$$(h_\Lambda, \eta_{\Lambda^c})_x = \begin{cases} h_x & \text{, if } x \in \Lambda \\ \eta_x & \text{, if } x \notin \Lambda \end{cases} \tag{1.6}$$

We set, for any finite volume Λ

$$H_{J,\Lambda}(h_\Lambda, \eta_{\Lambda^c}) \equiv \sum_{x,y\in\Lambda:|x-y|=1} |h_x - h_y| + \sum_{\substack{x\in\Lambda, y\in\Lambda^c \\ |x-y|=1}} |h_x - \eta_y| + \sum_{x\in\Lambda} J_x(h_x) \tag{1.7}$$

This is of course always a finite sum. The *local specifications* (or *finite volume Gibbs measures*) are probability kernels on Ω such that for any Σ-measurable function f,

$$\mu^\eta_{\Lambda,\beta,J}(f) \equiv \frac{1}{Z^\eta_{\Lambda,\beta,J}} \int_{\Omega_\Lambda} f(h_\Lambda, \eta_{\Lambda^c}) e^{-\beta H_{J,\Lambda}(h_\Lambda,\eta_{\Lambda^c})} dh_\Lambda \tag{1.8}$$

where dh_Λ denotes the counting measure on Ω_Λ. The constant $Z^\eta_{\Lambda,\beta,J}$ is a normalization constant chosen such that $\mu^0_{\Lambda,\beta,J}(1) = 1$, usually called the *partition function.* Measures $\mu^\eta_{\beta,J}$ on (Ω,Σ) are in fact called *Gibbs measures* for η, if for all finite Λ, this measure conditioned on η_{Λ^c} coincides with $\mu^\eta_{\Lambda,\beta,J}$ (these are the so-called DLR-equations (see [Ge]). More important for us is the fact that (at least) the *extremal* Gibbs measures can be constructed as weak limit points of sequences $\mu^\eta_{\Lambda_n,\beta,J}$, for sequences Λ_n that increase to $\mathbb{Z}^d$ [Ge]. The problem of statistical mechanics is then to investigate the structure of the set of these limit points. Here, however, our ambitions are somewhat more modest: we want to show that for constant configurations $\eta_x \equiv k$, $k \in \mathbb{Z}$, and suitable sequences of volumes Λ_n, the sequences of measures $\mu^k_{\Lambda_n,\beta,J}$ converge to a limiting measure, for almost all J. It should be noted that for our models not even the existence of a limit point is a non-trivial question, since a priori a sequence of probability measures on Ω need not converge to a measure, due to the non-compactness of the space $\mathbb{Z}$! (As an example for such a situation, take the sequence of probability measures ρ_n on $\mathbb{Z}$, which assign mass $1/n$ to the atoms $\{1,\ldots,n\}$ and mass zero to all others. Clearly this sequence has no limit point in the space of measures (cf. [CT] chap. 1.5, ex.6)).

Finally we must mention that all the objects introduced above are of course random variables on some underlying probability space $(\Theta, \mathcal{F}, \mathbb{P})$ on which the $J_x(h)$ are defined. It should be noted in particular that due to the definition of $H_{J,\Lambda}$, the local specifications $\mu^\eta_{\Lambda,\beta,J}$ are measurable w.r.t. the sigma algebras $\mathcal{F}_\Lambda$ (the sub-sigma algebras generated by the functions $\{J_x(h)\}_{x\in\Lambda}$). Care should be taken that then limits are taken, neither Λ_n nor η should depend on J. It is frequently possible to produce pathological results by choosing random boundary conditions[1]. The central result of this paper is then the following

THEOREM 1: *Let $d \geq 3$ and assume that the random variables $J_x(h)$ satisfy the conditions detailed under (i). Let Λ_n denote cubes of side-length L^n centered at the origin. Then there exists $\beta_0 < \infty$, $\sigma_0 > 0$, such that for all $\beta \geq \beta_0$ and $\sigma \leq \sigma_0$, and for suitably chosen integer L, the sequence of measures $\mu^k_{\Lambda_n,\beta,J}$ converges to a unique Gibbs measure $\mu^k_{\beta,J}$, for $\mathbb{P}$-almost all J, and for $k \neq k'$, $\mu^k_{\beta,J}$ and $\mu^{k'}_{\beta,J}$ are disjoint.*

Remark: The condition that the sequence of volumes be a sequence of cubes is only made to simplify some technical aspects of the proof. It is not difficult to prove the theorem for any (non-random) sequence of increasing and absorbing volumes. The measures constructed in Theorem 1 are clearly the only extremal Gibbs measures corresponding to 'translational invariant' boundary conditions. To analyse the full structure of the set of Gibbs measures remains an interesting, but difficult question.

Before entering the details of the proof of this theorem, we would like to explain some of the main ideas and features of the RG approach. As always in statistical mechanics, the principal idea is to find a way of arranging the summations involved in the expression (1.8) for the local specifications in a suitable way as a convergent sum. In the low temperature phase, the usual way of doing this is by first finding the ground states (minima of H_Λ) and then representing all other configurations as (local) deformations of these ground states (often called 'contours' or 'Peierls

[1] Newman and Stein [NS] have recently investigated interesting phenomena of this type in the context of spin glass models.

contours'). Under favourable circumstances, one may arrange the sum over all these deformations as a convergent expansion ('low temperature expansion'). As opposed to many 'ordered' systems, the first (and in some sense *main*) difficulty in most disordered systems is that the ground state configuration depends in general on the particular realization of the disorder, and, worse, may in principle depend strongly on the shape and size of the finite volume Λ! In our particular model this means that a ground state for the infinite system may not even exist! This latter situation is actually expected to occur in dimensions $d \leq 2$.[2] In dimension $d \geq 3$, we expect, on the contrary, that a ground state in the infinite volume exists and moreover that this ground state itself may be seen as a 'small' deformation of the ground state of the ordered system. This property must, however, be proven in the course of the computation.

The crucial observation that forms the ideological basis for the renormalization group approach is that while for large volumes Λ we have no a priori control on the ground state, for sufficiently small volumes we can give conditions on the random variables J that are fulfilled with large probability under which the ground state in this volume is actually the same as the one without randomness. Moreover, the size of the regions for which this holds true will depend on the variance of the r.v.'s and increases to infinity as the latter decreases. This allows to find 'conditioned' ground states, where the conditioning is on some property of the configuration on this scale (e.g. mean height over a certain region), except in some small region of space. Re-summing then over the fluctuations about these conditioned ground states one obtains a new effective model for the conditions (the coarse grained variables) with effective random variables that (hopefully!!) have smaller variance than the previous ones. In this case, this procedure may be iterated, as now conditioned ground states on a larger scale can be found. This is the basic idea of the renormalization group.

To implement these ideas one has to overcome two major difficulties. The first is that one needs to find a formulation of the model, i.e. a representation of the degrees of freedom and of the interactions that is sufficiently general that its form remains invariant under the renormalization group transformation. There has been an extensive discussion recently in the literature (see [EFS]) on some 'pathological' aspects of the RG that indicates that a 'spin system' like formulation (like (1.1)) will in general be inadequate. We will see that a adequate solution of this problem can be given through a class of contour models. The second, and really the most fundamental difficulty is that the re-summation procedure as indicated above can only be performed outside a small, random region of space, called the 'bad region'. Now while in the first step this may look like no big problem, in the process of renormalization even a very thin region will 'infect' a larger and larger portion of space, if nothing is done. Moreover, in each step some more bad regions are created from

[2] It is expected that the methods of Aizenman and Wehr [AW] used to prove the uniqueness of the Gibbs state in the two-dimensional random field Ising model can be used to prove such a result.

regions in which the new effective random variables have bad properties. This requires to get some control also in the bad regions and to get a precise notion of how regions with a certain degree of badness can be regarded as 'harmless' and be removed on the next scale. For the method to succeed we must then find ourselves in a situation where the bad regions 'die out' over the scales much faster than new ones are produced. This will generally depend on the geometry of the system and in particular on the dimension.

The remainder of this paper is organized in three stages. In the next section we give a more detailed and more specific outline of the renormalization group method. This will serve to expose the conceptual framework and to introduce most of the notation for later use. It should give the reader who may not be bothered with the hard technical work a fairly good idea of what we are doing. Then, in Section III, these ideas are set to work for the analysis of the 'ground states' (i.e. the case of zero temperature) and to prove the corresponding special case of Theorem 1. Here again we have two purposes in mind: First, this case is still considerably less complicated than the case of finite temperature while already exhibiting most of the interesting features. Second, all of the estimates used here are also needed in the more general case and separating those pertinent to the ground states from those related to expansions about them may make things only more transparent. This section also contains (almost) all the probabilistic estimates that then apply (almost) unaltered in the finite temperature case. Section IV finally contains the analysis of the finite temperature Gibbs states and the proof of Theorem 1. In Section V we conclude with some remarks on possible future developments. An appendix contains the proofs of some estimates of geometric nature.

II. The renormalization group and contour models

This section is intended to serve two purposes. First, we want to describe the principal ideas behind the renormalization group approach for disordered systems in the low-temperature regime. We hope to give the reader an outline of what he is to expect before exposing him to the, admittedly, somewhat complicated technical details. Second, we want to present the particular types of contour models on which the renormalization group will act. In this sense the present section introduces the notation for the later chapters. Most of the basic ideas outlined here are contained explicitly or implicitly in [BK].

II.1. The renormalization group for measure spaces

Let us recall first what is generally understood by a renormalization group transformation in a statistical mechanics system. We consider a statistical mechanics system to be given by a probability space (Ω, Σ, μ), where μ is an (infinite volume) Gibbs measure. One may think for the moment of Ω as the 'spin'-state over the lattice $\mathbb{Z}^d$, but we shall need more general spaces later. What we shall, however, assume is that Ω is associated with the lattice $\mathbb{Z}^d$ in such a way that for any finite subset $\Lambda \subset \mathbb{Z}^d$ there exists a subset $\Omega_\Lambda \subset \Omega$ and sub-sigma algebras, Σ_Λ, relative to Ω_Λ that satisfy $\Sigma_\Lambda \subset \Sigma_{\Lambda'}$, if and only if $\Lambda \subset \Lambda'$. Note that in this case any increasing and absorbing sequence of finite volumes, $\{\Lambda_n\}_{n\in\mathbb{Z}_+}$, induces a filtration $\{\Sigma_n \equiv \Sigma_{\Lambda_n}\}_{n\in\mathbb{Z}_+}$ of Σ. It should always be kept in mind that in the situations we are interested in we have, a priori, no explicit knowledge of the measures μ, but only of their local specifications for finite volumes, i.e. the expectations of μ conditioned on Σ_{Λ^c} (finite volume Gibbs measures with 'boundary conditions'). The other important notion that should be kept in mind is that the measures μ are, by Kolmogorov's theorem [Ge], uniquely determined by their values on all cylinder functions on all finite volumes Λ ('local observables').

Ideally, a *renormalization group transformation* is a measurable map, $\mathcal{R}$, that maps $\mathbb{Z}^d \to \mathbb{Z}^d$ and $(\Omega, \Sigma) \to (\Omega, \Sigma)$ in such a way that for any $\Lambda \subset \mathbb{Z}^d$,

(i) $\mathcal{R}(\Lambda) \subset \Lambda$, and moreover $\exists_{n<\infty} : \mathcal{R}^n(\Lambda) = \{0\}$, where n of course may depend on Λ.

(ii) $\mathcal{R}(\Omega_\Lambda) = \Omega_{\mathcal{R}(\Lambda)}$

We will see later that these conditions are slightly too restrictive in general, but for the moment we will stick to them. Note that the use of the same name, $\mathcal{R}$, for the action of renormalization on the lattice and on the space Ω which should not create confusion. The action of $\mathcal{R}$ on space will generally be blocking[3], e.g. $\mathcal{R}(x) = \mathcal{L}^{-1}x \equiv \operatorname{int}(x/L)$. The action on Ω has to be compatible with this blocking but needs to be defined carefully. We should like to stress that in different situations

[3] We call the blocking operator $\mathcal{L}^{-1}$ for historical reasons.

it may be appropriate to use RG maps that are not based on the simple blocking operation $\mathcal{L}^{-1}$.

Having the action of $\mathcal{R}$ on the measurable space (Ω, Σ) we may of course lift it to the measures on (Ω, Σ) via

$$(\mathcal{R}\mu)(\mathcal{A}') = \mu(\mathcal{R}^{-1}(\mathcal{A}')) \tag{2.1}$$

for any Borel-set $\mathcal{A} \in \mathcal{R}(\Sigma)$. The fundamental relation of the renormalization group allows to decompose the measure μ into a conditioned expectation and the renormalized measure on the condition, i.e. for any Borel-set $\mathcal{A} \in \Sigma$ we have

$$\mu(\mathcal{A}) = (\mathcal{R}\mu)(\mu(\mathcal{A}|\cdot)) \equiv \int_{\mathcal{R}(\Omega)} \mu\left(\mathcal{A}|\mathcal{R}^{-1}(\omega')\right)(\mathcal{R}\mu)(d\omega') \tag{2.2}$$

A priori, all of the above are just trivialities. They are useful only if the map $\mathcal{R}$ can be chosen in such a manner that equation (2.2) and its iterates are useful in computing expectations of interest. This requires that the measure $\mathcal{R}\mu$ is 'simpler' in an appropriate sense than the measure μ itself, and that the conditioned expectations are more easy to control at least on a subspace that has large measure w.r.t. $\mathcal{R}\mu$. This has to be verified in explicit computations. An example in which this can be easily carried through in full detail is given for instance by the hierarchical model treated in [BoK2].

So far, we have not made reference to the specific situation in random systems. In a random system, the measure μ is itself random, i.e. is a measure valued random variable on some underlying probability space $(\Theta, \mathcal{F}, I\!\!P)$ that describes the randomness of the system. In such a situation it may – and will – turn out that the specific choice of the renormalization group transformation has to be adapted to the particular realization of the disorder, i.e. will itself have to be a – complicated – random function. In particular, in such a case the renormalization group transformation cannot be simply iterated since after each step the properties of the new measure have to be taken into account when specifying the new map. As a matter of fact we will even go one step further and allow the underlying spaces Ω to be random and to change under the application of the renormalization group map (although this point is to some extend a question of taste and convenience).

A final aspect that should be kept in mind is that of course the renormalized measures (or even their local specifications) can only *in principle* be computed exactly. In practice we must restrict our knowledge to certain bounds, and it is only on these that the renormalization maps may depend.

II.2. Contour models

The concept of *"contours"* has been fundamental in the analysis of low-temperature phases of spin-systems since its introduction by Robert Peierls in 1936 [P]. It formed the basis for the first proof of the existence of a phase transition in the Ising model, and as such, in any model of statistical mechanics. They play the fundamental role in the most powerful modern method to analyse phase transitions, the Pirogov-Sinai theory (see e.g. [Za2]). The basic idea behind contours is to make explicit the region in space where a spin-configuration deviates from a ground state configuration and to use the fact that these regions carry energy proportional to their volume and therefore are suppressed sufficiently to counter their entropy, if the temperature is sufficiently small. It is thus natural that contour models should constitute the proper context for our analysis as well.

As we have said before, the main new problem that arises in disordered systems is that the ground states are very difficult to come by, which seems to make the implementation of the contour idea impossible. However, in sufficiently simple situations – like the one we are studying –, one may guess that for sufficiently weak disorder the ground state should look *almost* like that of the ordered system. It is thus natural to build the contour model on the basis of the 'ideal' ground states and to let the contours themselves keep track of the deviations of the true ground state from these ideal ones. Section III will pinpoint this idea by dealing exclusively with the ground state problem while omitting the added complications of the thermal fluctuations. The basic notion here is that of the *"bad regions"* introduced in [BK1]. They are those regions in space where the randomness *locally* is sufficiently strong to potentially influence the ground state configuration. As long as these regions have a small density, they will be treated in a sense like deviations from ground states and kept track of with the contours.

We will now become more specific and give the precise definitions of contours in our situation.

DEFINITION 2.1: *A contour, Γ, is a pair $(\underline{\Gamma}, h)$, where $\underline{\Gamma}$ is a subset of $\mathbb{Z}^d$, called the support of Γ, and $h \equiv h(\Gamma) : \mathbb{Z}^d \to \mathbb{Z}$ is a map that is constant on connected components of $\underline{\Gamma}^c$.*

Note that h alone does not specify the contour as we do not require that $\underline{\Gamma}$ be restricted to the region where h is non-constant. We follow the usage of [BK1] in calling $\underline{\Gamma}$ the *support*, although this may be misleading. Of course, when mapping our SOS-model to a contour model, we must give a one-to-one map from heights to contours, but this information will always be assumed to be contained in the measures (in other words, we start with a model where only certain contours have non-zero measure). In the sequel, Ω shall denote the space of all contours. Also, Ω_Λ will denote the space of contours in the finite volume Λ. [4]

[4] Let us remark that the space of contours can also be described as a more general spin (h, σ), where σ_x takes the values zero if $x \in \underline{\Gamma}$ and zero otherwise.

We will also need to consider spaces of contours satisfying some further constraints. To explain this, we must introduce some notation. Let D denote some subset of $\mathbb{Z}^d \times \mathbb{Z}$. Such sets will later arise as the so-called 'bad regions'. Let us note already here that the fact that bad regions are subsets of $\mathbb{Z}^d \times \mathbb{Z}$, that is the basic lattice times the space of the 'heights' is a main difference to the random field case treated in [BK1], where bad regions are subsets of the base space $\mathbb{Z}^d$ only. It is the non-compactness of our 'spin-space' that makes this modification necessary. Given D, we denote by $D(h)$ the slices of D at height h, i.e.

$$D(h) \equiv \{(x, h') \in D | h' = h\} \tag{2.3}$$

Of course $D(h)$ can be identified in a natural way with a subset of $\mathbb{Z}^d$ by simply projecting. We must also define, for a given contour Γ the set

$$D(\Gamma) \equiv \{x \in \mathbb{Z}^d | (x, h_x(\Gamma) \in D\} \tag{2.4}$$

Given a set D, we will denote by $\Omega(D)$ all those contours whose support contains $D(\Gamma)$, i.e.

$$\Omega(D) \equiv \{\Gamma \in \Omega | D(\Gamma) \subset \underline{\Gamma}\} \tag{2.5}$$

Note that this definition is consistent since on the one hand $D(\Gamma)$ really only depends on the height part of Γ and the condition in the definition of (2.5) only affects the support of Γ. As a matter of fact, given any contour in $\Gamma' \in \Omega$ we can easily associate to it a contour in $\Omega(D)$ by first computing $D(\Gamma')$, and then setting $\underline{\Gamma} \equiv \underline{\Gamma}' \cup D(\Gamma')$. Then $\Gamma = (\underline{\Gamma}, h(\Gamma')$ is a contour in $\Omega(D)$.

As we have indicated above, a renormalization group transformation may depend on the starting measure. In particular, the transformations we will use depend on the set D (which of course will be chosen in accordance with the measure μ, or more precisely some of the random parameters μ depends on). The sets D will necessarily also be affected by the renormalization, so that we will have to construct maps $\mathcal{R}_D$, depending on D (as well as other parameters) that map the spaces $\Omega(D)$ into $\Omega(D')$ for a suitably computed D'. The resulting structure will then be a measurable map $\mathcal{R}_D : (\Omega(D), \Sigma(D)) \to (\Omega(D'), \Sigma(D'))$ that can be lifted to the measure μ s.t. for any $\mathcal{A} \in \Sigma(D')$,

$$(\mathcal{R}_D\mu)(\mathcal{A}) = \mu(\mathcal{R}_D^{-1}(\mathcal{A})) \tag{2.6}$$

Of course we want to iterate this procedure. At least here it becomes clear that it is necessary to find a parameterization of the measures we are dealing with that remains invariant under the RG transformations. As a first step, let us rewrite the original SOS model as a contour model.

II.3. The SOS model as a contour model

One of the basic ideas in the reformulation of our model in terms of contours is that the form in which the contour weights are written should make manifest how the energy is associated to the supports of the contours, and more specifically, to the connected components of the supports. To do this, we first need to introduce some more notation. First, we will always use the metric $d(x,y) = \max_{i=1}^{d} |x_i - y_i|$ for points in $\mathbb{Z}^d$. We will call a set $A \subset \mathbb{Z}^d$ connected, iff for all $x \in A$, $d(x, A\backslash\{x\}) \leq 1$. A maximal connected subset of a set A will be called a connected component of A. We will also use the notation $\overline{A}$ for the set of points whose distance from A is not bigger than 1, and we will write $\partial A \equiv \overline{A}\backslash A$ and call ∂A the boundary of A. A further important notion is that of the *interior* of a set A, int$A$. It is defined as follows: For any set $A \subset \mathbb{Z}^d$, let $\hat{A} \subset \mathbb{R}^d$ denote the set in $\mathbb{R}^d$ obtained by embedding A into $\mathbb{R}^d$ and surrounding each of its points by the unit cube in d dimensions. Then the complement of $\hat{A}$ may have finite connected components. Their union with $\hat{A}$ is called int$\hat{A}$, and the intersection of this set with $\mathbb{Z}^d$ is defined to be int$A$.

Important operations will consist of the cutting and glueing of contours. First, for any contour Γ we may decompose its support, $\underline{\Gamma}$, into connected components, $\underline{\gamma}_i$ in the sense described above. Note that a contour is uniquely described by specifying its support $\underline{\Gamma}$, the values of h on the support and the values h takes on each of the connected components of the boundary of the support.This makes it possible to associate with each connected component $\underline{\gamma}_i$ of the support a contour, γ_i, by furnishing the additional information of the heights on $\underline{\gamma}_i$ and on the connected components of $\partial\underline{\gamma}_i$. We will call a contour with connected support a *connected contour*. In the same spirit we call a connected contour γ_i obtained from a connected component of the support of a contour Γ a *connected component* of Γ. A collection, $\{\gamma_1, \ldots, \gamma_n\}$ of connected contours is called *compatible*, if there exists a contour Γ, such that $\gamma_i, \ldots, \gamma_n$ are the connected components of Γ. This contour will also be called $(\gamma_1, \ldots, \gamma_n)$. It is clear that the notion of compatibility has a simple expression in terms of the heights on the connected components of the boundaries of the $\underline{\gamma}_i$.

Notice that the connected components of a contour are not entirely independent of each other, since a connected component may lie within the interior of another and thus has to adjust its exterior height to the corresponding height of this outer contour. Therefore, we will have two use a second notion of connectedness that we call *weak connectedness.* We say that a set $\mathcal{A} \subset \mathbb{Z}^d$ is *weakly connected*, if int A is connected. All the notions of the previous paragraph then find their weak analogs.

Finally, let us define the *level sets* $V_h(\Gamma)$ of a contour by

$$V_h(\Gamma) \equiv \{x \in \mathbb{Z}^d | h_x(\Gamma) = h\} \tag{2.8}$$

For any function $F : \mathbb{Z}^d \times \mathbb{Z} \to I\!R$, we introduce the notion

$$(F, V(\Gamma)) \equiv \sum_{h \in \mathbb{Z}} \sum_{x \in V_h(\Gamma)} F_x(h) \tag{2.9}$$

This notation will receive a considerable generalization shortly.

We now come to the representation of the SOS-model as a contour model. Defining

$$E_s(\Gamma) = \sum_{\substack{x,y \in \underline{\Gamma} \\ |x-y|=1}} |h_x(\Gamma) - h_y(\Gamma)| \tag{2.9}$$

we could write

$$H(h) = E_s(\Gamma) + (J, V(\Gamma)) \tag{2.10}$$

with $\underline{\Gamma}$ defined for a given function h as the set of x that possess a nearest neighbor y for which $h_y \neq h_x$. Then the term $E_s(\Gamma)$ could be written as a sum over connected components, $E_s(\Gamma) = \sum_i E_s(\gamma_i)$; this would produce the contour activities and the term $(J, V(\Gamma))$ would play the role of a 'field'. This would be reasonable, if the configuration $h \equiv 0$ were indeed the true ground state. However, the fields terms here may and will alter the ground state configuration, although only locally in rare places. As indicated above, we want to take this into account by adopting the definition of contours more closely to the real situation. Of course, due to our limited control over the random terms, this can be achieved only on a given finite length scale. To implement this, we allow only $J_x(h)$ that are small enough to remain in the field term. For a fixed $\delta > 0$ that will be chosen appropriately later, we set

$$S_x(h) \equiv J_x(h) 1\!\!I_{|J_x(h)|<\delta} \tag{2.11}$$

Here and everywhere in the sequel $1\!\!I_X$ denotes the indicator function of the event X (i.e. $1\!\!I_X = 1$ if X holds and $1\!\!I_X = 0$ otherwise). For fields that are not small in this sense, we introduce a new field that only keeps roughly track of their size. It is defined as

$$N_x(h) \equiv \delta^{-1} 1\!\!I_{|J_x(h)|\geq\delta} |J_x(h)| \tag{2.12}$$

Here we chose the prefactor δ^{-1} so that the control fields have minimal size one. This is not particularly significant but makes some of our later estimates more easily comparable with those in [BK]. The region D, the *bad region* is then defined as

$$D \equiv \left\{ (x,h) | \sup_{k \in \mathbb{Z}} \left(N_x(h+k) - \frac{c}{L}|k| \right) > 0 \right\} \tag{2.13}$$

where L is a positive integer (it will be the blocking scale of the RG transformation) and $c > 0$ is a suitably determined positive constant. Now we define the mass of a contour Γ as

$$\mu(\Gamma) = \begin{cases} \rho(\Gamma) e^{-\beta(S, V(\Gamma))} & \text{, if } \underline{\Gamma} = \{x | \exists_{y:|x-y|=1} : h_y(\Gamma) \neq h_x(\Gamma)\} \cup D(\Gamma) \\ 0 & \text{, otherwise} \end{cases} \tag{2.14}$$

where

$$\rho(\Gamma) = e^{-\beta(E_s(\Gamma)+(J,V(\Gamma)\cap D(\Gamma)\cap \underline{\Gamma})} \tag{2.15}$$

The important fact here is that $\rho(\Gamma)$ factors over the connected components of Γ, i.e. if $\Gamma = (\gamma_1, \ldots, \gamma_n)$, then

$$\rho(\Gamma) = \prod_{i=1}^{n} \rho(\gamma_i) \tag{2.16}$$

(Note that it is to have (2.16) that we wrote $D(\Gamma) \cup \underline{\Gamma}$ rather than simply $D(\Gamma)$. We must note here that the connected components of a contour will, of course, not in general satisfy the constraint to contain the bad region of that contour!).

Note that (2.14) gives a one-to-one relation between height-configurations and contours with non-zero weight. All quantities of interest in the SOS-model can thus be computed in the contour model defined above.

In an ideal situation, we would hope that the form of the measures on the contours would remain in this form under renormalization, i.e. activities factorizing over connected components plus a 'small-field' contribution. Unfortunately, the truth will not be that simple, except in the case of zero temperature, as will be shown in Section III. In general, the renormalization will introduce non-local interactions between connected components of supports as well as a non-local 'small random field', $\{S_C\}$ indexed by the connected subsets C of $\mathbb{Z}^d$. The final structure of the contour measures will be the following:

$$\mu(\Gamma) = \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{\mathbb{Z}^d \supset G \supset \underline{\Gamma}} \rho(\Gamma, G) \tag{2.17}$$

where $\rho(\Gamma, G)$ are activities that factor over the connected components of G. Z is as usual the normalization constant that turns μ into a probability measure. For non-local fields the notion $(S, V(\Gamma))$ is extended to abbreviate

$$(S, V(\Gamma)) \equiv \sum_{h \in \mathbb{Z}} \sum_{C \subset V_h(\Gamma)} S_C(h) \tag{2.18}$$

the sum over C being here and always a sum over connected sets. The functions S, the activities ρ and the fields N will be the parameters on which the action of the renormalization group will finally be controlled. Of course these quantities will have to satisfy certain bounds that will be specified later, and it will be these bounds that eventually have to be controlled in the process of renormalization.

II.4. Renormalization of the contours

As the last point of this section we will now define the action of the renormalization group on the contours themselves. We should remark that this cannot yet be done completely, since, as indicated above, the renormalization group map will depend on the bad regions and even to some extend on the starting measure μ itself (basically through the fields $N_x(h)$). Thus at present we give the general outline while the details must be filled in the appropriate places of Sections III and IV.

The renormalization group transformation we shall construct consists of three distinct steps:

(i) Summation of small connected components of contours

(ii) Blocking of the remaining large contours

(iii) Dressing of the supports by the new bad region

Note that step (iii) is to some extent cosmetic and requires already the knowledge of the renormalized bad regions. We note that this causes no problem, as the bad regions may in practice already be computed after step (i).

Let us now give a brief description of the individual steps.

STEP 1: In principle we would like to sum in this step over all those classes of contours for which we can get a convergent expansion in spite of the random fields. In practice, we restrict ourselves to a much smaller, but sufficiently large, class of contours. Namely we define a connected component as 'small', if it is geometrically small (in the sense that $d(\underline{\gamma_i}) < L - 2$) and if its support does not intersect the bad region, with the exception of a suitably defined 'harmless' subset of the bad region. This latter point is important since it will allow us to forget about this harmless part in the next stage of the iteration and this will assure that the successive bad regions become sparser and sparser. Precise definitions (although certainly not the optimal ones) are given in Section III.

A contour which contains no small connected component is called large, and we denote by $\Omega^l(D)$ the subspace of large contours. The first step of RG transformation is then nothing but the canonical projection form $\Omega(D)$ to $\Omega^l(D)$, i.e. to any contour in Ω we associated the large contour composed of only the large components of Γ.

STEP 2: In this step the large contours are mapped to a coarse-grained lattice. We choose the simplest action of $\mathcal{R}$ on $\mathbb{Z}^d$, namely $(\mathcal{R}x)_i = \mathcal{L}^{-1} \equiv \text{ int}(x_i/L)$. We will denote by $\mathcal{L}x$ the set of all points y s.t. $\mathcal{L}^{-1}y = x$. Now the action of $\mathcal{L}^{-1}$ on height configurations of large contours is defined as averaging, i.e.

$$(\mathcal{L}^{-1}h)_y = \frac{1}{L^d} \sum_{x \in \mathcal{L}y} h_x \tag{2.19}$$

with this definition we have the action of $\mathcal{L}^{-1}$ on large contours as

$$\mathcal{L}^{-1}\Gamma \equiv (\mathcal{L}^{-1}\underline{\Gamma}, \mathcal{L}^{-1}h) \tag{2.20}$$

STEP 3: The action of $\mathcal{R}$ given by (2.19) does not yet give a contour in $\Omega(D')$. Thus, the last step in the RG transformation consists of enlarging the supports of the contours by the newly created bad regions, which of course requires first to compute those. This will in fact be the most subtle and important part of the entire renormalization program and will be explained later. Given a new region D', the effect on the contours is just to replace $\underline{\mathcal{L}^{-1}\Gamma}$ by $\underline{\mathcal{L}^{-1}\Gamma} \cup D'(\mathcal{L}^{-1}\Gamma)$, so that finally the full RG transformation on the contours can be written as

$$\mathcal{R}_D(\Gamma) \equiv (D'(\mathcal{L}^{-1}\Gamma^l(\Gamma)) \cup \mathcal{L}^{-1}\underline{\Gamma}^l(\Gamma), \mathcal{L}^{-1}h(\Gamma^l(\Gamma))) \tag{2.21}$$

We have now set up the basic formalism for our RG program. The remaining task is now to analyze the induced action on measures of the type described above and to show that this technique has the computational power to prove the theorem announced in the introduction.

III. The ground states

As we have indicated in the introduction, the crucial new feature in the analysis of the low temperature phase of disordered systems as opposed to that of ordered ones lies in the fact that even the analysis of the properties of the ground state becomes highly non-trivial and the result crucial for the structure of the low-temperature phases. But while the essential conceptual features are already present, on the technical level this is still much simpler than the situation at finite temperature, mainly due to the fact that we need not perform any cluster expansions. We think it is helpful for the understanding of this method to separate difficulties of diffferent origin and therefore we devote this section entirely to this particular case. The results obtained here will then prove useful in the general case that we will treat in Section IV.

