Linear Elliptic Boundary Value Problems with Non-smooth Data: Normal Solvability on Sobolev–Campanato Spaces

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Abstract

In this paper linear elliptic boundary value problems of second order with non-smooth data (L^∞-coefficients, Lipschitz domains, regular sets, non-homogeneous mixed boundary conditions) are considered. It is shown that such boundary value problems generate Fredholm operators between appropriate Sobolev–Campanato spaces, that the weak solutions are Hölder continuous up to the boundary and that they depend smoothly (in the sense of a Hölder norm) on the coefficients and on the right hand sides of the equations and boundary conditions.

1 Introduction

In this paper we consider weak solutions to boundary value problems for linear elliptic equations of the type

$$
(1.1) \qquad \begin{cases} -\nabla\cdot(A\nabla u + bu) + c\cdot\nabla u + du = -\nabla\cdot f + g & \text{in } \Omega, \\ (A\nabla u + bu)\cdot\nu + eu = f\cdot\nu + h & \text{on } \Gamma, \\ u = 0 & \text{on } \partial\Omega\setminus\Gamma, \end{cases}
$$

and for linear elliptic systems the principal part of which is close to be triangular.

In (1.1) $\Omega \subset \mathbb{R}^N$ is a bounded domain, Γ is a subset of the boundary $\partial\Omega$, and $\nu : \partial\Omega \to \mathbb{R}^N$ is the unit outward normal vector field on $\partial\Omega$. The coefficients A, b, c, d and e are bounded measurable maps, defined on Ω and Γ, respectively, A is real symmetric $N\times N$-matrix valued, b and c are $\mathbb{R}^N$ valued, and d and e are scalar valued. By ∇u and $\nabla\cdot f$ we denote the gradient of a function $u : \Omega \to \mathbb{R}$ and the divergence of a vector field $f : \Omega \to \mathbb{R}^N$, respectively, and for the Euclidean scalar product of two vectors we use a centered dot. Finally, it is supposed that there exists an $\varepsilon > 0$ such that

$$
(1.2) \qquad A(x)\xi\cdot\xi \geq \varepsilon\,\xi\cdot\xi \quad \text{for all } \xi\in\mathbb{R}^N \text{ and for almost all } x\in\Omega.
$$

It is well-known that each weak solution to the boundary value problem (1.1) is Hölder continuous up to the boundary if, for example,

$$
(1.3) \qquad f\in L^p(\Omega;\mathbb{R}^N)\ ,\ g\in L^{p/2}(\Omega) \quad \text{and} \quad h\in L^{p-1}(\Gamma) \text{ with } p > N,
$$

and if $\partial\Omega$ and Γ satisfy certain regularity assumptions (see, for instance, Gilbarg, Trudinger [14] for the case $\Gamma = \varnothing$, Troianiello [30] for the case that Γ is open and closed in $\partial\Omega$ and

$e = h = 0$, and STAMPACCHIA [29], MURTHY, STAMPACCHIA [21] for more general cases). Moreover, the weak solution to (1.1) – if it is unique – depends continuously in the sense of a Hölder space $C^{0,\alpha}(\overline{\Omega})$ on the right hand sides f, g and h in the sense of the Lebesgue spaces mentioned in (1.3).

In the present paper we will prove, among other things, that the weak solution to (1.1) – if it is unique – depends smoothly in the sense of a Hölder space $C^{0,\alpha}(\overline{\Omega})$ not only on the right hand sides f, g and h, but also on the coefficients A, b, c, d and e in the sense of L^∞-norms. This result seems to be new (in case of $N > 2$) even if $\Gamma = \varnothing$ (pure Dirichlet boundary conditions) or if $\Gamma = \partial\Omega$ (pure natural boundary conditions). Moreover, it has important consequences because it allows to apply theorems of the differential calculus (Implicit Function Theorem, Sard-Smale Theorem, Lyapunov-Schmidt procedure in bifurcation problems) to quasilinear elliptic boundary value problems with non-smooth data (cf. RECKE [26] and GRIEPENTROG [16]).

The main problem connected with such applications to quasilinear problems consists in the following: On the one hand, one has to work on sufficiently large function spaces such that weak solutions exist. On the other hand, one has to work on sufficiently small function spaces such that the appearing superposition operators are smooth. For example, suppose the coefficient matrix A in (1.1) to depend on u, i.e. $A = A(x, u)$, let $A(x, \cdot)$ be smooth for almost all x, and assume that there exists a constant $c > 0$ such that $|A(x, u)| \le c$ for all u and almost all x. Then the corresponding superposition operator

$$u \in W^{1,2}(\Omega) \longmapsto A(\cdot, u(\cdot)) \in L^\infty(\Omega; \mathbb{R}^{N\times N})$$

is not continuously differentiable (except that $N = 1$ or that $A(x, \cdot)$ is affine for almost all x), but its restriction to $W^{1,2}(\Omega) \cap L^\infty(\Omega)$, for example, is smooth. For differentiability of superposition operators see, e.g., VALENT [31], APPELL, ZABREJKO [3] and RUNST, SICKEL [28].

In the case of $N = 2$ the smooth dependence (in the sense of $C^{0,\alpha}(\overline{\Omega})$) of the weak solution of (1.1) on the coefficients follows from the paper of GRÖGER [18]. Moreover, in the case of $N = 2$ this result holds true for boundary value problems for general elliptic systems, which are not necessarily close to be triangular, as well.

Our paper is closely related with the results of RECKE [27] and XIE [33], which contain some of the results of Sections 5 and 6 of the present paper. [27] concerns the particular case of $e = h = 0$. In [33] it is supposed that e and h are Lipschitz continuous (in order to absorb the corresponding Γ-integrals in the variational formulation of (1.1) into Ω-integrals via the divergence theorem). Moreover, [33] does not concern the dependence of the weak solutions on the coefficients.

Most of the results of Sections 3 and 4 of the present paper, describing properties of regular sets (introduced by GRÖGER [18], cf. Definition 3.1 below) and trace properties of Sobolev-Campanato functions, seem to be new. The development of these rather technical results is motivated to some extent by the wrong claim of XIE [33, Remark 2.3] that all regular sets have Lipschitz boundaries. Because this claim is used repeatedly (cf., e.g., [33, Remark 4.1]), there are gaps in the proofs of that paper.

In the particular case of $e = h = 0$ the results of the present paper are essentially due to the second author. The generalizations to the case of nonzero e and h (and, especially, the Trace Theorem 4.4) belong to the first author.

The present paper is organized as follows:

In Section 2 we introduce some notation and results related to Sobolev–Campanato spaces.

Section 3 is devoted to the concept and the properties of regular sets.

In Section 4 we define Campanato spaces on the natural boundary part, and we prove a trace theorem for Sobolev–Campanato spaces.

In Section 5 we prove a regularity result for weak solutions to (1.1) for the case of $b = c = 0$ and $e = 0$ closely following the methods of Troianiello [30].

In Section 6 we show that the operator, associated with the boundary value problem (1.1), is a Fredholm operator (index zero) from $W_0^{1,2,\omega}(\Omega \cup \Gamma)$ into $W^{-1,2,\omega}(\Omega \cup \Gamma)$ for all $\omega \in [0, \bar{\omega})$, where $\bar{\omega}$ is a certain number which depends on ε and $\Omega \cup \Gamma$, only, and which is larger than $N - 2$. Moreover, this Fredholm operator depends linearly and continuously (in the sense of the operator norm) on A, b, c, d and e. Hence, if it is injective, the Implicit Function Theorem yields that the weak solution to (1.1) depends smoothly on A, b, c, d and e. Here $W_0^{1,2,\omega}(\Omega \cup \Gamma)$ consist of all elements u of the Sobolev space $W_0^{1,2}(\Omega \cup \Gamma)$ such that the gradient ∇u belongs to the Campanato space $\mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$, and $W^{-1,2,\omega}(\Omega \cup \Gamma)$ is the image of $W_0^{1,2,\omega}(\Omega \cup \Gamma)$ with respect to the duality map of the Hilbert space $W_0^{1,2}(\Omega \cup \Gamma)$. Remark that, for $\omega > N - 2$, the Sobolev-Campanato space $W_0^{1,2,\omega}(\Omega \cup \Gamma)$ is continuously embedded into the Hölder space $C^{0,\alpha}(\overline{\Omega})$ with $\alpha = (\omega - N + 2)/2$.

