On maximum of Gaussian non-centered fields indexed on smooth manifolds.

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Abstract

The double sum method of evaluation of probabilities of large deviations for Gaussian processes with non-zero expectations is developed. Asymptotic behaviors of the tail of non-centered locally stationary Gaussian fields indexed on smooth manifold are evaluated. In particular, smooth Gaussian fields on smooth manifolds are considered.

1 Introduction

The double-sum method is one of the main tools in studying asymptotic behavior of maxima distribution of Gaussian processes and fields, see [1], [7], [3] and references therein. Until recently only centered processes have been considered. It can be seen from [7] and the present paper that the investigation of non-centered Gaussian fields can be performed with similar techniques, which, however, are far from trivial. Furthermore, there are examples when the need for the asymptotic behaviour for non-centered fields arises. In [8], [9] statistical procedures have been introduced to test non-parametric hypotheses for multi-dimensional distributions. The asymptotic decision rules are based on tail distributions of maxima of Gaussian fields indexed on spheres or products of spheres. In order to estimate power of the procedures one might have to have asymptotic behaviour of tail maxima distributions for non-centered Gaussian fields.

In this paper we extend the double sum method to study Gaussian processes with non-zero expectations. We evaluate asymptotic behavior of the tail of non-centered locally $(\alpha_{\mathbf{t}}, D_{\mathbf{t}})$-stationary Gaussian field indexed on smooth manifold, as defined below. In particular, smooth Gaussian fields on smooth manifolds are considered.

2 Definitions, auxiliary results, main results

Let the collection $\alpha_1, ..., \alpha_k$ of positive numbers be given, as well as the collection $l_1, ..., l_k$ of positive integers such that $\sum_{i=1}^k l_i = n$. We set $l_0 = 0$. This two collections is called a *structure*, [7]. For any vector $\mathbf{t} = (t_1, ..., t_n)^\top$ its *structural module* is defined by

$$|\mathbf{t}|_\alpha = \sum_{i=1}^{k} \left(\sum_{j=E(i-1)+1}^{E(i)} t_j^2 \right)^{\frac{\alpha_i}{2}}, \tag{1}$$

where $E(i) = \sum_{j=0}^{i} l_j$, $j = 1, ..., k$. The structure defines a decomposition of the space R^n into the direct sum $\mathrm{R}^n = \bigoplus_{i=1}^k \mathrm{R}^{l_i}$, such that the restriction of the structural module on either of R^{l_i} is just Euclidean norm taken to the degree α_i, $i = 1, ..., k$, respectively. For $u > 0$ denote by G_u^i the homothety of the subspace R^{l_i} with the coefficient u^{-2/α_i}, $i = 1, ..., k$, respectively, and by g_u, the superposition of the homotheties, $g_u = \bigcirc_{i=1}^k G_u^i$. It is clear that for any $\mathbf{t} \in \mathrm{R}^n$,

$$|g_u \mathbf{t}|_\alpha = u^{-2} |\mathbf{t}|_\alpha. \tag{2}$$

Let $\chi(\mathbf{t})$, $\mathbf{t} \in \mathrm{R}^n$, be a Gaussian field with continuous paths, the expected value and the covariance function are given by

$$\mathbf{E}\chi(\mathbf{t}) = -|\mathbf{t}|_\alpha, \qquad \mathbf{Cov}(\chi(\mathbf{t}), \chi(\mathbf{s})) = |\mathbf{t}|_\alpha + |\mathbf{s}|_\alpha - |\mathbf{t} - \mathbf{s}|_\alpha, \tag{3}$$

respectively. Thus $\chi(\mathbf{t})$ can be represented as a sum of independent multi-parameter drifted fractional Brownian motions (Lévy-Shönberg fields) indexed on R^{l_i}, with parameters α_i.

To proceed, we need a generalization of the Pickands' constant. Define the function on measurable subsets of R^n,

$$H_\alpha(B) = \exp\left\{\sup_{\mathbf{t}\in B} \chi(\mathbf{t})\right\}. \tag{4}$$

Let D be a non-degenerated matrix $n \times n$, throughout we make no notation difference between a matrix and the corresponding linear transformation. Next, for any $S > 0$, we denote by

$$[0, S]^k = \{\mathbf{t}:\ 0 \le t_i \le S,\ i = 1, ..., k,\ t_i = 0,\ i = k+1, .., n\},$$

a cube of dimension k generated by the first k coordinates in R^n. In [2] it is proved that there exists a positive limit

$$0 < H_\alpha^{D R^k} := \lim_{S\to\infty} \frac{H_\alpha(D[0, S]^k)}{\mathrm{mes}_k(D[0, S]^k)} < \infty, \tag{5}$$

where $\mathrm{mes}_k(D[0, S]^k)$ denotes the k-dimensional Lebesgue measure of $D[0, S]^k$. We write shortly $H_\alpha^{(k)} = H_\alpha^{I R^k}$ with I is the unit matrix. The constant $H_\alpha = H_\alpha^{(n)}$ is the Pickands' constant. Denote

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{x^2}{2}} dx, \tag{6}$$

it is well known that

$$\Psi(u) = \frac{1}{\sqrt{2\pi} u} e^{-\frac{u^2}{2}} (1 + o(1)) \quad \text{as } u \to \infty. \tag{7}$$

Lemma 1 *Let $X(\mathbf{t})$, $\mathbf{t} \in \mathrm{R}^n$, be a Gaussian homogeneous centered field. Let for a non-degenerated matrix A and α-structure on R^n, the covariance function $r(\mathbf{t})$ of $X(\mathbf{t})$ satisfies*

$$r(\mathbf{t}) = 1 - |A\mathbf{t}|_\alpha + o(|A\mathbf{t}|_\alpha) \quad \text{as} \quad \mathbf{t} \to \mathbf{0}. \tag{8}$$

Then for any compact set $T \subset \mathrm{R}^n$ and any function $\theta(u)$ with $\theta(u) \to 1$ as $u \to \infty$,

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in g_u T} X(\mathbf{t}) > u\theta(u)\right\} = H_\alpha(AT)\Psi(u\theta(u))(1 + o(1)) \quad \text{as } u \to \infty. \tag{9}$$

Definition 1 *Let an α-structure is given on R^n. We say that $X(\mathbf{t})$, $\mathbf{t} \in T \subset \mathrm{R}^n$ has a local $(\alpha, D_{\mathbf{t}})$-stationary structure, or $X(\mathbf{t})$ is locally $(\alpha, D_{\mathbf{t}})$-stationary, if for any $\varepsilon > 0$ there exists a positive $\delta(\varepsilon)$ such that for any $\mathbf{s} \in T$ one can find a non-degenerated matrix $D_{\mathbf{s}}$ such that the covariance function $r(\mathbf{t}_1, \mathbf{t}_2)$ of $X(\mathbf{t})$ satisfies*

$$1 - (1+\varepsilon)|D_{\mathbf{s}}(\mathbf{t}_1 - \mathbf{t}_2)|_\alpha \leq r(\mathbf{t}_1, \mathbf{t}_2) \leq 1 - (1-\varepsilon)|D_{\mathbf{s}}(\mathbf{t}_1 - \mathbf{t}_2)|_\alpha \tag{10}$$

provided $||\mathbf{t}_1 - \mathbf{s}|| < \delta(\varepsilon)$ and $||\mathbf{t}_2 - \mathbf{s}|| < \delta(\varepsilon)$.