III.1 Formalism and set-up

We will try to make this section as self-contained as possible, but refer to notations introduced in Section II. Let Γ denote a contour as defined in Definition 2.1. and let Ω be the space of all contours. For a given energy function $H : \Omega \to I\!R$, we must define the proper notion of a ground state contour; in particular, we are interested in ground states corresponding to 'boundary conditions zero at infinity'. In the sequel, let Λ always denote a *finite* subset of $\mathbb{Z}^d$. We need to define restrictions H_Λ of H to finite volumes that are finite functions from Ω into $I\!R$. The precise definition of these restrictions for contour models will be given in a moment. Now define the sets $\mathcal{G}_\Lambda^{(\Gamma)}$ to be the contours of lowest energy in Λ for given external configuration Γ_{Λ^c}, i.e.

$$\mathcal{G}_\Lambda^{(\Gamma)} \equiv \left\{ \Gamma^* \in \Omega \middle| \Gamma^*_{\Lambda^c} = \Gamma_{\Lambda^c} \wedge H_\Lambda(\Gamma^*) = \inf_{\Gamma':\Gamma'_{\Lambda^c}=\Gamma_{\Lambda^c}} H_\Lambda(\Gamma') \right\} \tag{3.1}$$

Here Γ_{Λ^c} denotes the restriction of Γ to Λ^c. The set of all infinite volume ground states is usually defined (see [AL]) as

$$\mathcal{G}_\infty \equiv \left\{ \Gamma \in \Omega \,|\, \forall_{\Lambda \subset \mathbb{Z}^d} \Gamma \in \mathcal{G}_\Lambda^{(\Gamma)} \right\} \tag{3.2}$$

Remark: Under some weak smoothness assumptions on the measure $I\!P$ the sets $\mathcal{G}_\Lambda^{(\Gamma)}$ consist $I\!P$-a.s. of single elements.

The problem of determining the entire set of ground states is, at least in our situation, far too ambitious and we will content ourselves with proving the existence of groundstates corresponding to roughly flat interfaces with a given typical height. More precisely, we will proceed as follows. Let Λ_n denote the cube centered at the origin of sidelength L^n. Then define the cylinder set

$$\hat{\mathcal{G}}_{\Lambda_n,\Lambda}^{(h)} \equiv \left\{ \Gamma \in \Omega \,|\, \exists_{\Gamma^* \in \mathcal{G}_{\Lambda_n}^{(\emptyset, h_n \equiv h)}} : \Gamma_\Lambda = \Gamma^*_\Lambda \right\} \tag{3.3}$$

(this is the set of all contours that within Λ look like a ground state for the finite volume Λ_n with boundary condition $(\emptyset, h_x \equiv h)$. Now set

$$\mathcal{G}_\infty^{(h)} \equiv \bigcap_{\Lambda \subset \mathbb{Z}^d} \bigcup_{n_0=0}^{\infty} \bigcap_{n \geq n_0} \hat{\mathcal{G}}_{\Lambda_n,\Lambda}^{(h)} \tag{3.4}$$

It is easy to convince oneself that

$$\mathcal{G}_\infty^{(h)} \subset \mathcal{G}_\infty \tag{3.5}$$

(this is in fact analogous to the observation that weak limit points of local specifications yield infinite volume Gibbs states). The purpose of the present section is to prove that $\mathcal{G}_\infty^{(h)}$ is non-empty if $d \geq 3$ and the disorder weak enough. It will also follow that these sets are disjoint for different values of h. By stationarity, it will in fact be enough to consider the case $h = 0$ which we will do from now on.

To construct an element of $\mathcal{G}_\infty^{(h)}$ we have to study elements Γ^* of the sets $\mathcal{G}_{\Lambda_n}^{(0)}$ and show that for any fixed finite volume Λ the restriction Γ_Λ^* becomes independent of n for n sufficiently large. Let us introduce the abbreviation $\Omega_n \equiv \Omega_{\Lambda_n}^{(0)}$. The analysis of ground states via the renormalization group method then consists of the following inductive procedure. Let $\mathcal{R}$ be a map $\mathcal{R} : \Omega_n \to \Omega_{n-1}$. Then clearly

$$\inf_{\Gamma \in \Omega_n} H_{\Lambda_n} = \inf_{\tilde{\Gamma} \in \Omega_{n-1}} \left(\inf_{\Gamma \in \mathcal{R}^{-1}\tilde{\Gamma}} H_{\Lambda_n}(\Gamma) \right) \tag{3.6}$$

which suggests to define

$$(\mathcal{R}H_{\Lambda_{n-1}})(\tilde{\Gamma}) \equiv \inf_{\Gamma \in \mathcal{R}^{-1}\tilde{\Gamma}} H_{\Lambda_n}(\Gamma) \tag{3.7}$$

Since $\mathcal{R}$ is in general not invertible, $\mathcal{R}^{-1}\tilde{\Gamma}$ denotes the set pre-images of $\tilde{\Gamma}$ in Ω_n. Then, defining $\mathcal{R}\mathcal{G}_{\Lambda_{n-1}}^{(0)}$ to be the set of ground states with respect to the energy function $\mathcal{R}H$. Then we have that

$$\mathcal{G}_{\Lambda_n}^{(0)} = \left\{ \Gamma^* \,\middle|\, H_{\Lambda_n}(\Gamma^*) = \inf_{\Gamma \in \mathcal{R}^{-1}(\mathcal{R}\mathcal{G}_{\Lambda_{n-1}}^{(0)})} H_{\Lambda_n}(\Gamma) \right\} \tag{3.8}$$

that is, if we can determine the ground states with respect to $\mathcal{R}H$ in the smaller volume Λ_{n-1}, then we have to search for the true ground state only within the inverse image of this set. The proper setting up of a RG scheme consists of finding maps $\mathcal{R}$ such that both tasks become simpler than the original one.

We will now give a precise description of the class of admissible energy functions. The original energy function describing the SOS-model was already introduced in the introduction and adopted to the contour formulation in Section II. In that section the relation between the original random fields $J_x(h)$ and the 'small field' $S_x(h)$ and the auxiliary field $N_x(h)$ and the 'bad region' D was explained. To describe the general class of models that will appear in the RG process, we begin with the auxiliary or 'control' fields N. Thus we let $\{N_x(h)\}_{x \in \Lambda_n}^{h \in \mathbb{Z}}$ be a family of non-negative real numbers. In fact, they will later be assumed to be a random field satisfying certain specific probabilistic assumption. Given such N, we may now define the 'bad region' corresponding to it, namely

DEFINITION 3.1: *Given a control field N, the set*

$$D \equiv D(N) \equiv \left\{ (x,h) \in \Lambda_n \times \mathbb{Z} \Big| \sup_{h' \in \mathbb{Z}} \left(N_x(h+h') - \frac{c}{L}|h'| \right) > 0 \right\} \tag{3.9}$$

is called the 'bad region'. Here c is a constant that may be chosen e.g. $c = 1/16$. Given a contour $\Gamma \in \Omega$, we denote by $D(\Gamma)$ the set

$$D(\Gamma) \equiv \{x \in \Lambda_n | (x, h_x(\Gamma)) \in D\} \tag{3.10}$$

and we denote by $\Omega_n(D) \subset \Omega_n$ the space

$$\Omega_n(D) \equiv \{\Gamma \in \Omega_n | D(\Gamma) \subset \underline{\Gamma}\} \tag{3.11}$$

DEFINITION 3.2: *An N-bounded contour energy ϵ of level k is a map $\epsilon : \Omega_n(D) \to \mathbb{R}$, s.t.*

(1) If $\gamma_1, \ldots, \gamma_m$ are the connected components of Γ, then

$$\epsilon(\Gamma) = \sum_{i=1}^{m} \epsilon(\gamma_i) \tag{3.12}$$

(ii) If γ is a connected contour in $\Omega_n(D)$ then

$$\epsilon(\gamma) \geq E_s(\gamma) + L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| - (N, V(\gamma) \cap \underline{\gamma}) \tag{3.13}$$

where $E_s(\gamma)$ is the strictly deterministic surface energy defined in (2.9).

(iii) Let $C \subset D(h)$ be connected and $\gamma = (C, h_x \equiv h)$ be the connected component of a contour $\Gamma \subset \Omega_n(D)$. Then

$$\epsilon(\gamma) \leq \sum_{x \in C} N_x(h) \tag{3.14}$$

An N-bounded energy function of level k is a map $H_{\Lambda_n} : \Omega_n \to \mathbb{R}$ of the form

$$H_{\Lambda_n}(\Gamma) = \epsilon(\Gamma) + (S, V(\Gamma)) \tag{3.15}$$

where $S_x(h)$ are bounded random fields (see e.g. (2.11)) and ϵ is a N-bounded contour energy of level k.

Remark: The 'level' k in the definition refers to the fact that some properties of energy function change under the application of the RG transformation. A energy function for the SOS-model will be a energy function of level k after k iterations of the RG.

Remark: The appearance of the dimension and k dependent constant in the lower bound (3.14) is due to the fact that in the RG process no uniform constant suppressing supports of contours outside the bad region is maintained. The specific form of this constant is somewhat technical.

Remark: Restrictions of energy functions to more general finite volumes Λ are defined as

$$H_\Lambda(\Gamma) = \sum_{i:\gamma_i \cap \Lambda \neq \emptyset} \epsilon(\gamma_i) + (S, V_h(\Gamma) \cap \Lambda) \tag{3.16}$$

We are now ready to define what we mean specifically be a RG transformation

DEFINITION 3.3: *For a given control field N, a proper renormalization group transformation, $\mathcal{R}^{(N)}$, is a map from $\Omega_n(D(N))$ into $\Omega_{n-1}(D(N'))$, such that if H_{Λ_n} is of the form (3.16) with ϵ a N-bounded contour energy of level k, then $H'_{\Lambda_{n-1}} \equiv \mathcal{R}^{(N)} H_{\Lambda_n}$ is of the form*

$$H'_{\Lambda_{n-1}}(\Gamma) = \epsilon'(\Gamma) + (S', V(\Gamma)) \tag{3.17}$$

where ϵ' is a N'-bounded contour energy of level $k+1$, and S' is a new bounded random field and N' is a new control field.

From the above definition it is obvious that in order to make use of a RG transformation, it is crucial to be able to compute N' and S', i.e. to study the action of the RG on the random and control fields. As both are random fields, this control will be probabilistic, i.e. consist of statements an the effective probability distributions. We must therefore specify more precisely the corresponding assumptions.

Recall that the energy functions H are random functions on a probability space $(\Theta, \mathcal{F}, I\!\!P)$ and that H_{Λ_n} is assumed to be $\mathcal{F}_{\Lambda_n}$-measurable (this is evident e.g. in the original SOS-model, where H_{Λ_n} is a function of the stochastic sequences $J_x(h)$ with $x \in \Lambda_n$ only, and $\mathcal{F}_{\Lambda_n}$ is the sigma-algebra generated by these sequences). Of course, the renormalized energy functions are still random variables on this same probability space. It is useful to consider an action of the RG map on the sigma-algebras and to introduce $\mathcal{F}^{(k)} = \mathcal{R}^k \mathcal{F}$, where in particular $\mathcal{R}\mathcal{F}_\Lambda^{(k)} \subset \mathcal{F}_{\mathcal{L}\Lambda}^{(k-1)}$, such that after k iterations of the RG the resulting energy function is $\mathcal{F}^{(k)}$-measurable. Naturally, $\mathcal{F}^{(k)}$ is endowed with a filtration with respect to the renormalized lattice. In the general step we will drop the reference to the level in the specification of this sigma-algebra and write simply $\mathcal{F}$. We need to maintain certain such locality properties that we state as follows:

(i) The stochastic sequences $\{N_x(h)\}_{h \in \mathbb{Z}}$ and $\{S_x(h)\}_{h \in \mathbb{Z}}$ are measurable w.r.t. the sigma-algebras $\mathcal{F}_{\overline{x}}$.

(ii) For any connected contour $\gamma \in \Omega_n(D)$, $\epsilon(\gamma)$ is measurable w.r.t. $\mathcal{F}_{\underline{\gamma}}$

A further important assumption on the random quantities is that of stationarity w.r.t. shift in the height variables:

(iii) The stochastic sequences $\{N_x(h)\}_{h\in\mathbb{Z}}$ and $\{S_x(h)\}_{h\in\mathbb{Z}}$ are stationary w.r.t. the shifts $h \to h+h'$, $h' \in \mathbb{Z}$.

(iv) $\epsilon(\gamma)$ is stationary w.r.t. the shift $(\underline{\gamma}, \{h_x\}_{x\in\Lambda_n}) \to (\underline{\gamma}, \{h_x + h'\}_{x\in\Lambda_n})$, $h' \in \mathbb{Z}$. (Note that for components touching the boundary of Λ_n, stationarity holds only if the apply the shift also to the external height).

Finally, we need assumptions on the smallness of the disorder. Here the S-fields are centered and bounded, i.e.

(v) $\mathbb{E} S_x(h) = 0$

(vi) $|S_x(h)| \leq \delta$, with δ some suitable small constant (for instance $\delta = \frac{1}{16L^{(1+\alpha)}}$, with $1 \gg \alpha > 0$ will work).

The distribution of the the S satisfies the bounds

(vii)

$$\begin{aligned} \mathbb{P}\left[S_x(h) \geq \epsilon\right] &\leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \quad \text{and} \\ \mathbb{P}\left[S_x(h) \leq -\epsilon\right] &\leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \end{aligned} \tag{3.18}$$

Here the constants σ_k are parameters that will change in the course of renormalization (we will prove later that $\sigma_k^2 = L^{d-2-\eta}\sigma_0^2$) and those flow will have to be controlled. The control fields $N_x(h)$ should also satisfies bounds like (3.18), but actually the situation there is quite more complicated. Notice that in the original model the N-fields as defined in (2.12) satisfy bounds $\mathbb{P}(N_x(h) > z) \leq 2\exp\left(-\frac{z^2}{2\sigma^2}\right)$, and moreover the smallest non-zero value they take is δ. This latter fact is crucial in that it ensures that $D(N)$ is a fairly sparse set! It will be important to maintain such a property in the course of the RG process. As the exact form of these constraints is fairly complicated and difficult to motivate a priori, we postpone the precise formulation to Section III.5.

We have now a sufficient description of the general class of models on which the RG is to be performed. The RG transformation is now performed in three steps, as indicated in Section II.

III.2 Absorbtion of small contours

In Section II we explained that the first part of the RG map consists of the re-summing of so-called 'small contours'. These can be defined as connected components of small size (on scale L) with support outside the bad regions. Now the definition of the bad regions is such that they exclude in fact the existence of such small components in a (conditioned) ground state contour. Actually, there is even a large portion of the bad region that may be removed if we are willing to allow for the appearance of 'flat' small contours, i.e contours with non-empty supports but constant height even on their support. It will be crucial to take advantage of this fact. The following definition describes this 'harmless' part of the bad region.

DEFINITION 3.4: *Let $D_i(h)$ denote the $L^{1/2}$-connected[5] components of $D(h)$. Such a connected component is called* small, *on level k, if*

(i) $|D_i(h)| < L^{(1-\alpha)/2}$

(ii) $d(D_i(h)) \leq L/4$

(iii) $\sum_{y \in D_i(h)} \sup_{h' \in \mathbb{Z}} (N_y(h+h') - \frac{c}{L}|h'|) < L L^{-(d-2)k} \sigma^2$

Here $\alpha > 0$ is a constants that will be fixed later and $\sigma^2 \equiv \sigma_0^2$ refers to the variance of the original random fields, not to those at level k. Define now

$$\begin{aligned} \mathcal{D}(h) &\equiv \bigcup_{D_i(h)\,\mathrm{small}} D_i(h) \quad \textit{and} \\ \mathcal{D} &\equiv \bigcup_{h \in \mathbb{Z}} (\mathcal{D}(h) \times \{h\}) \end{aligned} \tag{3.19}$$

Remark: The definition of the slices $\mathcal{D}(h)$ follows closely that used in [BK], allowing us to carry over many of the geometric estimates used there. It is certainly not optimal. An important aspect of the definition of $\mathcal{D}$ is that it is 'local' in the following sense: If we consider a fixed point x and a set $E \subset \Lambda_n$ containing x, then the event $\{E$ is a component of $\mathcal{D}(h)\}$ can only depend on $N_{x'}$-fields within the region $d(x,x') \leq L/4 + 2L^{(1-\alpha)/2}$, i.e. the sigma-algebra generated by such events is independent of $\{N_{x'}(h)\}_{x':x' \in A, h \in \mathbb{Z}}$ if $d(x,A) > L/4 + 2L^{(1-\alpha)/2}$.

In the light of this definition, we may now define the 'small contours':

DEFINITION 3.5: *A connected contour $\gamma \in \Omega_n(D)$ is called* small (given N), *iff*

(i) $d(\underline{\gamma}) < L - 2$, *and*

(ii) $(D(h_\gamma) \backslash \mathcal{D}(h_\gamma)) \cap \ \mathrm{int}\underline{\gamma} = \emptyset$

[5] It should be clear what is meant by $L^{1/2}$-connectedness: A set A is said to be $L^{1/2}$-connected, if there exists a path in A with steps of length less that or equal to $L^{1/2}$ joining each point in A.

Here h_γ is defined as the height $h(\gamma)$ on the boundary of $\mathrm{int}\underline{\gamma}$. A contour Γ is called small iff all its connected components are small and a contour which contains no small connected component is called large. We denote by $\Omega_n^s(D)$ the set of small contours and by $\Omega_n^l(D)$ the set of large contours.

Remark: Notice that $\Omega_n^l(D) \subset \Omega_n(D\backslash\mathcal{D})$, but in general it is *not* a subset of $\Omega_n(D)$!

DEFINITION 3.6: *The first step in the RG transformation is a map T_1 that is nothing but the canonical projection from $\Omega_n(D)$ onto $\Omega_n^l(D)$, i.e. if $\Gamma = (\gamma_1, \ldots, \gamma_r, \gamma_{r+1}, \ldots, \gamma_q)$ with γ_i large for $i = 1, \ldots, t$ and small for $i = r+1, \ldots, q$, then*

$$T_1(\Gamma) \equiv \Gamma^l \equiv (\gamma_1, \ldots, \gamma_r) \tag{3.20}$$

To give a precise description the conditioned ground states under the projection T_1, we need to define the following sets. First let $\overline{\mathcal{D}_i(h)}$ denote the *ordinary* connected components of $\overline{\mathcal{D}(h)}$ (in contrast to the definition of $D_i(h)$!). Given a contour $\Gamma^l \in \Omega_n^l(D)$ we write $\mathcal{B}_i(\Gamma^l, h) \equiv \overline{\mathcal{D}_i(h)}\backslash\underline{\Gamma^l}$ for all those components such that $\overline{\mathcal{D}_i(h)} \subset V_h(\Gamma^l)\backslash\underline{\Gamma^l}$. Let $\mathcal{B}(\Gamma^l) \equiv \bigcup_{i,h} \mathcal{B}_i(\Gamma^l, h) = \overline{\mathcal{D}}(\Gamma^l)\backslash\underline{\Gamma^l}$. Finally we set $\mathcal{D}_i(h) = \overline{\mathcal{D}_i(h)} \cap \mathcal{D}(h)$. Note that these $\mathcal{D}_i(h)$ need not be connected.

Let us denote by $\mathcal{G}_{\Gamma^l,1}$ the set of contours in $\Omega_n(D)$ that minimize H_n under the condition that $T_1\Gamma = \Gamma^l$. We have the following characterization of this set:

LEMMA 3.1: *Let $\Gamma^l \in \Omega_\Lambda^l(D)$ Then, for any $\Gamma \in \mathcal{G}_{1,\Gamma^l}$*

(i) $\underline{\Gamma}\backslash\underline{\Gamma^l} \subset \mathcal{B}(\Gamma^l)$, and

(ii) For all x, $h_x(\Gamma) \equiv h_x(\Gamma^l)$.

Remark: This Lemma is the crucial result of the first step of the RG transformation. It makes manifest that fluctuations on length scale L can only arise due to 'large fields in the bad regions'. Since this statement will hold in each iteration of the RG, it shows that any fluctuations of the surface are localized in the bad regions. We will come back to this more specifically later.

Before giving the proof of this Lemma, the following Lemma gives a formula for the renormalized energy function under T_1. We set

$$\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) \equiv \inf_{\gamma:\mathcal{D}_i(h)\subset\underline{\gamma}\subset\mathcal{B}_i(\Gamma^l,h),\gamma=(\underline{\gamma},h_\gamma\equiv h)} \epsilon(\gamma) \tag{3.21}$$

(Note that γ here is not necessarily connected).

LEMMA 3.2: *Let for any $\Gamma^l \in \Omega_n^l(D)$ denote*

$$(T_1 H_n)(\Gamma^l) \equiv \inf_{\Gamma\in\Omega_n(D):T_1(\Gamma)=\Gamma^l} H_n(\Gamma) \tag{3.22}$$

Then

$$(T_1 H_n)(\Gamma^l) - H_n(\Gamma^l) = \sum_{i,h} \epsilon^h(\mathcal{B}_i(\Gamma^l, h)) \tag{3.23}$$

Note that in the expression $H_n(\Gamma^l)$, we view Γ^l as a contour in $\Omega_n(D\backslash\mathcal{D})$; that is, the contributions to the energy in the regions $\mathcal{D}\backslash\underline{\Gamma}^l$ is ignored.

Proof: (Of Lemmas 3.1 and 3.2) We first prove Lemma 3.1. We will show that for any Γ s.t. $T_1(\Gamma) = \Gamma^l$ the quantity

$$E(\Gamma) - \left(E(\Gamma^l) + \sum_{j,h} \epsilon^h(\mathcal{B}_j(\Gamma^l, h)) \right) > 0 \tag{3.24}$$

unless Γ obeys the conditions stated in Lemma 3.1. Let $\gamma_1, \dots, \gamma_r$ denote the small *weakly* connected components of Γ. Clearly, the difference $H_n(\Gamma) - H_n(\Gamma^l)$ can be written as a sum over these weakly connected components, namely

$$H_n(\Gamma) - H_n(\Gamma^l) = \sum_{i=1}^{r} \left(\epsilon(\gamma_i) + (S, V(\Gamma) \cap \ \mathrm{int}\, \underline{\gamma}_i) - (S, V(\Gamma^l) \cap \ \mathrm{int}\, \underline{\gamma}_i) \right) \tag{3.25}$$

Similarly, we may split the sum over the $\mathcal{B}_j(\Gamma^l, h)$ into sums over those contained in a given $\mathrm{int}\, \underline{\gamma}_i$ (Note that all $\mathcal{B}_j(\Gamma^l, h)$ must be contained in some such connected component, as $\underline{\Gamma}$ is constrained to contain all of D, and thus the supports of the small connected components of it must contain $\mathcal{D}\backslash\underline{\Gamma}^l$). Thus we are left to show that for any small weakly connected component γ of Γ, the quantities

$$\Delta E(\gamma) \equiv \epsilon(\gamma) + (S, V(\Gamma) \cap \ \mathrm{int}\, \underline{\gamma}) - (S, V(\Gamma^l) \cap \ \mathrm{int}\, \underline{\gamma}) - \sum_{j,h: \mathcal{B}_j(\Gamma^l, h) \subset \mathrm{int}\, \underline{\gamma}} \epsilon^h(\mathcal{B}_j(\Gamma^l, h)) \tag{3.26}$$

are strictly positive, unless γ has constant height and support contained in $\overline{\mathcal{D}}$. To show this, we insert the lower bounds (3.14) on ϵ for $\epsilon(\gamma)$, and an upper bound on $\epsilon^h(\mathcal{B}_j(\Gamma^l, h))$ obtained by bounding the infimum in (3.21) by the value obtained with the flat contour whose support is $\mathcal{D}_j(\Gamma^l, h)$ and bounding the result through the upper bound (3.15). Using moreover that $|S_x(h)| \leq \delta$, this yields

$$\begin{aligned} \Delta E(\gamma) \geq E_s(\gamma) + L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| - (N, V(\gamma) \cap \underline{\gamma}) - \sum_{j,h: \mathcal{B}_j(\Gamma^l, h) \subset \mathrm{int}\, (\underline{\gamma})} \ \sum_{x \in \mathcal{D}_j(\Gamma^l, h)} N_x^{(k)}(h) \\ - \delta \sum_{x \in \mathrm{int}\, \underline{\gamma}} \mathbb{I}_{h_x(\Gamma) \neq h_x(\Gamma^l)} \end{aligned} \tag{3.27}$$

To continue, we need the following geometrical Lemma:

LEMMA 3.3: (local lower bound on the surface energy) *Let γ be a weakly connected contour s.t. $d(\text{ int}\underline{\gamma}) \leq L$. Let h_γ denote the height of γ on ∂ int$\underline{\gamma}$. Then*

$$E_s(\gamma) \geq \frac{2d}{L} \sum_{x \in \text{int}\,(\underline{\gamma})} |h_x(\gamma) - h_\gamma| \tag{3.28}$$

The proof of this Lemma will be given in appendix G, where all geometrical estimates of this type are collected and proven. Using this bound, we get

$$\begin{aligned} \Delta E(\gamma) \geq \sum_{x \in \text{int}\,\underline{\gamma}} \Big(& \frac{2d}{L} |h_x(\gamma) - h_x(\Gamma^l)| - \delta \mathbb{I}_{h_x(\gamma) \neq h_x(\Gamma^l)} \\ & - N_x(h(\gamma)) - N_x(h(\Gamma^l)) \Big) + L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| \end{aligned} \tag{3.29}$$

Now using condition (iii) from Definition 3.4 of $\mathcal{D}$, we see that, if $c \leq \frac{d}{2}$ and if $\delta \leq \frac{d}{2L}$,

$$\begin{aligned} \sum_{x \in \text{int}\,\underline{\gamma}} \Big(& \frac{d + L\delta}{L} |h_x(\gamma) - h_x(\Gamma^l)| - \delta \mathbb{I}_{h_x(\gamma) \neq h_x(\Gamma^l)} - N_x(h(\gamma)) - N_x(h(\Gamma^l)) \Big) + \frac{1}{2} L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| \\ & \geq -2LL^{-(d-2)k} \sigma^2 \end{aligned} \tag{3.30}$$

Note that the lower bound corresponds to such γ those support contains a single component $D_i(h)$; for flat contours containing several such components must support of volume of the order of $L^{1/2}$ outside of $\overline{\Delta(\gamma)}$, due to the fact that the $D_i(h)$ are the $L^{1/2}$-connected components of $D(h)$. Thus

$$\Delta E(\gamma) \geq \sum_{x \in \text{int}\,\underline{\gamma}} \frac{d - L\delta}{L} |h_x(\gamma) - h_x(\Gamma^l)| + \frac{1}{2} L^{-(d-2)k} |\underline{\gamma} \backslash \overline{D}(\gamma)| - 2LL^{-(d-2)k} \sigma^2 \tag{3.31}$$

Now if $\frac{d-L\delta}{L} > 2LL^{-(d-2)k}\sigma^2$ and $2L\sigma^2 < 1$, the lower bound in (3.31) is strictly positive unless $h_x(\Gamma) \equiv h_x(\Gamma^l)$ and $\underline{\gamma} \backslash \overline{D}(\gamma) = \emptyset$, which proves Lemma 3.1. Lemma 3.2 now follows immediately. $\diamond$

Remark: Note that the prove imposes a smallness condition on σ^2 w.r.t. L and a condition on the constant c' in Definition 3.4.

From the previous Lemmas, and the Definition 3.4, we finally obtain the following uniform bounds on the ϵ^h.

LEMMA 3.4: *For any Γ^l, and any component $\mathcal{B}_i(\Gamma^l, h)$*

$$|\epsilon^h(\mathcal{B}_i(\Gamma^l, h)| \leq LL^{-(d-2)k} \sigma^2 \tag{3.32}$$

Here we see an additional rational for the definition of the harmless part of the large field region, namely that the ground state contours supported in then only introduce an extremely small correction to the energy which can, as we will see in the next step, be absorbed locally in the small fields.

III.3 The blocking

We now come to the crucial step in the RG transformation, that is the mapping of the configuration space Ω_n to Ω_{n-1}. The corresponding operator, T_2, will be chosen as $T_2 \equiv \mathcal{L}^{-1}$, with $\mathcal{L}^{-1}$ defined in Section II (c.f. eq. (2.19 ,20)). We will generally use the name $\mathcal{L}^{-1}$ when referring to the purely geometric action of T_2. Notice that $\mathcal{L}^{-1}$ is naturally a map from $\Omega_n(D\backslash\mathcal{D})$ into $\Omega_{n-1}(\mathcal{L}^{-1}(D\backslash\mathcal{D}))$, where $\mathcal{L}^{-1}(D\backslash\mathcal{D})$ is naturally defined as the union of the sets $\mathcal{L}^{-1}(D(h)\backslash\mathcal{D}(h))$. We must now construct the induced action of this map on the energy functions and on the random fields S and N. We consider first the small fields. Recall that we wanted to absorb the contributions of the small contours into the renormalized small fields. This would be trivial, if there were no interaction between the small contours and the supports of the large ones, i.e. if the $\mathcal{B}_i(\Gamma^l, h)$ did not depend on Γ^l. To take this effect into account, we proceed by writing

$$\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) = \epsilon^h(\overline{\mathcal{D}_i(h)}) + \left(\epsilon^h(\mathcal{B}_i(\Gamma^l, h)) - \epsilon^h(\overline{\mathcal{D}_i(h)})\right) \tag{3.33}$$

and adding the first term to the small fields while the second is non-zero only for $\mathcal{D}_i(h)$ that touch the contours of Γ^l and will later be absorbed in the new contour energies. Thus we define the (preliminary) new small fields by

$$\tilde{S}'_y(h) \equiv L^{-(d-1-\alpha)} \left(\sum_{x\in\mathcal{L}y} S_x(h) + \sum_{i:\overline{\mathcal{D}_i(h)}\cap\mathcal{L}y\neq\emptyset} \frac{\epsilon^h(\overline{\mathcal{D}_i(h)})}{|\mathcal{L}^{-1}\overline{\mathcal{D}_i(h)}|} \right) \tag{3.34}$$

The pre-factor in this definition anticipates the scaling factor of the surface energy term under blocking. Note here that the $\tilde{S}'_y$ satisfy the locality conditions (i): $\tilde{S}'_y$ and $\tilde{S}'_{y'}$ are independent stochastic sequences if $|y-y'|>1$, since the $\overline{\mathcal{D}_i(h)}$ cannot extend over distances larger than L.

The (preliminary) new control field is defined as

$$\tilde{N}'_y(h) \equiv L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}y\backslash\mathcal{D}(h)} \sup_{h'\in\mathbb{Z}} \left(N_x(h+h') - \frac{1}{8L}|h'| \right) \tag{3.35}$$

Note here that the summation over x excludes the regions $\mathcal{D}$, as the contributions there are dealt with elsewhere. This is crucial, as otherwise the regions with positive $\tilde{N}'$ would grow rather than shrink in the RG process. Note that the constant $\frac{1}{8L}$ in the definition is to some extend arbitrary.

We now define the induced energy function $T_2T_1H_n$ on $\Omega_{n-1}(\mathcal{L}^{-1}(D\backslash\mathcal{D}))$ by

$$(T_2T_1H_n)(\Gamma') \equiv \inf_{\Gamma^l:\mathcal{L}^{-1}\Gamma^l=\Gamma'} (T_1H_n)(\Gamma^l) \tag{3.36}$$

The following Lemma states that this energy function is essentially of the same form as H_n:

LEMMA 3.5: *For any $\Gamma' \in \Omega_{n-1}(\mathcal{L}^{-1}(D\backslash\mathcal{D}))$ we have*

$$(T_2T_1H_n)(\Gamma') = L^{d-1-\alpha}\left(\sum_{i=1}^{q}\bar{\epsilon}(\gamma_i') + (\tilde{S}', V(\Gamma'))\right) \tag{3.37}$$

where the γ_i' are the connected components of Γ', and $\bar{\epsilon}$ is a $\tilde{N}'$-bounded contour energy of level $k+1$.