Finally, in Section 7 we show that our results about the boundary value problems for linear elliptic equations of type (1.1) hold for linear elliptic systems the principal part of which is close to be triangular as well.

2 Notation and some Results on Sobolev-Campanato Spaces

By $\mathfrak{M}_N$ and $\mathfrak{S}_N$ we denote the spaces of all real $N \times N$-matrices and real symmetric $N \times N$-matrices, respectively. The symbol $|\cdot|$ is used for the absolute value, the Euclidean norm in $\mathbb{R}^N$ and for the Euclidean operator norm in $\mathfrak{M}_N$, respectively, i.e.

$$\begin{aligned} |\xi| &:= \sqrt{\xi \cdot \xi} \quad \text{for } \xi \in \mathbb{R}^N, \\ |A| &:= \max\{|A\xi| : \xi \in \mathbb{R}^N,\ |\xi| \le 1\} \quad \text{for } A \in \mathfrak{M}_N. \end{aligned}$$

For $x \in \mathbb{R}^N$ and $r > 0$ we denote by $B(x, r) := \{\xi \in \mathbb{R}^N : |\xi - x| < r\}$ the open ball around x with radius r.

As usual, for subsets G of $\mathbb{R}^N$ we write G°, $\overline{G}$ and ∂G for the interior, the closure and the (topological) boundary of G, respectively.

A bijective map Φ between two subsets of $\mathbb{R}^N$ such that Φ and Φ^{-1} are Lipschitz continuous is called Lipschitz transformation.

A subset M of $\mathbb{R}^N$ is called Lipschitz hypersurface in $\mathbb{R}^N$ if for each $x_0 \in M$ there exist open neighborhoods U of x_0 and V of zero in $\mathbb{R}^N$ and a Lipschitz transformation Φ from U onto V such that $\Phi(x_0) = 0$ and

$$U \cap M = \{x \in U :\ \Phi_N(x) = 0\}. \tag{2.1}$$

Here $\Phi_N : U \to \mathbb{R}$ is the N-th component of the map Φ (and similar notation will be used later on). The map

$$\varphi : V_N := \{\xi \in V :\ \xi_N = 0\} \to M, \quad \varphi(\xi) := \Phi^{-1}(\xi)$$

is called embedding chart of M in x_0. It is a Lipschitz continuous map from the neighborhood V_N of zero in $\mathbb{R}^{N-1}$ into $\mathbb{R}^N$. Its functional matrix $D\varphi(\xi)$ exists for λ^{N-1}-almost all $\xi \in V_N$, and for such ξ the absolute value of the corresponding Jacobian $J\varphi(\xi)$ is defined by

$$(2.2)\qquad |J\varphi(\xi)|^2 := \begin{cases} \text{sum of squares of all } (N-1)\times(N-1)\text{-} \\ \text{subdeterminants of } D\varphi(\xi), \end{cases}$$

cf., e.g., Evans, Gariepy [10, Section 3.3.4]. Here λ^{N-1} denotes the $(N-1)$-dimensional Lebesgue measure on $\mathbb{R}^{N-1}$. Analogously, by λ^N we will denote the N-dimensional Lebesgue measure on $\mathbb{R}^N$.

Let M be a Lipschitz hypersurface in $\mathbb{R}^N$. By λ_M we denote the $(N-1)$-dimensional Lebesgue measure on M. Thus, on the algebra of Lebesgue measurable subsets of M it is equal to the (suitably normalized) $(N-1)$-dimensional Hausdorff measure (cf. [10]). Using embedding charts, we have

$$(2.3)\qquad \lambda_M(U\cap M) = \int_{V_N} |J\varphi(\xi)|\,\mathrm{d}\lambda^{N-1}(\xi)$$

and, for integrable functions $u : M \to \mathbb{R}$,

$$(2.4)\qquad \int_{U\cap M} u(x)\,\mathrm{d}\lambda_M(x) = \int_{V_N} u(\varphi(\xi))|J\varphi(\xi)|\,\mathrm{d}\lambda^{N-1}(\xi).$$

A subset M_0 of M is called Lipschitz hypersurface in M if for each $x_0 \in M_0$ there exist open neighborhoods U of x_0 and V of zero in $\mathbb{R}^N$ and a Lipschitz transformation Φ from U onto V such that $\Phi(x_0) = 0$ and (2.1) as well as

$$U \cap M_0 = \{x \in U :\ \Phi_N(x) = \Phi_1(x) = 0\}.$$

Let Ω be a bounded open subset of $\mathbb{R}^N$. We write $L^\infty(\Omega)$, $L^\infty(\Omega;\mathbb{R}^N)$ and $L^\infty(\Omega;\mathfrak{S}_N)$ for the spaces of bounded measurable maps from Ω into $\mathbb{R}$, $\mathbb{R}^N$ and $\mathfrak{S}_N$, respectively. The norms of these spaces are denoted by $\|\cdot\|_{L^\infty}$. Analogously, for $1 \le p < \infty$ we write $\|\cdot\|_{L^p}$ for the norms in the Lebesgue spaces $L^p(\Omega)$ and $L^p(\Omega;\mathbb{R}^N)$, respectively. The Sobolev space $W^{1,p}(\Omega)$ will be equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} := \Big(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega;\mathbb{R}^N)}\Big)^{1/p}.$$

For $1 \le p < \infty$, $0 \le \omega \le N + p$ we denote by $\mathfrak{L}^{p,\omega}(\Omega)$ the Campanato space, i.e. the space of all $u \in L^p(\Omega)$ such that

$$(2.5)\qquad [u]_{\mathfrak{L}^{p,\omega}(\Omega)} := \left(\sup_{\substack{x\in\Omega\\ r>0}} \Big(r^{-\omega} \int_{\Omega(x,r)} |u(y) - u_{\Omega(x,r)}|^p\,\mathrm{d}\lambda^N(y)\Big)\right)^{1/p} < \infty.$$

In (2.5) we used the notation

$$(2.6)\qquad \Omega(x,r) := \Omega \cap B(x,r),\quad u_{\Omega(x,r)} := \frac{1}{\lambda^N(\Omega(x,r))}\int_{\Omega(x,r)} u(y)\,\mathrm{d}\lambda^N(y).$$

The space $\mathfrak{L}^{p,\omega}(\Omega)$ is a Banach space with the norm

$$(2.7)\qquad \|u\|_{\mathfrak{L}^{p,\omega}(\Omega)} := \Big(\|u\|^p_{L^p(\Omega)} + [u]^p_{\mathfrak{L}^{p,\omega}(\Omega)}\Big)^{1/p}.$$

Analogously, by $\mathfrak{L}^{p,\omega}(\Omega;\mathbb{R}^N)$ we denote the space of all $f \in L^p(\Omega,\mathbb{R}^N)$ with components in $\mathfrak{L}^{p,\omega}(\Omega)$, and the norm in $\mathfrak{L}^{p,\omega}(\Omega;\mathbb{R}^N)$ is defined similarly to (2.7). Finally, for the sake of simplicity, for $\omega \le 0$ we will use the notation $\mathfrak{L}^{p,\omega}(\Omega) := L^p(\Omega)$.