It is convenient for the reader to cite here four theorems which are in our use, in suitable to our purposes forms. Before that we need some notations. Let L be a k-dimensional subspace of R^n, for fixed orthogonal coordinate systems in R^n and in L, let $(x_1, ..., x_k)^\top$ be the coordinate presentation of a point $\mathbf{x} \in L$, and $(x_1', ...x_n')^\top$ be its coordinate presentation in R^n. Denote by $M = M(L)$ the corresponding transition matrix,

$$(x_1', ...x_n')^\top = M(x_1, ..., x_k)^\top,$$

that is $M = (\partial x_i' / \partial x_j, , \ i = 1, ..., n \, , \, j = 1, ..., k)$.

Next, for a matrix G of size $n \times k$ we denote by $V(G)$, the square root of the sum of squares of all minors of order k. This invariant transforms the volume when the dimension of vectors is changed, that is $d\mathbf{t} = V(G)^{-1} dG\mathbf{t}$. Note that since both coordinate systems in L and R^n are orthogonal, $V(M) = 1$.

Theorem 1 *(*Theorem 7.1, [7]*) Let $X(\mathbf{t})$, $\mathbf{t} \in \mathrm{R}^n$, be a Gaussian homogeneous centered field such that for some α, $0 < \alpha \leq 2$ and a non-degenerated matrix D its covariance function satisfies*

$$r(\mathbf{t}) = 1 - ||D\mathbf{t}||^\alpha + o(||D\mathbf{t}||^\alpha) \quad as \ \mathbf{t} \to 0, \tag{11}$$

Then for any k, $0 < k \leq n$, every subspace L of R^n with $\dim L = k$, *any Jordan set $A \subset L$, and every function $w(u)$ with $w(u)/u = o(1)$ as $u \to \infty$,*

$$\mathbf{P}\left\{\sup_{\mathbf{t} \in A} X(\mathbf{t}) > u + w(u)\right\} = \tag{12}$$

$$= H_\alpha^{(k)} V(DM(L)) \mathrm{mes}_L(A) u^{\frac{2k}{\alpha}} \Psi(u + w(u))(1 + o(1)) \tag{13}$$

as $u \to \infty$, provided

$$r(\mathbf{t} - \mathbf{s}) < 1 \quad for\ all \ \ \mathbf{t}, \mathbf{s} \in \bar{A}, \ \mathbf{t} \neq \mathbf{s}, \tag{14}$$

with $\bar{A}$ the closure of A.

Theorem 2 (Theorem 1, [4]). *Let $X(\mathbf{t})$, $\mathbf{t} \in \mathrm{R}^n$, be a Gaussian centered locally $(\alpha, D_{\mathbf{t}})$-stationary field, with $\alpha > 0$ and a continuous matrix function $D_{\mathbf{t}}$. Let $\mathcal{M} \subset \mathrm{R}^n$ be a smooth compact of dimension k, $0 < k \leq n$. Then for any c,*

$$\mathbf{P}\left\{\sup_{\mathbf{t} \in \mathcal{M}} X(\mathbf{t}) > u - c\right\} = \tag{15}$$

$$= H_\alpha^{(k)} u^{\frac{2k}{\alpha}} \Psi(u - c) \int_{\mathcal{M}} V(D_{\mathbf{t}} M_{\mathbf{t}}) \, d\mathbf{t} (1 + o(1)) \tag{16}$$

as $u \to \infty$, where $M_{\mathbf{t}} = M(T_{\mathbf{t}})$ with $T_{\mathbf{t}}$ the tangent subspace taken to $\mathcal{M}$ at the point $\mathbf{t}$ and $d\mathbf{t}$ is an element of volume of $\mathcal{M}$.

Theorem 3 *(*The Borell-Sudakov-Tsirelson inequality.*) Let $X(t)$, $t \in T$, be a measurable Gaussian process indexed on an arbitrary set T, and let numbers σ, m, a be defined by relations,*

$$\sigma^2 = \sup_{t\in T} \mathbf{Var} X(t) < \infty, \quad m = \sup_{t\in T} \mathbf{E} X(t) < \infty,$$

and

$$\mathbf{P}\left\{\sup_{t\in T} X(t) - \mathbf{E}X(t) \geq a\right\} \leq \frac{1}{2}. \tag{17}$$

Then for any x,

$$\mathbf{P}\left\{\sup_{t\in T} X(t) > x\right\} \leq 2\Psi\left(\frac{x-m-a}{\sigma}\right). \tag{18}$$

Theorem 4 *(*Slepian inequality.*) Let $X(t)$, $Y(t)$, $t \in T$, be separable Gaussian processes indexed on an arbitrary set T, and suppose that for all $t, s \in T$,*

$$\begin{aligned} \mathbf{Var} X(t) = \mathbf{Var} Y(t), &\qquad \mathbf{E}X(t) = \mathbf{E}Y(t), \\ &\textit{and} \\ \mathbf{Cov}(X(t), X(s)) &\leq \mathbf{Cov}(Y(t), Y(s)). \end{aligned} \tag{19}$$

Then for all x,

$$\mathbf{P}\left\{\sup_{t\in T} X(t) < x\right\} \leq \mathbf{P}\left\{\sup_{t\in T} Y(t) < x\right\}. \tag{20}$$

We turn now to our main results.