Proof: Let us write the energy of a large contour in the form

$$\begin{aligned}&(T_1E(\Gamma^l) - (S, V(\Gamma^l)) = \\ &\sum_i \epsilon(\gamma_i) + \sum_{\substack{i,h:\overline{\mathcal{D}_i(h)}\cap\underline{\Gamma^l}\neq\emptyset \\ \overline{\mathcal{D}_i(h)}\subset V_h(\Gamma)\backslash\underline{\Gamma^l}}}\left(\epsilon^h(\mathcal{B}_i(\Gamma^l,h)) - \epsilon^h(\overline{\mathcal{D}_i(h)})\right) + \sum_{i,h:\overline{\mathcal{D}_i(h)}\subset V_h(\Gamma^l)\backslash\underline{\Gamma^l}}\epsilon^h(\overline{\mathcal{D}_i(h)})\end{aligned} \tag{3.38}$$

Note that here the condition that $\overline{\mathcal{D}_i(h)}\cap\underline{\Gamma^l}\neq\emptyset$ in the second sum makes explicit the fact that the summand vanishes otherwise. Let us introduce the notation

$$S_C(h) \equiv \sum_i \epsilon^h(\overline{\mathcal{D}_i(h)})\mathbb{I}_{C=\overline{\mathcal{D}_i(h)}} \tag{3.40}$$

for any finite subset $C\subset\Lambda_n$. Let us note that while here these object have only a transitory significance, in the finite temperature case they will acquire the meaning of non-local random fields. Notice that here $S_C(h)$ is strictly zero whenever $d(C)\geq\frac{L}{4}$.

Using this notation, we may write that

$$\begin{aligned}&(T_2T_1H_n)(\Gamma') - L^{d-1-\alpha}(\tilde{S}', V(\Gamma')) \\ &= \inf_{\Gamma^l:\mathcal{R}\Gamma^l=\Gamma'}\left(\sum_i\epsilon(\gamma_i) + \sum_{\substack{i,h:\overline{\mathcal{D}_i(h)}\cap\underline{\Gamma^l}\neq\emptyset \\ \overline{\mathcal{D}_i(h)}\subset V_h(\Gamma^l)\backslash\underline{\Gamma^l}}}\left(\epsilon^h(\mathcal{B}_i(\Gamma^l,h)) - \epsilon^h(\overline{\mathcal{D}_i(h)})\right)\right. \\ &\left.+\sum_{x\in\Lambda_n}\left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma')\right) + \sum_{\substack{h\in Z,C\subset\Lambda_n \\ |C|>1}} S_C(h)\left(\mathbb{I}_{C\subset V_h(\Gamma)\backslash\underline{\Gamma}} - \sum_{y\in\mathcal{L}^{-1}C}\frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|}\right)\right)\end{aligned} \tag{3.41}$$

Notice that the first term in the last line gives a non-vanishing contribution only from x s.t. $\mathcal{L}^{-1}x\in\underline{\Gamma}'$ and the second one only from such C that intersect $\overline{\underline{\Gamma^l}}\cup\mathcal{L}\underline{\Gamma}'$. Therefore, the expression in the infimum can be unambiguously split into a sum over the connected components of Γ' and moreover, the infimum may be taken separately in all the terms of the sum. Thus, if Γ' can be decomposed in connected components as $\Gamma' = (\gamma_1',\ldots,\gamma_q')$, we get that

$$(T_2T_1H_n)(\Gamma') - L^{d-1-\alpha}(\tilde{S}', V(\Gamma')) = \sum_{i=1}^{q}L^{d-1-\alpha}\bar{\epsilon}(\gamma') \tag{3.42}$$

where

$$L^{d-1-\alpha}\overline{\epsilon}(\gamma') = \inf_{\Gamma^l:\mathcal{L}^{-1}\Gamma^l=\gamma'} \Bigg(\epsilon(\Gamma^l) + \sum_{\substack{i,h:\overline{\mathcal{D}_i(h)}\cap\underline{\Gamma^l}\neq\emptyset \\ \overline{\mathcal{D}_i(h)}\subset V_h(\Gamma^l)\setminus\underline{\Gamma^l}}} \left(\epsilon^h(\mathcal{B}_i(\Gamma^l,h)) - \epsilon^h(\overline{\mathcal{D}_i(h)})\right)$$
$$+ \sum_{x\in\Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma'))\right) + \sum_{h\in\mathbb{Z},C\subset\Lambda_n} S_C(h) \left(\mathbb{I}_{C\subset V_h(\Gamma)\setminus\underline{\Gamma}} - \sum_{y\in\mathcal{L}^{-1}C} \frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|} \right)\Bigg) \tag{3.43}$$

Notice that the locality condition for $\overline{\epsilon}$ is obvious from the above remarks, i.e. $\overline{\epsilon}(\gamma')$ is measurable w.r.t. the sigma-algebra $\mathcal{F}_{\mathcal{L}\underline{\overline{\gamma'}}}$.

Let us now prove the lower bound on $\overline{\epsilon}$. Inserting the lower bound on $\epsilon(\Gamma^l)$ into (3.43) and noting that trivially $\epsilon^h(\mathcal{B}_i(\Gamma^l,h)) \geq \epsilon^h(\overline{\mathcal{D}_i(h)})$, we get

$$L^{d-1-\alpha}\overline{\epsilon}(\gamma') \geq \inf_{\Gamma^l:\mathcal{L}^{-1}\Gamma^l=\gamma'} \Bigg(E_s(\Gamma^l) + L^{-(d-2)k}|\underline{\Gamma^l}\setminus\overline{D}(\Gamma^l)| - (N, V(\Gamma^l)\cap\underline{\Gamma^l})$$
$$+ \sum_{x\in\Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma'))\right) + \sum_{h\in\mathbb{Z},C\subset\Lambda_n} S_C(h) \left(\mathbb{I}_{C\subset V_h(\Gamma)\setminus\underline{\Gamma^l}} - \sum_{y\in\mathcal{L}^{-1}C} \frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|} \right)\Bigg) \tag{3.44}$$

Now the terms in the second line are all small and will be bounded uniformly against some fraction of the surface energy and volume energy of the support of the Γ^l, while the remaining surface and volume energies together with the N-term will give the effective renormalized bounds. To see this, we rearrange the terms in (3.44) in the following form:

$$L^{d-1-\alpha}\overline{\epsilon}(\gamma') \geq \inf_{\Gamma^l:\mathcal{L}^{-1}\Gamma^l=\gamma'} \Bigg(\frac{1}{4}E_s(\Gamma^l)$$
$$+ \frac{1}{4}E_s(\Gamma^l) + L^{-(d-2)k}|\underline{\Gamma^l}\setminus\overline{D}(\Gamma^l)| - \sum_{h\in\mathbb{Z},C\subset V_h(\Gamma^l):C\cap\underline{\Gamma^l}\neq\emptyset} S_C(h)$$
$$- \left((N, V(\Gamma^l)\cap\underline{\Gamma^l}) - \frac{1}{4}E_s(\Gamma^l) \right)$$
$$+ \frac{1}{4}E_s(\Gamma^l) + \sum_{x\in\Lambda_n} \left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\gamma'))\right) + \sum_{h\in\mathbb{Z},C\subset\Lambda_n} S_C(h)\left(\mathbb{I}_{C\subset V_h(\Gamma^l)} - \mathbb{I}_{C\in\mathcal{R}V_h(\gamma')}\right)\Bigg) \tag{3.45}$$

Note that we have split the $S_C(h)$ terms in such a way that the term appearing in the last line vanishes if Γ^l is a flat contour.

To treat (3.45) further, we need the following geometrical Lemmas, those proof will again be given the appendix.

LEMMA 3.6: *Let $h' = \mathrm{Rnd}\left(\overline{h}\right)$ where $\overline{h} = L^{-d}\sum_{x\in\mathcal{L}0} h_x$. Then*

$$\sum_{<x,y>:x,y\in\mathcal{L}0} |h_x - h_y| \geq \frac{1}{L}\sum_{x\in\mathcal{L}0} |h_x - h'| \tag{3.46}$$

From this Lemma and the boundedness of the small fields S we see immediately that e.g.

$$\begin{aligned}
&\frac{1}{8}E_s(\Gamma^l) + \sum_{x\in\Lambda_n}\left(S_x(h_x(\Gamma^l)) - S_x(h_{\mathcal{L}^{-1}x}(\Gamma'))\right) \\
&\geq \sum_{x\in\mathcal{R}\gamma'}\left(S_x(h_x(\Gamma^l)) - (S_x(h_{\mathcal{L}^{-1}x}(\gamma')) - \frac{1}{8L}|h_x(\Gamma^l)) - h_{\mathcal{L}^{-1}x}(\gamma')|\right) \geq 0
\end{aligned} \tag{3.47}$$

provided only $\delta \leq \frac{1}{16L}$. In much the same way we can deal with the remaining terms in the last line of (3.45). Just notice that only such C in that sum give a non-zero contribution that contain at least one site x for which $h_x(\Gamma^l) \neq h_{\mathcal{L}^{-1}x}(\gamma')$ and for each such site only the C that equals the $\overline{D_i(h)}$ that contains x gives a contribution, which in turn is bounded by $LL^{-(d-2)k}\sigma^2$. Therefore, provided this quantity is smaller than $\frac{1}{16L}$, the remainder of the last line of (3.45) is also non-negative. Note that the last condition holds true for $k \geq 1$ if $\sigma^2 \leq \frac{1}{8L}$; if $k = 0$, i.e. in the first RG step these terms even do not exist since $\mathcal{D}$ is empty in this case.

Thus we have shown that the last line in (3.45) is uniformly non-negative.

Let us now consider the third line in (3.45). We split the N-term as

$$(N, V(\Gamma^l)\cap\underline{\Gamma^l}) = \sum_{x\in\underline{\Gamma^l}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}_s}(\gamma'))} N_x(h_x(\Gamma^l)) + \sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}_s}(\gamma'))} N_x(h_x(\Gamma^l)) \tag{3.48}$$

With the help of Lemma 3.6, the first term from (3.48) together with a piece of the surface energy can be bounded by the new large fields:

$$\begin{aligned}
&\sum_{x\in\underline{\Gamma^l}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}_s}(\gamma'))} N_x(h_x(\Gamma^l)) - \frac{1}{8}E_s(\Gamma^l) \\
&\leq \sum_{x\in\underline{\Gamma^l}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}_s}(\gamma'))}\left(N_x(h_x(\Gamma^l)) - \frac{1}{8L}|h_x(\Gamma^l) - h_{\mathcal{L}^{-1}x}(\gamma')|\right) \\
&\leq \sum_{x\in\mathcal{R}\underline{\gamma'}:\, x\notin\mathcal{D}(h_{\mathcal{L}^{-1}_s}(\gamma'))}\sup_{h'\in\mathbb{Z}}\left(N_x(h') - \frac{1}{8L}|h' - h_{\mathcal{L}^{-1}x}(\gamma')|\right) \\
&= L^{d-1-\alpha}(\tilde{N}', V(\gamma')\cap\underline{\gamma'})
\end{aligned} \tag{3.49}$$

The second term in (3.48) gives for flat contours Γ^l a small contribution proportional to $|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|$;

taking advantage of the remaining $\frac{1}{8}E_s(\Gamma^l)$ we get the same bound for an arbitrary contour

$$\begin{aligned}
&\sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} N_x(h_x(\Gamma^l)) - \frac{1}{8}E_s(\Gamma)\\
&\leq \sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} \left(N_x(h_x(\Gamma^l)) - \frac{1}{16L}|h_x(\Gamma) - h_{\mathcal{L}^{-1}x}(\gamma')| - \frac{1}{16}E_s(\Gamma^l)\right)\\
&\leq \sum_{x\in\underline{\Gamma^l}:\, x\in\mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))} \left(LL^{-(d-2)k}\sigma^2 - \frac{1}{16L}|h_x(\Gamma) - h_{\mathcal{L}^{-1}x}(\gamma')|\right)\\
&\leq LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|
\end{aligned} \tag{3.50}$$

where the last inequality was obtained by noting that in the previous sum only such sites x for which $h_x(\Gamma^l) = h_{\mathcal{L}^{-1}x}(\gamma')$ give a positive contribution and that for such sites $x \in \mathcal{D}(h_{\mathcal{L}^{-1}x}(\gamma'))$ implies $x \in \mathcal{D}(\Gamma^l)$. The final bound in (3.50) is of course just a tiny fraction of the volume term in the second line and will be absorbed in it.

The S_C term in the second line is in fact dominated by the same bound, i.e.

$$\sum_{h\in\mathbb{Z},C\subset V_h(\Gamma^l):C\cap\underline{\Gamma^l}\neq\emptyset} |S_C(h)| \leq LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}| \tag{3.51}$$

where the volume term arises as a trivial upper bound on the number of connected components of $\mathcal{D}(\Gamma^l)$ that may intersect $\underline{\Gamma^l}$. Putting theses results together, we have arrived at

$$L^{d-1-\alpha}\overline{\epsilon}(\gamma') \geq \inf_{\Gamma^l\in\mathcal{L}\gamma'} \left(\frac{1}{4}E_s(\Gamma^l) + \frac{1}{4}E_s(\Gamma^l) + L^{-(d-2)k}|\underline{\Gamma^l}\backslash\overline{\mathcal{D}}(\Gamma^l)| - 2LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma^l}|\right) \tag{3.52}$$

The first quarter of surface energy here yields the new surface energy by the following Lemma (whose proof may again be found in the appendix):

LEMMA 3.7: *Let $\Gamma \in \mathcal{L}^{-1}\gamma'$. Then*

$$E_s(\Gamma) \geq \frac{L^{d-1}}{d+1}E_s(\gamma') \tag{3.53}$$

Note that in order to use this bound, we must choose α such that $L^\alpha \geq 4(d+1)$.

To bound the remaining terms, let us set $\tilde{D}' \equiv D(\tilde{N}')$. Clearly we want to bound the remaining surface and volume energies by a term proportional to $|\underline{\gamma'}\backslash\overline{\tilde{D}'}(\gamma')|$. The tricky part here is that the original estimate is only in the volume of $\underline{\Gamma^l}$ *outside* the bad region, while the new estimate involves the new bad region, which is smaller than the image of the bad region under $\mathcal{L}^{-1}$ since the harmless part, $\mathcal{D}$, has been excluded in the definition of the $\tilde{N}'$. In fact, the geometric constraints in the definition of $\mathcal{D}$ were essentially made in order to get the desired estimate nonetheless, as we will see from the following two Lemmas, that are again proven in the appendix.

LEMMA 3.8: *Let γ be connected and large. Then*

$$L^{(1-\alpha)/2}E_s(\gamma)+|\underline{\gamma}\backslash\overline{D}(\gamma)|\geq\frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(\gamma)| \tag{3.54}$$

LEMMA 3.9: *Let $\Gamma\in\mathcal{L}^{-1}\gamma'$. Then there exists a constant $c_6>0$ s.t.*

$$LE_s(\Gamma)+|\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))|\geq c_6L|\underline{\gamma}'\backslash\overline{\tilde{D}'}(\gamma')| \tag{3.55}$$

where $\tilde{D}'\equiv D(\tilde{N}')$.

To use these Lemmas, we just have to rearrange terms slightly: Setting $q\equiv\frac{1}{5}$, we may write

$$\begin{aligned}
&\frac{1}{4}E_s(\Gamma^l)+L^{-(d-2)k}|\underline{\Gamma}^l\backslash\overline{D}(\Gamma^l)|-2LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma}^l|\\
&\geq\frac{1}{8}E_s(\Gamma^l)+\frac{1}{8}E_s(\Gamma^l)+(1-q)L^{-(d-2)k}|\underline{\Gamma}^l\backslash\overline{D}(\Gamma^l)|+qL^{-(d-2)k}|\underline{\Gamma}^L\backslash(\overline{D}(\Gamma^l)\backslash\overline{\mathcal{D}}(\Gamma^l))|\\
&\quad-qL^{-(d-2)k}|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma}^l|-2LL^{-(d-2)k}\sigma^2|\overline{\mathcal{D}}(\Gamma)\cap\underline{\Gamma}^l|\\
&\geq\frac{1}{8}E_s(\Gamma^l)+qL^{-(d-2)k}|\underline{\Gamma}^l\backslash(\overline{D}(\Gamma^l)\backslash\overline{\mathcal{D}}(\Gamma^l))|\\
&\quad+\left(L^{-(d-2)k}(\frac{1}{2}(1-q)-q)-2LL^{-(d-2)k}\sigma^2\right)|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma}^l|\\
&\geq c_6qL^{-(d-2)k}|\underline{\gamma}'\backslash\overline{\tilde{D}'}(\gamma')|
\end{aligned} \tag{3.56}$$

Here the last inequality was obtained assuming that σ^2 is sufficiently small, so that the term proportional to $|\overline{\mathcal{D}}(\Gamma^l)\cap\underline{\Gamma}^l|$ in the one but last line is positive. We have also assumed that $\frac{1}{8}L^{-(1-\alpha)/2}\geq L^{-(d-2)k}(1-q)$.

Collecting everything, we get the desired lower bound

$$\overline{\epsilon}(\gamma')\geq E_s(\gamma')+c_6\frac{1}{5}L^\alpha LL^{-(d-2)(k+1)}|\underline{\gamma}'\backslash\overline{\tilde{D}'}(\gamma')|-(\tilde{N}',V(\gamma')\cap\underline{\gamma}') \tag{3.57}$$

which is the desired lower bound if L is large enough s.t. $c_6\frac{1}{5}L^\alpha\geq1$.

To conclude the proof of the Lemma, it remains to prove the upper bound for flat contours γ'. Thus let $C'\subset\tilde{D}'(h)$ be connected and set $\gamma'\equiv(C',h_x\equiv h)$. Then we bound the infimum over Γ^l from above by the term for which $\Gamma=((D(h)\backslash\mathcal{D}(h))\cap\mathcal{L}C',h_x\equiv h)$. It is easy to check that for this contour, only the term $\epsilon(\Gamma^l)$ in (3.43) gives a non-zero contribution. It is then a trivial matter to conclude from the upper bound on $\epsilon(\Gamma^l)$ and the definition of $\tilde{N}'$, that

$$L^{d-1-\alpha}\overline{\epsilon}(\gamma')\leq\sum_{x\in\mathcal{L}C'}N_x^{(k)}(h)\leq L^{d-1-\alpha}\sum_{y\in C'}\tilde{N}'_y(h) \tag{3.58}$$

This concludes the proof of Lemma 3.5. $\diamondsuit$

III.4 Final shape-up

The hard part of the RG transformation is now done. However, not all of the properties of the original model are yet shared by the renormalized quantities; in particular, the renormalized weak field $\tilde{S}'$ is not centered and it may have become too large. Both defaults are, however, easily rectified. We define

$$S'_y(h) \equiv \tilde{S}'_y(h)1_{|\tilde{S}'_y(h)|<\delta} - I\!E\left(\tilde{S}'_y(h)1_{|\tilde{S}'_y(h)|<\delta}\right) \tag{3.59}$$

It is important here that due to the stationarity assumptions, the expectation in (3.59) does not depend on h, i.e $I\!E\left(\tilde{S}'_y(h)1_{|\tilde{S}'_y(h)|<\delta}\right) = I\!E\left(\tilde{S}'_y(0)1_{|\tilde{S}'_y(0)|<\delta}\right)$. What is left, i.e. the large part of the small field is taken account of through the redefined control field, i.e. we set

$$N'_y(h) \equiv \tilde{N}'_y(h) + |\tilde{S}'_y(h)|1_{|\tilde{S}'_y(h)|>\delta'} \tag{3.60}$$

Given N', we may now define $D' \equiv D(N')$ as in Definition 3.1. Then let T_3 (given N') be the map from Ω_{n-1} to $\Omega_{n-i}(D')$ defined through

$$T_3(\Gamma) = (h(\Gamma), \underline{\Gamma} \cup D'(\Gamma)) \tag{3.61}$$

Defining as before

$$\begin{aligned}(T_3T_2T_1E)(\Gamma') &\equiv \inf_{\Gamma:T_3\Gamma=\Gamma'}(T_2T_1E)(\Gamma)\\ &= \inf_{\Gamma:T_3\Gamma=\Gamma'} L^{d-1-\alpha}\left(\bar{\epsilon}(\Gamma) + (\tilde{S}', V(\Gamma))\right)\end{aligned} \tag{3.62}$$

we have the final

LEMMA 3.10: *For any $\Gamma' \in \Omega_{\Lambda'}(N')$ we have*

$$T_3T_2T_1E(\Gamma') = L^{d-1-\alpha}\left(\epsilon'(\Gamma') + (S', V(\Gamma')) + \sum_{y\in\Lambda'} I\!E(\tilde{S}'_y(0)1_{|\tilde{S}'_y(0)|>\delta'})\right) \tag{3.63}$$

where ϵ' is a N'-bounded contour energy of level $k+1$.

Proof: Notice that for all $\Gamma \in T_3^{-1}\Gamma'$ $(S', V(\Gamma)) = (S', V(\Gamma'))$. Thus, for any connected γ',

$$\epsilon'(\gamma') = \inf_{\Gamma\in T_3^{-1}\gamma'}\left(\bar{\epsilon}(\Gamma) + (\tilde{S}'1_{|S'|>\delta'}, V(\gamma')\cap\underline{\gamma'})\right) \tag{3.64}$$

Thus, noting that $|\underline{\Gamma}\backslash\widetilde{D'}(\gamma')| \geq |\underline{\gamma'}\backslash D'(\gamma')|$ and that for any $\Gamma \in T_3^{-1}\gamma'$,

$$(\tilde{N}', V(\Gamma)\cap\underline{\Gamma}) + (|\tilde{S}'|1_{|S'|>\delta'}, V(\gamma')\cap\underline{\gamma'}) = (N', V(\gamma')\cap\underline{\gamma'}) \tag{3.65}$$

we get immediately from Lemma 3.5 that

$$\epsilon'(\gamma') \geq E_s(\gamma') + L^{-(d-2)k}|\underline{g}'\backslash D'(\gamma')| - (N', V(\gamma') \cap \underline{\gamma}') \tag{3.66}$$

The upper bound on ϵ' follows also easily from Lemma 3.5. For $C \subset D'(h)$ connected and $\gamma' = (C, h_x \equiv h)$, we bound the infimum in (3.64) by the value for the contour $\Gamma = (C \cap \tilde{D}'(h), h_x \equiv h) \in T_3^{-1}\gamma'$. Then from (3.58) it follows that

$$\epsilon(\gamma') \leq \sum_{x \in \underline{\gamma}'} \tilde{N}'_x(h) + \sum_{x \in \underline{\gamma}'} |\tilde{S}'_x(h)| 1_{|\tilde{S}'_x(h)| > \delta'} = \sum_{x \in \underline{\gamma}'} N'_x(h) \tag{3.67}$$

The locality condition on ϵ' being trivially verified, this concludes the proof of Lemma 3.10. $\diamondsuit$

This concludes the construction of the entire RG transformation. We may summarize the results of the previous three subsections in the following

PROPOSITION 3.1: *Let $\mathcal{R}^{(N)} \equiv T_3 T_2 T_1 : \Omega_n(D(N)) \to \Omega_{n-1}(D(N'))$ with T_1, T_2 and T_3 defined above; let N' and S' and ϵ' be defined as above and define $H'_{n-1} \equiv L^{-(d-1-\alpha)}(\mathcal{R}^{(N)} H_n)$ through*

$$H'_{n-1}(\Gamma) = \epsilon'(\Gamma) + (S', V(\Gamma)) \tag{3.68}$$

Then, if H_n is a N-bounded energy function of level k, then H'_{n-1} is a N'-bounded energy function of level $k+1$.

This proposition allows us to control the flow of the RG transformation on the energies through its action on the random fields S and N. Let us remark here that the stationarity assumptions on the fields and contour energies made for the original fields are trivially verified for the renormalized quantities, due to the 'translation invariant' way we have constructed these quantities. What is now left to do is to study the evolution of the probability distributions of these random fields under the RG map. This will be done in the next subsection.

III.5 Probabilistic estimates

Our task is now to control the action of the RG transformation on the random fields S and N, i.e. given the probability distribution of these random fields, we would like to compute the distribution of the renormalized random fields S' and N' as defined through eqs.. (3.34), (3.35) and (3,59), (3.60). Of course, rather than the precise probability distributions themselves we only compute certain bounds on these distributions.

Let us begin with the simpler small fields. In the k-th level of iteration, the distributions of the random fields are governed by a parameter σ_k^2 (essentially the variance of $S_x(h)$), that decreases exponentially fast to zero with k. We will set

$$\sigma_k^2 \equiv L^{-(d-2-\eta)}\sigma^2 \tag{3.69}$$

where η may be chosen as $\eta \equiv 3\alpha$. We denote by $S^{(k)}$ the small random field obtained from S after k iterations of the RG map $\mathcal{R}$. (Where the action of $\mathcal{R}$ on S is defined through (3.59) and (3.34)). We then have the following

PROPOSITION 3.2: *Let $d \geq 3$. Assume that the initial S satisfies assumptions (i), (iii) and (v)-(vii) with σ^2 sufficiently small. Then, for all $k \in \mathbb{N}$ and for all $\epsilon \geq 0$,*

$$\begin{aligned} &\mathbb{P}\left[S_y^{(k)}(h) \geq \epsilon\right] \leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \quad \text{and} \\ &\mathbb{P}\left[S_y^{(k)}(h) \leq -\epsilon\right] \leq \exp\left(-\frac{\epsilon^2}{2\sigma_k^2}\right) \end{aligned} \tag{3.70}$$

with σ_k defined through (3.69), and $S^{(k)}$ satisfy assumptions (i),(iii), (v), and (vi).

Remark: This Lemma is essentially equivalent to Proposition 1 of [BK]. We give a slightly different proof for the convenience of the reader.

Proof: The proof of this proposition relies on a general probabilistic result on convolutions of random variables satisfying Gaussian estimates. It reads:

LEMMA 3.11: *Let $\{X_j\}_{j=1,\dots,N}$ be a family of random variables and define $X \equiv \sum_{j=1}^N X_j$. Assume that there is a decomposition $\{1,\dots,N\} = \bigcup_{i=1}^m V_i$ s.t.*

(i) For each i, $\{X_j\}_{j\in V_i}$ is a family of independent random variables

(ii) $\mathbb{E}X_j = 0$ for all j

(iii) For all $\epsilon \geq 1$, $\mathbb{P}[X_j > \epsilon] \leq e^{-\frac{\epsilon^2}{2}}$ and $\mathbb{P}[X_j < -\epsilon] \leq e^{-\frac{\epsilon^2}{2}}$.

Then there exists a constant $C > 1$ independent of N and m, s.t. for all $\epsilon > 0$, $\mathbb{P}[X > \epsilon] \leq e^{-\frac{\epsilon^2}{2CmN}}$ and $\mathbb{P}[X < -\epsilon] \leq e^{-\frac{\epsilon^2}{2CmN}}$

Proof: Notice first that the information on $\mathbb{P}$ in the assumptions of the Lemma are completely symmetric, so that it suffices to prove the bound on $\mathbb{P}[X > \epsilon]$. To do this, we will first prove a bound on the Laplace transform $\mathbb{E}e^{tX}$ for $t \geq 0$ from which the desired estimate will follow by the exponential Markov inequality (see e.g. [CT]). To do this, we need a bound on $\mathbb{E}e^{tX_j}$ first. This will be derived from the assumptions as in [BoK]: We distinguish the cases $t \geq 1$ and $t < 1$. For $t \geq 1$ we have

$$\begin{aligned} \mathbb{E}e^{tX_j} &= t\int_{-\infty}^{\infty} e^{tx}\mathbb{P}[X_j \geq x]dx \\ &\leq t\int_{-\infty}^{1} e^{tx}dx + t\int_{1}^{\infty} e^{tx}\mathbb{P}[X \geq x]dx \\ &\leq e^t + t\int_1^\infty e^{tx}e^{-\frac{x^2}{2}}dx \leq e^{\frac{C_1^2t^2}{2}} \end{aligned} \tag{3.71}$$

with some constant C_1^2. For $t<1$ we use

$$\begin{aligned} \mathbb{E} e^{tX_j} &\leq 1+\frac{t^2}{2}\left(\mathbb{E}[X_j^2 \mathbb{I}_{X_j<0}]+\mathbb{E}[X_j^2 e^{tX_j}\mathbb{I}_{X_j\geq 0}]\right) \\ &\leq \exp\left[\frac{t^2}{2}\left(\mathbb{E}[X_j^2 \mathbb{I}_{X_j<0}]+\mathbb{E}[X_j^2 e^{X_j}\mathbb{I}_{X_j\geq 0}]\right)\right] \end{aligned} \tag{3.72}$$

where in the last line we have estimated the second term in the argument of the exponential by its value for $t=1$. Using the bounds (iii), it is easy to see that the expectations in the second line of (3.72) are bounded by universal numerical constants, so that we see that there exists a universal constant C s.t.

$$\mathbb{E} e^{tX_j} \leq e^{\frac{C^2 t^2}{2}} \quad \text{for all} \quad t\geq 0 \tag{3.73}$$

Using this, we can now bound the Laplace Transform of X

$$\begin{aligned} &\mathbb{E}\left[e^{tX}\right]=\mathbb{E}\left[e^{\sum_{i=1}^m \sum_{j\in V_i} tX_j}\right] \leq \prod_{i=1}^m \left(\mathbb{E}\left[e^{m\sum_{j\in V_i} tX_j}\right]\right)^{\frac{1}{m}} \\ &=\prod_{j=1}^N \left(\mathbb{E}\left[e^{mtX_j}\right]\right)^{\frac{1}{m}} \leq e^{\frac{mNCt^2}{2}} \end{aligned} \tag{3.74}$$

where the first inequality is an application of the Hölder inequality. The bound on $\mathbb{P}[X>\epsilon]$ is now an immediate consequence of (3.74) by the exponential Markov inequality. $\diamondsuit$

To prove the proposition, recall that

$$S_y^{(k+1)}(h) \equiv \tilde{S}_y^{(k+1)}(h)\mathbb{I}_{|\tilde{S}_y^{(k+1)}(h)|<\delta}-\mathbb{E}\left(\tilde{S}_y^{(k+1)}(h)\mathbb{I}_{|\tilde{S}_y^{(k+1)}(h)|<\delta}\right) \tag{3.75}$$

where

$$\tilde{S}_y^{(k+1)}(h) \equiv L^{-(d-1-\alpha)}\left(\sum_{x\in\mathcal{L}y} S_x^{(k)}(h)+\sum_{i:\overline{\mathcal{D}_i(h)}\cap\mathcal{L}y\neq\emptyset}\frac{\epsilon^h(\overline{\mathcal{D}_i(h)})}{|\mathcal{L}^{-1}\overline{\mathcal{D}_i(h)}|}\right) \tag{3.76}$$