The following well-known (cf., e.g., Troianiello [30, Section 1.4.1]) property of Campanato spaces will be used repeatedly in our paper: If $r_0 > 0$ is fixed and if the supremum in (2.5) is taken over $0 < r < r_0$, only, then the corresponding r_0-depending norm, defined analogously to (2.7), is equivalent to the original norm in $\mathfrak{L}^{p,\omega}(\Omega)$. Moreover, we will use the following theorem (cf. Kufner, John, Fučik [19], Giaquinta [13] and Troianiello [30]) that describes embedding and transformation properties of Campanato spaces.

Theorem 2.1 (i) *Let $1 \le p_1 \le p_2 < \infty$ and $\omega_1, \omega_2 \in \mathbb{R}$ such that it holds $(\omega_1 - N)/p_1 \le (\omega_2 - N)/p_2$. Then $\mathfrak{L}^{p_2,\omega_2}(\Omega)$ is continuously embedded into $\mathfrak{L}^{p_1,\omega_1}(\Omega)$.*

(ii) *Let Φ be a Lipschitz transformation from Ω into $\mathbb{R}^N$ and $\omega < N + 2$. Then there exists a constant $c > 0$ such that for all $u \in \mathfrak{L}^{2,\omega}(\Phi(\Omega))$ it holds $u \circ \Phi \in \mathfrak{L}^{2,\omega}(\Omega)$ and*

$$\|u \circ \Phi\|_{\mathfrak{L}^{2,\omega}(\Omega)} \le c\, \|u\|_{\mathfrak{L}^{2,\omega}(\Phi(\Omega))}.$$

For $0 \le \omega \le N+2$ we denote by $W^{1,2,\omega}(\Omega)$ the Sobolev–Campanato space, i.e. the space of all $u \in W^{1,2}(\Omega)$ such that $\nabla u \in \mathfrak{L}^{2,\omega}(\Omega;\mathbb{R}^N)$. The space $W^{1,2,\omega}(\Omega)$ is a Banach space with the norm

$$\|u\|_{W^{1,2,\omega}(\Omega)} := \Big(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{\mathfrak{L}^{2,\omega}(\Omega;\mathbb{R}^N)}\Big)^{1/2}.$$

In order to formulate further properties of Campanato spaces (equivalence to Morrey and Hölder spaces, multiplier and embedding properties) we have to suppose certain minimal regularity of the boundary $\partial\Omega$. Hence, let us introduce the following usual terminology (using notation (2.6)):

Definition 2.2 Let $a > 0$. An open set $\Omega \subset \mathbb{R}^N$ is said to have property (a) if for all sufficiently small $r > 0$ we have $\lambda^N(\Omega(x,r)) \ge a\, \lambda^N(B(x,r))$ for all $x \in \Omega$.

The results, summarized in the following theorem, are classical (cf. Campanato [4, 5, 6, 7], Chen, Wu [8]). Remark, however, that in some references (Kufner, John, Fučik [19], Nečas [22], Troianiello [30] and Giaquinta [12, 13]) they are formulated and posed partially under stronger regularity assumptions on $\partial\Omega$.

Theorem 2.3 *Let Ω have property* (a). *Then the following is true:*

(i) *Let $0 \le \omega < N$ and $u \in L^2(\Omega)$. Then it holds $u \in \mathfrak{L}^{2,\omega}(\Omega)$ if and only if*

$$\left(\sup_{\substack{x\in\Omega\\ r>0}} \Big(r^{-\omega} \int\limits_{\Omega(x,r)} |u(y)|^2 \,\mathrm{d}\lambda^N(y)\Big)\right)^{1/2} < \infty, \tag{2.8}$$

and (2.8) *is an equivalent norm in $\mathfrak{L}^{2,\omega}(\Omega)$.*

(ii) *Let $0 \le \omega < N$. Then for all $u \in \mathfrak{L}^{2,\omega}(\Omega)$ and $v \in L^\infty(\Omega)$ the product uv belongs to $\mathfrak{L}^{2,\omega}(\Omega)$, again, and there exists a constant $c > 0$ such that*

$$\|uv\|_{\mathfrak{L}^{2,\omega}(\Omega)} \le c\, \|u\|_{\mathfrak{L}^{2,\omega}(\Omega)}\, \|v\|_{L^\infty(\Omega)}$$

for all such u and v.

(iii) *Let* $N < \omega \leq N + 2$. *Then* $\mathfrak{L}^{2,\omega}(\Omega)$ *is isomorphic to the Hölder space* $C^{0,\alpha}(\overline{\Omega})$ *with* $\alpha = (\omega - N)/2$.

(iv) *Let* $\omega < N$. *Then* $W^{1,2,\omega}(\Omega)$ *is continuously embedded into* $\mathfrak{L}^{2,\omega+2}(\Omega)$.

Remark 2.4 For more complicated multiplier properties of Campanato functions and applications to interior solution regularity of elliptic equations with unbounded coefficients see Di Fazio [9] and Ragusa [23]. Using these results, it should be possible to generalize the results of the present paper to equations with suitable unbounded coefficients b, c, d and e.

3 Regular Sets

To define the concept of regular sets let us denote for $x_0 \in \mathbb{R}^N$ and $r > 0$

$$\begin{aligned} E_1(x_0, r) &:= \{x \in \mathbb{R}^N : |x - x_0| < r,\, x_N - x_{0N} < 0\}, \\ E_2(x_0, r) &:= \{x \in \mathbb{R}^N : |x - x_0| < r,\, x_N - x_{0N} \leq 0\}, \\ E_3(x_0, r) &:= \{x \in E_2(x_0, r) : x_1 - x_{01} > 0 \text{ or } x_N - x_{0N} < 0\}. \end{aligned}$$

Here and later on in the case of $x_0 = 0$ and $r = 1$ we shortly write E_1, E_2 and E_3, respectively. The following terminology is essentially due to Gröger [18]:

Definition 3.1 A set $G \subset \mathbb{R}^N$ is called regular if it is bounded and if for each $x \in \partial G$ there exist an open neighborhood U of x in $\mathbb{R}^N$ and a Lipschitz transformation Φ of U into $\mathbb{R}^N$ such that

$$\Phi(U \cap G) \in \{E_1, E_2, E_3\}. \tag{3.1}$$

Lemma 3.2 *Let* $G \subset \mathbb{R}^N$ *be regular and* $V \subset \mathbb{R}^N$ *be open. Then for each point* $x \in V \cap \partial G$ *there exist an open neighborhood* U *of* x *in* V *and a Lipschitz transformation* Φ *of* U *into* $\mathbb{R}^N$ *with* (3.1) *and* $\Phi(x) = 0$.

Proof By definition, there exist an open neighborhood U_0 of x in $\mathbb{R}^N$ and a Lipschitz transformation Φ_0 of U_0 into $\mathbb{R}^N$ with $\Phi_0(U_0 \cap G) = E_k$ and $k = 1, 2$ or 3. Moreover, the Theorem of Invariance of Domain implies $\Phi_0(x) \in \partial E_k$. Hence, in order to prove the lemma it is sufficient to show the following:

$$\left\{ \begin{array}{l} \text{For all } l \in \{1, 2, 3\}, \text{ all } y \in \partial E_l \text{ and all sufficiently small } r > 0 \\ \text{there exists a Lipschitz transformation } \Psi \text{ from } B(y, r) \text{ into } \mathbb{R}^N \\ \text{such that } \Psi(y) = 0 \text{ and } \Psi(B(y, r) \cap E_l) \in \{E_1, E_2, E_3\}. \end{array} \right. \tag{3.2}$$

Indeed, take $r > 0$ sufficiently small such that $U := \Phi_0^{-1}(B(y, r)) \subset U_0 \cap V$, take Ψ corresponding to (3.2) with $l = k$, $y = \Phi_0(x)$ and the chosen r, then $\Phi := \Psi \circ \Phi_0$ is the Lipschitz transformation sought-after.