Theorem 5 *Let $X(\mathbf{t})$, $\mathbf{t} \in \mathrm{R}^n$, be a Gaussian locally $(\alpha, D_{\mathbf{t}})$-stationary field, with some $\alpha > 0$ and continuous matrix function $D_{\mathbf{t}}$. Let $\mathcal{M} \subset \mathrm{R}^n$ be a smooth k-dimensional compact, $0 < k \leq n$. Let the expectation $m(\mathbf{t}) = \mathbf{E}X(\mathbf{t})$ is continuous on $\mathcal{M}$ and attains its maximum on $\mathcal{M}$ at the only point $\mathbf{t}_0$, with*

$$m(\mathbf{t}) = m(\mathbf{t}_0) - (\mathbf{t}-\mathbf{t}_0)B(\mathbf{t}-\mathbf{t}_0)^\top + O(||\mathbf{t}-\mathbf{t}_0||^{2+\beta}) \quad \textit{as } \mathbf{t} \to \mathbf{t}_0, \tag{21}$$

for some $\beta > 0$ and positive matrix B. Then

$$\begin{aligned} &\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}} X(\mathbf{t}) > u\right\} = \\ &= \frac{\pi^{k/2}}{\sqrt{\mathbf{det} M^\top B M}} V(D_{\mathbf{t}_0} M)\, H_\alpha^{(k)} u^{\frac{2k}{\alpha} - \frac{k}{2}} \Psi(u - m(\mathbf{t}_0))(1 + o(1)) \end{aligned} \tag{22}$$

as $u \to \infty$, where $M = M(T_{\mathbf{t}_0})$ and $T_{\mathbf{t}_0}$ is the tangent subspace to $\mathcal{M}$ taken at the point $\mathbf{t}_0$.

Theorem 6 *Let $\mathcal{M} \subset \mathrm{R}^n$ be a smooth k-dimensional compact, $0 < k \leq n$. Let $X(\mathbf{t})$, $\mathbf{t} \in \mathrm{R}^n$, be a differentiable in square mean sense Gaussian field with $\mathbf{Var}X(\mathbf{t}) = 1$ for* *all* $\mathbf{t} \in \mathcal{M}$ *and* $r(t,s) < 1$ *for all* $\mathbf{t}, \mathbf{s} \in \mathcal{M}$, $\mathbf{t} \neq \mathbf{s}$. *Let the expectation* $m(\mathbf{t}) = \mathbf{E}X(\mathbf{t})$ *is same as in Theorem 5. Then*

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}} X(\mathbf{t}) > u\right\} = \tag{23}$$
$$= \frac{\sqrt{V(\frac{1}{2}A_{\mathbf{t}_0}M)}}{\sqrt{\mathbf{det} M^\top BM}} u^{\frac{k}{2}} \Psi(u - m(\mathbf{t}_0))(1 + o(1))$$

as $u \to \infty$, *with* M *as in Theorem 5 and* $A_{\mathbf{t}_0}$ *the covariance matrix of the orthogonal projection of the gradient vector of the field* $X(\mathbf{t})$ *in point* $\mathbf{t}_0$ *onto the tangent subspace to the* $\mathcal{M}$ *taken at the point* $\mathbf{t}_0$.

3 Proofs.

Proof of Lemma 1. First, observe that if one changes g_u on $g_{u\theta(u)}$, the lemma immediately follows from Lemma 6.1, [7]. Second, observe that we can write $g_u T = g_{u\theta(u)}(I_u T)$, where I_u is a linear transformation of R^n, which also is a superposition of homotheties of R^{k_i} with coefficients tending to 1 as $u \to \infty$. Thus I_u tends to identity, and $I_u T$ tends to T in Euclidean distance. Third, note that $H_\alpha(T)$ is continuous in T in the topology of the space of measurable subsets of a compact, say K, generated by Euclidean distance. To prove that, observe that χ is a.s. continuous and $H_\alpha(T) \leq H_\alpha(K) < \infty$, for all $T \subset K$, and use the dominated convergence theorem. These observations imply the Lemma assertion. □

Proof of Theorem 5. Let $T_{\mathbf{t}_0}$ be the tangent plane to $\mathcal{M}$ taken at the point $\mathbf{t}_0$. Let $\mathcal{M}_0$ be a neighbourhood of $\mathbf{t}_0$ in $\mathcal{M}$, so small that it can be one-to-one projected on $T_{\mathbf{t}_0}$. We denote by P the corresponding one-to-one projector so that $P\mathcal{M}_0$ is the image of $\mathcal{M}_0$. The field $X(\mathbf{t})$, $\mathbf{t} \in \mathcal{M}$, generates on $P\mathcal{M}_0$ a field $\tilde{X}(\tilde{\mathbf{t}}) = X(\mathbf{t})$, $\tilde{\mathbf{t}} = P\mathbf{t}$. It is clear, that $\mathbf{E}\tilde{X}(\tilde{\mathbf{t}}) = m(\mathbf{t}) = m(P^{-1}\tilde{\mathbf{t}})$. We denote by $\tilde{r}(\tilde{\mathbf{t}}, \tilde{\mathbf{s}}) = r(\mathbf{t}, \mathbf{s})$, the covariance function of $\tilde{X}(\tilde{\mathbf{t}})$. Choose an arbitrary $\varepsilon \in (0, \frac{1}{2})$. Due to the local stationary structure, one can find $\delta_0 = \delta(\varepsilon) > 0$ such that for all $\tilde{\mathbf{t}}_1, \tilde{\mathbf{t}}_2 \in T_{\mathbf{t}_0} \cap S(\delta_0, \mathbf{t}_0)$, where $S(\delta_0, \mathbf{t}_0)$ is centered at $\mathbf{t}_0$ ball with radius δ_0, we have

$$\exp\left\{-(1+\varepsilon)||D_{\mathbf{t}_0}(\tilde{\mathbf{t}}_1 - \tilde{\mathbf{t}}_2)||^\alpha\right\} \leq \tilde{r}(\tilde{\mathbf{t}}_1, \tilde{\mathbf{t}}_2) \leq \exp\left\{-(1-\varepsilon)||D_{\mathbf{t}_0}(\tilde{\mathbf{t}}_1 - \tilde{\mathbf{t}}_2)||^\alpha\right\}. \tag{24}$$

We also can assume δ_0 to be so small that we could let $\mathcal{M}_0 = P^{-1}[T_{\mathbf{t}_0} \cap S(\delta_0, \mathbf{t}_0)]$ and think of $P\mathcal{M}_0$ as of a ball in $T_{\mathbf{t}_0}$ centered at $\tilde{\mathbf{t}}_0 = \mathbf{t}_0$, with the same radius. Denote $\mathcal{M}_1 = \mathcal{M} \setminus \mathcal{M}_0$. Since $m(\mathbf{t})$ is continuous,

$$\sup_{\mathbf{t}\in\mathcal{M}_1} m(\mathbf{t}) = m(\mathbf{t}_0) - c_0,$$

with $c_0 > 0$. By Theorem 2, for $X_0(\mathbf{t}) = X(\mathbf{t}) - m(\mathbf{t})$ we have,

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}_1} X(\mathbf{t}) > u\right\} = \mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}_1} X_0(\mathbf{t}) + m(\mathbf{t}) > u\right\} \leq$$

$$\leq \mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}_1} X_0(\mathbf{t}) > u - m(\mathbf{t}_0) + c_0\right\} =$$
$$= H_\alpha^{(k)} u^{\frac{2k}{\alpha}} \Psi(u - m(\mathbf{t}_0) + c_0)(1+o(1)) \int_{\mathcal{M}_1} V(D_\mathbf{t} M_\mathbf{t})\, d\mathbf{t} =$$
$$= o(\Psi(u - m(\mathbf{t}_0) + c_1)), \tag{25}$$

for any c_1 with $0 < c_1 < c_0$.