We will first prove bounds for $\tilde{S}^{(k+1)}$. Note that the second term in (3.76) is uniformly bounded by

$$L^{-(d-1-\alpha)}\sum_{i:\overline{\mathcal{D}_i(h)}\cap\mathcal{L}y\neq\emptyset}\frac{\epsilon^h(\overline{\mathcal{D}_i(h)})}{|\mathcal{L}^{-1}\overline{\mathcal{D}_i(h)}|} \leq L^{1+\alpha-d}C_1 L L^{-(d-2)k}\left(L^{\frac{1}{2}}\right)^d\sigma^2 \leq L^{-(d-4-2\alpha)/2-k\eta}C_1\sigma_k^2 \tag{3.77}$$

with some constant $C_1>0$. Now notice that, for $k\geq 1$,

$$\begin{aligned} &L^{-(d-4-2\alpha)/2}C_1 L^{-k\eta}\sigma_k^2=\sigma_{k+1}L^{1-\eta/2-k\eta}C_1\sigma_k \\ &\leq \sigma_{k+1}L^{-\eta+\alpha+(3-d)/2}C_1 L^{1/2}\sigma \leq \frac{1}{2}\sigma_{k+1} \end{aligned} \tag{3.78}$$

if $d > 3 - \eta + \alpha$, $L^{1/2}\sigma \leq 1$ (which has been assumed before) and η and L are large enough s.t. $L^{-\eta+(3-d)/2}C_1 \leq 1$. Under these assumptions we have that for all $\epsilon \geq \sigma_{k+1}$

$$\mathbb{P}\left[\tilde{S}_y^{(k+1)}(h) \geq \epsilon\right] \leq \mathbb{P}\left[L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y} S_x^{(k)}(h) \geq \frac{\epsilon}{2}\right] \tag{3.79}$$

Now, since the S_x are independent for x's that are separated by distances greater than one, one may easily group the L^d variables in the sum into 2^d sets of mutually independent ones. This allows us to apply Lemma 3.11 which yields

$$\mathbb{P}\left[L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y} S_x^{(k)}(h) \geq \frac{\epsilon}{2}\right] \leq e^{-\frac{\epsilon^2}{2^d \cdot 8CL^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \tag{3.80}$$

With this bound for $\tilde{S}^{(k+1)}$, it is now easy to derive the desired bound on $S^{(k+1)}$. Note first that

$$\mathbb{E}\left(\tilde{S}_y^{(k+1)}(h)\mathbb{1}_{|\tilde{S}_y^{(k+1)}(h)|<\delta}\right) \leq \left(\mathbb{E}\left[\tilde{S}_y^{(k+1)}(h)\right]^2\right)^{1/2} \leq 2\sigma_{k+1} \tag{3.81}$$

where the last bound uses (3.80) and assumes $8dCL^{-(\eta-2\alpha)} \leq 1$. Using this bound, we see easily that for $\epsilon > 3\sigma_{k+1}$

$$\mathbb{P}\left[S_y^{(k+1)}(h) > \epsilon\right] \leq P\left[\tilde{S}_y^{(k+1)}(h) > \epsilon - 2\sigma_{k+1}\right] \leq e^{-\frac{(\epsilon-2\sigma_{k+1})^2}{2^d 8CL^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \leq e^{-\frac{\epsilon^2}{2^d \cdot 9 \cdot 8CL^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \tag{3.82}$$

Using once more Lemma 3.11, we get from this that *for all* $\epsilon > 0$,

$$\mathbb{P}\left[S_y^{(k+1)}(h) > \epsilon\right] \leq e^{-\frac{\epsilon^2}{2^d \cdot 9 \cdot 8C^2L^{-(\eta-2\alpha)}\sigma_{k+1}^2}} \tag{3.83}$$

and assuming that L and η are large enough s.t. $2^d 36C^2L^{-(\eta-2\alpha)} \leq 1$, this gives the desired bound. The bound on $\mathbb{P}\left[S_y^{(k+1)}(h) < -\epsilon\right]$ follows in exactly the same way. The centeredness of the S is true by definition, the locality and stationarity properties have already been established in the course of the previous sections. Thus Proposition 3.2 is proven. $\diamondsuit$

Now we come to the central estimate on the distribution of the control fields. In the same spirit as above, we denote by $N_x^{(k)}(h)$ the fields obtained after k iterations of the RG transformation from a starting field $N^{(0)}$, where the iterative steps are defined by equations (3.35) and (3.60). In the same spirit we will denote by $D^{(k)}$ and $\mathcal{D}^{(k)}$ the bad regions and harmless bad regions in the k-th RG step. What we need to prove for these control fields are two types of results: First of all, they must be large only with very small probabilities; second, and more important, they must be equal to zero with large and larger probability, as k increases. This second fact is crucial for the 'bad regions' to become smaller and smaller in each iteration of the RG group (remember that we have good control over the ground states only outside these bad regions!). The proof of this second fact must take into account the absorbtion of parts of the bad regions, the $\mathcal{D}$, in each step. Morally, what is happening is that once a large field has been scaled down sufficiently, it will actually drop to zero, since it finds itself in the region $\mathcal{D}$. Actually, due to the complications arising from interactions between neighboring blocks, this is not quite true, as the field really drops to zero only if the fields at neighboring sites, too, are small. This has been taken into account in [BK] by considering an upper bound on the control field that is essentially the sum of the original N over small blocks. We follow their procedure for simplicity by defining

$$
\begin{aligned}
&\bar{N}_y^{(0)}(h) = N_y^{(0)}(h) \\
&\bar{N}_y^{(k+1)}(h) = \left(L^{-(d-1-\alpha)} \sum_{z: d(y-z)\leq 1} \sum_{x \in \mathcal{L}z \setminus \mathcal{D}^{(k)}(h)} \sup_{h' \in \mathbb{Z}} \left(\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'| \right) \right) \mathbb{I}_{y \in D^{(k+1)}(h)} \\
&\qquad + \mathbb{I}_{|\tilde{S}_y^{(k+1)}(h)| > \delta}
\end{aligned}
\tag{3.84}
$$

with c as in Definition 3.4. (Note that $|\tilde{S}_y^{(k+1)}(h)| \leq L^{(1+\alpha)}\delta \leq 1$, by the choice of δ). Quite obviously,

$$
\bar{N}_x^{(k)}(h) \geq N_x^{(k)}(h) \tag{3.85}
$$

The only difference between our definition and that of [BK] is the appearance of the $\sup_h$, which will introduce some slight modifications. In particular, in the case treated by [BK], the $\bar{N}$ could be shown to be either equal to zero or greater than $L^{-(d-3/2)k}\sigma^2$. In our case this is not strictly true, but we will still be able to show that they lie in this region only with extremely small probability. The reason for this is that the bad regions D are not defined simply through the non-vanishing of the N themselves, but through the non-vanishing of the $\sup_{h'}(N_x^{(k)}(h+h') - \frac{c}{L}|h'|)$! Of course, with large probability theses sup's are non-zero because $N_x^{(k)}(h)$ is non-zero and we will have to take advantage of this fact. We will now prove the following

PROPOSITION 3.3: *Let* $f_d(z) \equiv z^2 \mathbb{I}_{z\geq 1} + z^{\frac{d-2}{d-1}} \mathbb{I}_{u<1}$ *and let* $\gamma_k \equiv L^{k(1-\eta)/2}\sigma^{-2}$. *Then*

$$
\mathbb{P}\left[L^{-(d-3/2)k}\sigma > \bar{N}_y^{(k)}(h) > 0 \right] \leq e^{-\gamma_k}, \tag{3.86}
$$

and

$$
\mathbb{P}\left[\bar{N}_y^{(k)}(h) \geq z \right] \leq e^{-f_d(z)\frac{\delta^2}{\sigma_k^2}}, \quad \text{if } z \geq L^{-(d-3/2)k}\sigma \tag{3.87}
$$

Proof: The proof of this proposition will be by induction over k. Note that it is trivially verified for $k = 0$. Thus we assume (3.86) and (3.87) for k. The first and crucial observation is that then the variables

$$
\mathcal{N}_x^{(k)}(h) \equiv \sup_{h' \in \mathbb{Z}} \left(\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'| \right) \tag{3.88}
$$

satisfy essentially the same bounds as $\bar{N}$ itself. Namely, we have

LEMMA 3.12: *Assume that $\bar{N}$ satisfies (3.86) and (3.87) and let $\bar{\mathcal{N}}$ be defined through (3.88). Then, for σ large enough, there exists a constant $b > 1$ (close to 1), such that*

$$\mathbb{P}\left[L^{-(d-3/2)k}\sigma > \bar{\mathcal{N}}_y^{(k)}(h) > 0\right] \leq e^{-\gamma_k/b}, \tag{3.89}$$

and

$$\mathbb{P}\left[\bar{\mathcal{N}}_y^{(k)}(h) \geq z\right] \leq e^{-f_d(z)\frac{\delta^2}{b\sigma_k^2}}, \quad \text{if } z \geq L^{-(d-3/2)k}\sigma \tag{3.90}$$

Proof: Consider first (3.89). Obviously,

$$\begin{aligned}
&\mathbb{P}\left[0 < \sup_{h'}(\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'|) < L^{-(d-3/2)k}\sigma\right] \leq \mathbb{P}\left[0 < N_x^{(k)}(h) < L^{-(d-3/2)k}\sigma\right] \\
&+ \sum_{|h'|\geq 1} \mathbb{P}\left[0 < (\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'|) < L^{-(d-3/2)k}\sigma\right] \\
&\leq e^{-\gamma_k} + \sum_{|h'|\geq 1} e^{-f_d(\frac{c}{L}|h'|)\frac{\delta^2}{\sigma_k^2}}
\end{aligned} \tag{3.91}$$

Now the sum in the second term converges and is of the order of $e^{-f_d(\frac{c}{L})\frac{\delta^2}{\sigma_k^2}}$ which is much smaller than the first term, which gives (3.89). To prove (3.90) we proceed in exactly the same way, noting that

$$\begin{aligned}
&\mathbb{P}\left[\sup_{h'}(\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'|) \geq z\right] \leq \mathbb{P}\left[N_x^{(k)}(h) \geq z\right] \\
&+ \sum_{|h'|\geq 1} \mathbb{P}\left[\bar{N}_x^{(k)}(h+h') \geq z + \frac{c}{L}|h'|\right] \\
&\leq e^{-f_d(z)\frac{\delta^2}{\sigma_k^2}} + \sum_{|h'|\geq 1} e^{-f_d(z+\frac{c}{L}|h'|)\frac{\delta^2}{\sigma_k^2}}
\end{aligned} \tag{3.92}$$

from which (3.90) follows by the same argument as before. $\diamondsuit$

Let us now prove (3.86) for $k+1$. Notice first that the event under consideration cannot occur if $|\tilde{S}_y^{(k+1)}(h)| > \delta$. Therefore, unless $\bar{N}_y^{(k+1)}(h) = 0$, the site y must lie within $\tilde{D}^{(k+1)}(h)$. But this implies that

$$\mathcal{L}y \cap \left(D^{(k)}(h)\backslash \mathcal{D}^{(k)}(h)\right) \neq \emptyset \tag{3.93}$$

and hence there must exist a $L^{\frac{1}{2}}$-connected component $D_i(h) \subset D^{(k)}(h)$ intersecting $\mathcal{L}y$ that violates one of the conditions of 'smallness' from Definition 3.4. Assume first that condition (iii) is violated. In this case, $D_i(h)$ is so small that it is contained in $\mathcal{L}\bar{y}$ and therefore contributes a term larger than $L^{-(d-2)(k+1)+\alpha}\sigma^2$ to $\bar{N}_y(h)$, and since $\sigma^2 \sim 1/L$, this already exceeds $L^{-(d-3/2)(k+1)}\sigma$. Thus, either condition (i) or (ii) must be violated. In both cases, this implies that the number of sites in $D_i(h)$ exceeds $L^{(1-\alpha)/2}$. Thus, there exists of necessity a set $E \subset \mathcal{L}\bar{y}$ s.t. for all $x \in E$, $\sup_{h'}(N_x^{(k)}(h+h') - \frac{c}{L}|h'|) > 0$, and

$$\bar{N}_y^{(k+1)}(h) \geq L^{-(d-1-\alpha)} \sum_{x\in E} \sup_{h'}(\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'|) \tag{3.94}$$

Now if the sup for all sites in E was larger than $L^{-(d-3/2)k}\sigma$, the sum would be larger than $L^{-(d-3/2)(k+1)+\alpha/2}\sigma$ and the event would not take place. As a matter of fact, the event does also not hold if the sup is larger than this bound on all of E except some subset $E' \subset E$ of size $|E'| \geq L^{(1-2\alpha)/2}$.

In other words, there must exist a subset $E'' \subset \mathcal{L}\bar{y}$ of size at least $|E| - L^{(1-2\alpha)/2} \geq L^{(1-\alpha)/2}(1 - L^{-\alpha/2})$ on which the sup is larger than zero but strictly smaller than $L^{-k(d-3/2)}\sigma$ Thus, assuming for simplicity $L^{-\alpha/2} \leq 1/2$),

$$\begin{aligned}
&I\!P\left[L^{-(d-3/2)(k+1)}\sigma > \bar{N}_y^{(k+1)}(h) > 0\right] \\
&\leq I\!P\left[\exists_{E''\subset\mathcal{L}\bar{y}:\,|E''|\geq\frac{1}{2}L^{(1-\alpha)/2}} : 0 < \sup_{h'}(\bar{N}_x^{(k)}(h+h') - \frac{c}{L}|h'|) < L^{-(d-3/2)k}\sigma\right] \\
&\leq \sum_{m=L^{(1-\alpha)/2}/2}^{(3L)^{dk}} \binom{(3L)^{dk}}{m} \left(I\!P\left[0 < \mathcal{N}_x^{(k)}(h) < L^{-(d-3/2)k}\sigma\right]\right)^{\frac{m}{5^d}}
\end{aligned} \tag{3.95}$$

where in the last line we used the independence of the $\bar{N}_x^{(k)}$ for sites farther apart than a distance 4 (i.e. that a set of m $N_x^{(k)}$'s contains at least $m/5^d$ independent elements). Using now (3.89) we get that

$$\begin{aligned}
I\!P\left[L^{-(d-3/2)k}\sigma > \bar{N}_y^{(k+1)}(h) > 0\right] &\leq \sum_{m=L^{(1-\alpha)/2}/2}^{(3L)^{dk}} \frac{1}{m!}(3L)^{mdk}e^{-\gamma_k m/(b5^d)} \\
&\leq \frac{1}{(L^{(1-\alpha)/2}/2)!}e^{-(\gamma_k - d\ln(3L))L^{(1-\alpha)/2}/2 + e^{-(\gamma_k - d\ln(3L)}} \\
&= e^{-\gamma_k L^{(1-\eta)/2}}
\end{aligned} \tag{3.96}$$

with η chosen e.g. $\eta = 3\alpha$ to absorb the constants. Thus (3.86) is proven for $k+1$.

We now turn to (3.87). Before proving it, let us point out that it is crucial to have the function $f_d(z)$ rather than simply z^2; namely, our goal is to show that $\bar{N}_x^{(k)}(h)$ is non-zero with very small probability which is true if $f_d(L^{-(d-3/2)k}\sigma)\frac{\delta^2}{\sigma_k^2}$ is large and grows with k. Apparently, this is true if f_d, for small values of its argument, is strictly smaller than linear! The way f_d can be chosen is governed by the following Lemma, as we will see shortly.

LEMMA 3.13: *The function f_d defined in proposition 3.3 satisfies*

$$\sum_{x\in\mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}(h)) \geq L^{d-2-\eta} f_d\left(L^{-(d-1-\alpha)} \sum_{x\in\mathcal{L}\bar{y}} \bar{N}_x^{(k)}(h)\right) \tag{3.97}$$

Proof: We distinguish the cases $L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\leq 1$ and $L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\geq 1$. In the first case, just notice that

$$
\begin{aligned}
\sum_{x\in\mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}(h)) &\geq \sum_{x\in\mathcal{L}\bar{y}}\left(\bar{N}_x^{(k)}(h)\right)^{\frac{d-2}{d-1}} \geq \left(\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\right)^{\frac{d-2}{d-1}} \\
&= L^{(d-1-\alpha)\frac{d-2}{d-1}}\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\right)^{\frac{d-2}{d-1}} \\
&= L^{d-2-\alpha\frac{d-2}{d-1}} f_d\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\right)
\end{aligned}
\tag{3.98}
$$

In the second case, use the Schwarz inequality to get

$$
\begin{aligned}
\sum_{x\in\mathcal{L}\bar{y}} f_d(\bar{N}_x^{(k)}(h)) &\geq \sum_{x\in\mathcal{L}\bar{y}}\left(\bar{N}_x^{(k)}(h)\right)^2 \geq \frac{\left(\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\right)^2}{(3L)^d} \\
&= \frac{L^{2(d-1-\alpha)}}{(3L)^d}\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\right)^2 \\
&= L^{d-2-2\alpha-\frac{d\ln 3}{\ln L}} f_d\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}}\bar{N}_x^{(k)}(h)\right)
\end{aligned}
\tag{3.99}
$$

from which the Lemma follows, for $\alpha\geq\frac{d\ln 3}{\ln L}$. $\diamondsuit$

Let us now derive (3.87). Obviously,

$$
\begin{aligned}
I\!P\left[\bar{N}_y^{(k+1)}(h)\geq z\right] &\leq I\!P\left[L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}\backslash\mathcal{D}(h)}\bar{\mathcal{N}}_x^{(k)}(h)\geq z\right] \\
&+ I\!P\left[L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}\backslash\mathcal{D}(h)}\bar{\mathcal{N}}_x^{(k)}(h)\geq z-1\ \wedge\ |\tilde{S}_y^{(k)}(h)|>\delta\right]
\end{aligned}
\tag{3.100}
$$

Let us consider the first term in (3.100). By Lemma 3.13,

$$
\begin{aligned}
&P\left[L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}\backslash\mathcal{D}(h)}\bar{\mathcal{N}}_x^{(k)}(h)\geq z\right] = I\!P\left[f_d\left(L^{-(d-1-\alpha)}\sum_{x\in\mathcal{L}\bar{y}\backslash\mathcal{D}(h)}\bar{\mathcal{N}}_x^{(k)}(h)\right)\geq f_d(z)\right] \\
&\leq I\!P\left[\sum_{x\in\mathcal{L}\bar{y}} f_d\left(\bar{\mathcal{N}}_x^{(k)}(h)\right)\geq L^{d-2-\eta}f_d(z)\right]
\end{aligned}
\tag{3.101}
$$

Now the variables $f_d(N_x^{(k)}(h))$ are essentially exponentially distributed in their tails. Moreover, we can bound their Laplace transform by

$$
\begin{aligned}
I\!E\left(e^{tf_d(N_x^{(k)}(h))}\right) &\leq I\!P\left[N_x^{(k)}(h)=0\right]+e^{-\gamma_k}e^{tf_0}+t\int_{f_0}^{\infty}e^{tf}e^{-f\alpha_k}df \\
&\leq 1+e^{tf_0-\gamma_k}+t\frac{e^{(t-\alpha_k)f_0}}{\alpha_k-t}
\end{aligned}
\tag{3.102}
$$

where we have set $f_0 \equiv f_d(L^{-(d-3/2)k}\sigma)$ and $\alpha_k \equiv \frac{\delta^2}{\sigma^2}$. We will now bound the Laplace transform *uniformly* for all $t \leq t^* \equiv (1-\epsilon)\alpha_k$, for some small $\epsilon > 0$. Noticing that $\gamma_k \gg (1-\epsilon)\alpha_k f_0$ (check!), we get in this range

$$
I\!E\left(e^{tf_d(N_x^{(k)}(h))}\right) \leq 1+\frac{1-\epsilon}{\epsilon}e^{-\epsilon f_0\alpha_k}
\tag{3.103}
$$

Now using the independence of well-separated $\bar{N}_x^{(k)}$, We find

$$
\begin{aligned}
I\!P\left[\sum_{x\in\mathcal{L}\bar{y}} f_d\left(\bar{\mathcal{N}}_x^{(k)}(h)\right) \geq L^{d-2-\eta}f_d(z)\right] &\leq e^{-L^{d-2-\eta}f_d(z)\frac{t^*}{5^d}}I\!E\left(e^{\frac{t^*}{5^d}\sum_{x\in\mathcal{L}\bar{y}}f_d(\bar{\mathcal{N}}_x^{(k)}(h))}\right) \\
&\leq e^{-L^{d-2-\eta}f_d(z)\frac{\alpha_k(1-\epsilon)}{5^d}}\left[I\!E\left(e^{t^*f_d(\bar{\mathcal{N}}_x^{(k)}(h))}\right)\right]^{\frac{L^d}{5^d}} \\
&\leq e^{-L^{d-2-\eta}f_d(z)\frac{\alpha_k(1-\epsilon)}{5^d}}\left[1+\frac{1-\epsilon}{\epsilon}e^{-\epsilon f_0\alpha_k}\right]^{\frac{L^d}{5^d}}
\end{aligned}
\tag{3.104}
$$

The last factor in (3.104) is in fact close to one and may be absorbed in a constant in the exponent, since we want a bound only for $z \geq L^{-(d-3/2)(k+1)}\sigma$. This gives a bound of the desired form for the first term in (3.100). The second term in (3.100) is simply bounded by the minimum of the probabilities of the two events, making use of Proposition 3.2. This leads to a bound of the same type. We leave it to the readers to check the details themselves or to consult [BK]. The proof of Proposition 3.3 is now finished. $\diamondsuit$

III.6 Applications

With the probabilistic estimates on the random fields obtained in the last subsection, we are now ready to make use of the RG transformations to analyse the infinite volume ground states. The first and quite immediate consequence of proposition 3.3 is that for any given point $x \in \mathbb{Z}^d$ it is quite unlikely to be contained in $D(0)$ in any iteration of the RG. Namely, we get from proposition 3.3 the

COROLLARY 3.4 *Let $d \geq 3$, σ^2 small enough. Then, there exists a constant c' (of order unity) such that for any $x \in \mathbb{Z}^d$*

$$
I\!P\left[\exists_{k\geq 0}\,:\,\sup_{h'\in\mathbb{Z}}\left(N_{\mathcal{L}^{-k}x}^{(k)}(h+h')-\frac{c}{L}|h'|\right)\neq 0\right] \leq \exp\left(-\frac{\delta^2}{c'\sigma^{2-\frac{d-2}{d-1}}}\right)
\tag{3.105}
$$

Proof: For (3.105), just use (3.89) and (3.90) with $z = L^{-d-3/2)k}\sigma$ and notice that the contribution $e^{-\gamma_k}$ is negligibly small compared to $\exp\left(-f_d(L^{-d-3/2)k}\sigma)\frac{\delta^2}{\sigma_k^2}\right)$, so that

$$I\!P\left[\exists_{k\geq 0}\bar{\mathcal{N}}^{(k)}_{\mathcal{L}^{-k}x}(h) \neq 0\right] \leq \sum_{k=0}^{\infty} c'' \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)k}\frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{3.106}$$

and since the last sum converges rapidly, and $\bar{\mathcal{N}}$ bounds the sup in (3.105) from above, this proves the corollary. $\diamondsuit$

Let us denote by $D^{(k)}(h) \subset \mathbb{Z}^d$ the sets

$$D^{(k)}(h) \equiv \left\{x \in \mathbb{Z}^d \,|\, \sup_{h'\in\mathbb{Z}}\left(N^{(k)}_{\mathcal{L}^{-k}x}(h+h') - \frac{c}{L}|h'|\right) > 0\right\} \tag{3.107}$$

and set

$$\Delta^{(k)}(h) \equiv \bigcup_{i=0}^{k} \overline{D^{(k)}(h)} \tag{3.108}$$

In this terminology corollary 3.4 states that even $\widetilde{D}^{(\infty)}(h)$ is a very sparse set, for any h. But this statement has an immediate implication for the ground states, via the following

PROPOSITION 3.5: *Let $\Lambda_n \equiv \mathcal{L}^n 0$, and let $\mathcal{G}^{(0)}_{\Lambda_n}$ be defined through (3.1). Then for any $\Gamma^* \in \mathcal{G}^{(0)}_{\Lambda_n}$,*

$$\underline{\Gamma^*} \subset \Delta^{(n)}(0) \cap \Lambda_n \tag{3.109}$$

Proof: Let γ_i^* denote the maximal weakly connected components of Γ^*. It is clear that for all these components $h_{\partial \operatorname{int} \underline{\gamma_i^*}} = 0$. Let $\tilde{\gamma}_i^*$ denote the 'outer' connected component of γ_i^*, i.e. the connected component of γ_i^* s.t. the supports of all its other connected components are contained in the interior of its support (by the definition of weak connectivity, such a component must exist). If $\tilde{\gamma}_i^*$ is 'small' (in the sense of definition 3.5), since it occurs in a ground state, by Lemma 3.1, it is 'flat' (i.e. $h_x(\tilde{\gamma}_i^*) \equiv 0$) and its support is contained in $\overline{D(0)}$. Then all the other connected components of γ_i^* are also small, so that γ_i^* is flat and its support is contained in $\overline{D(0)}$ (in fact, the support is contained in $\overline{\mathcal{D}(0)} \subset \overline{D(0)}$, which in the first step is even empty). Thus $\underline{\Gamma^*} \subset \underline{\Gamma^{*,l}} \cup D^{(0)}(0)$. On the other hand, $\underline{\Gamma^{*,l}} \subset \mathcal{L}(\underline{\mathcal{R}\Gamma^*})$; again the support of the small components of $\mathcal{R}\Gamma^*$ will be contained, by the same argument in the closure of the small parts of the new bad regions, and so $\mathcal{L}(\underline{\mathcal{R}\Gamma^{*,s}}) \subset D^{(1)}$, while $\mathcal{L}(\underline{\mathcal{R}\Gamma^{*,l}}) \subset \mathcal{L}^2(\underline{\mathcal{R}^2\Gamma^{*,l}})$. This may be iterated as long as the renormalized contours still have non-empty supports; in the worst case, after n steps, we are left with $\mathcal{R}^n\Gamma^*$, whose support, if not empty, may only consist of the single point 0, and this only if 0 is in the n-th level bad set. But this proves the proposition. $\diamondsuit$

This proposition, together with corollary 3.4 already gives a very nice information about the ground states, namely that for any site x, the value of the height is zero with probability that is exponentially close to one. From this it is almost evident that a 'flat' ground state in the sense of (3.6) will exist, i.e. we will now proof the main theorem of this section:

THEOREM 3.1: *Let $d \geq 3$ and σ^2 small enough. Then, for any integer $h \in \mathbb{Z}$, the set $\mathcal{G}_\infty^{(0)} \neq \emptyset$, $\mathbb{P}$-a.s.*

Remark: It is clear that the same holds height 0 replaced by any other height $h \in \mathbb{Z}$; from the previous observation it is also clear that these ground states for different heights are different.

Proof: Let Λ' be any finite box, let $\widetilde{D}_i^{(\infty)}(0)$ denote the connected components of $\Delta^{(\infty)}(0)$ and set

$$\Delta(\Lambda') \equiv \bigcup_{i:\Delta_i^{(\infty)}(0)\cap\Lambda'\neq\emptyset} \Delta_i^{(\infty)}(0) \tag{3.110}$$

i.e. the union of all connected components of $D^{(\infty)}(0)$ that intersect Λ'. Now *if* this set is finite, then, by the preceeding proposition, for all Λ containing this set, the restriction of any $\Gamma^* \in \mathcal{G}_\Lambda^{(0)}$ to Λ' is independent of Λ and depends only on the 'local' random fields. Therefore, there exists Λ_0 s.t. $\bigcap_{\Lambda\supset\Lambda_0} \mathcal{G}_{\Lambda,\Lambda'}^{(0)} \neq \emptyset$, and if indeed this holds for all Λ', this means that $\mathcal{G}_\infty^{(0)} \neq \emptyset$.

Now given the estimates of corollary 3.4, it is almost obvious that $\Delta^{(\infty)}(0)$ contains non infinite connected components attached to Λ'. For, consider the probability that $D^{(k)}(0)$ contains a connected component, C, of size M containing, say, the origin. Using the locality properties of the N, we easily see that

$$\begin{aligned} &\mathbb{P}\left[\exists_{C:|C|=M,0\in C} : C \subset D^{(k)}(0)\right] \\ &\leq \begin{cases} L^{kd} K^{\frac{M}{L^{dk}}} \left[\exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)k} \frac{\delta_k^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right)\right]^{\frac{M}{5^d L^{dk}}}, & \text{if } M > L^{dk} \\ L^{dk} \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)k} \frac{\delta_k^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right), & \text{if } M \leq L^{dk} \end{cases} \end{aligned} \tag{3.111}$$

Here K is a geometrical constant, such that the number of connected sets of size M containing the origin is bounded by K^M. Notice that for given M, this probability is largest for the smallest k s.t. $L^{dk} > M$, and in fact

$$\mathbb{P}\left[\exists_{C:|C|=M,C\cap\Lambda'\neq\emptyset} : C \subset \widetilde{D}^{(\infty)}(0)\right] \leq |\Lambda'| \exp\left(-M^{\frac{d-2}{2(d-1)}-\eta'} \frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{3.112}$$

where $\eta' > \eta$ is chosen such as to absorb the various constants, for large enough M. Since this probability is summable over M, from the Borel-Cantelli it follows that the event considered occurs only a finite number of times, a.s. which is what we want to prove. Finally, only a finite number of connected components can be attached to a finite Λ', which concludes the proof of the theorem. $\diamondsuit\diamondsuit$

The main result for the $T = 0$ case is now proven: There exist 'flat' infinite volume ground states in dimension $d \geq 3$ if the disorder is sufficiently weak. It is, by the way, also clear that this flat ground state is unique under some weak assumptions on the distribution of the J that excludes local degeneracies; continuity, for instance is sufficient (but not necessary). Also, we already know how the ground state looks outside a very sparse region, but so far we have not said anything about how it may look like within that region. To get a precise knowledge of about the ground states in the bad regions would require a more careful analysis of the RG map, keeping track on more parameters in the renormalized contours within the bad region than we do (for example, if in the blocking procedure the height on a block has large fluctuations about its (maybe) small average, we simply forget about them, although they must be accompanied by large excess surface energy. One might thus for instance carry through the estimates an extra parameter keeping track of maximal and minimal heights that occurred in the history of a block. We will not go into such a detailed analysis here). However, even without doing this, the results we already have can give rough estimates on the probability distribution of the height, say at 0 of the ground state contour and we will present them as a last result of this section.

PROPOSITION 3.6: *Let Γ^* be an element of $\mathcal{G}_\infty^{(0)}$. Then,*

$$\mathbb{P}\left[|h_0(\Gamma^*)| \geq h\right] \leq \exp\left(-\frac{h^{\gamma_d}}{\sigma^{2-\mu_d}}\right) \tag{3.113}$$

where γ_d, μ_d are positive, d dependent constants for $d \geq 3$.