Obviously, in order to prove (3.2) it is sufficient to consider only a finite collection of pairs $(l, y) \in \{1, 2, 3\} \times \partial E_1$ such that for each other pair $(l^*, y^*) \in \{1, 2, 3\} \times \partial E_1$ the set $B(y^*, r) \cap E_{l^*}$

is equal to one of the sets $B(y,r)\cap E_l$ after translations, reflections and rotations in $\mathbb{R}^N$. Such a collection is, for example, the following one:

$$l=1,\ y=0, \tag{3.3}$$

$$l=2,\ y=0, \tag{3.4}$$

$$l=3,\ y=0, \tag{3.5}$$

$$l=1,\ y=-e_N, \tag{3.6}$$

$$l=1,\ y=e_1, \tag{3.7}$$

$$l=2,\ y=e_1, \tag{3.8}$$

$$l=3,\ y=e_2. \tag{3.9}$$

In (3.6), ..., (3.9), by $e_1,\dots,e_N$ we denoted the standard orthonormal base in $\mathbb{R}^N$.

Assertion (3.2) is obvious in the cases (3.3), ..., (3.5).

In order to handle the remaining cases (3.6), ..., (3.9) it is sufficient to show that there exist Lipschitz transformations Ψ from $B(0,1/2)$ into $\mathbb{R}^N$ with $\Psi(0)=0$, which map $B(0,1/2)\cap E_1(e_N,1)$ onto E_1 or $B(0,1/2)\cap E_1(-e_1,1)$ onto E_1 or $B(0,1/2)\cap E_2(-e_1,1)$ onto E_3 or $B(0,1/2)\cap E_3(-e_2,1)$ onto E_3. It is not hard to check out the existence of such Lipschitz transformations, but to write down them explicitly is quite complicated. □

Remark 3.3 Let $G\subset\mathbb{R}^N$ be regular and $V\subset\mathbb{R}^N$ be open. Then, as a conclusion of the preceding Lemma, for each point $x\in V\cap\partial G$ there exist an open neighborhood U_* of x in V and a Lipschitz transformation Φ_* of U_* into $\mathbb{R}^N$ with (3.1) and $\Phi_*(x)=0$. We want to state a slight improvement of this result. Obviously, zero is an inner point of $\Phi_*(U_*)$. Therefore, we can choose a small $0<r<1$ such that $B(0,r)\subset\Phi_*(U_*)$. Hence, $U:=\Phi_*^{-1}(B(0,r))$ is an open neighborhood of x in V and $\Phi:=r^{-1}\Phi_*|_U$ is a Lipschitz transformation of U but now onto $B(0,1)$ with (3.1) and $\Phi(x)=0$.

Remark 3.4 It is easy to verify that there exists a Lipschitz transformation of $\mathbb{R}^N$ onto $\mathbb{R}^N$ which maps E_3 onto E_2. Hence, Definition 3.1 would not be changed if one would replace condition (3.1) by

$$\Phi(U\cap G)\in\{E_1,E_2\}. \tag{3.10}$$

But for a regular $G\subset\mathbb{R}^N$, an open $V\subset\mathbb{R}^N$ and an $x\in V\cap\partial G$ there do not exist an open $U\subset V$ with $x\in U$ and a Lipschitz transformation Φ from U into $\mathbb{R}^N$ with (3.10) and $\Phi(x)=0$, in general.

In order to simplify subsequent notation we introduce the following

Definition 3.5 Let $G\subset\mathbb{R}^N$ be regular, $x\in\partial G$, $U\subset\mathbb{R}^N$ be open with $x\in U$ and Φ a Lipschitz transformation of U onto $B(0,1)$ with (3.1) and $\Phi(x)=0$. Then the pair (Φ,U) is called a chart of ∂G in x.

Lemma 3.6 *Let $G\subset\mathbb{R}^N$ be regular. Then the following is true:*

(i) *G° and $\overline{G}$ are regular.*

(ii) *If V is an open neighborhood of $\overline{G}$ in $\mathbb{R}^N$ and Ψ a Lipschitz transformation of V into $\mathbb{R}^N$ then $\Psi(G)$ is regular.*

(iii) *∂G is a Lipschitz hypersurface in $\mathbb{R}^N$.*

(iv) *G° satisfies property* (a) *for some $a > 0$.*

Proof (i) Let $x \in \partial G$, and let (Φ, U) be a chart of ∂G in x. By invariance of domain, we have

$$\Phi(U \cap G^\circ) = E_1. \tag{3.11}$$

Hence, G° is a regular set.

By $\Phi(U) = B(0,1)$, we get

$$\Phi(U \cap \overline{G}) = \Phi(U \cap \overline{U \cap G}) = B(0,1) \cap \overline{\Phi(U \cap G)} = E_2.$$

Thus, $\overline{G}$ is regular.

(ii) Let $x \in \partial\Psi(G)$. Then $\Psi^{-1}(x) \in \partial G$, and, hence, there exists a chart (Φ, U) of ∂G in $\Psi^{-1}(x)$ with $U \subset V$ (cf. Remark 3.3). Therefore, $(\Phi \circ \Psi^{-1}, \Psi(U))$ is a chart of $\partial\Psi(G)$ in x.

(iii) Let $x \in \partial G$, and let (Φ, U) be a chart of ∂G in x. For small $0 < r < r_0$ it holds $B(x, r_0) \subset U$ and

$$\Phi(B(x,r) \cap \partial G) = \Phi(B(x,r)) \cap \partial E_1 = \{y \in \Phi(B(x,r)) : y_N = 0\}. \tag{3.12}$$

Hence, ∂G is a Lipschitz hypersurface in $\mathbb{R}^N$.

(iv) By the compactness of ∂G, there exist points $x^{(1)}, \ldots, x^{(n)} \in \partial G$ and charts (Φ_j, U_j) of ∂G in $x^{(j)}$ such that $\partial G \subset U_1 \cup \ldots \cup U_n$. Moreover, there exists an $r_0 > 0$ such that for all $x \in G^\circ$ it holds $B(x, r_0) \subset G^\circ$ or $B(x, r_0) \subset U_j$ for a certain j.

Let $x \in G^\circ$. In the case $B(x, r_0) \subset G^\circ$ the conclusion is trivial. Otherwise there exists an index $j \in \{1, \ldots, n\}$ such that $B(x, r_0) \subset U_j$. Then, because of (3.11), for $0 < r < r_0$ it holds

$$\begin{aligned}
&\lambda^N(G^\circ \cap B(x,r)) \geq L^{-N}\,\lambda^N(\Phi_j(G^\circ \cap B(x,r))) = \\
&= L^{-N}\,\lambda^N(E_1 \cap \Phi_j(B(x,r))) \geq L^{-N}\,\lambda^N(E_1 \cap B(\Phi_j(x), r/L)) \geq \\
&\geq 1/2\, L^{-N}\,\lambda^N(B(\Phi_j(x), r/L)) \geq 1/2\, L^{-2N}\,\lambda^N(B(x,r)),
\end{aligned}$$

where $L > 0$ is a common Lipschitz constant for all the maps Φ_j and Φ_j^{-1}. Hence, G° has property (a). □

Lemma 3.7 *Let $G \subset \mathbb{R}^N$ be bounded, and suppose that for each $x \in \partial G$ there exists an open neighborhood V of x in $\mathbb{R}^N$ such that $V \cap G$ is regular. Then G is regular.*

Proof Let $x \in \partial G$. Take the open neighborhood V of x such that $V \cap G$ is regular. Because of $x \in \partial(V \cap G)$ there exists a chart (Φ, U) of $\partial(V \cap G)$ in x with $U \subset V$ (cf. Lemma 3.6). Hence, $\Phi(U \cap G) = \Phi(U \cap V \cap G) \in \{E_1, E_2, E_3\}$, and (Φ, U) is a chart of ∂G in x. □

Remark 3.8 Lemma 3.6(iv) shows that the set of all regular subsets in $\mathbb{R}^N$ is not too large. Nevertheless, Lemma 3.6(i) and (ii) and Lemma 3.7 give a feeling that there exist quite a lot of regular sets. Of course, there exist other sufficient conditions for a set to be regular, for example the following:

If $\Omega \subset \mathbb{R}^N$ is bounded and open and has a Lipschitz boundary (this condition is stronger than $\partial\Omega$ to be a Lipschitz hypersurface in $\mathbb{R}^N$, see Grisvard [15, Section 1.2.1]), then Ω is regular.