Now turn to $\mathcal{M}_0$. Note that

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}_0} X(\mathbf{t}) > u\right\} = \mathbf{P}\left\{\sup_{\tilde{\mathbf{t}}\in P\mathcal{M}_0} \tilde{X}(\tilde{\mathbf{t}}) > u\right\}. \tag{26}$$

Introduce a Gaussian stationary centered field $X_H(\mathbf{t})$, $\mathbf{t}\in \mathrm{R}^n$, with covariance function

$$r_H(\mathbf{t}) = \exp\{-(1+2\varepsilon)||D_{\mathbf{t}_0}\mathbf{t}||^\alpha\}.$$

Since (24) by Slepian inequality,

$$\mathbf{P}\left\{\sup_{\tilde{\mathbf{t}}\in P\mathcal{M}_0} \tilde{X}(\tilde{\mathbf{t}}) > u\right\} \leq \mathbf{P}\left\{\sup_{\mathbf{t}\in P\mathcal{M}_0} X_H(\tilde{\mathbf{t}}) + \tilde{m}(\tilde{\mathbf{t}}) > u\right\}. \tag{27}$$

Clear that without loss of generality we can put the origin of R^n at the point $\mathbf{t}_0$, so that the tangent plane $T_{\mathbf{t}_0}$ is now a tangent subspace and $\mathbf{t}_0 = \tilde{\mathbf{t}}_0 = \mathbf{0}$. From this point on we restrict ourselves by the k-dimensional subspace $T_{\mathbf{t}_0}$ and will drop the "tilde". Let now $S = S(\mathbf{0}, \delta)$ be a ball in $T_{\mathbf{t}_0}$ centered at zero with radius δ with $\delta = \delta(u) = u^{-1/2}\log^{1/2} u$, this choice will be clear later on. For all sufficiently large u we have $S \subset P\mathcal{M}_0$, and there exists a positive c_1, such that

$$\mathbf{P}\left\{\sup_{\mathbf{v}\in S^c\cap P\mathcal{M}_0} X_H(\mathbf{v}) + \tilde{m}(\mathbf{v}) > u\right\} \leq$$
$$\leq \mathbf{P}\left\{\sup_{\mathbf{v}\in S^c\cap P\mathcal{M}_0} X_H(\mathbf{v}) > u - \tilde{m}(\mathbf{t}_0) + c_1\delta^2(u)\right\} \leq$$
$$\leq \mathbf{P}\left\{\sup_{\mathbf{v}\in P\mathcal{M}_0} X_H(\mathbf{v}) > u - \tilde{m}(\mathbf{t}_0) + c_1\delta^2(u)\right\}. \tag{28}$$

Applying Theorem 1 to the latter probability and making elementary calculations we get

$$\mathbf{P}\left\{\sup_{\mathbf{v}\in S^c\cap P\mathcal{M}_0} X_H(\mathbf{v}) + \tilde{m}(\mathbf{v}) > u\right\} = o\left(\Psi(u - m(\mathbf{t}_0))\right) \quad \text{as } u\to\infty. \tag{29}$$

Turn now to the ball S. Let $\mathbf{v}_1 = (v_{11}, ..., v_{n1})$, ...,$\mathbf{v}_k = (v_{1k}, ..., v_{nk})$ be an orthonormal basis in $T_{\mathbf{t}_0}$ given in the coordinates of R^n. In the coordinate system, consider the cubes

$$\Delta_\mathbf{0} = u^{-2/\alpha}[0,T]^k,\ \Delta_\mathbf{l} = u^{-2/\alpha} \times_{\nu=1}^k [l_\nu T, (l_\nu+1)T],$$
$$\mathbf{l} = (l_1, ..., l_k) \in \mathrm{Z}^k,\ T > 0$$

We have,

$$\sum_{\mathbf{i}\in L}\mathbf{P}\left\{A_{\mathbf{i}}\right\}-\sum_{\mathbf{i},\mathbf{j}\in L',\mathbf{i}\neq\mathbf{j}}\mathbf{P}\left\{A_{\mathbf{i}}A_{\mathbf{j}}\right\}\leq\mathbf{P}\left\{\sup_{\mathbf{v}\in S}X_H(\mathbf{v})+\tilde{m}(\mathbf{v})>u\right\}\leq\sum_{\mathbf{i}\in L'}\mathbf{P}\left\{A_{\mathbf{i}}\right\}, \quad (30)$$

where $A_{\mathbf{i}}=\left\{\sup_{\mathbf{v}\in\Delta_{\mathbf{i}}}X_H(\mathbf{v})+\tilde{m}(\mathbf{v})>u\right\}$, L' is the set of multi-indexes $\mathbf{i}$ with $\Delta_{\mathbf{i}}\cap S\neq\emptyset$ and L is the set of multi-indexes $\mathbf{i}$ with $\Delta_{\mathbf{i}}\subset S$. Using (21), we have

$$\mathbf{P}\left\{\sup_{\mathbf{v}\in\Delta_{\mathbf{i}}}X_H(\mathbf{v})+\tilde{m}(\mathbf{v})>u\right\}\leq \quad (31)$$

$$\leq\mathbf{P}\left\{\sup_{\mathbf{v}\in\Delta_{\mathbf{l}}}X_H(\mathbf{v})+m(\mathbf{t}_0)-\min_{\mathbf{v}\in\Delta_{\mathbf{i}}}||\sqrt{B}\mathbf{v}||^2+w_1(u)>u\right\}.$$

Here $uw_1(u)\to 0$ as $u\to\infty$ because of the choice of $\delta(u)$ and the remainder in (21). By Lemma 1 and the equivalence

$$\mathbf{t}=\tilde{\mathbf{t}}+O(||\tilde{\mathbf{t}}||^2)\ \text{ as }\ \mathbf{t}\to\mathbf{0}$$