Proof: Let us denote by Γ_n^* an element of $\mathcal{G}_{\Lambda_n}^{(0)}$, and let A_n be the event that $h_0(\Gamma^*) = h_0(\Gamma_n^*)$. The point is that $h_0(\Gamma_n^*)$ can be estimated in probability a priori, for n not too large (depending on h), while it is quite unlikely that A_n occurs only for very large n. Thus let $B_n \equiv \cap_{k\geq n}^{\infty} A_n$. Then, for any n we have that

$$\begin{aligned}\mathbb{P}\left[|h_0(\Gamma^*)| \geq h\right] =& \mathbb{P}\left[|h_0(\Gamma^*)| \geq h \wedge (B_n \vee B_n^c)\right] \\ &\leq \mathbb{P}\left[|h_0(\Gamma_n^*)| \geq h \wedge B_n\right] + \mathbb{P}\left[|h_0(\Gamma^*)| \geq h \wedge B_n^c\right] \\ &\leq \mathbb{P}\left[|h_0(\Gamma_n^*)| \geq h\right] + \mathbb{P}\left[B_n^c\right]\end{aligned} \tag{3.114}$$

The desired bound will be obtained by choosing n in dependence on h such as to minimize the right-hand side of (3.114). Now B_n^c occurs only if the site zero is contained in the interior of the support of a connected component of Γ^* that depasses Λ_n. By proposition 3.5 and the estimates used in the proof of the theorem, it is clear that the probability of this to happen is bounded by

$$\mathbb{P}\left[B_n^c\right] \leq \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta'\right)n}\frac{\delta^2}{\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{3.115}$$

To bound the other contribution in (3.114), notice that for any possible height-function that vanishes outside Λ_n, by Lemma 3.3,

$$E_s(h(\Gamma_n^*)) \geq \frac{|h_0(\Gamma_n^*)|}{2}\frac{d}{L^n}\sum_{x\in\mathcal{L}^n 0}|h_x(\Gamma_n^*)| \tag{3.116}$$

Therefore,

$$I\!P\left[|h_0(\Gamma_n^*)| \geq h\right] \leq I\!P\left[\sum_{x\in\mathcal{L}^n 0}\inf_{h_x\in\mathbb{Z}}\left(\frac{d}{L^n}|h_x| + J_x(h_x) - J_x(0)\right) < -\frac{h}{2}\right] \tag{3.117}$$

Let us set

$$I_x^{(n)} \equiv \inf_{h\in\mathbb{Z}}\left(\frac{d}{L^n}|h| + J_x(h) - J_x(0)\right) \tag{3.118}$$

Given the bounds on J, it is easy to show that

$$I\!P\left[I_x^{(n)} \leq -z\right] \leq \frac{\pi\sigma L^n}{d}\exp\left(-\frac{z^2}{4\sigma^2}\right) \tag{3.119}$$

which in turn is bounded by $\exp\left(-\frac{z^2}{8\sigma^2}\right)$, for $z \geq z_0 \equiv \sigma\sqrt{8\ln(\frac{2\pi\omega L^n}{d})}$. From that a simple computation shows that

$$0 \geq I\!E(I_x^{(n)}) \geq -\sigma\sqrt{8\ln\frac{2\pi\omega L^n}{d}} \tag{3.120}$$

and with $\tilde{I}_x^{(n)} \equiv I_x^{(n)} - I\!E(I_x^{(n)})$,

$$\begin{aligned} I\!P\left[\tilde{I}_x^{(n)} \leq -z\right] &\leq \exp\left(-\frac{z^2}{8\sigma^2}\right) \\ I\!P\left[\tilde{I}_x^{(n)} \geq z\right] &= 0 \leq \exp\left(-\frac{z^2}{8\sigma^2}\right) \end{aligned} \tag{3.121}$$

for $z \geq z_0$. Using essentially Lemma 3.11, some rescaling and some rather rough overestimation, we get from this that

$$I\!P\left[\sum_{x\in\mathcal{L}^n 0}\tilde{I}_x^{(n)} \leq -z\right] \leq \begin{cases} \exp\left(-\frac{z^2}{8\sigma^2 L^{nd}}\right), & \text{if } z_0 \leq 1 \\ \exp\left(-\frac{z^2}{8\sigma^2 L^{nd} z_0^2}\right), & \text{if } z_0 \geq 1 \end{cases} \tag{3.122}$$

and so

$$I\!P\left[\sum_{x\in\mathcal{L}^n 0} I_x^{(n)} \leq -\frac{h}{2}\right] \leq \begin{cases} \exp\left(-\frac{(h/2 - L^{dn}z_0)^2}{8\sigma^2 L^{nd}}\right), & \text{if } z_0 \leq 1 \\ \exp\left(-\frac{(h/2 - L^{nd}z_0)^2}{8\sigma^2 L^{nd} z_0^2}\right), & \text{if } z_0 \geq 1 \end{cases} \tag{3.123}$$

We see that to make use of this, we must choose n small enough, s.t. $h/2 \geq L^{dn}z_0$. The optimal choice of n is now in principal found by equating the bound from (3.123) with the probability of B_n^c; a rough estimate of the yields the solution

$$I\!P\left[|h_0(\Gamma^*)| \geq h\right] \leq \exp\left(-\frac{h^{\rho/d}\delta^2}{\sigma^{1+\rho/d-\gamma}\left(\ln(\frac{2\pi\sigma L^n}{d})\right)^{\rho/d}}\right) \tag{3.124}$$

with $\rho = \frac{d-2}{2(d-1)} - \eta'$ and $\gamma = \frac{d-2}{d-1}$ which proves the proposition. $\diamondsuit$

With this bound on the height we conclude our analysis of the ground state. It should be clear that further and more detailed information can in principle be extracted from the RG and our estimates. The task of the next section will be to carry over these results to the finite temperature case and the Gibbs measures.

IV. The Gibbs states at finite temperature

In this section we repeat the construction and analysis of the renormalization maps from Section 3 for the finite temperature Gibbs measures. The steps will follow closely those of the previous section and we will be able to make use of many of the results obtained there. In fact, no new 'serious' problems will have to be dealt with here, and in particular the probabilistic analysis of Section 3.5 will mostly carry over. The difficulties here lie mostly in the technicalities of the various expansions that we will have to use.

IV.1 Set-up and inductive assumptions

Just as in Section 3 an object of crucial importance will be the control field $N_x(h)$; given such a field, the corresponding bad regions $D \equiv D(N)$ are defined just as there, with one exception: rather than considering the $\sup_{h'\in\mathbb{Z}}\left(N_x(h+h') - \frac{c}{L}|h'|\right)$, we introduce

$$\mathcal{N}_x(h) \equiv \begin{cases} \sup_{h'\in\mathbb{Z}}\left(N_x(h+h') - \frac{c}{2L}|h'|\right), & \text{if } \sup_{h'\in\mathbb{Z}}\left(N_x(h+h') - \frac{c}{2L}|h'|\right) \geq \frac{c}{2L} \\ N_x(h), & \text{otherwise} \end{cases} \tag{4.1}$$

The point here is that $\mathcal{N}_x(h) \geq \sup_{h'\in\mathbb{Z}}\left(N_x(h+h') - \frac{c}{L}|h'|\right)$, so that we may substitute it for the original sup without harm (and we may be generous regarding the constant c), while with this definition, the recursively defined $\bar{N}_x^{(k)}$ (see Section 3.4) will strictly (and not only with large probability) have the property to be either zero or large than $L^{-(d-3/2)k}\sigma$. This will turn out to be, if not strictly necessary, at least convenient in the case of finite temperatures. The specific reasons for this will be detailed at the end of Subsection 4.3.

Now to define the bad set, Eq. (3.10) in Definition 3.1 is simply replaced by

$$D \equiv D(N) \equiv \left\{(x,h) \in \Lambda_n \times \mathbb{Z} \middle| \mathcal{N}_x(h) > 0\right\} \tag{4.2}$$

Now analogously to Definition 3.2 we will now define an N-bounded contour measure:

DEFINITION 4.1: *A N-bounded contour measure is a probability measure on $\Omega_n(D)$ of the form*

$$\mu(\Gamma) = \frac{1}{Z} e^{-\beta(S,V(\Gamma))} \sum_{\Lambda_n \supset G \supset \underline{\Gamma}} \rho(\Gamma, G) \tag{4.3}$$

where

(i) S is a non-local small random field, that is a map that assign to each connected (non-empty) set $C \subset \Lambda_n$ and each height h a real number $S_C(h)$ that satisfy

$$|S_C(h)| \leq e^{-\bar{\beta}|C|}, \quad \text{if} \quad |C| > 1 \tag{4.4}$$

and for sets made of a single point x,

$$|S_x(h)| \leq \delta \tag{4.5}$$

The notion $(S, V(\Gamma))$ is shorthand for

$$(S, V(\Gamma)) \equiv \sum_{h \in \boldsymbol{Z}} \sum_{C \subset V_h(\Gamma)} S_C(h) \tag{4.6}$$

(ii) $\rho(\Gamma, G)$ are positive activities factorizing over connected components of G, i.e. if $(G_1, \ldots, G_l)$ are the connected components of G and if Γ_i denotes the contour made from those connected components of Γ those supports are contained in G_i, then

$$\rho(\Gamma, G) = \prod_{i=1}^{l} \rho(\Gamma_i, G_i) \tag{4.7}$$

They satisfy the upper bound

$$0 \leq \rho(\Gamma, G) \leq e^{-\beta E_s(\Gamma) - \tilde{\beta}|G \setminus \overline{D(\Gamma)}| + \beta(N, V(\Gamma) \cap \underline{\Gamma})} \tag{4.8}$$

Let $C \subset D(h)$ be connected and $\gamma = (C, h_x(\Gamma) \equiv h)$ be a connected component of a contour $\Gamma \subset \Omega_n(D)$. Then

$$\rho(\gamma, C) \geq e^{-\beta(N, V(\gamma) \cap C)} \tag{4.9}$$

Z is of course the partition function that turns μ into a probability measure.

Here β and $\tilde{\beta}$ are parameters ('temperatures') that will be renormalized in the course of the iterations. Note that we have not adorned the μ and ρ's with them as indices, nor with the finite volumes Λ_n, although of course they depend on these parameters as well as on others, in order to keep notations as streamlined as possible.

We must remark on some differences between our assumptions and those used in [BK]. Loosely speaking, the sets G are what [BK] call the 'outer supports'; however, in their method, a renormalization of the normal supports is not maintained. They are, in fact, forgotten after each RG step and the outer supports become the new inner supports, while a new outer support is created. This allows to perform the RG really only on spin configurations but not on contours. We felt that a formulation that allows to renormalize contour models more appealing, particularly in view of the analysis of the ground states. Also, [BK] keep track of an extra non-local interaction, called $W(\Gamma)$. It turns out this is unnecessary and disturbing.

The probabilistic assumptions on stationarity and locality of the quantities appearing here are completely analogous to those in Section III and we will not restate them; all quantities depending on sets C are of course supposed to be measurable w.r.t. $\mathcal{B}_{\overline{C}}$, etc.

The definition of a proper RG transformation will now be adopted to this set-up.

DEFINITION 4.2: *For a given control field N, a proper renormalization group transformation, $\mathcal{R}^{(N)}$, is a map from $\Omega_n(D(N))$ into $\Omega_{n-1}(D(N')$, such that if μ is a N-bounded contour measure on $\Omega_n(D(N))$ with 'temperatures' β and $\tilde{\beta}$ and small field S (of level k), then $\mu'_{\Lambda_{n-1}} \equiv \mathcal{R}^{(N)}\mu_{\Lambda_n}$ is a N'-bounded contour measure on $\Omega_{n-1}(D(N')$ for some control field N', with temperatures β' and $\tilde{\beta}'$ and small field S' (of level $k+1$).*

Remark: We will see (and anticipate) that $\beta' = L^{d-1-\alpha}\beta$, $\tilde{\beta}' = L^{1-\alpha}\tilde{\beta}$; N' will be defined as in Section 3.

IV.2 Absorption of small contours

The construction of the map T_1 on the level of contours proceeds now exactly as before, i.e. Definition 3.4 still defines the harmless large field region, Definition 3.5 the 'small' contours and Definition 3.6 the map T_1. What we have to do is to control the induced action of T_1 on the contour measures. Let us for convenience denote by $\hat{\mu} \equiv Z\mu$ the non-normalized measures; this only simplifies notations since T_1 leave the partition functions invariant (i.e. $T_1\mu = \frac{1}{Z}T_1\hat{\mu}$).

Of course we have for any $\Gamma^l \in \Omega^l_N(D)$

$$
\begin{aligned}
(T_1\hat{\mu})(\Gamma^l) &\equiv \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} \hat{\mu}(\Gamma) \\
&= \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} e^{-\beta(S,V(\Gamma))} \sum_{G\supset\underline{\Gamma}} \rho(\Gamma, G)
\end{aligned}
\tag{4.10}
$$

Now we write

$$
(S,V(\Gamma)) = (S,V(\Gamma^l)) + \left[(S,V(\Gamma)) - (S,V(\Gamma^l))\right]
\tag{4.11}
$$

Here the first term is of course what we would like to have; the second reads explicitly

$$
\begin{aligned}
\left[(S,V(\Gamma)) - (S,V(\Gamma^l))\right] &= \sum_{h\in\mathbb{Z}} \left[\sum_{x\in V_h(\Gamma)\cap \operatorname{int}\underline{\Gamma^s}} S_x(h) - \sum_{x\in V_h(\Gamma^l)\cap \operatorname{int}\underline{\Gamma^s}} S_x(h) \right] \\
&+ \sum_{h\in\mathbb{Z}} \left[\sum_{\substack{C\subset V_h(\Gamma) \\ C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset}} S_C(h) - \sum_{\substack{C\subset V_h(\Gamma^l) \\ C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset}} S_C(h) \right] \\
&\equiv \delta S_{loc}(\Gamma,\Gamma^l) + \delta S_{nl}(\Gamma,\Gamma^l)
\end{aligned}
\tag{4.12}
$$

where we used the suggestive notation $\underline{\Gamma^s} \equiv \underline{\Gamma}\backslash\underline{\Gamma^l}$. Note also that all sets C are assumed to have volume at least 2 and are assumed to be connected. The conditions on C (resp. x) to intersect $\underline{\Gamma^s}$ just make manifest that otherwise the two contributions cancel. Thus all these unwanted terms are attached to the supports of the 'small' components of Γ. The local piece, δS_{loc} thus poses no particular problem. The non-local piece, however, may join up 'small' and 'large' components, which spoils the factorization properties of ρ. To overcome this difficulty, we apply a cluster-expansion, a trick that will be used again later. It is useful to introduce the notation

$$\tilde{\sigma}_{\Gamma,\Gamma^l}(C) \equiv \sum_{h\in\mathbb{Z}} S_C(h)\left(\mathbb{I}_{C\subset V_h(\Gamma)} - \mathbb{I}_{C\subset V_h(\Gamma^l)}\right) \tag{4.13}$$

so that

$$\delta S_{nl}(\Gamma,\Gamma^l) = \sum_{C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset} \tilde{\sigma}_{\Gamma,\Gamma^l}(C) \tag{4.14}$$

Unfortunately the $\tilde{\sigma}_{\Gamma,\Gamma^l}(C)$ have arbitrary signs. Therefore expanding $\exp(-\beta\delta S_{nl})$ directly would produce a polymer systems with possibly negative activities (see below). However, by assumption,

$$|\tilde{\sigma}_{\Gamma,\Gamma^l}(C)| \leq \sup_{h\in\mathbb{Z}} |S_C(h)| \leq e^{-\tilde{\beta}|C|} \equiv f(C) \tag{4.15}$$

Therefore, $\tilde{\sigma}_{\Gamma,\Gamma^l}(C) - f(C) \leq 0$ and setting

$$F(\ \operatorname{int}\underline{\Gamma^s}) \equiv \sum_{C\cap \operatorname{int}\underline{\Gamma^s}\neq\emptyset} f(C) \tag{4.16}$$

we get

$$e^{-\beta\delta S_{nl}(\Gamma,\Gamma^l)} = e^{-\beta F(\operatorname{int}\underline{\Gamma^s})} e^{\beta\sum_{C\cap\operatorname{int}\underline{\Gamma^s}\neq\emptyset}(f(C)-\tilde{\sigma}_{\Gamma,\Gamma^l}(C))} \tag{4.17}$$

where the second exponential could be expanded in a sum over positive activities. The first exponential is not yet quite what we would like, since it does not factor over connected components. However, it is dominated by such a term, and the remainder may be added to the σ-terms. This will follow from the next Lemma.

LEMMA 4.1: *Let $A\subset\mathbb{Z}^d$ and let $(A_1,\dots,A_l)$ be its connected components. Let $F(A)$ be as defined in (4.17) and set*

$$\delta F(A) \equiv F(A) - \sum_{i=1}^{l} F(A_i) \tag{4.18}$$

Then

$$\delta F(A) = -\sum_{C\cap A\neq\emptyset} k(A,C) f(C) \tag{4.19}$$

where

$$0 \leq k(A,C)f(C) \leq e^{-\tilde{\beta}(1-\kappa)|C|} \tag{4.20}$$

for $\kappa = \tilde{\beta}^{-1}$

Proof: The proof of this lemma is very simple. Obviously, $\sum_{i=1}^{l} F(A_i)$ counts all C that intersect k connected components of A exactly k times, whereas in $F(A)$ such a C appears only once. Thus

(4.19) holds with $k(A,C) = \#\{A_i : A_i \cap C \neq \emptyset\} - 1$. Furthermore, if C intersects k components, then certainly $|C| \geq k$, from which the upper bound in (4.20) follows. ◊

Now we can bring the non-local terms in their final form:

LEMMA 4.2: *Let $\delta S_{nl}(\Gamma, \Gamma^l)$ be defined in (4.12). Then*

$$
\begin{aligned}
e^{-\beta\delta S_{nl}(\Gamma,\Gamma^l)} &= r(\underline{\Gamma}^s)\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{\substack{C_1,\dots,C_l \\ C_i \cap \operatorname{int}\underline{\Gamma}^s \neq \emptyset \\ C_i \neq C_j}}\prod_{i=1}^{l}\phi_{\Gamma,\Gamma^l}(C_i) \\
&\equiv r(\underline{\Gamma}^s)\sum_{C: C\cap \operatorname{int}\underline{\Gamma}^s \neq \emptyset}\phi_{\Gamma,\Gamma^l}(C)
\end{aligned}
\tag{4.21}
$$

where $\phi_{\Gamma,\Gamma^l}(C)$ satisfies

$$
0 \leq \phi_{\Gamma,\Gamma^l}(C) \leq e^{-\tilde\beta|C|/2}
\tag{4.22}
$$

$r(\underline{\Gamma}^s)$ is a non-random positive activity factoring over connected components of $\operatorname{int}\underline{\Gamma}^s$; for a weakly connected component γ^s,

$$
1 \geq r(\underline{\gamma}^s) \equiv e^{-\beta F(\operatorname{int}\underline{\gamma}^s)} \geq e^{-\beta|\operatorname{int}\underline{\gamma}^s|e^{-\beta}}
\tag{4.23}
$$

Proof: Define for $|C| \geq 2$

$$
\sigma_{\Gamma,\Gamma^l}(C) \equiv \tilde\sigma_{\Gamma,\Gamma^l}(C) - f(C)(k(\operatorname{int}\underline{\Gamma}^s, C) + 1)
\tag{4.24}
$$

Then we may write

$$
\begin{aligned}
e^{-\beta\sum_{C\cap \operatorname{int}\underline{\Gamma}^s\neq\emptyset}\sigma_{\Gamma,\Gamma^l}(C)} &= \prod_{C\cap \operatorname{int}\underline{\Gamma}^s\neq\emptyset}\left(e^{-\beta\sigma_{\Gamma,\Gamma^l}(C)} - 1 + 1\right) \\
&= \sum_{l=0}^{\infty}\sum_{\substack{C_1,\dots,C_l \\ C_i \cap \operatorname{int}\underline{\Gamma}^s \neq \emptyset \\ C_i \neq C_j}}\prod_{i=1}^{l}\left(e^{-\beta\sigma_{\Gamma,\Gamma^l}(C_i)} - 1\right)
\end{aligned}
\tag{4.25}
$$

which gives (4.21). But since $|\sigma_{\Gamma,\Gamma^l}(C)| \leq 2e^{-\tilde\beta(1-\kappa)|C|}$ by (4.20) and the assumption on $S_C(h)$, (4.22) follows if only $2\beta \leq e^{\tilde\beta(1-2\kappa)/2}$. Let us remark that given the behaviour of β and $\tilde\beta$ as given in the remark after Definition 4.2, this relation holds if it holds initially. The initial choice will be $\tilde\beta = \beta/L$, and with this relation we must only choose β large enough, e.g. $\beta \geq L(\ln L)^2$ will do.

The properties of $r(\underline{\Gamma}^s)$ follow from Lemma 4.1. Note that these activities depend only on the geometry of the support of Γ^s and are otherwise non-random. ◊

We can now write

$$
\begin{aligned}
(T_1\hat{\mu})(\Gamma^l) &= e^{-\beta(S,V(\Gamma^l))} \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} r(\underline{\Gamma^s}) \sum_{G\supset\underline{\Gamma}} \rho(\Gamma,G) e^{-\beta\delta S_{loc}(\Gamma,\Gamma^l)} \sum_{\mathcal{C}:\mathcal{C}\cap \mathrm{int}\,\underline{\Gamma^s}\neq\emptyset} \phi_{\Gamma,\Gamma^l}(\mathcal{C}) \\
&= e^{-\beta(S,V(\Gamma^l))} \sum_{\Gamma:T_1(\Gamma)=\Gamma^l} \sum_{K\supset\underline{\Gamma}} \sum_{\underline{\Gamma}\subset G\subset K} \sum_{\substack{\mathcal{C}\subset K\\ \mathcal{C}\cap\mathrm{int}\,\underline{\Gamma^s}\neq\emptyset\\ \mathcal{C}\cup G=K}} \\
&\quad\times r(\underline{\Gamma^s})\rho(\Gamma,G)e^{-\beta\delta S_{loc}(\Gamma,\Gamma^l)}\phi_{\Gamma,\Gamma^l}(\mathcal{C})
\end{aligned}
\tag{4.26}
$$

Now we may decompose the set K into its connected components and call K_1 the union of those components that contain components of $\underline{\Gamma^l}$. Naturally we call $K_2 = K\backslash K_1$. Note that everything factorizes over these two sets, including the sum over Γ (the possible small contours that can be inserted into Γ^l being independent from each other in these sets). We can make this explicit by writing

$$
\begin{aligned}
(T_1\hat{\mu})(\Gamma^l) = e^{-\beta(S,V(\Gamma^l))} &\sum_{K_1\supset\underline{\Gamma^l}} \sum_{\Gamma_1:T_1(\Gamma_1)=\Gamma^l} \sum_{\underline{\Gamma_1}\subset G_1\subset K_1} \sum_{\substack{\mathcal{C}_1\subset K_1\\ \mathcal{C}_1\cap\mathrm{int}\,\underline{\Gamma_1^s}\neq\emptyset\\ \mathcal{C}_1\cup G_1=K_1}} \\
&r(\underline{\Gamma_1^s})\rho(\Gamma_1,G_1)e^{-\beta\delta S_{loc}(\Gamma_1,\Gamma^l)}\phi_{\Gamma_1,\Gamma^l}(\mathcal{C}_1) \\
&\times \sum_{K_2:K_2\cap\overline{K_1}=\emptyset} \sum_{\Gamma_2:T_1(\Gamma_2)=\Gamma^l} \sum_{\underline{\Gamma_2}\subset G_2\subset K_2} \sum_{\substack{\mathcal{C}_2\subset K_2\\ \mathcal{C}_2\cap\mathrm{int}\,\underline{\Gamma_2^s}\neq\emptyset\\ \mathcal{C}_2\cup G_2=K_2}} \\
&r(\underline{\Gamma_2^s})\rho(\Gamma_2,G_2)e^{-\beta\delta S_{loc}(\Gamma_2,\Gamma^l)}\phi_{\Gamma_2,\Gamma^l}(\mathcal{C}_2) \\
&\equiv e^{-\beta(S,V(\Gamma^l))} \sum_{K_1\supset\underline{\Gamma^l}} \hat{\rho}(\Gamma^l,K_1) \sum_{K_2:K_2\cap\overline{K_1}=\emptyset} \tilde{\rho}(\Gamma^l,K_2)
\end{aligned}
\tag{4.27}
$$

Here, of course, the contours Γ_1 and Γ_2 are understood to have small components with supports only within the sets K_1 and K_2, respectively. Also, of course, the set K_2 must contain $D(\Gamma^l)\cap K_1^c$. Now the final form of (4.27) is almost the original one, except for the sum over K_2. This latter will give rise to an additional (non-local) field term, as we will explain now.[5]

Notice that the sum over K_2 can be factored over the connected components of K_1^c. In these components, $\tilde{\rho}$ depends on Γ^l only through the (constant) height $h(\Gamma^l)$ in this component. Let Y denote such a connected component and let h be the corresponding height. We have

LEMMA 4.3: *Let $\tilde{\rho}$ be defined in (4.27). Then*

$$
\sum_{\mathcal{D}(h)\cap Y\subset K\subset Y} \tilde{\rho}(h,K) = e^{\beta\sum_{\mathcal{C}\subset Y}\psi_{\mathcal{C}}(h)}
\tag{4.28}
$$

[5] The fact that a non-local field is produced here even then initially no such field is present is of course the reason to include such fields in the inductive assumptions

where the sum is over connected sets C and

$$|\psi_C(h)| \leq \begin{cases} e^{-g\tilde{\beta}|C\backslash\mathcal{D}(h)|}, & \text{if } C\backslash\mathcal{D}(h) \neq \emptyset \\ \sigma^2\frac{\tilde{\beta}}{\beta} + \frac{L^{(1-\alpha)/2}}{\beta}e^{-\tilde{\beta}}, & \text{if } C \subset \mathcal{D}(h) \end{cases} \tag{4.29}$$

where $0 < g < 1$ is some constant. Note that $\psi_C(h)$ are random and depend in particular on the geometry of $\mathcal{D}(h)$.

Proof: Naturally, the form (4.28) will be obtained through a Mayer-expansion, provided we get the necessary estimates on the activities $\tilde{\rho}$. Let K be connected and recall that

$$\tilde{\rho}(h,K) = \sum_{\Gamma:T_1(\Gamma)=(\emptyset,h}\sum_{\underline{\Gamma}\subset G\subset K}\sum_{\substack{\mathcal{C}\subset K\\ \mathcal{C}\cap\mathrm{int}\,\underline{\Gamma'}\neq\emptyset\\ \mathcal{C}\cup G=K}} e^{-\beta\sum_{x\in\underline{\Gamma}}[S_x(h_x(\Gamma))-S_x(h)]}r(\underline{\Gamma})\rho(\Gamma,G)\phi_{\Gamma,h}(\mathcal{C}) \tag{4.30}$$

Note that by assumption,

$$\left|\sum_{x\in\underline{\Gamma}}[S_x(h_x(\Gamma))-S_x(h)]\right| \leq 2\delta|\underline{\Gamma}| \tag{4.31}$$

and

$$|\rho(\Gamma,G)| \leq e^{-\beta E_s(\Gamma)-\tilde{\beta}|G\backslash(\overline{\mathcal{D}(h)}|+\beta(N,V(\Gamma)\cap\underline{\Gamma})} \tag{4.32}$$

Let us now estimate the contribution from the sum over the $\mathcal{C}$. We will prove the following

LEMMA 4.4: *For $\tilde{\beta}$ large enough, there exists a finite constant $0 < g < 1$,*

$$\sum_{\substack{\mathcal{C}\subset K,\mathcal{C}\cup G=K\\ \mathcal{C}\cap\mathrm{int}\,\underline{\Gamma}\neq\emptyset}} \phi_{\Gamma,h}(\mathcal{C}) \leq e^{|\mathrm{int}\,\underline{\Gamma}|e^{-g\tilde{\beta}}}e^{-g\tilde{\beta}|K\backslash G|} \tag{4.33}$$

Proof: Let us first consider the case where $G = K$. The the condition $\mathcal{C}\cup G = K$ is void, and we may therefore just reverse the process used to get Lemma 4.2, getting

$$\sum_{\substack{\mathcal{C}\subset K\\ \mathcal{C}\cap\mathrm{int}\,\underline{\Gamma'}\neq\emptyset}} \phi_{\Gamma,h}(\mathcal{C}) = e^{-\beta\sum_{C\subset K,C\cap\mathrm{int}\,\underline{\Gamma}\neq\emptyset}\sigma_{\Gamma,\Gamma'}(C)} \tag{4.34}$$

Now

$$\begin{aligned}\left|\sum_{C\subset K,C\cap\mathrm{int}\,\underline{\Gamma}\neq\emptyset}\sigma_{\Gamma,\Gamma'}(C)\right| &\leq \sum_{v=2}^{|K|}\sum_{x\in\underline{\Gamma}}\sum_{\substack{C\ni x\\ |C|=v}} e^{-\tilde{\beta}v} \\ &\leq |\;\mathrm{int}\,\underline{\Gamma}|\sum_{v=2}^{\infty}b^v e^{-\tilde{\beta}v} = |\;\mathrm{int}\,\underline{\Gamma}|\frac{b^2e^{-2\tilde{\beta}}}{1-be^{-\tilde{\beta}}}\end{aligned} \tag{4.35}$$

Here b is some dimension-dependent constant such that b^v bounds the number of connected subsets of volume v that contain a specified point. For $\tilde{\beta}$ somewhat large, this yields the upper and lower bounds

$$e^{-|\mathrm{int}\,\underline{\Gamma}|e^{-\tilde{\beta}}} \leq \sum_{\substack{\mathcal{C}\subset K\\ \mathcal{C}\cap\mathrm{int}\,\underline{\Gamma'}\neq\emptyset}} \phi_{\Gamma,h}(\mathcal{C}) \leq e^{+|\mathrm{int}\,\underline{\Gamma}|e^{-\tilde{\beta}}} \tag{4.36}$$

We now turn to the case where $K\backslash G \equiv M \neq \emptyset$. In this case, $\mathcal{C}$ must have volume at least $|M|$. We may estimate

$$\begin{aligned}\left|\sum_{\substack{\mathcal{C}\subset K,\,\mathcal{C}\cup G=K\\ \mathcal{C}\cap \operatorname{int}\underline{\Gamma}^s\neq\emptyset}} \phi_{\Gamma,h}(\mathcal{C})\right| &\leq \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{\substack{C_1,\dots,C_l\\ C_i\cap\operatorname{int}\underline{\Gamma}\neq\emptyset,\,C_i\subset K\\ \cup_{i=1}^{l}C_i\cap G=K}}\prod_{i=1}^{l}e^{-\tilde\beta|C_i|/2}\\ &\leq \sum_{l=1}^{\infty}\frac{1}{l!}\sum_{v\geq|M|}\sum_{\substack{v_1,\dots,v_l\geq 2\\ \sum_i v_i=v}} |\operatorname{int}\underline{\Gamma}|^l b^v e^{-\tilde\beta v/2}\end{aligned} \tag{4.37}$$

where we have overestimated by ignoring the constraint that $C_i \neq C_i$ and have bounded the sum over connected sets C_i of fixed volume just like before. Now the sum over the v_i can easily be computed,

$$\sum_{\substack{v_1,\dots,v_l\geq 2\\ \sum_i v_i=v}} 1 = \binom{v-l-1}{l-1} \tag{4.38}$$

So we have to estimate the sum

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{|\operatorname{int}\underline{\Gamma}|^l}{l!}\sum_{v=m}^{\infty}\binom{v-l-1}{l-1}c^v &= \sum_{l=1}^{m/2}\frac{|\operatorname{int}\underline{\Gamma}|^l}{l!}\sum_{v=m}^{\infty}\binom{v-l-1}{l-1}c^v\\ &+\sum_{l=m/2+1}^{\infty}\frac{|\operatorname{int}\underline{\Gamma}|^l}{l!}\sum_{v=m}^{\infty}\binom{v-l-1}{l-1}c^v\end{aligned} \tag{4.39}$$

where $m \equiv |M|$ and $c \equiv be^{-\tilde\beta/2}$, and $m/2$ is understood to be the largest integer less than or equal $m/2$. Now for $2l \leq m$, $\binom{v-l-1}{l-1} \leq \binom{v-m/2-1}{m/2-1}$, so that in the first sum we may use

$$\begin{aligned}\sum_{v=m}^{\infty}\binom{v-l-1}{l-1}c^v &\leq c^{m/2}\sum_{v=m/2}^{\infty}\binom{v-1}{m/2-1}c^v\\ &\leq c^{m/2}\sum_{v=m/2}^{\infty}\frac{v^{m/2}c^v}{(m/2)!}\leq\frac{c^{m/2}}{|\ln c|^{m/2}}\end{aligned} \tag{4.40}$$

where the last estimate involves bounding the sum by an integral and recalling the definition of the Gamma-function. In the second sum we use similarly that

$$\sum_{v=2l}^{\infty}\binom{v-l-1}{l-1}c^v = c^l\sum_{v=l}^{\infty}\frac{v^{l-1}c^v}{(l-1)!}\leq\frac{c^l}{|\ln c|^l} \tag{4.41}$$