However, the reversal of this claim is not true: There exist open regular subsets of $\mathbb{R}^N$ which do not have a Lipschitz boundary. This is because the image under a Lipschitz transformation of a bounded open set with Lipschitz boundary can be without Lipschitz boundary (cf. [15, Section 1.2]). Even the claim of Xie [33, Remark 3.1], that such an image has the interior cone property, is wrong, in general. Nevertheless, for regular subsets $G \subset \mathbb{R}^N$ we have embedding theorems (Theorem 2.3(iv) and Lemma 3.9) and trace theorems (Theorem 4.4).

Lemma 3.9 *Let $G \subset \mathbb{R}^N$ be regular. Then the embedding $W^{1,2}(G) \hookrightarrow L^2(G)$ is completely continuous.*

Proof By the compactness of ∂G, there exist points $x^{(1)}, \dots, x^{(m)} \in \partial G$ and charts (Φ_j, U_j) of ∂G in $x^{(j)}$ such that $\partial G \subset U_1 \cup \dots \cup U_m$. Moreover, there exist balls $U_{m+1}, \cdots, U_{m+n} \subset G^\circ$ such that

$$G \subset U_1 \cup \dots \cup U_{m+n}.$$

Let $\alpha_1, \cdots, \alpha_{m+n}$ be a smooth partition of unity subordinate to this covering of G. Further, let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $W^{1,2}(G)$. It holds

$$u_j = \sum_{k=1}^{m+n} \alpha_k u_j \quad \text{for all} \quad j.$$

We have to show that there exists a subsequence $\{j_i\}_{i\in\mathbb{N}}$ such that for all k the products $\alpha_k u_{j_i}$ converge in $L^2(G)$ for $i \to \infty$. Taking subsequences of subsequences, it suffices to show that for each k one can find such a subsequence.

First take $k \in \{1, \cdots, m\}$. The restrictions to U_k of the products $\alpha_k u_j$ form a bounded sequence in $W^{1,2}(U_k \cap G)$. Hence, the functions $(\alpha_k u_j) \circ \Phi_k^{-1}$ form a bounded sequence in $W^{1,2}(E_1)$. Here we used the fact that a Lipschitz coordinate transformation induces a continuous map between the $W^{1,2}$-spaces on the corresponding bounded domains, without any requirements concerning the boundaries of the domains (cf., e.g., Morrey [20, Theorem 3.1.7]). By the classical Rellich Embedding Theorem, there exists a subsequence $\{(\alpha_k u_{j_i}) \circ \Phi_k^{-1}\}_{i\in\mathbb{N}}$ which converges in $L^2(E_1)$. Hence, the restrictions to U_k of the products $\alpha_k u_{j_i}$ converge in $L^2(U_k \cap G)$. Therefore, the zero extensions to G of these restrictions, which are nothing but the functions $\alpha_k u_{j_i}$, converge in $L^2(G)$ for $i \to \infty$.

Now take $k \in \{m+1, \cdots, m+n\}$. The restrictions to U_k of the products $\alpha_k u_j$ form a bounded sequence in $W^{1,2}(U_k)$. By the classical Rellich Embedding Theorem, again, there is a subsequence $\{j_i\}_{i\in\mathbb{N}}$ such that the restrictions to U_k of $\alpha_k u_{j_i}$ converge in $L^2(U_k)$ for $i \to \infty$. Taking the the zero extensions to G of these restrictions, again, we get the desired result. □

Remark 3.10 In Fraenkel [11, Theorem 5.3] one can find a similar approach to get a Rellich-type theorem with minimal boundary smoothness assumptions.

The applications of Definition 3.1 to mixed boundary value problems are the motivation for defining abstractly the Dirichlet and the natural boundary part of a regular set $G \subset \mathbb{R}^N$ as well as the corresponding separating manifold by

$$
\left\{
\begin{array}{rcl}
\partial_{\mathcal{N}} G & := & G \cap \partial G, \\
\partial_{\mathcal{D}} G & := & \partial G \setminus \overline{\partial_{\mathcal{N}} G}, \\
\partial_0 G & := & \overline{\partial_{\mathcal{D}} G} \cap \overline{\partial_{\mathcal{N}} G}.
\end{array}
\right.
\tag{3.13}
$$

Lemma 3.11 *Let $G \subset \mathbb{R}^N$ be regular, then the following holds:*
(i) *$\partial_{\mathcal{D}} G$ and $\partial_{\mathcal{N}} G$ are relatively open in ∂G.*
(ii) *$\partial_0 G$ is a Lipschitz hypersurface in ∂G.*

Proof (i) $\partial_{\mathcal{D}} G$ is relatively open by definition.

Let (Φ, U) be a chart of ∂G in a point $x \in \partial_{\mathcal{N}} G$. Then, obviously, $\Phi(U \cap G) = E_2$. Moreover, by definition we have $\partial_{\mathcal{N}} G = G \setminus G^\circ$ and, hence,

$$\Phi(U \cap \partial_{\mathcal{N}} G) = \Phi(U \cap G) \setminus \Phi(U \cap G^\circ) = E_2 \setminus E_1$$

(cf. (3.11)). But $E_2 \setminus E_1 = E_2 \setminus (E_2)^\circ$ is relatively open in ∂E_2. Therefore, the set $\Phi^{-1}(E_2 \setminus (E_2)^\circ) = U \cap \partial_{\mathcal{N}} G$ is relatively open in

$$\Phi^{-1}(\partial E_2) = \partial \Phi^{-1}(E_2) = \partial(U \cap G)$$

and, all the more, in $U \cap \partial G$.

(ii) Let $x \in \partial_0 G$, and let (Φ, U) be a chart of ∂G in x. By definition, a point $\xi \in \partial G$ belongs to $\partial_0 G$ if and only if for all $r > 0$ $B(\xi, r) \cap \partial G \cap G \neq \varnothing$ and $(B(\xi, r) \cap \partial G) \setminus G \neq \varnothing$. Hence, (3.12) yields that a point $\xi \in U \cap \partial G$ belongs to $\partial_0 G$ if for all sufficiently small $r > 0$ we have $\{y \in \Phi(B(\xi, r)) \cap \Phi(U \cap G) : y_N = 0\} \neq \varnothing$ and $\{y \in \Phi(B(\xi, r)) \setminus \Phi(U \cap G) : y_N = 0\} \neq \varnothing$. This provides $\Phi(U \cap G) = E_3$ and $\Phi(U \cap \partial_0 G) = \{y \in \Phi(U) : y_N = y_1 = 0\}$. Therefore, $\partial_0 G$ is a Lipschitz hypersurface in ∂G. $\square$

Let $G \subset \mathbb{R}^N$ be a regular set. We will work with the following notation, which is usual in the theory of mixed boundary value problems (cf., e.g., TROIANIELLO [30], GRÖGER [18]). By $W_0^{1,2}(G)$ we denote the closure in $W^{1,2}(G^\circ)$ of the set

$$C_0^\infty(G) := \{u|_{G^\circ} : u \in C_0^\infty(\mathbb{R}^N), \operatorname{supp}(u) \cap (\overline{G} \setminus G) = \varnothing\}. \tag{3.14}$$