(recall that we have assumed $\mathbf{t}_0=\tilde{\mathbf{t}}_0=\mathbf{0}$), there exists a function $\gamma_1(u)$, with $\gamma_1(u)\to 0$ as $u\to\infty$, such that for all sufficiently large u and every $\mathbf{i}\in L'$,

$$\mathbf{P}\left\{\sup_{\mathbf{v}\in\Delta_{\mathbf{l}}}X_H(\mathbf{v})+\tilde{m}(\mathbf{v})>u\right\}\leq(1+\gamma_1(u))H_\alpha\left((1+\varepsilon)^{1/\alpha}D_{\mathbf{t}_0}[0,T]^k\right)\times$$

$$\times\Psi\left(u-m(\mathbf{t}_0)+\min_{\mathbf{v}\in\Delta_{\mathbf{l}}}||\sqrt{B}\mathbf{v}||^2+w_1(u)\right). \quad (32)$$

Using similar arguments, we get, that there exists $\gamma_2(u)$ with $\gamma_2(u)\to 0$ as $u\to\infty$, such that for all sufficiently large u and every $\mathbf{i}\in L$,

$$\mathbf{P}\left\{\sup_{\mathbf{v}\in\Delta_{\mathbf{l}}}X_H(\mathbf{v})+\tilde{m}(\mathbf{v})u\right\}\geq(1-\gamma_2(u))H_\alpha\left((1+\varepsilon)^{1/\alpha}D_{\mathbf{t}_0}[0,T]^k\right)\times$$

$$\times\Psi\left(u-m(\mathbf{t}_0)+\min_{\mathbf{v}\in\Delta_{\mathbf{l}}}||\sqrt{B}\mathbf{v}||^2+w_2(u)\right), \quad (33)$$

where $uw_2(u)\to 0$ as $u\to\infty$.

Now, in accordance with (30), we sum right-hand parts of (32) and (33) over L' and L, respectively. Using (7), we get for all sufficiently large u,

$$\sum_{\mathbf{i}\in L'}\Psi\left(u-m(\mathbf{t}_0)+\min_{\mathbf{v}\in\Delta_{\mathbf{i}}}||\sqrt{B}\mathbf{v}||^2+w_1(u)\right)\leq$$

$$\leq(1+\gamma_1'(u))\Psi(u-m(\mathbf{t}_0))T^{-k}u^{2k/\alpha}\times$$

$$\times\sum_{\mathbf{i}\in L'}\exp\left\{-u\min_{\mathbf{v}\in\Delta_{\mathbf{i}}}||\sqrt{B}\mathbf{v}||^2+o\left(1/u\right)\right\}T^ku^{-2k/\alpha}, \quad (34)$$

where $\gamma_1'(u)\to 0$ as $u\to\infty$. Changing variables $\mathbf{w}=\sqrt{u}\mathbf{t}$ and using the dominated convergence, we get

$$\sum_{\mathbf{i}\in L'}\exp\left\{-u\min_{\mathbf{v}\in\Delta_{\mathbf{i}}}||\sqrt{B}\mathbf{v}||^2+o\left(1/u\right)\right\}=$$

$$=T^{-k}\int_{T_{\mathbf{t}_0}}\exp\{-B\mathbf{w},\mathbf{w}\}d\mathbf{w}u^{2k/\alpha-k/2}(1+o(1)), \quad (35)$$

as $u\to\infty$. Note that dw means here k-dimensional volume unite in $T_{\mathbf{t}_0}$. Similarly,

$$\sum_{\mathbf{i}\in L}\exp\left\{-u\min_{\mathbf{v}\in\Delta_{\mathbf{i}}}||\sqrt{B}\mathbf{v}||^2+o\left(1/u\right)\right\}=$$
$$=T^{-k}\int_{T_{\mathbf{t}_0}}\exp\{-B\mathbf{w},\mathbf{w}\}d\mathbf{w}u^{2k/\alpha-k/2}(1+o(1)) \tag{36}$$

as $u\to\infty$. In order to compute the integral $\int_{T_{\mathbf{t}_0}}\exp\{-B\mathbf{w},\mathbf{w}\}d\mathbf{w}$ we note that $\mathbf{w}=M\mathbf{t}$, where $\mathbf{t}$ denotes the vector $\mathbf{w}$ presented in the orthogonal coordinate system of $T_{\mathbf{t}_0}$, recall that in this case $V(M)=1$. Hence

$$\int_{T_{\mathbf{t}_0}}\exp\{-B\mathbf{w},\mathbf{w}\}d\mathbf{w}=\int_{T_{\mathbf{t}_0}}\exp\{-BM\mathbf{t},M\mathbf{t}\}\,d\mathbf{t}=$$
$$=\frac{\pi^{k/2}}{\sqrt{\det(M^\top BM)}}=:e^*. \tag{37}$$

Thus for all sufficiently large u,

$$\sum_{\mathbf{i}\in L'}\mathbf{P}\{A_{\mathbf{i}}\}\le(1+\gamma_1''(u))H_\alpha\left((1+\varepsilon)^{1/\alpha}D_{\mathbf{t}_0}[0,t]^k\right)e^*T^{-k}u^{2k/\alpha-k/2}\Psi(u-m(\mathbf{t}_0)) \tag{38}$$

and

$$\sum_{\mathbf{i}\in L}\mathbf{P}\{A_{\mathbf{i}}\}\ge(1-\gamma_1''(u))H_\alpha\left((1+\varepsilon)^{1/\alpha}D_{\mathbf{t}_0}[0,t]^k\right)e^*T^{-k}u^{2k/\alpha-k/2}\Psi(u-m(\mathbf{t}_0)), \tag{39}$$

where $\gamma_1''(u)\to0$ as $u\to\infty$.

Now we are in a position to analyze the double sum in the left-hand part of (30). We begin with the estimation of the probability

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\Delta_1}X_H(\mathbf{t})+\tilde{m}(\mathbf{t})>u,\sup_{\mathbf{t}\in\Delta_2}X_H(\mathbf{t})+\tilde{m}(\mathbf{t})>u\right\},$$

with

$$\Delta_1=u^{-2/\alpha}\times_{\nu=1}^k[S_\nu^1,T_\nu^1],\quad S_\nu<T_\nu,\ \nu=1,...,k,$$
$$\Delta_2=u^{-2/\alpha}\left(\mathbf{w}+\times_{\nu=1}^k[S_\nu^1,T_\nu^1]\right),\quad S_\nu^1<T_\nu^1,\ \nu=1,...,k,$$

where $\mathbf{w}$, T_ν, S_ν are such that $\rho(\Delta_1,\Delta_2)>0$, with $\rho(\cdot,\cdot)$ is the Euclidean distance in R^k. Recall that $\Delta_i\cap S(\mathbf{0},\delta(u))\neq\emptyset$, $i=1,2$. Estimations of this probability follow the proof of Lemma 6.3, [7], but since the expectation of the field variates, more details have to be discussed, therefore we give complete computations. Denote

$$K_1=\times_{\nu=1}^k[S_\nu,T_\nu],\ K_2=\mathbf{w}+K_1,\ c(u)=\max_{\mathbf{t}\in\Delta_1\cup\Delta_2}\tilde{m}(\mathbf{t}),\ \theta(u)=1-\frac{c(u)}{u}.$$