Inserting these bounds into (4.39), we arrive at

$$\begin{aligned}\sum_{l=1}^{\infty}\frac{|\operatorname{int}\underline{\Gamma}|^l}{l!}\sum_{v=m}^{\infty}\binom{v-l-1}{l-1}c^v &= \sum_{l=1}^{m/2}\frac{|\operatorname{int}\underline{\Gamma}|^l}{l!}\frac{c^{m/2}}{|\ln c|^{m/2}}\\ &+\sum_{l=m/2+1}^{\infty}\frac{|\operatorname{int}\underline{\Gamma}|^l}{l!}\frac{c^l}{|\ln c|^l}\\ &\leq\frac{2}{1-\frac{c^{1/2}}{|\ln c|^{1/2}}}e^{\frac{c^{1/2}}{|\ln c|^{1/2}}|\operatorname{int}\underline{\Gamma}|}\frac{c^{m/4}}{|\ln c|^{m/4}}\end{aligned} \tag{4.42}$$

where to obtain the last line we have used the Schwarz-inequality, splitting $\frac{c^l}{|\ln c|^l} = \frac{c^{l/2}}{|\ln c|^{l/2}} \frac{c^{l/2}}{|\ln c|^{l/2}}$. The bounds in the two cases may now be unified to give the estimate claimed in the lemma. $\diamondsuit$

We are now ready to estimate the sum over all the non-local terms in the definition of $\tilde{\rho}$. Remembering the assumption (4.8), we get that

$$
\begin{aligned}
&\sum_{\underline{\Gamma} \subset G \subset K} \sum_{\substack{\mathcal{C} \subset K \\ \mathcal{C} \cap \operatorname{int} \underline{\Gamma}^s \neq \emptyset \\ \mathcal{C} \cup G = K}} r(\underline{\Gamma}) \rho(\Gamma, G) \phi_{\Gamma,h}(\mathcal{C}) \\
&\leq \sum_{\underline{\Gamma} \subset G \subset K} r(\underline{\Gamma}) e^{-\beta E_s(\Gamma) + \beta(N, V(\Gamma) \cap \underline{\Gamma}) - \tilde{\beta}|G \setminus \overline{\mathcal{D}(\Gamma)}|} e^{|\operatorname{int} \underline{\Gamma}| e^{-g\tilde{\beta}}} e^{-g\tilde{\beta}|K \setminus G|} \\
&\leq r(\underline{\Gamma}) e^{|\operatorname{int} \underline{\Gamma}| e^{-g\tilde{\beta}}} e^{-\beta E_s(\Gamma) + \beta(N, V(\Gamma) \cap \underline{\Gamma}) - g\tilde{\beta}|K \setminus \underline{\Gamma}| - \tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(\Gamma)}|} \sum_{\underline{\Gamma} \subset G \subset K} e^{-(\tilde{\beta} - g\tilde{\beta})|G \setminus \underline{\Gamma}|} \\
&\leq r(\underline{\Gamma}) e^{|\operatorname{int} \underline{\Gamma}| e^{-g\tilde{\beta}}} e^{-\beta E_s(\Gamma) + \beta(N, V(\Gamma) \cap \underline{\Gamma}) - \tilde{\beta}|\Gamma \setminus \overline{\mathcal{D}(\Gamma)}| - g\tilde{\beta}|K \setminus \underline{\Gamma}|} e^{|K \setminus \underline{\Gamma}| e^{-(\tilde{\beta} - g\tilde{\beta})}} \\
&\leq r(\underline{\Gamma}) e^{|\operatorname{int} \underline{\Gamma}| e^{-g\tilde{\beta}}} e^{-\beta E_s(\Gamma) + \beta(N, V(\Gamma) \cap \underline{\Gamma}) - \tilde{\beta}|\underline{\Gamma} \setminus \overline{\mathcal{D}(\Gamma)}| + g\tilde{\beta}''|\underline{\Gamma} \setminus \overline{\mathcal{D}(h)}|} e^{-g\tilde{\beta}''|K \setminus \overline{\mathcal{D}(h)}|} \\
&\equiv u(\Gamma) e^{-g\tilde{\beta}''|K \setminus \overline{\mathcal{D}(h)}|}
\end{aligned}
\tag{4.43}
$$

where $\tilde{\beta}'' \equiv g\tilde{\beta} - e^{-(\tilde{\beta} - g\tilde{\beta})}$. Notice that the function $u(\Gamma)$ factorizes over connected components of Γ. Next we have to estimate the sum over the small contours within G. Here we may make use of the estimates used in the proof of Lemmas 3.1 and 3.2 of Section III. Let us consider the sum over Γ within a connected G. Of course, the interior of the support of Γ must contain all of $\mathcal{D}(h) \cap G$. Let $\{\mathcal{D}_j(h)\}_{j=1,\ldots,q}$ be the $L^{1/2}$-connected components of $\mathcal{D}(h)$ in G. From the analysis of the ground state we know that flat contours with supports contained in the $\overline{\mathcal{D}_i(h)}$ will give the dominant contribution. We would like to group all small components of Γ into clusters attached to the $\mathcal{D}_i(h)$ and 'free' ones. Unfortunately, since connected components of Γ may join several components of $\mathcal{D}(h)$, this is not immediately possible. However, for any Γ, we may break its support into the set $\underline{\tilde{\Gamma}} \equiv \underline{\Gamma} \setminus \overline{\mathcal{D}(h)}$ and $\underline{\hat{\Gamma}} \equiv \underline{\Gamma} \cap \overline{\mathcal{D}(h)}$. Note that the two sets may or may not be connected. Let us now introduce the notion of D-connectedness:

DEFINITION 4.3: *For any $B \subset \mathbb{Z}^d$, we will denote by B_D the union of the connected components of $B \cup \overline{\mathcal{D}(h)}$ that intersect B. A set B is then said to be D-connected, if and only if B_D is connected. A collections of sets $B_1, \ldots, B_l$ is said to be D-disjoint, if $B_1 \cup \overline{\mathcal{D}(h)}, \ldots, B_l \cup \overline{\mathcal{D}(h)}$ is a collection of disjoint connected sets.*

This allows us to write

$$
\begin{aligned}
\tilde{\rho}(h, K) \leq &\sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{\underline{\gamma_1}, \ldots, \underline{\gamma_l} \subset K \setminus \overline{\mathcal{D}(h)} \\ \underline{\gamma_i}\ D-\text{disjoint}}} \sum_{\mathcal{D}(h) \subset \underline{\hat{\Gamma}} \subset \overline{\mathcal{D}(h)}} \\
&\times \prod_j \left(\sum_{h_x,\, x \in \operatorname{int} \underline{\gamma_i}} u(\gamma_i) e^{-\beta \sum_{x \in \operatorname{int} \underline{\gamma_i}} (S_x(h_x(\gamma_i)) - S_x(h))} \right) e^{-\tilde{\beta}''|K \setminus \overline{\mathcal{D}(h)}|}
\end{aligned}
\tag{4.44}
$$

where the product is over the connected components of $\underline{\tilde{\Gamma}} \cup \underline{\hat{\Gamma}}$, which are denoted by $\underline{\gamma_i}$. Given a configuration of heights h_x, we denote the by γ_i the resulting connected components of the entire contour. We may now perform the sum over the heights in each connected components. The estimates used here are similar to those used in the proofs of Lemmas 3.1 and 3.2. Let γ be a weakly connected component of Γ. Then

$$
\begin{aligned}
& u(\bar{\gamma}) e^{-\beta \sum_{x \in \operatorname{int} \underline{\gamma}} (S_x(h_x(\gamma)) - S_x(h))} \\
& \leq e^{|\operatorname{int} \underline{\gamma}| e^{-\tilde{\beta}} - \beta E_s(\gamma) - \tilde{\beta} |\underline{\gamma} \backslash \overline{D(\gamma)}| + \tilde{\beta}'' |\underline{\gamma} \backslash \overline{D(h)}| + \beta(N, V(\gamma) \cup \underline{\gamma}) - \beta \sum_{x \in \operatorname{int} \underline{\gamma}} (S_x(h_x(\gamma)) - S_x(h))} \\
& \leq e^{|\operatorname{int} \gamma| e^{-\tilde{\beta}} - \frac{\beta}{2} E_s(\gamma) - \frac{\tilde{\beta} - \tilde{\beta}''}{2} |\underline{\gamma} \backslash \overline{D(h)}|} \\
& \quad \times e^{-\frac{\beta}{4} + (\tilde{\beta} - \tilde{\beta}'')(|\underline{\gamma} \backslash \overline{D(\gamma)}| - |\underline{\gamma} \backslash \overline{D(h)}|)} \\
& \quad \times e^{-\frac{\beta}{4} E_s(\gamma) - \frac{\tilde{\beta} - \tilde{\beta}''}{2} |\underline{\gamma} \backslash \overline{D(h)}| + \beta(N, V(\gamma) \cup \underline{\gamma}) + 2\beta\delta \sum_{x \in \operatorname{int} \underline{\gamma}} \mathbb{1}_{h_x(\gamma) \neq h}}
\end{aligned}
\tag{4.45}
$$

Recalling Lemma 3.3, we see that the first exponential can be used to control the summation over h_x (yielding a term $e^{|\operatorname{int} \gamma| e^{-\beta/(2L)}}$), while the second and third exponential will be bounded uniformly. For, by the same arguments as used in section III, under just slightly altered conditions on the parameters δ, L and c, the exponents in the last two factors are *negative* whenever $h_x(\gamma) \not\equiv h$; they may in fact even be bounded from above by $-const.\beta$ in this case. Moreover, they are also negative whenever $\underline{\gamma} \not\subset \mathcal{D}(h)$, which in particular requires $\tilde{\gamma} = \emptyset$. On the other hand, if γ is flat and $\gamma \subset \mathcal{D}(h)$, by definition of the set $\mathcal{D}$, the exponent is still bounded from above by $\tilde{\beta} L \sigma^2$ for each connected component of Γ.

The resulting bounds can now be factored over the $\underline{\tilde{\gamma}_j}$ and the components of $\underline{\hat{\Gamma}}$ that make up $\underline{\tilde{\gamma}_i}$. Inserting it into (4.44) we get therefore

$$
\begin{aligned}
\tilde{\rho}(h, K) \leq & \; e^{-\tilde{\beta}'' |K \backslash \overline{\mathcal{D}(h)}|} \sum_{l=0}^{\infty} \frac{1}{l!} \sum_{\substack{\underline{\gamma_1}, \ldots, \underline{\gamma_l} \subset K \backslash \overline{\mathcal{D}(h)} \\ \underline{\gamma_i} \; \mathcal{D}\text{-disjoint}}} \prod_{j=1}^{l} e^{|\operatorname{int} \underline{\tilde{\gamma}_j}| (e^{-\tilde{\beta}} + e^{-\beta/(2L)}) - (\tilde{\beta} - \tilde{\beta}'') |\underline{\tilde{\gamma}_j}|} \\
& \times \prod_{i:\, \mathcal{D}_i(h) \subset K} \left(\sum_{\mathcal{D}_i(h) \subset \underline{\gamma_i} \subset \overline{\mathcal{D}_i(h)}} e^{|\operatorname{int} \underline{\gamma_i}| (e^{-\tilde{\beta}} + e^{-\beta/(2L)}) + \beta \sum_{x \in \overline{\mathcal{D}_i(h)}} \mathcal{N}_x(h)} \right) \\
\leq & \; e^{-\tilde{\beta}'' |K \backslash \overline{\mathcal{D}(h)}| + a e^{-\tilde{\beta}''} |K \backslash \overline{\mathcal{D}(h)}|} \prod_{i:\, \mathcal{D}_i(h) \subset K} \mathcal{Z}^h(\mathcal{D}_i(h))
\end{aligned}
\tag{4.46}
$$

To obtain the last inequality we have ignored the constraint on the $\underline{\tilde{\gamma}_i}$ to be $\mathcal{D}$-disjoint and written the resulting expression as an exponential. We have also used the trivial fact that for small γ_i, the $|\operatorname{int} \underline{\tilde{\gamma}_i}| \leq L |\underline{\tilde{\gamma}_i}|$, and that thus, e.g. $|\operatorname{int} \underline{\tilde{\gamma}_j}| (e^{-\tilde{\beta}} + e^{-\beta/(2L)}) - (\tilde{\beta} - \tilde{\beta}'') |\underline{\tilde{\gamma}_j}| \leq -\tilde{\beta}'' |\underline{\tilde{\gamma}_j}|$, if $\tilde{\beta}''$ had been chosen not too large. The constant a in the bound is again just sum geometrical constant taking into account the entropy in the sum over $\tilde{\gamma}$ in the exponent. Also we have set

$$\mathcal{Z}^h(\mathcal{D}_i(h)) = \sum_{\mathcal{D}(h)\subset\hat{\underline{\Gamma}}\subset\overline{\mathcal{D}(h)}} e^{|\text{int}\,\underline{\gamma_i}|(e^{-\beta}+e^{-\beta/(2L)})+\beta\sum_{x\in\overline{\mathcal{D}_i(h)}}\mathcal{N}_x(h)} \leq c(L)e^{\tilde{\beta}\sigma^2+e^{-\beta}L^{(1-\alpha)/2}} \tag{4.47}$$

These are in fact the desired bounds on the activities $\tilde{\rho}$ to exponentiate the sum over K in Y by a Mayer expansion. However, as in the sum over the small Γ above, we must take into account the constraint that K must contain the set $\mathcal{D}(h)\cap Y$. This will be dealt with similarly as before. First, put

$$\bar{\rho}(\tilde{K}_j) \equiv \left(\prod_{i:\mathcal{D}_i(h)\subset\bar{K}_j} \mathcal{Z}^h(\mathcal{D}_i(h))\right)^{-1} \tilde{\rho}(h,\bar{K}_j) \tag{4.48}$$

Here $\bar{K}_i$ stands for the connected subset of $\tilde{K}_i\cup\overline{\mathcal{D}(h)}$ that contains $\tilde{K}_i$ and is of course uniquely computable from $\tilde{K}_j$ for given $\mathcal{D}(h)$. Then

$$\begin{aligned}\sum_{Y\cap\mathcal{D}(h)\subset K\subset Y} \tilde{\rho}(h,K) &= \prod_{i:\mathcal{D}_i(h)\subset Y} \mathcal{Z}^h(\mathcal{D}_i(h))\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{\substack{\tilde{K}_1,\dots,\tilde{K}_l\subset Y\setminus\overline{\mathcal{D}(h)}\\ \tilde{K}_j\ \mathcal{D}-\text{disjoint}}}\prod_{j=1}^{l}\bar{\rho}(\tilde{K}_j)\\ &= e^{\beta\sum_{i:\mathcal{D}_i(h)\subset Y}\left(\frac{1}{\beta}\ln\left(\mathcal{Z}^h(\mathcal{D}_i(h))\right)\right)}e^{\beta\psi^c(Y\setminus\overline{\mathcal{D}(h)})}\end{aligned} \tag{4.49}$$

where

$$\psi^c(Y\setminus\overline{\mathcal{D}(h)}) = \frac{1}{\beta}\sum_{l=1}^{\infty}\frac{1}{l!}\sum_{\tilde{K}_1,\dots,\tilde{K}_l\subset Y\setminus\overline{\mathcal{D}(h)}}\Psi^c(\tilde{K}_1,\dots,\tilde{K}_l)\prod_{j=1}^{l}\bar{\rho}(\tilde{K}_j) \tag{4.50}$$

and $\bar{K}_i$ stands for the connected subset of $\tilde{K}_i\cup\overline{\mathcal{D}(h)}$ that contains $\tilde{K}_i$. The function $\Psi^c(\tilde{K}_1,\dots,\tilde{K}_l)$ forces the sets of $\tilde{K}_j$ occurring in the sum to form a $\mathcal{D}$-connected cluster. For a detailed description of the Mayer expansion and explicit formulas for Ψ^c we refer to the literature (see e.g. [GJ,Br]). For us it is important that with $\bar{\rho}$ satisfying the bounds given through (4.46), (4.50) represents an absolutely convergent sum; thus, grouping terms by the total volume of the union of the sets and proceeding just like in the proof of Lemma 4.4, we see that the terms arising from ψ^c satisfy the bounds claimed in Lemma 4.3 (But note that the constant g in Lemma 4.3 is not the same as that in Lemma 4.4, but a slightly smaller one). The same holds true for those arising from the $\mathcal{Z}^h$. ◊

Remark: Note that the quantities $\mathcal{Z}^h$ are defined from bounds; this differs from the procedure of [BK] where the exact contributions from the potential 'ground state contours' are factored out (see also our treatment of the ϵ^h in Section III). This does not bring, however, particular advantages.

Next we need to control the activities $\hat{\rho}(\Gamma^l, K)$. Our aim here is to show that it satisfies essentially the same bounds as the original ρ. With the work to prove Lemma 4.3 already done, this will not be too difficult.

LEMMA 4.5: *The activities $\hat{\rho}$ defined through (4.27) satisfy the following relations:*

$$0 \leq \hat{\rho}(\Gamma^l, K)| \leq e^{-\beta E_s(\Gamma^l)-\tilde{\beta}'|K\setminus\overline{D(\Gamma^l)}|+\beta(N,V(\Gamma^l)\cap\underline{\Gamma}^l)} \prod_{i,h:\,\mathcal{D}_i(h)\subset V_h(\Gamma^l)} \mathcal{Z}^h(\mathcal{D}_i(h)) \tag{4.51}$$

and for flat contours $\Gamma^l = (C, h_x \equiv h)$, $C \subset D(h)$,

$$\hat{\rho}(\Gamma^l, \underline{\Gamma}^l) \geq e^{-\beta(N,V(\Gamma^l)\cap\underline{\Gamma}^l)} \tag{4.52}$$

Proof: Notice first that $\hat{\rho}(\Gamma^l, \underline{\Gamma}^l) = \rho(\Gamma^l, \underline{\Gamma}^l)$ so that (4.52) is trivial from the assumptions on ρ. The upper bound (4.51) is proven in exactly the same way as the upper bound on $\tilde{\rho}$, since small contours can be summed over in each connected component of the complement of $\underline{\Gamma}^l$ in K. We do not repeat the the details of the estimations. $\diamondsuit$

Our expression for $T_1\hat{\mu}$ can be brought into a slightly more convenient form, namely

$$\begin{aligned}(T_1\hat{\mu})(\Gamma^l) = &\prod_{h,i:\,\mathcal{D}_i(h)\subset V_h(\Gamma^l)} \mathcal{Z}^h(\mathcal{D}_i(h)) e^{-\beta(S,V(\Gamma^l))} \\ &\sum_{K\supset\underline{\Gamma}^l} \hat{\rho}'(\Gamma^l, K) e^{-\beta(\psi,V(\Gamma^l)\setminus K)}\end{aligned} \tag{4.53}$$

where

$$\hat{\rho}'(\Gamma^l, K) \equiv \frac{\hat{\rho}(\Gamma^l, K)}{\prod_{h,i:\,\mathcal{D}_i(h)\subset V_h(\Gamma^l)} \mathcal{Z}^h(\mathcal{D}_i(h))} \tag{4.54}$$

The point here is that the $\mathcal{Z}^h(\overline{\mathcal{D}_i(h)})$ are independent of the contours and K' and thus can be exponentiated to give random non-local field, and $\hat{\rho}'$ satisfies the more pleasant estimates

LEMMA 4.5': *The activities $\hat{\rho}'(\Gamma^l, K)$ satisfy the bounds*

$$0 \leq \hat{\rho}'(\Gamma^l, K) \leq e^{-\beta E_s(\Gamma^l)-\tilde{\beta}'|K\setminus\overline{D(\Gamma^l)}|+\beta(N,V(\Gamma^l)\cap\underline{\Gamma}^l)} \tag{4.55}$$

and for $\Gamma^l = (C, H_x \equiv h)$, with $C \subset D(h)$,

$$\hat{\rho}'(\Gamma^l, \underline{\Gamma}^l) \geq e^{-\beta(N,V(\Gamma^l)\cap\underline{\Gamma}^l)} \tag{4.55}$$

Proof: (4.55) are evident from Lemma 4.4. To get (4.57), just notice that a connected subset of $D(h)$ cannot contain a component of $\mathcal{D}(h)$, so that in this case $\hat{\rho}'(\Gamma^l, \underline{\Gamma}^l) = \hat{\rho}(\Gamma^l, \underline{\Gamma}^l) = \rho(\Gamma^l, \underline{\Gamma}^l)$. $\diamondsuit$

This concludes the summation over small contours.

IV.3 The blocking

We now turn to the main step of the RG transformation, the blocking. As before, nothing changes as far as the action of $\mathcal{R}$ on contours is concerned and all we have to do is to study the effect on the contour measures.

First we exponentiate all terms in (4.52) that give rise to the new random fields. We set

$$z_C(h) \equiv \sum_i \mathbb{I}_{C=\overline{\mathcal{D}_i(h)}} \left(-\frac{1}{\beta} \ln \left(\mathcal{Z}^h(\overline{\mathcal{D}_i(h)}) \right) \right) \tag{4.56}$$

Setting now

$$\tilde{S}_C(h) \equiv S_C(h) + z_C(h) + \psi_C(h) \tag{4.57}$$

and noticing that

$$(\psi, V(\Gamma^l) \cap K) = (\psi, V(\Gamma^l)) - \sum_{h\in\mathbb{Z}} \sum_{\substack{C\subset V_h(\Gamma^l)\\ C\cap K\neq\emptyset}} \psi_C(h) \tag{4.58}$$

We have that

$$(T_1\hat{\mu})(\Gamma^l) = e^{-\beta(\tilde{S}, V(\Gamma^l))} \sum_{K\supset \underline{\Gamma^l}} \hat{\rho}'(\Gamma^l, K) e^{-\beta \sum_{h\in Z} \sum_{C\subset V_h(\Gamma^l),\, C\cap K\neq\emptyset} \psi_C(h)} \tag{4.59}$$

where now the random field and the activity-like contributions are almost well separated. We first prepare the field term for blocking. For given $\Gamma' \subset \Omega_{n-1}(\mathcal{L}^{-1}D)$, we can split the term into three parts

$$(\tilde{S}, V_h(\Gamma^l)) = L^{d-1-\alpha}(\tilde{S}', V_h(\Gamma')) + \delta\tilde{S}_{loc}(\Gamma^l, \Gamma') + \delta\tilde{S}_{nl}(\Gamma^l, \Gamma') \tag{4.60}$$

where for single points y

$$\tilde{S}'_y(h) \equiv L^{-(d-1-\alpha)} \left(\sum_{x\in\mathcal{L}y} \tilde{S}_x(h_x(\Gamma')) + \sum_{\substack{C\subset V_h(\Gamma'):\, C\cap\mathcal{L}y\neq\emptyset\\ d(C)<L/4 \vee C\subset\mathcal{L}y}} \frac{\tilde{S}_C(h)}{|\mathcal{L}^{-1}(C)|} \right) \tag{4.61}$$

and for $|C'| > 1$,

$$\tilde{S}'_{C'}(h) \equiv L^{-(d-1-\alpha)} \sum_{\substack{C:\mathcal{L}^{-1}(C)=C'\\ d(C)\geq L/4}} \tilde{S}_C(h) \tag{4.62}$$

Eqs. (4.61) and (4.62) are the analogues of (3.34) and almost the final definitions of the renormalized 'small random fields'. Furthermore

$$\begin{aligned} \delta\tilde{S}_{loc}(\Gamma^l, \Gamma') \equiv \sum_{y\in\Lambda_{n-1}} \Bigg[& \sum_{x\in\mathcal{L}y} \left(\tilde{S}_x(h_x(\Gamma^l)) - \tilde{S}_x(h_{\mathcal{L}^{-1}x}(\Gamma')) \right) \\ & + \sum_{h\in\mathbb{Z}} \sum_{\substack{C:\, C\cap\mathcal{L}y\neq\emptyset\\ d(C)<L/4 \vee C\subset\mathcal{L}y}} \tilde{S}_C(h) \left[\mathbb{I}_{C\subset V_h(\Gamma^l)} - \frac{\mathbb{I}_{h_y(\Gamma')=h}}{|\mathcal{L}^{-1}C|} \right] \Bigg] \end{aligned} \tag{4.63}$$

and

$$\delta\tilde{S}_{nl}(\Gamma^l,\Gamma') \equiv \sum_{h\in\mathbb{Z}} \sum_{\substack{C:C\subset\Lambda_{n-1}\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \tilde{S}_C(h)\left[\mathbb{I}_{C\subset V_h(\Gamma^l)} - \mathbb{I}_{\mathcal{L}^{-1}C\subset V_h(\Gamma^l)}\right] \tag{4.64}$$

The point here is that the contributions from $\delta\tilde{S}_{loc}$ will factor over the connected components of the blocked K, while the non-local $\delta\tilde{S}_{nl}$ can be expanded and gives only very small contributions, due to the minimal size condition on the C occurring in it. We split the remaining ψ-term in (4.59) in the same way in a local and a non-local part,

$$\begin{aligned}\sum_{h\in\mathbb{Z}} \sum_{\substack{C\subset V_h(\Gamma^l)\\ C\cap K\neq\emptyset}} \psi_C(h) &= \sum_{h\in\mathbb{Z}} \sum_{y\in\Lambda_{n-1}} \sum_{\substack{C\subset V_h(\Gamma^l):\,C\cap\mathcal{L}_y\neq\emptyset\\ C\cap K\neq\emptyset\\ d(C)<L/4\vee C\subset\mathcal{L}_y}} \frac{\psi_C(h)}{|\mathcal{L}^{-1}C|}\\ &+\sum_{h\in\mathbb{Z}} \sum_{\substack{C\subset V_h(\Gamma^l)\\ C\cap K\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \psi_C(h)\\ &\equiv \delta\psi_{loc}(\Gamma^l,K) + \delta\psi_{nl}(\Gamma^l,K)\end{aligned} \tag{4.65}$$

In all of the non-local terms only sets C give a contribution for which $C\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset$, $d(C)\geq L/4$ and $|\mathcal{L}^{-1}C|\geq 2$. In analogy to Lemma 4.2 we can therefore expand these contributions to get

$$\begin{aligned}e^{-\beta(\delta\tilde{S}_{nl}(\Gamma^l,\Gamma')+\delta\psi_{nl}(\Gamma^l,K)} &= R(K)\sum_{l=0}^{\infty}\frac{1}{l!} \sum_{\substack{C_1,\dots,C_l:\,C_i\neq C_j\\ C_i\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \prod \Phi_{\Gamma',\Gamma^l,K}(C_i)\\ &\equiv \sum_{\substack{\mathcal{C}:\mathcal{C}\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \Phi_{\Gamma',\Gamma^l,K}(\mathcal{C})\end{aligned} \tag{4.66}$$

where the activities Φ satisfy

$$0\leq \Phi_{\Gamma',\Gamma^l,K}(C)\leq e^{-\tilde{\beta}''|C|} \tag{4.67}$$

And $R(K)$ are non-random activities factoring over connected components of $\mathcal{L}(\mathcal{L}^{-1}K)$, satisfying, for a connected component,

$$1\geq R(K)\equiv \exp\left(-\sum_{\substack{C\cap\mathcal{L}(\mathcal{L}^{-1}K)\neq\emptyset\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}} \tilde{f}(C)\right)\geq e^{-|\mathcal{L}(\mathcal{L}^{-1}K)|e^{-\frac{L}{4}\beta''}} \tag{4.68}$$

With these preparations we can now write dow the blocked contour measures in the form

$$(\mathcal{R}T_1\hat{\mu})(\Gamma') = e^{-\beta L^{d-1-\alpha}(\tilde{S}',V(\Gamma')}\sum_{G'\supset\underline{\Gamma'}}\rho'(\Gamma',G') \tag{4.69}$$

where

$$\begin{aligned}\rho'(\Gamma',G') \equiv \sum_{G'\supset K'\supset\underline{\Gamma'}} \sum_{C':C'\cup K'=G'} \sum_{\Gamma^l:\,\mathcal{R}(\Gamma^l)=\Gamma'} \sum_{\substack{K\supset\underline{\Gamma^l}\\ \mathcal{L}^{-1}K=K'}} \sum_{\substack{\mathcal{C}:\mathcal{L}^{-1}\mathcal{C}=\mathcal{C}'\\ d(C)\geq L/4\wedge|\mathcal{L}^{-1}C|\geq 2}}\\ e^{-\beta(\delta\tilde{S}_{loc}(\Gamma',\Gamma^l)+\delta\psi_{loc}(\Gamma',K))}R(K)\hat{\rho}'(\Gamma^l,K)\Phi_{\Gamma',\Gamma^l,K}(\mathcal{C})\end{aligned} \tag{4.70}$$

Notice that by construction the C occurring in the local fields $\delta\tilde{S}$ and $\delta\psi$ cannot connect disconnected components of G', and therefore $\rho'(\Gamma', G')$ factorizes over connected components of G'. The main task that is left is to prove that ρ' yields a N' bounded contour measure for a suitably defined N'. In analogy to (3.35) of section III, we define the preliminary new control field by

$$\tilde{N}'_y(h) \equiv L^{-(d-1-\alpha)} \sum_{x \in \mathcal{L}y \setminus \mathcal{D}(h)} \mathcal{N}_x(h) \tag{4.71}$$

where $\mathcal{N}$ was defined already in (4.1). We will now proof the following

Lemma 4.6: *Let $\tilde{N}'$ be defined in (4.71) and set $\tilde{D}' \equiv D(\tilde{N}')$. Then the activities ρ' defined in (4.70) factor over connected components of G' and for any connected G'*

$$0 \leq \rho'(\Gamma,' G') \leq e^{-\beta' E_s(\Gamma') - \tilde{\beta}' |G' \setminus \overline{\tilde{D}'(\Gamma')}| + \beta'(\tilde{N}', V(\Gamma') \cap G')} \tag{4.72}$$

and for $\Gamma' = (C, h_y \equiv h)$, with $C \subset \tilde{D}'$ connected,

$$\rho'(\Gamma', \underline{\Gamma'}) \geq e^{-\beta'(\tilde{N}', V(\Gamma'))} \tag{4.73}$$

with $\beta' = L^{d-1-\alpha}\beta$ and $\tilde{\beta}' = L^{1-\alpha}\tilde{\beta}$, for some $\alpha > 0$ that can be chosen as small as desired, provided L and β are large enough.