In (3.14) $u|_{G^\circ}$ is the restriction of the function u to G° . Furthermore, for $0 \le \omega \le N + 2$ we consider subspaces of the Sobolev–Campanato spaces defined as

$$W_0^{1,2,\omega}(G) := W_0^{1,2}(G) \cap W^{1,2,\omega}(G^\circ)$$

and equipped with the norm of $W^{1,2,\omega}(G^\circ)$. Finally, let $W^{-1,2}(G)$ be the dual space to $W_0^{1,2}(G)$, $\langle \cdot, \cdot \rangle_G$ the dual pairing between these two spaces, and let $J_G : W_0^{1,2}(G) \to W^{-1,2}(G)$ be the duality map of $W_0^{1,2}(G)$, defined as

$$\langle J_G w, v \rangle_G := \int_G (\nabla w \cdot \nabla v + wv) \, \mathrm{d}\lambda^N \quad \text{for all } w, v \in W_0^{1,2}(G). \tag{3.15}$$

By $W^{-1,2,\omega}(G)$ we denote the subspace of all functionals $\phi \in W^{-1,2}(G)$, which belong to the image of the space $W_0^{1,2,\omega}(G)$ under the duality map J_G, with the norm

$$\|J_G u\|_{W^{-1,2,\omega}(G)} := \|u\|_{W^{1,2,\omega}(G^\circ)} \text{ for } u \in W_0^{1,2,\omega}(G). \tag{3.16}$$

For the sake of simplicity we will denote $\mathfrak{L}^{2,\omega}(G) := \mathfrak{L}^{2,\omega}(G^\circ)$ and $\mathfrak{L}^{2,\omega}(G; \mathbb{R}^N) := \mathfrak{L}^{2,\omega}(G^\circ; \mathbb{R}^N)$.

Lemma 3.12 *Let $G \subset \mathbb{R}^N$ be regular, V an open neighborhood of $\overline{G}$ in $\mathbb{R}^N$ and Ψ a Lipschitz transformation of V into $\mathbb{R}^N$. Then v belongs to $W_0^{1,2}(\Psi(G))$ if and only if $v \circ \Psi$ is an element of $W_0^{1,2}(G)$.*

Proof By Lemma 3.6(ii), $H := \Psi(G)$ is a regular set. Let $v \in W_0^{1,2}(H)$. Then there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^n)$ with

$$\operatorname{supp}(v_j) \cap (\overline{H} \setminus H) = \varnothing \quad \text{and} \quad \lim_{j\to\infty} \|v - v_j|_{H^\circ}\|_{W_0^{1,2}(H)} = 0.$$

Hence, it holds $v \circ \Psi \in W^{1,2}(G^\circ)$,

$$\lim_{j\to\infty} \|v \circ \Psi - v_j \circ \Psi\|_{W^{1,2}(G^\circ)} = 0 \tag{3.17}$$

and

$$\operatorname{supp}(v_j \circ \Psi) \cap (\overline{G} \setminus G) \subset \Psi^{-1}(\operatorname{supp}(v_j) \cap (\overline{H} \setminus H)) = \varnothing.$$

Because $\overline{G} \setminus G = \partial G \setminus \partial_{\mathcal{N}} G$ and $\operatorname{supp}(v_j \circ \Psi)$ are closed sets, there must be a positive distance between these two sets. We denote

$$\delta_j := {}^1\!/\!_4 \operatorname{dist}(\overline{G} \setminus G, \operatorname{supp}(v_j \circ \Psi))$$

and extend $v_j \circ \Psi$ by zero to a function $u_j \in L^\infty(\mathbb{R}^n)$. Now, we define for $i, j \in \mathbb{N}, i > 1/\delta_j$ functions w_{ij} by convolution with mollifiers $\zeta_i \in C_0^\infty(\mathbb{R}^N)$

$$w_{ij}(x) = (\zeta_i * u_j)(x) \quad \text{for } x \in \mathbb{R}^n,$$

where

$$\operatorname{supp}(\zeta_i) \subset B(0, 1/i) \quad \text{and} \quad \int\limits_{\mathbb{R}^N} \zeta_i \,\mathrm{d}\lambda^N = 1.$$

Obviously, it holds $w_{ij} \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp}(w_{ij}) \cap (\overline{G} \setminus G) = \varnothing$ for all $i > 1/\delta_j$. Using the convergence properties of convolutions with mollifiers we get

$$\lim_{i\to\infty} \|v_j \circ \Psi - w_{ij}|_{G^\circ}\|_{W^{1,2}(G^\circ)} = 0,$$

and, therefore, $v_j \circ \Psi \in W_0^{1,2}(G)$ for all $j \in \mathbb{N}$. Because $W_0^{1,2}(G)$ is a closed subspace of $W^{1,2}(G^\circ)$, (3.17) yields $v \circ \Psi \in W_0^{1,2}(G)$.

Analogously, it follows $v \in W_0^{1,2}(H)$, if we suppose $v \circ \Psi \in W_0^{1,2}(G)$. □

4 Campanato Spaces on the Natural Boundary Part

Throughout this section G is a fixed regular subset of $\mathbb{R}^N$.

Because of Lemma 3.6(iii) and Lemma 3.11(i) the natural boundary part $\partial_{\mathcal{N}} G$ is a Lipschitz hypersurface in $\mathbb{R}^N$. Hence, the $(N-1)$-dimensional Lebesgue measure $\lambda_{\partial G}$ can be introduced by (2.2), (2.3) and (2.4), and for $1 \le p \le \infty$ we denote by $L^p(\partial_{\mathcal{N}} G)$ the corresponding Lebesgue spaces. The norms are defined as

$$\|u\|_{L^p(\partial_{\mathcal{N}} G)} := \left(\int\limits_{\partial_{\mathcal{N}} G} |u(\xi)|^p \,\mathrm{d}\lambda_{\partial G}(\xi) \right)^{1/p} \quad \text{for } 1 \le p < \infty.$$

Further, for $0 \le \omega \le N-1+p$ we denote by $\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}} G)$ the corresponding Campanato space, i.e. the space of all $u \in L^p(\partial_{\mathcal{N}} G)$ such that

$$[u]_{\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}} G)} := \left(\sup_{\substack{x\in\partial_{\mathcal{N}} G \\ r>0}} \left(r^{-\omega} \int\limits_{M(x,r)} |u(y) - u_{M(x,r)}|^p \,\mathrm{d}\lambda_{\partial G}(y) \right) \right)^{1/p} < \infty. \tag{4.1}$$

In (4.1) we used the notation

$$M(x,r) := B(x,r) \cap \partial_{\mathcal{N}} G, \quad u_{M(x,r)} := \frac{1}{\lambda_{\partial G}(M(x,r))} \int\limits_{M(x,r)} u(y)\,\mathrm{d}\lambda_{\partial G}(y).$$

The space $\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}} G)$ is a Banach space with the norm

$$\|u\|_{\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}} G)} := \left(\|u\|^p_{L^p(\partial_{\mathcal{N}} G)} + [u]^p_{\mathfrak{L}^{p,\omega}(\partial_{\mathcal{N}} G)} \right)^{1/p}.$$

In order to prove certain properties of functions from $\mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)$ we will work with chart representations. Hence, let us introduce the corresponding terminology.

Definition 4.1 (i) Let (Φ, U) be a chart of ∂G in $x_0 \in \partial G$. If $\Phi(U \cap G) = E_k$ ($k = 1, 2$ or 3) then (Φ, U) is called a chart of type k.

(ii) A finite set of charts $\{(\Phi_j, U_j) : j = 1, \ldots, n\}$ such that

$$\partial_{\mathcal{N}} G \subset \bigcup_{j=1}^{n} U_j$$

is called atlas of $\partial_{\mathcal{N}} G$.