We have

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\Delta_1}X_H(\mathbf{t})+\tilde{m}(\mathbf{t})>u,\sup_{\mathbf{t}\in\Delta_2}X_H(\mathbf{t})+\tilde{m}(\mathbf{t})>u\right\}\le$$
$$\le\mathbf{P}\left\{\sup_{\mathbf{t}\in\Delta_1}X_H(\mathbf{t})>u\theta(u),\sup_{\mathbf{t}\in\Delta_2}X_H(\mathbf{t})>u\theta(u)\right\}. \tag{40}$$

Introduce a scaled Gaussian homogeneous field $\xi(\mathbf{t}) = X_H((1+2\varepsilon)^{-1/\alpha} D_{\mathbf{t}_0}^{-1}\mathbf{t})$. Note that

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in\Delta_1} X_H(\mathbf{t}) > u\theta(u), \sup_{\mathbf{t}\in\Delta_2} X_H(\mathbf{t}) > u\theta(u)\right\} =$$
$$= \mathbf{P}\left\{\sup_{\mathbf{t}\in(1+\varepsilon)^{1/\alpha} D_{\mathbf{t}_0} K_1} \xi(\mathbf{t}) > u\theta(u), \sup_{\mathbf{t}\in(1+\varepsilon)^{1/\alpha} D_{\mathbf{t}_0} K_2} \xi(\mathbf{t}) > u\theta(u)\right\}. \qquad (41)$$

We have for the covariance function of ξ,

$$r_\xi(\mathbf{t}) = 1 - ||\mathbf{t}||^\alpha + o(||\mathbf{t}||^\alpha) \qquad \text{as } \mathbf{t} \to \mathbf{0}.$$

Hence there exists ε_0, $\varepsilon_0 > 0$, such that for all $\mathbf{t} \in B(\varepsilon_0/5) = \{\mathbf{t} : ||\mathbf{t}||^\alpha < \varepsilon_0/5\}$,

$$1 - 2||\mathbf{t}||^\alpha \le r_\xi(\mathbf{t}) \le 1 - \frac{1}{2}||\mathbf{t}||^\alpha. \qquad (42)$$

Let u be as large as

$$K_1' = (1+2\varepsilon)^{1/\alpha} D_{\mathbf{t}_0} K_1 \subset B(\varepsilon_0/5) \quad \text{and} \quad K_2' = (1+2\varepsilon)^{1/\alpha} D_{\mathbf{t}_0} K_2 \subset B(\varepsilon_0/5).$$

We have for the field $Y(\mathbf{t},\mathbf{s}) = \xi(\mathbf{t}) + \xi(\mathbf{s})$,

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in K_1'} \xi(\mathbf{t}) > u\theta(u), \sup_{\mathbf{t}\in K_2'} \xi(\mathbf{t}) > u\theta(u)\right\} \le \mathbf{P}\left\{\sup_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} Y(\mathbf{t},\mathbf{s}) > 2u\theta(u)\right\}. \qquad (43)$$

For all $\mathbf{t} \in K_1'$, $\mathbf{s} \in K_2'$, we have $||\mathbf{t}-\mathbf{s}||^\alpha \le 2||\mathbf{t}||^\alpha + 2||\mathbf{s}||^\alpha < \varepsilon_0$. Since $D_{\mathbf{t}_0}$ is non-degenerated, for some $\kappa > 0$ and all $\mathbf{t}$, $||D_{\mathbf{t}_0}\mathbf{t}|| \ge \kappa||\mathbf{t}||$. The variance of Y equals $\sigma_Y^2(\mathbf{t},\mathbf{s}) = 2 + 2r_\xi(\mathbf{t}-\mathbf{s})$, hence for all $\mathbf{t} \in K_1'$, $\mathbf{s} \in K_2'$ we have,

$$4 - 4||\mathbf{t}-\mathbf{s}||^\alpha \le \sigma^2(\mathbf{t},\mathbf{s}) \le 4 - ||\mathbf{t}-\mathbf{s}||^\alpha. \qquad (44)$$

This follows that

$$\inf_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} \sigma^2(\mathbf{t},\mathbf{s}) \ge 4 - 4\varepsilon_0 > 2, \qquad (45)$$

provided ε_0 is sufficiently small, and

$$\sup_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} \sigma^2(\mathbf{t},\mathbf{s}) \le 4 - u^{-2}(1+2\varepsilon)\kappa^\alpha\rho^\alpha(K_1,K_2) =: h(u,K_1,K_2) \qquad (46)$$

For the standardised field $Y^*(\mathbf{t},\mathbf{s}) = Y(\mathbf{t},\mathbf{s})/\sigma(\mathbf{t},\mathbf{s})$ we have,

$$\mathbf{P}\left\{\sup_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} Y(\mathbf{t},\mathbf{s}) > 2u\theta(u)\right\} \le$$
$$\le \mathbf{P}\left\{\sup_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} Y^*(\mathbf{t},\mathbf{s}) > 2u\theta(u)h^{-1/2}(u,K_1,K_2)\right\}. \qquad (47)$$

Algebraic calculations give

$$\mathbf{E}(Y^*(\mathbf{t},\mathbf{s}) - Y^*(\mathbf{t}_1,\mathbf{s}_1))^2 \le 16(||\mathbf{t}-\mathbf{t}_1||^\alpha + ||\mathbf{s}-\mathbf{s}_1||^\alpha). \tag{48}$$

Let $\eta_1(\mathbf{t})$, $\eta_2(\mathbf{t})$, $\mathbf{t}\in \mathrm{R}^n$ be two independent identically distributed homogeneous Gaussian fields with expectations equal zero and covariance functions equal

$$r^*(\mathbf{t}) = \exp(-32||\mathbf{t}||^a).$$

Gaussian field

$$\eta(\mathbf{t},\mathbf{s}) = \frac{1}{\sqrt{2}}\left(\eta_1(\mathbf{t}) + \eta_2(\mathbf{s})\right), \qquad (\mathbf{t},\mathbf{s}) \in \mathrm{R}^n \times \mathrm{R}^n.$$

is homogeneous, its covariance function is

$$r^{**}(\mathbf{t},\mathbf{s}) = \frac{1}{2}\left(\exp(-32||\mathbf{t}||^a + \exp(-32||\mathbf{s}||^a\right). \tag{49}$$