Proof: Let us first proof the upper bound. The sum over the C can first be estimated just as in Lemma 4.4, but taking into account the restrictions on the minimal size of the sets C; that is, we get that

$$\begin{aligned} \sum_{C' : C' \cup K' = G'} \sum_{\substack{C : \mathcal{L}^{-1} C = C' \\ d(C) \geq L/4 \wedge |\mathcal{L}^{-1} C| \geq 2}} \Phi_{\Gamma', \Gamma^l, K}(C) &\leq e^{|\mathcal{L}(\mathcal{L}^{-1} K| e^{-\beta''(L/4)}} e^{-\tilde{\beta}' \min\left(|V| : \mathcal{L}^{-1} V = G' \setminus K' \wedge d(V) \geq L/4\right)} \\ &\leq e^{|K'| L^d e^{-L/4}} e^{-|G' \setminus K'| \tilde{\beta}' L/4} \end{aligned} \tag{4.74}$$

The two local field terms, $\delta\tilde{S}_{loc}$ and $\delta\psi_{loc}$ will be dealt with differently: $\delta\tilde{S}_{loc}$ is only present for locally non-flat Γ^l and will thus be estimated against some fraction of the surface energy (just like in Section III), while for the $\delta\psi_{loc}$ we get

$$\delta\psi_{loc}(\Gamma^l, K) = \sum_{h \in \mathbb{Z}} \sum_{y \in K'} \sum_{\substack{C \subset V_h(\Gamma^l) : C \cap \mathcal{L}y \neq \emptyset \\ C \cap K \neq \emptyset \\ d(C) < L/4 \vee C \subset \mathcal{L}y}} \frac{\psi_C(h)}{|\mathcal{L}^{-1} C|} \leq |K'| L^d e^{-\tilde{\beta}''} \tag{4.75}$$

(in both formulas $\tilde{\beta}''$ stands for $\tilde{\beta}$ times some geometric constant (less than one)). Using this, we get that

$$\begin{aligned} \rho'(\Gamma', G') \leq \sum_{G' \supset K' \supset \underline{\Gamma'}} \sum_{\Gamma^l : \mathcal{R}(\Gamma^l) = \Gamma'} \sum_{\substack{K \supset \underline{\Gamma^l} \\ \mathcal{L}^{-1} K = K'}} & e^{-\beta \sum_{y \in K'} \sum_{x \in \mathcal{L}y} \left(\tilde{S}_x(H_x(\Gamma^l) - \tilde{S}_x(h_y(\Gamma'))\right)} \\ & \times e^{-\beta E_s(\Gamma^l) - \tilde{\beta}' |K \setminus \overline{D(\Gamma^l)}| + \beta(N, V(\Gamma^l) \cap \underline{\Gamma^l})} e^{-\tilde{\beta}'' L |G' \setminus K'| + L^d |K'| e^{-\beta''}} \end{aligned} \tag{4.76}$$

Now we may use the estimates of Section III.3 (in particular (3.57)) to estimate

$$
\begin{aligned}
&\beta \sum_{y\in K'} \sum_{x\in\mathcal{L}y} \left(\tilde{S}_x(H_x(\Gamma^l) - \tilde{S}_x(h_y(\Gamma'))\right) + \beta E_s(\Gamma^l) + \tilde{\beta}'|K\backslash\overline{D(\Gamma^l)}| - \beta(N, V(\Gamma^l)\cap\underline{\Gamma^l})\\
&\geq \beta\frac{L^{d-1}}{16(d+1)}E_s(\Gamma') + \tilde{\beta}''\frac{Lc_6}{2}|K'\backslash\overline{\tilde{D}'(\Gamma')}| - \beta L^{d-1-\alpha}(\tilde{N}', V(\Gamma')\cap\underline{\Gamma'})\\
&+\frac{\beta}{2}E_s(\Gamma^l) + \frac{\tilde{\beta}''}{2}|K\backslash D(\Gamma^l)|
\end{aligned}
\tag{4.77}
$$

The idea here is that the first three terms in the lower bound provide essentially the bound for the new activities, while remaining terms suffice to control the convergence of the sums over K', K and Γ^l. To see this, let us introduce, for given Γ', the set

$$
Y(\Gamma^l) \equiv \{x|h_x(\Gamma^l) \neq h_{\mathcal{L}^{-1}x}(\Gamma')\}
\tag{4.78}
$$

It is clear that for Γ^l such that $\mathcal{L}^{-1}\Gamma^l = \Gamma'$, $Y(\Gamma^l) \subset \mathcal{L}\underline{\Gamma'}$. Let us also write $D(\Gamma')$ for the bad region of the contour those height h_x is given by $h_x = h_{\mathcal{L}^{-1}x}(\Gamma')$. One sees then that $\underline{\Gamma^l}\cap Y(\Gamma^l)^c \supset D(\Gamma')\cap Y(\Gamma^l)^c$. Using this notation and Lemma 3.7, we get that

$$
\begin{aligned}
&\sum_{\Gamma^l:\,\mathcal{R}(\Gamma^l)=\Gamma'} \sum_{\substack{K\supset\underline{\Gamma^l}\\ \mathcal{L}^{-1}K=K'}} e^{-\frac{\beta}{2}E_s(\Gamma^l)-\frac{\beta''}{2}|K\backslash D(\Gamma^l)|}\\
&\leq \sum_{\Gamma^l:\,\mathcal{R}(\Gamma^l)=\Gamma'} \sum_{\substack{K\supset\underline{\Gamma^l}\\ \mathcal{L}^{-1}K=K'}} e^{-\frac{\beta}{2L}\sum_{x\subset Y(\Gamma^l)}\left|h_x(h(\Gamma^l)-h_{\mathcal{L}^{-1}x}(\Gamma')\right|-\frac{\beta''}{2}|K\cap Y(\Gamma^l)\backslash D(\Gamma^l)\cap Y(\Gamma^l)|}\\
&\quad\times e^{-\frac{\beta''}{2}|K\cap Y(\Gamma^l)^c\backslash D(\Gamma^l)\cap Y(\Gamma^l)^c|}\\
&\leq \sum_{Y\subset\mathcal{L}\underline{\Gamma'}} \sum_{K:\,\mathcal{L}^{-1}K=K'} \sum_{\underline{\Gamma^l}\cap Y^c\supset D(\Gamma')\cap Y^c} \sum_{\underline{\Gamma^l}\cap Y\subset K\cap Y} \prod_{x\in Y}\left(\sum_{h_x\neq h_{\mathcal{L}^{-1}x}(\Gamma')} e^{-\frac{\beta}{2L}\left|h_x-h_{\mathcal{L}^{-1}x}(\Gamma')\right|}\right)\\
&\quad\times e^{-\frac{\beta''}{2}|K\cap Y(\Gamma^l)^c\backslash D(\Gamma^l)\cap Y(\Gamma^l)^c|}\\
&\leq \sum_{K:\,\mathcal{L}^{-1}K=K'} \sum_{Y\subset\mathcal{L}\underline{\Gamma'}} e^{-\frac{\beta}{3L}|Y|-\frac{\beta''}{2}|K\cap Y(\Gamma^l)^c\backslash D(\Gamma^l)\cap Y(\Gamma^l)^c|}2^{|K\cap Y(\Gamma^l)^c\backslash D(\Gamma^l)\cap Y(\Gamma^l)^c|}\\
&\leq e^{e^{-\beta/3L}|\mathcal{L}\underline{\Gamma'}|+e^{-(\beta''-\ln 2)}|D(\Gamma')|}
\end{aligned}
\tag{4.79}
$$

Inserting this bound together with (4.77) into (4.76), we arrive at

$$
\begin{aligned}
\rho'(\Gamma', G') \leq &\sum_{G'\supset K'\supset\underline{\Gamma'}} e^{-\frac{L^{d-1}}{16(d+1)}E_s(\Gamma')-\tilde{\beta}''\frac{Lc_6}{2}|K'\backslash\overline{\tilde{D}'(\Gamma')}|-L\tilde{\beta}''|G'\backslash K'|}\\
&\times e^{L^d|K'|e^{-\tilde{\beta}''}+L^d|\underline{\Gamma'}|e^{-\beta/3L}+L^d|\tilde{D}'(\Gamma')|e^{-(\tilde{\beta}''-\ln 2)}+\beta L^{d-1-\alpha}(\tilde{N}',V(\Gamma')\cap\underline{\Gamma'})}\\
&\leq e^{-\frac{L^{d-1}}{16(d+1)}E_s(\Gamma')-\tilde{\beta}''Lc_7|G'\backslash\overline{\tilde{D}'(\Gamma')}|+\beta L^{d-1-\alpha}(\tilde{N}',V(\Gamma')\cap\underline{\Gamma'})+c_8L^d|\tilde{D}'(\Gamma')|e^{-(\tilde{\beta}''-\ln 2)}}
\end{aligned}
\tag{4.80}
$$

where c_7 and c_8 are numerical constants of order unity. This is almost the desired form, except for the term proportional to $|\overline{\tilde{D}'(\Gamma')}|$. Of course, the part of this area where Γ' is non-flat can easily be absorbed by a tiny fraction of the surface energy; in flat regions, on the contrary, we are able to absorb this term into the $\mathcal{N}'$-term. In fact we have the following

LEMMA 4.7: *Assume that N is a control field of level k. Then for any contour Γ,*

$$(N, V(\Gamma) \cap \underline{\Gamma}) - 2c_d L^{-(d-3/2)k} \sigma E_s(\Gamma') \geq c_d L^{-(d-3/2)k} \sigma |\overline{D(\Gamma)} \cap \underline{\Gamma}| \tag{4.81}$$

The proof of this Lemma will be postponed to section IV.5.

Assuming this Lemma and the fact that in level k $\beta = L^{(d-1-\alpha)k}\beta^{(0)}$, we see that

$$\begin{aligned} &+ c_8 L^d |\tilde{D}'(\Gamma')| e^{-(\tilde{\beta}''-\ln 2)} + \beta L^{d-1-\alpha}(\tilde{N}', V(\Gamma') \cap \underline{\Gamma}') \\ &\leq \left(1 + \frac{L^d c_8 e^{-(\tilde{\beta}''-\ln 2)}}{c_d \sigma \beta^{(0)} L^{(1/2-\alpha)k}}\right) \beta L^{d-1-\alpha}(\tilde{N}', V(\Gamma') \cap \underline{\Gamma}') \end{aligned} \tag{4.82}$$

Inserting this bound into (4.80) and setting $\beta' = L^{d-1-\alpha}\beta$ and $\tilde{\beta}' = L^{1-\alpha}\tilde{\beta}$, for suitable $\alpha > 0$, we get the upper bound (4.72). (Strictly speaking, due to (4.82) there is a factor of order $1 - o(e^{-L\tilde{\beta}'})$ in front of the $\tilde{N}'$-term. To get strictly the form claimed in the lemma, we should slightly modify the definition of $\tilde{N}'$, e.g. by choosing the α in (4.71) slightly different from the one in the definition of β; this has no influence on the probabilistic estimates, and the effect is so ridiculously small (even after iteration over the hierarchies) that we will ignore it).

Remark: Note that σ appears in the denominator in (4.82), so that the bounds appear not to be uniform in σ, for small σ. But note that this appear only for $\sigma \ll e^{-\tilde{\beta}}$, and no real harm is done by replacing such a minimal value for σ in our bounds, whenever σ might be smaller, since thermal fluctuations of the interface will in any case appear on such a scale.

Finally let us prove the lower bound (4.73). By just picking particular contributions and using the positivity of all activities, it is trivial to get that for $\Gamma' = (C, h_y \equiv h)$,

$$\rho(\Gamma', \underline{\Gamma}') \geq R(\underline{\Gamma}') e^{-\beta'(\tilde{N}', C)} \geq e^{-\beta'(\tilde{N}', C) - e^{-\beta' \frac{L}{4}} L^d |C|} \tag{4.83}$$

Now just as before, and since $C \subset \tilde{D}'(h)$, the second term is just a tiny correction to the first one and can be absorbed by a redefinition of β' to give (4.73). Given Lemma 4.7, this concludes the proof of Lemma 4.6. $\diamondsuit$

IV.4 Final shape up

Just as in section III we must make some final changes in the definition of the small and control fields and in the definition of the contours to recover the exact form of N'-bounded contour models. In fact the definitions (3.59), (3.60) and (3.61) remain unchanged. Of course, we also will center the non-local fields, i.e. we put

$$S'_{C'}(h) \equiv \tilde{S}'_{C'}(h) - I\!E\tilde{S}'_{C'}(h) \tag{4.84}$$

The centering has of course no effect on the contour measures, as the effect cancels with the partition functions (which are *not* invariant under this last part of the RG map.

The final RG map will then be given by $\mathcal{R}^{(N)} \equiv T_3T_2T_1$. By Lemma 4.6, this map transforms contour activities in the desired way; the last point we have to check is that also the non-local field $S'_C(h)$ satisfies the required uniform bound (4.4) with the renormalized $\tilde{\beta}'$. But by definition of $\tilde{S}'_{C'}(h)$ only C with $d(C) \geq L/4$ contribute, and since the $\tilde{S}_C(h)$ satisfy uniform bounds of the form $|\tilde{S}_C(h)| \leq e^{-\tilde{\beta}''|C|}$, the sum in (4.62) gives, for instance a bound

$$|S'_{C'}(h)| \leq e^{-c_9 L\tilde{\beta}''|C'|} \leq e^{-\tilde{\beta}'|C'|} \tag{4.85}$$

after a possible redefinition of α.

Collecting the results of the preceeding subsections we have the

PROPOSITION 4.1: *Let $\mathcal{R}^{(N)} \equiv T_3T_2T_1 : \Omega_n(D(N)) \to \Omega_{n-1}(D(N'))$ with T_1, T_2 and T_3 defined above; let N' and S' and ρ' be defined as above and let μ be a N-bounded contour measure at temperatures β and $\tilde{\beta}$ of level k. Define*

$$\mu'\Gamma' \equiv (\mathcal{R}\mu)(\Gamma') \equiv \frac{1}{Z'}e^{-\beta'(S',V(\Gamma'))} \sum_{G' \supset \underline{\Gamma'}} \rho'(\Gamma', G) \tag{4.86}$$

Then μ' is a N'-bounded contour measure with temperatures $\beta' = L^{d-1-\alpha}\beta$ and $\tilde{\beta}' = L^{1-\alpha}\tilde{\beta}$ of level $k+1$, for suitably chosen $\alpha > 0$.

Remark: Note that the condition on α is of the form $L^\alpha \geq$ *const.* for some geometrical constant independent of L, while L must satisfy a condition like $L \leq 1/\sigma^2$. Thus, α may be chosen of the order $\alpha \approx \frac{\ln(const.)}{|\ln \sigma^2|}$.

This concludes the construction of the RG map.

IV.5 Probabilistic estimates

Fortunately, for the probabilistic estimates of the flow of the small fields and the control fields, almost nothing has to be changed in the finite temperature case compared to section III. The only pertinent remark is that we should not try to insist on bounds on the fields with $\sigma^2 < e^{-\beta}$. This is partly due to our sometimes poor estimates, but it should be clear that thermal fluctuations alone lead to local height fluctuations of this order, and there is no point in getting better control over the disorder induced fluctuations. With this in mind, the control of the flow of the fields $S_x^{(k)}(h)$ is controlled as in Sect. III.5; the only difference is that the bound (3.77) is replaced by a bound on the fields $\tilde{S}_C(h)$ contributing in (4.61) which introduces an extra term of the order $e^{-\tilde{\beta}^{(k)}}$ compared to (3.77). This clearly does not alter anything in the proof, and we recover Proposition 3.2. The second point is that due to the modification of the recursive definition of the control fields, we get a slightly sharper control on the absence of 'very small' non-zero $\bar{N}$ and this will allow us to prove Lemma 4.7. Thus with definition (4.1) for the $\mathcal{N}$ in mind, we define here, in analogy to (3.84)

$$
\begin{aligned}
\bar{N}_y^{(0)}(h) &= \sum_{x:|y-x|\leq 1} N_x^{(0)}(h) \\
\bar{N}_y^{(k+1)}(h) &= \left(L^{-(d-1-\alpha)} \sum_{x:|y-x|\leq 1} \sum_{x\in\mathcal{L}y\setminus\mathcal{D}(h)} \bar{\mathcal{N}}_x^{(k)}(h) \right) \mathbb{I}_{y\in D^{(k+1)}(h)} \\
&\quad + \mathbb{I}_{|\tilde{S}_y^{(k+1)}(h)|>\delta_{(k+1)}}
\end{aligned}
\tag{4.87}
$$

where

$$
\bar{\mathcal{N}}_x(h) \equiv \begin{cases} \sup_{h'\in\mathbb{Z}} \left(\bar{N}_x(h+h') - \frac{c}{2L}|h'|\right), & \text{if } \sup_{h'\in\mathbb{Z}} \left(\bar{N}_x(h+h') - \frac{c}{2L}|h'|\right) \geq \frac{c}{2L} \\ \bar{N}_x(h), & \text{otherwise} \end{cases}
\tag{4.88}
$$

These fields satisfy equally the estimates of Proposition 3.3, but instead of (3.86) we have the sharper

LEMMA 4.8: For all $k \in \mathbb{N}$, we have that if $\bar{N}_x^{(k)} > 0$, then $\bar{N}_x^{(k)} \geq L^{-(d-3/2)k}\sigma$.

Proof: Note that $\bar{\mathcal{N}}_x^{(k)}(h)$ is defined in such a way that if $\bar{N}_x^{(k)}(h)$ satisfies the claim of the lemma for some k, then so does $\bar{\mathcal{N}}_x^{(k)}(h)$, in contrast to the situation in Lemma 3.12 where even then we would only get the probabilistic bound (3.89). But given that $\bar{\mathcal{N}}_x^{(k)}(h)$ cannot take values in the interval $(0, L^{-(d-3/2)k})$, the arguments used to prove (3.86) of Proposition 3.3 immediately imply that the claim of Lemma 4.8 is true for $k+1$, and since for $k = 0$ it is trivially true, the Lemma is proven by induction. $\diamondsuit$

Let us now prove Lemma 4.7.

Proof: (of Lemma 4.7) For any contour Γ let us define the set

$$
F(\Gamma) = \left\{ x \subset \underline{\Gamma} \,\middle|\, \forall_{y:|x-y|=1} h_y(\Gamma) = h_x(\Gamma) \right\}
\tag{4.89}
$$

This is clearly the set within the support where Γ is 'flat'. Now

$$|\overline{D(\Gamma)}| = |\overline{D(\Gamma)} \cap F(\Gamma)| + |\overline{D(\Gamma)} \backslash F(\Gamma)| \tag{4.90}$$

Now

$$E_s(\Gamma) \geq \frac{1}{2} |\underline{\Gamma} \backslash F(\Gamma) \tag{4.91}$$

while

$$\begin{aligned} \sum_{x \in \underline{\Gamma}} N_x(h_x(\Gamma)) &\geq c_d \sum_{x \in F(\Gamma)} \left(N_x(h_x(\Gamma)) + \sum_{y:\, |x-y|=1} N_y(h_x(\Gamma)) \right) \\ &\leq c_d \sum_{x \in F(\Gamma)} \bar{N}_x(h_x(\Gamma)) \geq c_d L^{-(d-3/2)k} \sigma |Y \cap \overline{D(\Gamma)}| \end{aligned} \tag{4.92}$$

where $c_d \geq \frac{1}{2d}$ is some geometrical constant. Putting these two estimates together gives the estimate claimed in Lemma 4.7. $\diamondsuit$

IV.6 Proof of the main Theorem

In this last subsection we can finally give the proof of our main theorem. This will show just one major application of estimates on the sequence of renormalized measures we have established above. The main technical estimate needed to prove Theorem 1 is contained in the following

PROPOSITION 4.2: *Let $\mathcal{E}_k$ denote the event*

$$\mathcal{E}_k \equiv \left\{\Gamma \in \Omega_n \;:\; int\underline{\Gamma} \supset \mathcal{L}^k(0) \wedge d(\gamma_0) > L^{k+1}\right\} \tag{4.93}$$

where γ_0 denotes the largest connected component of Γ for which $0 \in int\underline{\gamma_0}$. Then, for β large enough, σ small enough and the parameters L, α and η chosen such that the preceeding results are all valid, there exist positive constants a and b such that

$$I\!P\left[\mu_\Lambda\left(1\!\!1_{\mathcal{E}_k}\right) \geq e^{-b\tilde{\beta}^{(k)}}\right] \leq L^d \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)k}\frac{\delta_k^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{4.94}$$

where μ_Λ denotes the finite volume Gibbs measure with zero boundary conditions of the level zero model (the dependence on the parameters temperature, random fields, etc. is again suppressed)

Proof: Let us denote by $\mu_{n-k}^{(k)}$ the measures on Ω_{n-k}^0 obtained by k-fold iteration of the RG map (generically denoted by $\mathcal{R}$; we also write $\mathcal{R}^k$ for k iterations, even though the maps $\mathcal{R}$ in each step are not the same). Now, by definition of the event $\mathcal{E}_k$, for all $\Gamma \in \mathcal{E}_k$ $\mathrm{int}\left(\underline{\mathcal{R}^k(\Gamma)}\right) \supset 0$ (since the component γ_0, by its shear size, cannot have become 'small' in only $k-1$ iterations and thus the support of its k-th image still encloses zero. Therefore

$$\mu_n\left(1\!\!1_{\mathcal{E}_k}\right) \leq \mu_{n-k}^{(k)}\left(1\!\!1_{\mathrm{int}\,\underline{\Gamma}\supset 0}\right) \tag{4.95}$$

Now for each level and for each Γ it is uniquely defined what Γ^l and Γ^s means. Thus we may insert in the right hand side of (4.95) the identity $1\!\!1_{\mathrm{int}\,\underline{\Gamma^l}\supset 0} + 1\!\!1_{\mathrm{int}\,\underline{\Gamma^l}\not\supset 0}$ to get

$$\mu_n\left(1\!\!1_{\mathcal{E}_k}\right) \leq \mu_{n-k}^{(k)}\left(1\!\!1_{\mathrm{int}\,\underline{\Gamma^l}\supset 0}\right) + \mu_{n-k}^{(k)}\left(1\!\!1_{\mathrm{int}\,\underline{\Gamma^s}\supset 0}1\!\!1_{\mathrm{int}\,\underline{\Gamma^l}\not\supset 0}\right) \tag{4.96}$$

We will see later that the second term in (4.96) is fairly easy to handle. To deal with the first one, observe that if $\mathrm{int}\,\underline{\Gamma^l} \supset 0$, then the component of Γ whose interior surrounds 0 cannot disappear in the summation over small contours, and thus the image contour, $T\Gamma$, will still have a support those interior contains 0. Thus

$$\mu_{n-k}^{(k)}\left(1\!\!1_{\mathrm{int}\,\underline{\Gamma^l}\supset 0}\right) \leq \mu_{n-k-1}^{(k+1)}\left(1\!\!1_{\mathrm{int}\,\underline{\Gamma}\supset 0}\right) \tag{4.97}$$

This argument can thus be iterated to give

$$\mu_n\left(1\!\!1_{\mathcal{E}_k}\right) \leq \sum_{l=k}^{N-1}\mu_{n-l}^{(l)}\left(1\!\!1_{\mathrm{int}\,\underline{\Gamma^s}\supset 0}1\!\!1_{\mathrm{int}\,\underline{\Gamma^l}\not\supset 0}\right) + \mu_0^{(n)}\left(1\!\!1_{\underline{\Gamma}\supset 0}\right) \tag{4.98}$$

The last term will again be easy to treat, since at this stage we have a measure on a single site system. In order to deal with the terms in the sum, let us introduce the following objects, called 'restricted contour measures' We set

$$\hat{\nu}(\Gamma) \equiv e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{G \supset 0} \tag{4.99}$$

For all G contributing to (4.99), we denote by G_0 the connected component of G that contains the origin. Inserting in the summation the identity $\mathbb{I}_{G_0 \cap \underline{\Gamma^l} = \emptyset} + \mathbb{I}_{G_0 \cap \underline{\Gamma^l} \neq \emptyset}$, we can write

$$\hat{\nu}(\Gamma) = \hat{\nu}^l(\Gamma) + \hat{\nu}^s(\Gamma) \tag{4.100}$$

where

$$\begin{aligned} \hat{\nu}_s(\Gamma) &\equiv e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{G \supset 0} \mathbb{I}_{G_0 \cap \underline{\Gamma^l} = \emptyset} \quad \text{and} \\ \hat{\nu}_l(\Gamma) &\equiv e^{-\beta(S,V(\Gamma))} \sum_{G \supset \underline{\Gamma}} \rho(\Gamma, G) \mathbb{I}_{G \supset 0} \mathbb{I}_{G_0 \cap \underline{\Gamma^l} \neq \emptyset} \end{aligned} \tag{4.101}$$

With Z the usual partition function (i.e. the one normalizing μ!), we also set

$$\nu_{s,l}(\Gamma) \equiv \frac{1}{Z} \hat{\nu}_{s,l}(\Gamma) \tag{4.102}$$

The following Lemma will be important:

LEMMA 4.9: *Let ν' be defined like ν, but with the renormalized ρ', S' and the corresponding partition function Z'. Let T denote the corresponding RG map. Then*

$$\sum_{\Gamma : T(\Gamma) = \Gamma'} \nu_l(\Gamma) \leq \nu'(\Gamma') \tag{4.103}$$

Proof: Clearly we have that

$$\sum_{\Gamma : T_2 T_1(\Gamma) = \tilde{\Gamma}'} \hat{\nu}_l(\Gamma) \leq e^{-\beta'(\tilde{S}', V(\Gamma'))} \sum_{G' \supset \underline{\tilde{\Gamma}'}} \rho'(\tilde{\Gamma}', G') \mathbb{I}_{G \supset 0} \tag{4.104}$$

due to the fact that the sets G_0 contributing to $\hat{\nu}_l$, since they contain large connected components of Γ cannot be exponentiated in the summation over the small contours; moreover, in the blocking procedure, they contribute only to terms associated with sets G' such that $\mathcal{L}G' \supset G \supset 0$, which in turn implies $G' \supset 0$. Applying the third RG step only produces a constant (from the centering) which cancels against the same constant in partition function. This yields the lemma. $\diamondsuit$

We will see later that ν_s is very small with large probability. The Lemma states that ν_l can be pushed to the next level. We have to show now how the quantities $\mu^{(l)}_{n-l}\left(\mathbb{I}_{\text{int}\,\underline{\Gamma^s} \supset 0} \mathbb{I}_{\text{int}\,\underline{\Gamma^l} \not\supset 0}\right)$ are related to $\nu^{(l)}$. We write (we drop the superscripts l indicating the level for the moment)

$$
\begin{aligned}
&\mu\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)\\
&= \frac{1}{Z^{(l)}}\sum_{\Gamma} e^{-\beta(S,V(\Gamma))}\sum_{G\supset\underline{\Gamma}}\rho(\Gamma,G)\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\left(\mathbb{I}_{G_0\cap\Gamma^{l}=\emptyset}+\mathbb{I}_{G_0\cap\Gamma^{l}\neq\emptyset}\right)\\
&\equiv \mu_s\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)+\mu_l\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)
\end{aligned}
\tag{4.105}
$$

where G_0 stands for the connected component of G that contains support of the small component of Γ those interior contains the origin (this makes sense, since only such Γ contribute in the sum). Now again the term with the subscript s will cause no problem, while for the other we have

$$
\mu_l\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)\le \nu'(\mathbb{I})
\tag{4.106}
$$

(where $\mathbb{I}$ here means the function one. This fact follows almost by the same argument as Lemma 4.7. One should note that the component G_0 does not disappear, since it is constraint to contain large support; on the other hand, it contains the support of a small component, $\underline{\gamma_0}$, those interior contains the origin, and by the condition on the maximal size of a small component, $\mathcal{L}^{-1}\underline{\gamma_0}=0$, and so $\mathcal{L}^{-1}G\supset 0$. (4.106) is then obtained by just forgetting about any other possible constraints.

Adding now the level-indicating superscripts and iterating Lemma 4.9 we get that

$$
\nu^{(l+1)}(\mathbb{I})\le\sum_{j=l+1}^{n-1}\nu_s^{(j)}(\mathbb{I})+\nu^{(n)}(\mathbb{I})
\tag{4.107}
$$

and finally

$$
\mu_n(\mathcal{E}_k)\le\sum_{l=k}^{n-1}\left[\mu_{s,n-l}^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)+\sum_{j=l+1}^{n-1}\nu_s^{(j)}(\mathbb{I})+\nu^{(n)}(\mathbb{I})\right]+\mu_0^{(n)}\left(\mathbb{I}_{\underline{\Gamma}\ni 0}\right)
\tag{4.108}
$$

Finally we must estimate the s-subscripted terms in (4.108).

LEMMA 4.10: *There exists a constants $a>0$ and $b>0$ such that*

$$
I\!P\left[\mu_{s,n-l}^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)\ge e^{-b\bar\beta^{(l)}}\right]\le L^d\exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)l}\frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right)
\tag{4.109}
$$

Proof: The proof of this Lemma is reminiscent of the classical Peierl's argument [P] and makes (finally!!) use of the lower bounds on certain of the activities that were proven in the RG procedure. Making explicit the constraints we write

$$
\begin{aligned}
\mu_{s,n-l}^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{s}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)=\frac{1}{Z}&\sum_{\substack{\gamma_0,\,\text{small}\\ \mathrm{int}\,\underline{\gamma_0}\ni 0}}\ \sum_{G_0\supset\underline{\gamma_0}}\ \sum_{\substack{\Gamma_0^s:\underline{\Gamma_0^s}\subset G_0\\ \gamma_0\subset\Gamma_0^s}}\rho(\Gamma_0^s,G_0)\\
\times&\sum_{G\cap G_0=\emptyset}\ \sum_{\substack{\Gamma:\underline{\Gamma}\subset G\\ \mathrm{int}\,\underline{\Gamma}\not\ni G_0}}\rho(\Gamma,G)e^{-\beta(S,V(\Gamma\cup\Gamma_0^s))}
\end{aligned}
\tag{4.110}
$$

(we have again suppressed the level indicating superscripts l at the obvious places to keep the notation readable). Note that the second line almost reconstitutes a partition function outside the region G_0, except for the (topological) constraint on the support of Γ and the fact that the field term is not the correct one. This latter problem can be repaired by noting that

$$(S, V(\Gamma \cup \Gamma_0^s)) = (S, V(\Gamma)\backslash G_0) + (S, V(\Gamma \cup \Gamma_0^s))_{G_0} \tag{4.111}$$

where

$$(S, V(\Gamma \cup \Gamma_0^s))_{G_0} \equiv \sum_{h \in \mathbb{Z}} \sum_{\substack{C \subset V_h(\Gamma \cup \Gamma_0^s) \\ C \cap G_0 \neq \emptyset}} S_C(h) \tag{4.112}$$

This last term consists of a local term (i.e. involving only C consisting of a single site x) which depends only on Γ_0^s, and the non-local one, which as in the previous instances is very small, namely

$$|(S_{nl}, V(\Gamma \cup \Gamma_0^s))_{G_0}| \leq const.|G_0|e^{-\tilde{\beta}} \tag{4.113}$$

Thus

$$\begin{aligned} \mu_{s,n-l}^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma^s} \ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma^l} \not\ni 0}\right) \leq & \sum_{\substack{\gamma_0,\, \mathrm{small} \\ \mathrm{int}\,\underline{\gamma_0} \ni 0}} \sum_{G_0 \supset \underline{\gamma_0}} \sum_{\substack{\Gamma_0^s : \underline{\Gamma_0^s} \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} \rho(\Gamma_0^s, G_0) e^{-\beta(S_{loc}, V(\Gamma_0^s) \cap G_0)} e^{const.|G_0|e^{-\beta}} \\ & \times \frac{1}{Z} \sum_{G \cap G_0 = \emptyset} \sum_{\substack{\Gamma : \underline{\Gamma} \subset G \\ \mathrm{int}\,\underline{\Gamma} \not\supset G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma) \backslash G_0)} \end{aligned} \tag{4.114}$$

Now the last line has the desired form. A slight problem here is that the contours contributing to the denominator are not (in general) allowed to have empty support in G_0, as the support of any Γ must contain $D(\Gamma)$. Note however that G_0 is necessarily such that $D(0) \cap G_0 \subset \mathcal{D}(0)$, as otherwise G_0 would have to contain support from large contours. Thus for given G_0, we may bound the partition function from below by summing only over such contours that within G_0 have $h_x(\Gamma) \equiv 0$ and those have support in G_0 is exactly given by $\mathcal{D}(0) \cap G_0$. Treating the small-field term as above this gives the bound

$$\begin{aligned} Z \geq & \prod_{i: \mathcal{D}_i(0) \subset G_0} \rho(\mathcal{D}_i(0), \mathcal{D}_i(0)) e^{-\beta(S_{loc}, G_0)} e^{-cont.|G_0|e^{-\beta}} \\ & \times \sum_{G \cap G_0 = \emptyset} \sum_{\substack{\Gamma : \underline{\Gamma} \subset G \\ \mathrm{int}\,\underline{\Gamma} \not\supset G_0}} \rho(\Gamma, G) e^{-\beta(S, V(\Gamma) \backslash G_0)} \end{aligned} \tag{4.115}$$

Thus

$$\begin{aligned} \mu_{s,n-l}^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma^s} \ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma^l} \not\ni 0}\right) \leq & \sum_{\substack{\gamma_0,\, \mathrm{small} \\ \mathrm{int}\,\underline{\gamma_0} \ni 0}} \sum_{G_0 \supset \underline{\gamma_0}} \sum_{\substack{\Gamma_0^s : \underline{\Gamma_0^s} \subset G_0 \\ \gamma_0 \subset \Gamma_0^s}} e^{+2const.|G_0|e^{-\beta}} e^{-\beta(S_{loc}, V(\Gamma_0^s) \cap G_0) + \beta(S_{loc}, G_0)} \\ & \times \frac{\rho(\Gamma_0^s, G_0)}{\prod_{i: \mathcal{D}_i(0) \subset G_0} \rho(\mathcal{D}_i(0), \mathcal{D}_i(0))} \end{aligned} \tag{4.116}$$

Here the ρ's appearing in the denominator are exactly those for which we have lower bounds. Note that for this reason we could not deal directly we expressions in which G_0 is allowed to contain also large components of Γ. The estimation of the sums in (4.116) is now performed just like in the absorbtion of small contours RG step. Γ_0^s with non-constant heights give essentially no contribution, and due to the separatedness of the components $\mathcal{D}_i(0)$, and the smallness of the total control field on one such component, the main contribution comes from the term where Γ_0^s has support in only one component $\mathcal{D}_i(0)$. If there is such a component which surrounds 0, this could of course give a contribution of order one. However, if we assume that the origin is surrounded by a ball of radius $L/4$ such that for all points x within this ball $\mathcal{N}_x(0) = 0$, then G_0 cannot be contained in $\mathcal{D}(0)$ and therefore

$$\mu_{s,n-l}^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{\bullet}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right) \leq const.e^{-\tilde{\beta}^{(l)}} \tag{4.117}$$

On the other hand, the probability of this event is clearly greater than

$$1-(L/4)^d \mathbb{P}[\mathcal{N}_0^{(l)}(0)\neq 0] \geq 1-(L/4)\exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)l}\frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{4.118}$$

from which the Lemma follows. $\diamondsuit$

In almost exactly the same way one proves also the analogous bound on the $\nu_s^{(j)}(\mathbb{I})$:

LEMMA 4.11: *There exists a constants $a>0$ and $b>0$ such that*

$$\mathbb{P}\left[\nu_s^{(l)}\left(\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{\bullet}\ni 0}\mathbb{I}_{\mathrm{int}\,\underline{\Gamma}^{l}\not\ni 0}\right)\geq e^{-b\tilde{\beta}^{(l)}}\right] \leq \exp\left(-L^{\left(\frac{d-2}{2(d-1)}-\eta\right)l}\frac{\delta^2}{a\sigma^{2-\frac{d-2}{d-1}}}\right) \tag{4.119}$$

(Note that the factor L^d is missing here, since the event whose probability is considered occurs this time only if the origin is in $D(0)$).