Because for every $x \in \partial_{\mathcal{N}} G$ there exists a chart (Φ, U) of type 2 we can find an atlas $\{(\Phi_{j,k}, U_{j,k}) : k = 2, 3;\ j = 1, \ldots, n_k\}$ of $\partial_{\mathcal{N}} G$, where the charts $(\Phi_{j,k}, U_{j,k})$ are of type k. Obviously, it holds $\partial_{\mathcal{N}} E_2 = \{x \in \mathbb{R}^N : |x| < r,\ x_N = 0\}$ and $\partial_{\mathcal{N}} E_3 = \{x \in \partial_{\mathcal{N}} E_2 : x_1 > 0\}$, and we define embedding charts $\varphi_{j,k} : \partial_{\mathcal{N}} E_k \to \partial_{\mathcal{N}} G$ by $\varphi_{j,k}(\xi) := \Phi^{-1}_{j,k}(\xi)$. Further, let $\{\alpha_{j,k} : k = 2, 3;\ j = 1, \ldots, n_k\}$ be a smooth partition of unity subordinate to the covering $\{U_{j,k} : k = 2, 3;\ j = 1, \ldots, n_k\}$ of $\partial_{\mathcal{N}} G$. Then for all $u \in L^1(\partial_{\mathcal{N}} G)$ we have

$$\int\limits_{\partial_{\mathcal{N}} G} u(x)\,\mathrm{d}\lambda_{\partial G}(x) = \sum_{k=2}^{3} \sum_{j=1}^{n_k} \int\limits_{\partial_{\mathcal{N}} E_k} \alpha_{j,k}(\varphi_{j,k}(\xi))\, u(\varphi_{j,k}(\xi))\, |J\varphi_{j,k}(\xi)|\,\mathrm{d}\lambda^{N-1}(\xi). \tag{4.2}$$

In (4.2) $J\varphi_{j,k}$ is the Jacobian of the embedding chart $\varphi_{j,k}$, cf. (2.2).

Let Φ be a Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^N$. Then, because of Lemma 3.6(ii), $\Phi(G)$ is regular, and (cf. (3.13))

$$\partial_{\mathcal{N}} \Phi(G) = \Phi(\partial_{\mathcal{N}} G)$$

is a Lipschitz hypersurface in $\mathbb{R}^N$. Moreover, (4.2) yields

$$\int\limits_{\partial_{\mathcal{N}} G} u(x)\,|J\Phi_{\mathcal{N}}(x)|\,\mathrm{d}\lambda_{\partial G}(x) = \int\limits_{\partial_{\mathcal{N}} \Phi(G)} u(\Phi^{-1}_{\mathcal{N}}(y))\,\mathrm{d}\lambda_{\partial \Phi(G)}(y) \tag{4.3}$$

for all $u \in L^1(\partial_{\mathcal{N}} G)$. In (4.3), $\Phi_{\mathcal{N}} : \partial_{\mathcal{N}} G \to \partial_{\mathcal{N}} \Phi(G)$ is the restriction of Φ to $\partial_{\mathcal{N}} G$, and $J\Phi_{\mathcal{N}}$ is the Jacobian of $\Phi_{\mathcal{N}}$ in $x \in \partial_{\mathcal{N}} G$. By means of embedding charts, we get

$$(4.4) \qquad \begin{cases} J\Phi_{\mathcal{N}}(\varphi_{j,k}(\xi)) = \\ = J\psi_{j,k}((\psi_{j,k})^{-1} \circ \Phi_{\mathcal{N}} \circ \varphi_{j,k}(\xi))\, J[(\psi_{j,k})^{-1} \circ \Phi_{\mathcal{N}} \circ \varphi_{j,k}](\xi)\, [J\varphi_{j,k}(\xi)]^{-1} \end{cases}$$

for λ^{N-1}-almost all $\xi \in \partial_{\mathcal{N}} E_k$. In (4.4), $\psi_{j,k} : \partial_{\mathcal{N}} E_k \to \partial_{\mathcal{N}} \Phi(G)$ are embedding charts which are defined as $\psi_{j,k}(\eta) := \Psi_{j,k}^{-1}(\eta)$ for $\eta \in \partial_{\mathcal{N}} E_k$ by an atlas

$$\{(\Psi_{j,k}, V_{j,k}) : k = 2, 3;\ j = 1, \ldots, n_k\}$$

of $\Phi(G)$ such that $(\Psi_{j,k}, V_{j,k})$ is of type k and $\Phi(U_{j,k}) \subset V_{j,k}$. Moreover, $J\psi_{j,k}$ is the Jacobian of $\psi_{j,k}$, defined by (2.2), and $J[(\psi_{j,k})^{-1} \circ \Phi_{\mathcal{N}} \circ \varphi_{j,k}](\xi)$ is the determinant of the $(N-1) \times (N-1)$-dimensional functional matrix of the map

$$\psi_{j,k}^{-1} \circ \Phi_{\mathcal{N}} \circ \varphi_{j,k} : \partial_{\mathcal{N}} E_k \to \partial_{\mathcal{N}} E_k$$

in the point $\xi \in \partial_{\mathcal{N}} E_k$. Remark that for fixed $x \in \partial_{\mathcal{N}} G$ the right hand side of (4.4) does not depend on the choice of the charts $(\Phi_{j,k}, U_{j,k})$ and $(\Psi_{j,k}, V_{j,k})$.

By means of (4.2) and of Theorem 2.3(ii), the following lemma is easy to prove:

Lemma 4.2 *Let $\omega < N - 1$. Then for all $u \in \mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)$ and $v \in L^\infty(\partial_{\mathcal{N}} G)$ the product uv belongs to $\mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)$, and there exists a constant $c > 0$ such that*

$$\|uv\|_{\mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)} \le c\, \|u\|_{\mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)}\, \|v\|_{L^\infty(\partial_{\mathcal{N}} G)}$$

for all such u and v.

Analogously, from (4.3) and Theorem 2.1(ii) we get

Lemma 4.3 *Let Φ be a Lipschitz transformation from an open neighborhood of $\overline{G}$ into $\mathbb{R}^N$ and $\omega < N + 1$. Then, there exists a constant $c > 0$ such that for all $u \in \mathfrak{L}^{2,\omega}(\Phi(\partial_{\mathcal{N}} G))$ it holds $u \circ \Phi \in \mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)$ and*

$$\|u \circ \Phi\|_{\mathfrak{L}^{2,\omega}(\partial_{\mathcal{N}} G)} \le c\, \|u\|_{\mathfrak{L}^{2,\omega}(\Phi(\partial_{\mathcal{N}} G))}.$$

The main result of this section is the following theorem about traces of Sobolev–Campanato functions on the natural boundary part of regular sets:

Theorem 4.4 *Let $\omega < N$. Then there exists a linear bounded operator T from $W_0^{1,2,\omega}(G)$ into $\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}} G)$ such that, for all $u \in C_0^\infty(G)$, Tu is the restriction of u to $\partial_{\mathcal{N}} G$. Furthermore, the operator T maps $W_0^{1,2}(G)$ completely continuous into $L^2(\partial_{\mathcal{N}} G)$.*

Proof Step 1. Let $k = 2$ or 3, $0 < r < 1$, $0 < \varrho < \min\{r, 1 - r\}$ and $x_0 \in \partial_{\mathcal{N}} E_k(0, r)$ be fixed. For $u \in W_0^{1,2}(E_k)$ we denote by $u|_{\partial_{\mathcal{N}} E_k} \in L^2(\partial_{\mathcal{N}} E_k)$ the restriction of u to $\partial_{\mathcal{N}} E_k$ (in the usual sense of trace of a $W^{1,2}(E_1)$-function in $L^2(\partial E_1)$) and by

$$u_{\partial_{\mathcal{N}} E_k} := \frac{1}{\lambda^{N-1}(\partial_{\mathcal{N}} E_k)} \int\limits_{\partial_{\mathcal{N}} E_k} u|_{\partial_{\mathcal{N}} E_k}\, \mathrm{d}\lambda^{N-1}$$

the corresponding mean value. The notation

$$u|_{\partial_{\mathcal{N}} E_k(0,r)\cap B(x_0,\varrho)},\ u|_{\partial_{\mathcal{N}} E_k(x_0,\varrho)},\ u_{\partial_{\mathcal{N}} E_k(0,r)\cap B(x_0,\varrho)} \text{ and } u_{\partial_{\mathcal{N}} E_k(x_0,\varrho)}$$

will be used in a similar manner. Finally, for $v \in W_0^{1,2}(E_k(x_0,\varrho))$ we denote

$$(Hv)(x) := v\left(\frac{x-x_0}{\varrho}\right) \quad \text{for } \lambda^N\text{-almost all } x \in E_k.$$