As far as for the covariance function $r^{***}(\mathbf{t},\mathbf{s};\mathbf{t}_1,\mathbf{s}_1)$ of the field Y^* we have

$$r^{***}(\mathbf{t},\mathbf{s};\mathbf{t}_1,\mathbf{s}_1) \ge 1 - 8(||\mathbf{t}-\mathbf{t}_1||^a + ||\mathbf{s}-\mathbf{s}_1||^a), \tag{50}$$

for all $(\mathbf{t},\mathbf{s}), (\mathbf{t}_1,\mathbf{s}_1) \in K_1' \times K_2'$, for these $(\mathbf{t},\mathbf{s}), (\mathbf{t}_1,\mathbf{s}_1)$ we also have that

$$r^{***}(\mathbf{t},\mathbf{s};\mathbf{t}_1,\mathbf{s}_1) \ge r^{**}(\mathbf{t}-\mathbf{t}_1;\mathbf{s}-\mathbf{s}_1).$$

Thus by Slepian inequality,

$$\begin{aligned} &\mathbf{P}\left\{\sup_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} Y^*(\mathbf{t},\mathbf{s}) > 2u\theta(u)h^{-1/2}(u,K_1,K_2)\right\} \le \\ &\le \mathbf{P}\left\{\sup_{(\mathbf{t},\mathbf{s})\in K_1'\times K_2'} \eta(\mathbf{t},\mathbf{s}) > 2u\theta(u)h^{-1/2}(u,K_1,K_2)\right\}. \end{aligned} \tag{51}$$

Further, for sufficiently large u,

$$4u^2\theta^2(u)h^{-1}(u,K_1,K_2) \ge u^2\theta^2(u) + \frac{\kappa^\alpha}{5}\rho^\alpha(K_1,K_2). \tag{52}$$

Using the last two relations, Lemma 1 and (7) we get,

$$\begin{aligned} &\mathbf{P}\left\{\sup_{\mathbf{t}\in\Delta_1} X_H(\mathbf{t}) + \tilde{m}(\mathbf{t}_0+\mathbf{t})u, \sup_{\mathbf{t}\in\Delta_2} X_H(\mathbf{t}) + \tilde{m}(\mathbf{t}_0+\mathbf{t})u\right\} \le \\ &\le C\Psi(u\theta(u))H_\alpha(16(D_{\mathbf{t}_0}K_1 \times D_{\mathbf{t}_0}K_2))\exp\left(-\frac{\kappa^\alpha}{10}\rho^\alpha(K_1,K_2)\right) \le \\ &\le C_1 \prod_{\nu=1}^k (T_\nu^1 - S_\nu^1) \prod_{\nu=1}^k (T_\nu^2 - S_\nu^2)\exp\left(-\frac{\kappa^\alpha}{10}\rho^\alpha(K_1,K_2)\right)\Psi(u\theta(u)), \end{aligned} \tag{53}$$

which holds for all sufficiently large u and a constant C_1, independent of u, K_1, K_2. In order to estimate $H_\alpha(16(D_{\mathbf{t}_0}K_1 \times D_{\mathbf{t}_0}K_2))$ we use here Lemmas 6.4 and 6.2 from [6].

Now turn to the double sum $\sum_{\mathbf{i},\mathbf{j}\in L'} \mathbf{P}(A_\mathbf{i}A_\mathbf{j})$. We brake it into two sums. The first one, denote it by Σ_1, is the sum over all non-neighbouring cubes (that is, the distance between any two of them is are positive), and the second one, denote it by Σ_2, is the sum over all neighbouring cubes. Denote

$$x_\mathbf{i} = \min_{\mathbf{t}\in\Delta_\mathbf{i}} ||\sqrt{B}\mathbf{t}||, \quad \mathbf{i}\in L'.$$

Using (53) we get,

$$\mathbf{P}(A_\mathbf{i}A_\mathbf{j}) \quad \le C^\alpha T^{2k} \exp\left(-\frac{\kappa^\alpha}{10}T^\alpha (\max_{1\le\nu\le k} |i_\nu - j_\nu| - 1)^\alpha\right)\Psi(u\theta(u)) =: \theta_{\mathbf{i},\mathbf{j}}, \tag{54}$$

where $\theta(u) = 1 - c(u)$, $c(u) = \max\{\max_{\mathbf{t}\in\Delta_\mathbf{i}} \tilde{m}(\mathbf{t}_0+\mathbf{t}), \max_{\mathbf{t}\in\Delta_\mathbf{j}} \tilde{m}(\mathbf{t}_0+\mathbf{t})\}$. This estimation holds for all members of the first sum and all sufficiently large u. Using it and approximating the sum by an integral, we get

$$\Sigma_1 \le 2\sum_{\mathbf{i}\in L'} \sum_{\mathbf{j}\in L',\,\mathbf{i}\ne\mathbf{j},\,x_\mathbf{i}<x_\mathbf{j}} \theta_{\mathbf{i},\mathbf{j}} \le c^* e^* T^k \exp\left(-\frac{\kappa^\alpha}{10}T^\alpha\right) u^{\frac{2k}{\alpha}-\frac{k}{2}}\Psi(u-m(\mathbf{t}_0)). \tag{55}$$

Now consider Σ_2. We can assume that $\max_{\mathbf{t}\in\Delta_\mathbf{i}} \tilde{m}(\mathbf{t}_0+\mathbf{t}) > \max_{\mathbf{t}\in\Delta_\mathbf{j}} \tilde{m}(\mathbf{t}_0+\mathbf{t})$. Denote

$$\Delta'_\mathbf{i} = u^{\frac{2}{\alpha}}\left([i_1T, i_1T+\sqrt{T}] \times \times_{\nu=2}^k [i_\nu T, (i_n u+1)T]\right) \quad \text{and} \quad \Delta''_\mathbf{i} = \Delta_\mathbf{i}\setminus\Delta'_\mathbf{i}.$$

Clear,

$$\begin{aligned}\mathbf{P}\{A_\mathbf{i}A_\mathbf{j}\} \le \mathbf{P}&\left\{\sup_{\Delta_\mathbf{i}} X_H(\mathbf{t}) > u\theta(u)\right\}\\ &+\mathbf{P}\left\{\sup_{\Delta''_\mathbf{i}} X_H(\mathbf{t}) > u\theta(u), \sup_{\Delta_\mathbf{j}} X_H(\mathbf{t}) > u\theta(u)\right\}.\end{aligned} \tag{56}$$

Using now Lemma 1, (56), (53) and approximating the sum by an integral, we get for all sufficiently large u,

$$\Sigma_2 \le C_2^* T^{k-1/2} e^* u^{\frac{2k}{\alpha}-\frac{k}{2}}\Psi(u-m(\mathbf{t}_0)) + C_3^* T^k e^* u^{\frac{2k}{\alpha}-\frac{k}{2}} \exp\left\{-\frac{\kappa^\alpha T^\alpha}{10}\right\}\Psi(u-m(\mathbf{t}_0)). \tag{57}$$