Finally, the estimates for the $\mu^{(N)}$ of the same form are trivial, since only the single site 0 has survived the blocking at this stage. Putting these estimates together into (4.108), Proposition 4.2 follows immediately. $\diamondsuit\diamondsuit$

Now Proposition 4.2 implies Theorem 1 almost immediately. For, consider $\mu_n^0(\,\cdot\,|\mathcal{E}_k^c)$, the local specifications conditioned on the event that $\mathcal{E}_k$ does not occur. It is clear that the marginals on $\Omega_{\mathcal{L}^k(0)}$ of such sequences converge weakly; moreover, if we introduce the event

$$\mathcal{E}\equiv\bigcap_{k=0}^{\infty}\mathcal{E}_k \tag{4.120}$$

then

$$\mu_n^0\left(\,\cdot\,|\cup_{k=1}^n\mathcal{E}_k^c\right)\to\mu^0\left(\,\cdot\,|\mathcal{E}^c\right) \tag{4.121}$$

weakly, since convergence can only depend on the tail of the algebra generated by the $\mathcal{E}_k$. But from the estimates on the probabilities for $\mathcal{E}_k$ to occur, it is plain that

$$\lim_{n\uparrow\infty} \mu_n^0(\mathcal{E}) = 0, \ I\!P - \text{almost surely} \tag{4.122}$$

Therefore, in the limit, the conditioned and unconditioned probabilities coincide, almost surely, and hence the sequence of measures μ_n^0 converges $I\!P$-almost surely to some limiting measure μ^0 which is a Gibbs measure corresponding to height zero. Of course, the same construction can be carried through for any height $h \in \mathbb{Z}$, giving rise to a family of Gibbs measures labelled by the external height. It is evident from our construction that all these measures are disjoint; the reader will, for instance, easily verify that with large probability,

$$\mu^0(|h_0|) = 0(e^{-\beta}) \tag{4.123}$$

This concludes the proof of Theorem 1. $\diamond\diamond$

Remark: It is of course possible to obtain a more detailed characterization of the properties of the infinite volume Gibbs states (see e.g. our results in the $T = 0$ case). We will not go into this here. Some further estimates may be found in [K].

V. Concluding remarks

We have presented a renormalization group method suitable to prove the existence of low-temperature Gibbs states describing 'flat' interfaces in a SOS-model for surfaces in weakly random media in dimension $D \geq 4$. This consisted on realizing the SOS-model as a certain example of a class of contour models in dimension $d = D-1$ with non-compact state spaces. We have shown that we can construct, in an algorithmic way, RG maps that leave this class of models invariant. We have controlled the flow of the iterative application of these maps and have shown that under certain conditions on the initial model, this flow tends to a 'trivial' limit. This situation corresponds, in the language of classical probability theory, to a 'strong law of large numbers' type result. We have then shown how the control on this sequence of image systems can be used to obtain relevant information on the initial system.

It should be stressed that in a certain sense the situation we were dealing with is 'trivial' – in spite of the rather heavy machinery we needed to employ. What we mean by trivial here is that a single iteration of the RG brings the system much closer to the trivial one: temperatures are being reduced, variances of random fields are reduced, etc., so that, provided we can carry out *one* RG transformation, the subsequent iterations become more and more easy. For this reason, we could be rather generous in many of our estimates and even in the way we defined the RG maps themselves. It should be clear that there is a lot of room to improve things, if this is necessary for other applications or different models. One point, for instance, that may have annoyed some readers, is the appearance of the two inverse 'temperatures' β and $\tilde{\beta}$ that scale with different speeds to infinity. This is essentially due the way the 'coarse graining' step, or the absorption of small contours is performed, which still leaves untouched 'long and thin' contours as well as thin spikes emerging from fat contours, although such configurations cannot be provoked by 'small' random fields. A more extensive coarse graining could thus remove this artefact, if necessary or desired.

There are a number of direction to further generalize and develop this method. For one thing, one would like to prove the existence of Dobrushin states [Do] in the full-fledged disordered Ising model. We believe that such a proof is now actually within range. Another type of questions concerns systems with less 'symmetry', a simple example being already this SOS-model restricted to a half space, or other 'wetting-type' problems in disordered media. As mentioned in the introduction, this may be possible by merging in ideas from Pirogov-Sinai theory [Za1,Za2].

A particularly challenging problem is of course the analysis of the situation in lower dimension. Here one would no longer expect to have an infinite volume Gibbs or even ground state for the interface, but there should be some kind of scaling law for the interface fluctuation in finite volumes (a celebrated result of this type is the supposedly exact $L^{2/3}$ law in dimension $D = 2$ [FHH,KN]). One should say that the non-existence of a Gibbs state is likely to follow from arguments of Aizenman and Wehr [AW], but no formal proof has been given. An analysis of this regime through the RG method appears technically very hard, in that much sharper estimates would be required, but not entirely hopeless.

In the same spirit, the analysis of systems with genuinely 'strong' disorder remains a desideratum; here we have in mind in particular spin glass models. Although we are very far yet from treating such cases, the RG approach may prove a useful tool also there.

In conclusion, we hope that the present exposition of the RG method for disordered systems is convincing evidence for the power and flexibility of this technique and will help to make it a useful tool with numerous applications in this domain.

Appendix

This appendix contains the proofs of the four main geometrical lemmas used in Sections III and IV.

LEMMA 3.3: *Let γ be a weakly connected contour s.t. $d(\ \mathrm{int}\underline{\gamma}) \leq L$. Let h_γ denote the height of γ on $\partial\, \mathrm{int}\, \underline{\gamma}$. Then*

$$E_s(\gamma) \geq \frac{2d}{L} \sum_{x \in \mathrm{int}\,(\underline{\gamma})} |h_x(\gamma) - h_\gamma| \tag{A.1}$$

Proof: Without loss of generality we may assume that $h_\gamma = 0$ and that $\mathrm{int}\,\underline{\gamma}$ is contained in the cube $C_L \equiv [1, L]^d$. To prove the lemma, we then have to prove a lower bound on $\sum_{x \in C_L} |h_x|$ for any function h that vanishes outside this cube in terms of the surface energy. Let us write $x = (x_1, \ldots, x_d)$, let e_i denote the positive unit vectors in $I\!R^d$ and let $\hat{x}_i \equiv x - x_i e_i$ (i.e. the vector x with the i-th component set to zero. With this notation we have

$$\begin{aligned}
\sum_{x \in C_L} |h_x| &= \sum_{x \in C_L} \frac{1}{d} \sum_{i=1}^{d} \frac{1}{2} \left| \sum_{z_i=1}^{x_i} \left(h_{\hat{x}_i + z_i e_i} - h_{\hat{x}_i + (z_i - 1)e_i}\right) + \sum_{z_i = x_i + 1}^{L+1} \left(-h_{\hat{x}_i + z_i e_i} + h_{\hat{x}_i + (z_i - 1)e_i}\right) \right| \\
&\leq \sum_{x \in C_L} \frac{1}{d} \sum_{i=1}^{d} \frac{1}{2} \left[\sum_{z_i=1}^{x_i} \left|h_{\hat{x}_i + z_i e_i} - h_{\hat{x}_i + (z_i - 1)e_i}\right| + \sum_{z_i = x_i + 1}^{L+1} \left|-h_{\hat{x}_i + z_i e_i} + h_{\hat{x}_i + (z_i - 1)e_i}\right| \right] \\
&= \sum_{x \in C_L} \frac{1}{2d} \sum_{i=1}^{d} \sum_{z_i=1}^{L+1} \left|h_{\hat{x}_i + z_i e_i} - h_{\hat{x}_i + (z_i - 1)e_i}\right| \\
&= \frac{L}{2d} \sum_{<x,y> \in \overline{C_L}} |h_x - h_y|
\end{aligned} \tag{A.2}$$

where we have used the fact that the i-th term in the one-but-last line is independent of x_i; thus the summation over x_i gives a factor L, while the remaining sums together with the sum over z_i gives the part of the surface energy coming from the steps in the i-th direction. All terms together then yield the entire surface energy. This obviously proves the lemma.$\diamondsuit$

LEMMA 3.6: *Let h be any integer-valued height-function, and set $h' \equiv \mathrm{Rnd}\left(\overline{h}\right)$ where $\overline{h} \equiv L^{-d} \sum_{x \in \mathcal{R}0} h_x$. Then*

$$\sum_{<x,y>: x,y \in \mathcal{R}0} |h_x - h_y| \geq \frac{1}{L} \sum_{x \in \mathcal{L}0} |h_x - h'| \tag{A.3}$$

Proof: To prove this lemma, we will first proof that

$$\sum_{<x,y>: x,y \in \mathcal{R}0} |h_x - h_y| \geq \frac{2}{L} \sum_{x \in \mathcal{L}0} |h_x - \overline{h}| \tag{A.4}$$

for any function h (not necessarily integer-valued. Note that (A.3) follows immediately from (A.4) for integer valued h: By definition, h' is the integer closest to $\overline{h}$, so in particular for any integer h_x, $|h_x - h'| \leq |\overline{h} - h'|$. Thus $|h_x - h'| \leq |h_x - \overline{h}| + |\overline{h} - h'| \leq 2||\overline{h} - h'|$, which inserted into (A.4) gives (A.3).

We are thus left with proving (A.4). This will be done by induction over the dimension. Let first $d = 1$. Without loss of generality, we may assume $\overline{h} = 0$. Let $n_\pm$ denote the number of sites where h_x is positive or negative, respectively; set $\overline{h}_+ \equiv \frac{1}{n_+} \sum_{x=1}^{L} h_x \mathbb{I}_{h_x > 0}$ and define $\overline{h}_-$ analogously. Then

$$\begin{aligned} \sum_{x=1}^{L} |h_x| &= n_+ \overline{h}_+ - n_- \overline{h}_- = n_+ \left(\overline{h}_+ - \frac{n_+ \overline{h}_+}{L} - \frac{n_- \overline{h}_-}{L} \right) - n_- \left(\overline{h}_- - \frac{n_+ \overline{h}_+}{L} - \frac{n_- \overline{h}_-}{L} \right) \\ &= \frac{2}{L} n_+ n_- (\overline{h}_+ - \overline{h}_-) \leq \frac{L}{2} (h_{max} - h_{min}) \end{aligned} \tag{A.5}$$

where we have used that $n_+ \overline{h}_+ + n_- \overline{h}_- = L\overline{h} = 0$, that $n_+ n_- = n_+(L - n_-) \leq \frac{L^4}{4}$ and that $\overline{h}_+ - \overline{h}_- \leq h_{max} - h_{min}$. Now, obviously,

$$h_{max} - h_{min} \leq \sum_{x=2}^{L} |h_x - h_{x-1}| \tag{A.6}$$

which gives (A.4) for $d = 1$.

Assume now that (A.4) holds for $d - 1$. We will show that it holds for d. Let us write for $x \in \mathbb{Z}^d$, $x = (\hat{x}, t)$ with $\hat{x} \in \mathbb{Z}^{d-1}$, $t \in \mathbb{Z}$. Define $\overline{h}_t \equiv L^{1-d} \sum_{\hat{x} \in \{1,\ldots,L\}^{d-1}} h_{\hat{x},t}$. Then clearly

$$\begin{aligned} \sum_{x \in C_L} |h_x - \overline{h}| &= \sum_{t=1}^{L} \sum_{\hat{x} \in \{1,\ldots,L\}^{d-1}} |h_{\hat{x},t} - \overline{h}_t + \overline{h}_t - \overline{h}| \\ &\leq \sum_{t=1}^{L} \left(\sum_{\hat{x} \in \{1,\ldots,L\}^{d-1}} |h_{\hat{x},t} - \overline{h}_t| \right) + L^{d-1} \sum_{t=1}^{L} |\overline{h}_t - \overline{h}| \end{aligned} \tag{A.7}$$

Now in the first term, for each fixed t we may apply (A.4) for $d - 1$, so that

$$\sum_{t=1}^{L} \left(\sum_{\hat{x} \in \{1,\ldots,L\}^{d-1}} |h_{\hat{x},t} - \overline{h}_t| \right) \leq \frac{L}{2} \sum_{t=1}^{L} \sum_{<\hat{x},\hat{y}>} |h_{\hat{x},t} - h_{\hat{y},t}| \tag{A.8}$$

while for the second term the one-dimensional version of (A.4) can be applied, giving

$$\begin{aligned} L^{d-1} \sum_{t=1}^{L} |\overline{h}_t - \overline{h}| &\leq L^{d-1} \frac{L}{2} \sum_{t=2}^{L} |\overline{h}_t - \overline{h}_{t-1}| \\ &\leq \frac{L}{2} \sum_{\hat{x} \in \{1,\ldots,L\}^{d-1}} \sum_{t=2}^{L} |h_{\hat{x},t} - h_{\hat{x},t-1}| \end{aligned} \tag{A.9}$$

Obviously, the sum of the terms in (A.8) and (A.9) is bounded by $\frac{L}{2}\sum_{<x,y>}|h_x - h_y|$, which gives (A.4) for d and concludes the proof of the lemma. $\diamondsuit$

Note that the bounds given by the previous two lemmas are optimal since it is not difficult to construct configurations for which equality holds.

LEMMA 3.7: *Let $\Gamma \in \mathcal{R}^{-1}\gamma'$. Then*

$$E_s(\Gamma) \geq \frac{L^{d-1}}{d+1} E_s(\gamma') \tag{A.10}$$

Proof: Set $\overline{h}'_y = L^{-d}\sum_{x\in\mathcal{L}y} h_x$ and $E_s(\overline{\gamma}') = \sum_{<z,w>:z,w\in\mathcal{L}\underline{\overline{\gamma}'}} |\overline{h}'_z - \overline{h}'_w|$. We will first show that

$$E_s(\Gamma) \geq L^{d-1} E_s(\overline{\gamma}') \tag{A.11}$$

In fact, this is quite easy. Just write

$$E_s(\overline{\gamma}') = \sum_{i=1}^{d}\sum_{\hat{y}_i}\sum_{y_i}\left|\overline{h}'_{\hat{y}_i,y_i} - \overline{h}'_{\hat{y}_i,y_i-1}\right| \tag{A.12}$$

By an argument quite similar to the one used in the previous proof, we have that

$$\sum_{y_i}\left|\overline{h}'_{\hat{y}_i,y_i} - \overline{h}'_{\hat{y}_i,y_i-1}\right| \leq \sum_{y_i}\sum_{x_i=L(y_i)-1}^{Ly_i}\left|\tilde{h}_{\hat{y}_i,x_i} - \tilde{h}_{\hat{y}_i,x_i-1}\right| \tag{A.13}$$

where $\tilde{h}_{\hat{y}_i,x_i} \equiv L^{-d+1}\sum_{\hat{x}_i\in\mathcal{L}\hat{y}_i} h_{\hat{x}_i,x_i}$ with the obvious meaning of the notation for the summation range. From (A.13) (A.11) follows now simply by inserting this definition and using the triangle inequality.

We now have to cope with the fact that in $E_s(\gamma)$ enter the rounded means of the heights rather than the block means themselves. The basic idea here is that this may cause a problem only if these means are far from integers, in which case the height within such a block has been very non-constant. Indeed, using lemma 3.6, we may get another lower bound on $E_s(\Gamma)$, namely

$$\begin{aligned}E_s(\Gamma) &\geq \frac{2}{L}\sum_{y\in\underline{\overline{\gamma}'}}\sum_{x\in\mathcal{L}y}|h_x - \overline{h}'_y| \geq \frac{2}{L}\sum_{y\in\underline{\overline{\gamma}'}}\sum_{x\in\mathcal{L}y}|h'_y - \overline{h}'_y| \\ &= 2L^{d-1}\sum_{y\in\underline{\overline{\gamma}'}}|h'_y - \overline{h}'_y| = 2L^{d-1}\frac{1}{2d}\sum_{<y,z>\in\underline{\overline{\gamma}'}}|h'_y - \overline{h}'_y| + |h'_z - \overline{h}'_z|\end{aligned} \tag{A.14}$$

where we have again used that h'_y is the closest integer to $\overline{h}'_y$. Now

$$
\begin{aligned}
E_s(\Gamma) &= \frac{1}{d+1}\frac{d}{d+1}E_s(\Gamma) + \frac{d}{d+1}E_s(\Gamma) \\
&\geq \frac{L^{d-1}}{d+1} \sum_{<y,z>:y,z\in\mathcal{L}\underline{\gamma'}} \left| h'_y - h'_z - h'_y + \overline{h}'_y + h'_z - \overline{h}'_z \right| \\
&+ L^{d-1}\frac{1}{d+1} \sum_{<y,z>\in\underline{\gamma'}} |h'_y - \overline{h}'_y| + |h'_z - \overline{h}'_z| \\
&\geq \frac{L^{d-1}}{d+1} \sum_{<y,z>:y,z\in\mathcal{L}\underline{\gamma'}} |h'_y - h'_z|
\end{aligned}
\tag{A.15}
$$

using again the triangle inequality for the last inequality. This proves the lemma. ◇

LEMMA 3.8: *Let γ be connected and large. Then*

$$
L^{(1-\alpha)/2}E_s(\gamma) + |\underline{\gamma}\backslash\overline{\mathcal{D}}(\gamma)| \geq \frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(\gamma)| \tag{A.16}
$$

Proof: The following two properties of the sets $\mathcal{D}(h)$ are the essential ingredients in the proof of this lemma:

(i) (sparsity of $\mathcal{D}(h)$): If $C \subset \Lambda$ is connected and $d(C) \geq L^{\frac{1-\alpha/2}{2}}$ then $|C\backslash\overline{\mathcal{D}}(h)| \geq |C|/2$ for L large enough.

This follows from the definition of $\mathcal{D}(h)$ as a union of $L^{\frac{1}{2}}$-components each of which has a maximum volume of $L^{\frac{1-\alpha}{2}}$.

(ii) (separation of $\overline{\mathcal{D}}(h)$ and $\overline{D}(h)\backslash\overline{\mathcal{D}}(h)$): If $C \subset \Lambda$ is connected s.t. $C \cap \overline{\mathcal{D}}(h) \neq \emptyset$ and $|C \cap (\overline{\overline{D}}(h)\backslash\overline{\mathcal{D}}(h))| \neq \emptyset$ then $d(C) \geq L^{\frac{1-\alpha/2}{2}}$ and hence the conclusion of (i) holds for C.

This follows from the definition of $\mathcal{D}(h)$ as a union of $L^{\frac{1}{2}}$-components which implies that

$$
d(D(h)\backslash\mathcal{D}(h), \mathcal{D}(h)) \geq L^{\frac{1}{2}} \tag{A.17}
$$

We must now distinguish the cases where γ is flat or not. In the first case, the proof is in fact identical to the one given in [BK] and we repeat it here only for the convenience of the reader.

In this case we write $\gamma = (\underline{\gamma}, h_x \equiv h)$. Assume first that the set $\underline{\gamma}\backslash(\overline{D}(h)\backslash\overline{\mathcal{D}}(h))$ is nonempty and denote its connected components by C_i. Then

$$
\begin{aligned}
|\underline{\gamma}\backslash\overline{D}(h)| &= \sum_i |C_i\backslash\overline{\mathcal{D}}(h)| \geq \sum_{i:C_i\cap\overline{\mathcal{D}}(h)\neq\emptyset} |C_i\backslash\overline{\mathcal{D}}(h)| \\
&\geq \sum_{i:C_i\cap\overline{\mathcal{D}}(h)\neq\emptyset} \frac{1}{2}|C_i\cap\overline{\mathcal{D}}(h)| = \frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(h)|
\end{aligned}
\tag{A.18}
$$

where property (ii) was used.

Assume next that $|\underline{\gamma}\backslash(\overline{D}(h)\backslash\overline{\mathcal{D}}(h))| = \emptyset$. Now if $\underline{\gamma}\cap\overline{\mathcal{D}}(h) = \emptyset$, then there (A.16) is trivial. Thus we may assume the contrary. Now since γ is large, either $d(\underline{\gamma}) \geq L-2$ or $(D(\gamma)\backslash\mathcal{D}(\gamma))\cap V_i(\gamma) \neq \emptyset$. In the first case, (A.16) follows by property (i) while in the second it follows from property (ii). Thus, (A.16) is proven for flat contours.

Now consider the case that γ is not flat. We cannot say anything a priori about the sets $D(\gamma)$ and $\mathcal{D}(\gamma)$ for general γ since the defining geometrical properties of $\mathcal{D}$ only refer to the slices $D(h)$ at fixed height. However, all fluctuations in the heights introduce surface energy terms which ensure the validity of (A.16). Define $V_{h,i}(\gamma)$ to be the connected components of $V_h(\gamma)\cap\underline{\gamma}$. Notice that $E_s(\gamma)$ is clearly bounded from below by one-half times the number of such connected components Then,

$$\begin{aligned}
&L^{(1-\alpha)/2}E_s(\gamma)+|\underline{\gamma}\backslash\overline{D}(\gamma)| \\
&\geq \sum_{h,i:d(V_{h,i}(\gamma))\leq L^{1/2}} \frac{1}{2}L^{(1-\alpha)/2} + \sum_{h,i:d(V_{h,i}(\gamma))>L^{1/2}} |V_{h,i}(\gamma)\backslash\overline{D}(h)| \\
&\geq \frac{1}{2}\sum_{h,i:d(V_{h,i}(\gamma))\leq L^{1/2}} |V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)| + \frac{1}{2}\sum_{h,i:d(V_{h,i}(\gamma))>L^{1/2}} |V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)| \\
&= \frac{1}{2}|\underline{\gamma}\cap\overline{\mathcal{D}}(\gamma)|
\end{aligned} \tag{A.19}$$

Here, the estimation in the second last line follows since, for $d(V_{h,i}(\gamma)) \leq L^{1/2}$, $|V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)|$ contains at most $L^{(1-\alpha)/2}$ sites, due to the definition of $\mathcal{D}$ (definition 3.4), while for $d(V_{h,i}(\gamma)) > L^{1/2}$ we can apply the arguments of the flat case to obtain $|V_{h,i}(\gamma)\backslash\overline{D}(h)| \geq \frac{1}{2}|V_{h,i}(\gamma)\cap\overline{\mathcal{D}}(h)|$. This in fact concludes the proof of the lemma. $\Diamond$

LEMMA 3.9: *Let $\Gamma \in T_2{}^{-1}\gamma'$. Then there exists a constant $c_6 > 0$ s.t.*

$$LE_s(\Gamma) + |\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))| \geq c_6 L|\underline{\gamma'}\backslash\overline{\tilde{D}'}(\gamma')| \tag{A.20}$$

where $\tilde{D}' \equiv D(\tilde{N}')$. **Remark:** Lemma 3.9 also holds if $\mathcal{D} = \emptyset$, $\tilde{D}' = \mathcal{L}^{-1}D$ and the respective definition of largeness, given in chapter 3.

Proof: Let us define

$$\begin{aligned}
\hat{\mathcal{D}}'(h) &\equiv \mathcal{L}^{-1}\left(D(h)\backslash\mathcal{D}(h)\right) \\
\hat{\mathcal{D}}' &\equiv \bigcup_{h\in\mathbb{Z}}\left(\hat{\mathcal{D}}'(h)\times\{h\}\right)
\end{aligned} \tag{A.21}$$

We will in fact show that (A.20) holds with $\tilde{D}'$ replaced by for $\hat{D}'$, which implies the lemma since $\tilde{D}'\supset\hat{D}'$.

We consider the partition into connected components

$$\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma)) = \bigcup_j X_j \tag{A.22}$$

The idea of the proof is to write the l.h.s. of (A.20) as a sum over the X_j's and use for those parts of them which are 'well inside' the flat regions of Γ the same arguments as [BK], while for those parts which are 'near' to a region of the contour which is not entirely flat the surface energy will provide a sufficiently large contribution. Denote the set of blocks in which Γ has no constant height by $T(\Gamma)$, i.e.

$$T(\Gamma) \equiv \{y \in \underline{\gamma}', \nexists h \in \mathbb{Z} : \mathcal{L}y \subset V_h(\Gamma)\} \tag{A.23}$$

Now, for X_j such that $X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset$ with $d(X_j) \leq \frac{L}{4}$, we can apply the same arguments as [BK] to show that they are contained in $\mathcal{L}\hat{\overline{D}}'(\gamma')$ and thus do not contribute to the right-hand side of (A.20). Namely, let γ be a large connected component of Γ s.t. $X_j \subset \underline{\gamma}$. Then if $X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset$, then there is some $h \in \mathbb{Z}$ such that $h_x(\gamma') = h$ for $x \in \mathcal{L}(\mathcal{L}^{-1}X_j)$. We claim that this implies $\overline{X_j} \cap (\overline{D}(h)\backslash\overline{\mathcal{D}}(h)) \neq \emptyset$. For, if this was not true, since X_j is a component of $\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))$, we would have $\gamma = (\underline{\gamma}, h_x \equiv h)$ s.t. $\underline{\gamma} \subset \overline{X_j}$. But since $d(X_j) \leq \frac{L}{4}$ this would be in contradiction to the assumed largeness of γ. Now from the fact that $\overline{X_j} \cap (\overline{D}(h)\backslash\overline{\mathcal{D}}(h)) \neq \emptyset$ it follows obviously that $X_j \subset \mathcal{L}\hat{\overline{D}}'(\gamma')$. We remark here that we have written $\overline{T(\Gamma)}$ to ensure that the parts of $\hat{\overline{D}}'(\gamma')$ which absorb such X_j are in fact at the same uniform height as Γ is on X_j.

Next, consider the components X_j s.t. $X_j \cap \mathcal{L}\overline{T(\Gamma)} \neq \emptyset$. If $X_j \subset \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}}$, [1]we will forget its contribution to the second term on the l.h.s. of (A.20), but only use the surface energy term to estimate is contribution to the r.h.s. of (A.20) from above. If $X_j \not\subset \mathcal{L}\overline{\overline{\overline{T(\Gamma)}}}$, we decompose

$$X_j\backslash\mathcal{L}\overline{T(\Gamma)} = \bigcup_k Z_{j,k} \tag{A.24}$$

Note that for those $Z_{j,k}$ with $Z_{j,k}\backslash\mathcal{L}\overline{\overline{\overline{T(\Gamma)}}} \neq \emptyset$, we have $d(Z_{j,k}) \geq L$. Thus we obtain

$$\begin{aligned}|\underline{\Gamma}\backslash(\overline{D}(\Gamma)\backslash\overline{\mathcal{D}}(\Gamma))| &\geq \sum_{X_j : X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset, d(X_j) \geq L/4} |X_j| + \sum_{Z_{j,k} : Z_{j,k}\backslash\mathcal{L}\overline{\overline{\overline{T(\Gamma)}}} \neq \emptyset} |Z_{j,k}| \\ &\geq \bar{c}L \left(\sum_{X_j : X_j \cap \mathcal{L}\overline{T(\Gamma)} = \emptyset, d(X_j) \geq L/4} |\mathcal{L}^{-1}X_j| + \sum_{Z_{j,k} : Z_{j,k}\backslash\mathcal{L}\overline{\overline{\overline{T(\Gamma)}}} \neq \emptyset} |\mathcal{L}^{-1}Z_{j,k}| \right) \\ &\geq \bar{c}|(\underline{\gamma}'\backslash\hat{\overline{D}}'(\gamma'))\backslash\overline{\overline{\overline{T(\Gamma)}}}|\end{aligned} \tag{A.25}$$

with some constant $\bar{c}$, where the second inequality follows since the diameter of all sets involved is or the order L and the third inequality follows from the previous arguments. To get rid of $\overline{\overline{\overline{T(\Gamma)}}}$ we now use the surface energy. Obviously

$$|\overline{\overline{\overline{T(\Gamma)}}}| \leq c_d E_s(\Gamma) \tag{A.26}$$

[1] The double bar on a set really means the set of all points those distance to the set is less than or equal to two.

with $c_d = \frac{\#\{y \in \mathbb{Z}^d, d(y,0) \leq 2\}}{2d} = \frac{5^d}{2d}$. Hence we can finish the proof by

$$|\underline{\Gamma} \backslash (\overline{D}(\Gamma) \backslash \overline{\mathcal{D}}(\Gamma))| + L E_s(\Gamma) \geq c_6 L \left(|(\underline{\gamma}' \backslash \overline{\hat{D}'}(\gamma')) \backslash \overline{\overline{\overline{T(\Gamma)}}}| + |\overline{\overline{\overline{T(\Gamma)}}}| \right) \geq c_6 L |\underline{\gamma}' \backslash \overline{\hat{D}'}(\gamma')| \tag{A.27}$$

with $c_6 = \min\{\bar{c}, c_d^{-1}\}$.$\diamondsuit$

Remark: We would like to give an example which shows that the L in front of $E_s(\Gamma)$ is really necessary. Let Γ be defined by $h_x(\Gamma) = 1_{x=x_0}$ and $\underline{\Gamma} = \{x_0\}$ and assume that $\{x_0\} = D(\Gamma)$ and $x_0 \notin \overline{\mathcal{D}(\Gamma)}$. Then $|\underline{\Gamma} \backslash (\overline{D}(\Gamma) \backslash \overline{\mathcal{D}}(\Gamma))| = 0$ but $|\underline{\gamma}' \backslash \overline{\tilde{D}'}(\gamma')| = 1$. Hence we really need a factor of the order L to ensure the validity of (A.20).

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