Taking into account (38), (39), (55) and (57), we get for all positive T,

$$\begin{aligned}&\frac{H_\alpha\left((1+2\varepsilon)D_{\mathbf{t}_0}[0,t]^k\right)}{T^k} - \\ &\qquad -C_1^* T^k \exp\left\{-\frac{m^\alpha T^\alpha}{10}\right\} - C_2^* T^{-1/2} - C_3^* T^k u^{\frac{2k}{\alpha}-\frac{k}{2}} \exp\left\{-\frac{m^\alpha T^\alpha}{10}\right\}\\ &\le \liminf_{u\to\infty} \frac{\mathbf{P}\left\{\sup_{\mathbf{t}\in S} X_H(\mathbf{t}) + \tilde{m}(\mathbf{t}_0+\mathbf{t}) > u\right\}}{e^* u^{\frac{2k}{\alpha}-\frac{k}{2}}\Psi(u-m(\mathbf{t}_0))}\\ &\quad \le \limsup_{u\to\infty} \frac{\mathbf{P}\left\{\sup_{\mathbf{t}\in S} X_H(\mathbf{t}) + \tilde{m}(\mathbf{t}_0+\mathbf{t}) > u\right\}}{e^* u^{\frac{2k}{\alpha}-\frac{k}{2}}\Psi(u-m(\mathbf{t}_0))}\\ &\qquad\qquad \le \frac{H_\alpha\left((1+2\varepsilon)D_{\mathbf{t}_0}[0,t]^k\right)}{T^k}.\end{aligned} \tag{58}$$

Now, letting T go to infinity and using (29), we obtain, that

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in S} X_H(\tilde{\mathbf{t}}) + \tilde{m}(\tilde{\mathbf{t}}) > u\right\}$$
$$= (1+2\varepsilon)^k e^* V(D_{\mathbf{t}_0}M) H_\alpha^{(k)} u^{\frac{2k}{\alpha}-\frac{k}{2}} \Psi(u - m(\mathbf{t}_0))(1+o(1)), \tag{59}$$

as $u \to \infty$.

Let now $X_H^*(\mathbf{t}), \mathbf{t} \in \mathrm{R}^n$, be a homogeneous centered Gaussian field with the covariance function $r_H^*(\mathbf{t}) = \exp\left(-(1-2\varepsilon)||D_{\mathbf{t}_0}\mathbf{t}||^\alpha\right)$. From Theorem 4 we have,

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in S} \tilde{X}(\tilde{\mathbf{t}}) > u\right\} \geq \mathbf{P}\left\{\sup_{\mathbf{t}\in S} X_H^*(\tilde{\mathbf{t}}) + \tilde{m}(\tilde{\mathbf{t}}) > u\right\}. \tag{60}$$

Proceeding the above estimations for the latter probability, we get,

$$\mathbf{P}\left\{\sup_{\mathbf{t}\in \Sigma_\Pi} X_H^*(\tilde{\mathbf{t}}) + \tilde{m}(\tilde{\mathbf{t}} > u\right\}$$
$$= (1-2\varepsilon)^k e^* V(D_{\mathbf{t}_0}M) H_\alpha^{(k)} u^{\frac{2k}{\alpha}-\frac{k}{2}} \Psi(u - m(\mathbf{t}_0))(1+o(1)), \tag{61}$$

as $u \to \infty$.

Now we collect (25), (27), (59), (60) and (61), and get

$$\begin{aligned}(1-2\varepsilon)^k &\leq \liminf_{u\to\infty} \frac{\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}} X(\mathbf{t}) > u\right\}}{e^* V(D_{\mathbf{t}_0}M) H_\alpha^{(k)} u^{\frac{2k}{\alpha}-\frac{k}{2}} \Psi(u - m(\mathbf{t}_0))} \\ &\leq \limsup_{u\to\infty} \frac{\mathbf{P}\left\{\sup_{\mathbf{t}\in\mathcal{M}} X(\mathbf{t}) > u\right\}}{e^* V(D_{\mathbf{t}_0}M) H_\alpha^{(k)} u^{\frac{2k}{\alpha}-\frac{k}{2}} \Psi(u - m(\mathbf{t}_0))} \leq (1+2\varepsilon)^k. \end{aligned} \tag{62}$$

It follows from this the assertion of Theorem. □

Proof of Theorem 6. Let $\tilde{X}(\tilde{\mathbf{t}})$ be the field as it is defined in the proof of Theorem 5. Using Tailor expansion, we get

$$\tilde{X}(\tilde{\mathbf{t}}) = X(\mathbf{t}) = X(\mathbf{t}_0) + (\mathrm{grad} X(\mathbf{t}_0))^\top (\mathbf{t} - \mathbf{t}_0) + o(||\mathbf{t} - \mathbf{t}_0||), \quad \mathbf{t} \to \mathbf{t}_0. \tag{63}$$

From here it follows that

$$\tilde{X}(\tilde{\mathbf{t}}) - \tilde{X}(\tilde{\mathbf{t}}_0) = (\widetilde{\mathrm{grad}} X(\mathbf{t}_0))^\top (\tilde{\mathbf{t}} - \tilde{\mathbf{t}}_0) + o(||\tilde{\mathbf{t}} - \tilde{\mathbf{t}}_0||), \quad \tilde{\mathbf{t}} \to \tilde{\mathbf{t}}_0, \tag{64}$$

where $\widetilde{\mathrm{grad}}$ is the orthogonal projection of the gradient of the field X onto the tangent subspace $T_{\mathbf{t}_0}$ to the $\mathcal{M}$ at the point $\mathbf{t}_0$. From (64) by algebraic calculations it follows that

$$\tilde{r}(\tilde{\mathbf{t}} - \tilde{\mathbf{t}}_0) = 1 - \frac{1}{2}(\tilde{\mathbf{t}} - \tilde{\mathbf{t}}_0)^\top A_{\mathbf{t}_0} (\tilde{\mathbf{t}} - \tilde{\mathbf{t}}_0) + o(||\tilde{\mathbf{t}} - \tilde{\mathbf{t}}_0||), \quad \tilde{\mathbf{t}} \to \tilde{\mathbf{t}}_0, \tag{65}$$

where $A_{\mathbf{t}_0}$ is the covariance matrix of the vector $\widetilde{\mathrm{grad}} X(\mathbf{t}_0)$. Note that the matrix $\sqrt{A_{\mathbf{t}_0}/2}$ is just the matrix $D_{\mathbf{t}_0}$ from Theorem 5. Now the proof repeats up to all details the proof of Theorem 5. □

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