The usual trace theorem yields that there exists a constant $c > 0$ such that for all $u \in W_0^{1,2}(E_k)$ we have

$$\|u|_{\partial_{\mathcal{N}} E_2} - u_{\partial_{\mathcal{N}} E_2}\|^2_{L^2(\partial_{\mathcal{N}} E_2)} \le c\,\|\nabla u\|^2_{L^2(E_1;\mathbb{R}^N)}.$$

Hence, for such functions we get

$$\begin{aligned}
&\|u|_{\partial_{\mathcal{N}} E_2(x_0,\varrho)} - u_{\partial_{\mathcal{N}} E_2(x_0,\varrho)}\|^2_{L^2(\partial_{\mathcal{N}} E_2(x_0,\varrho))} = \\
&= \varrho^{N-1}\,\|H(u|_{\partial_{\mathcal{N}} E_2(x_0,\varrho)} - u_{\partial_{\mathcal{N}} E_2(x_0,\varrho)})\|^2_{L^2(\partial_{\mathcal{N}} E_2)} = \\
&= \varrho^{N-1}\,\|(Hu)|_{\partial_{\mathcal{N}} E_k} - (Hu)_{\partial_{\mathcal{N}} E_2}\|^2_{L^2(\partial_{\mathcal{N}} E_2)} \le \\
&\le c\varrho^{N-1}\,\|\nabla(Hu)\|^2_{L^2(E_1;\mathbb{R}^N)} = c\varrho\,\|\nabla u\|^2_{L^2(E_1(x_0,\varrho);\mathbb{R}^N)}.
\end{aligned}$$

Moreover, we have $\partial_{\mathcal{N}} E_k(0,r)\cap B(x_0,\varrho) \subset \partial_{\mathcal{N}} E_2(x_0,\varrho)$. Hence, the minimizing property of the mean value yields

$$\begin{aligned}
&\|u|_{\partial_{\mathcal{N}} E_k(0,r)\cap B(x_0,\varrho)} - u_{\partial_{\mathcal{N}} E_k(0,r)\cap B(x_0,\varrho)}\|^2_{L^2(\partial_{\mathcal{N}} E_2(0,r)\cap B(x_0,\varrho))} \le \\
&\le \|u|_{\partial_{\mathcal{N}} E_2(x_0,\varrho)} - u_{\partial_{\mathcal{N}} E_2(x_0,\varrho)}\|^2_{L^2(\partial_{\mathcal{N}} E_2(x_0,\varrho))} \le c\varrho\,\|\nabla u\|^2_{L^2(E_1(x_0,\varrho);\mathbb{R}^N)}.
\end{aligned}$$

Summarizing, we get: There exists a constant $c > 0$ such that for all $\omega < N$, $0 < r < 1$, $k = 2$ or 3 and $u \in W_0^{1,2,\omega}(E_k)$ we have $u|_{\partial_{\mathcal{N}} E_k(0,r)} \in \mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}} E_k(0,r))$ and

$$\|u|_{\partial_{\mathcal{N}} E_k(0,r)}\|_{\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}} E_k(0,r))} \le c\,\|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_1;\mathbb{R}^N)}.$$

Step 2. Let $\{(\Phi_{j,k},U_{j,k}) : j = 1,\dots,n_k,\ k = 2,3\}$ be an atlas of $\partial_{\mathcal{N}} G$, such that the charts $(\Phi_{j,k},U_{j,k})$ are of type k. Let $\{\alpha_{j,k} : j = 1,\dots,n_k,\ k = 2,3\}$ be a smooth partition of unity subordinate to the open covering $\{U_{j,k}\}$ of $\partial_{\mathcal{N}} G$. Because of the compactness of the support of $\alpha_{j,k}$ we can find a number $0 < r < 1$ such that

$$\operatorname{supp}(\alpha_{j,k}) \subset V_{j,k} := \Phi_{j,k}^{-1}(B(0,r))$$

and that the the sets $V_{j,k}$ form still an open covering of $\partial_{\mathcal{N}} G$.

For $u \in W_0^{1,2,\omega}(G)$ we set

$$u_{j,k}(\xi) := \alpha_{j,k}(\Phi_{j,k}^{-1}(\xi))\,u(\Phi_{j,k}^{-1}(\xi)) \quad \text{for } \lambda^N\text{-almost all } \xi \in E_1,$$

By construction, Theorems 2.1(ii), 2.3(ii) and Lemma 3.12, for each $u \in W_0^{1,2,\omega}(G)$ we have $u_{j,k} \in W_0^{1,2,\omega}(E_k)$ and

$$\|\nabla u_{j,k}\|_{\mathfrak{L}^{2,\omega}(E_1;\mathbb{R}^N)} \le c\,\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)},$$

where the constant c does not depend on u, j and k. Thus, Step 1 implies

$$\|u_{j,k}|_{\partial_{\mathcal{N}} E_k(0,r)}\|_{\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}} E_k(0,r))} \le c\,\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)}$$

with a modified constant $c > 0$. Let

$$v_{j,k}(x) := \begin{cases} u_{j,k}(\Phi_{j,k}(x)) & \text{for } \lambda_{\partial_{\mathcal{N}}G}\text{-almost all} \quad x \in \partial_{\mathcal{N}}G \cap V_{j,k}, \\ 0 & \text{for } \lambda_{\partial_{\mathcal{N}}G}\text{-almost all} \quad x \in \partial_{\mathcal{N}}G \setminus V_{j,k}. \end{cases}$$

Because of Lemma 4.2 and Lemma 4.3 it holds that $v_{j,k} \in \mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}G)$ and

$$\|v_{j,k}\|_{\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}}G)} \leq c\,\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)},$$

where the constant c does not depend on u, j and k, again. Finally,

$$u|_{\partial_{\mathcal{N}}G} = \sum_{k=2}^{3}\sum_{j=1}^{n_k} v_{j,k}$$

yields the sought-for estimate. Hence, the proof is finished.

For the compactness of T from $W_0^{1,2}(G)$ into $L^2(\partial_{\mathcal{N}}G)$ one has to proceed as in the proof of Lemma 3.9 and to use the usual trace theorem. □

5 Admissible Sets

The following terminology is essentially due to Recke [26]:

Definition 5.1 Let $G \subset \mathbb{R}^N$ be regular.

(i) For $0 < \varepsilon \leq 1$ we denote by $\mathfrak{A}(\varepsilon, G)$ the set of all pairs $(A, d) \in L^\infty(G; \mathfrak{S}_N) \times L^\infty(G)$, such that for λ^N-almost all $x \in G$

$$\varepsilon \leq d(x) \leq \frac{1}{\varepsilon} \quad \text{and} \quad \varepsilon\,|\xi|^2 \leq A(x)\xi\cdot\xi \leq \frac{1}{\varepsilon}\,|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N.$$

(ii) A regular subset G_0 of G is called G-admissible if for each $0 < \varepsilon \leq 1$ there exists an $\overline{\omega} > N-2$ such that for all $\omega < \overline{\omega}$, $(A,d) \in \mathfrak{A}(\varepsilon, G)$, $f \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(G)$, $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)$ and $u \in W_0^{1,2}(G)$ with

$$(5.1) \qquad \begin{cases} \displaystyle\int_G (A\nabla u\cdot\nabla v + duv)\,\mathrm{d}\lambda^N = \int_G (f\cdot\nabla v + gv)\,\mathrm{d}\lambda^N + \int_{\partial_{\mathcal{N}}G} hv\,\mathrm{d}\lambda_{\partial G} \\ \text{for all } v \in W_0^{1,2}(G), \end{cases}$$

it holds $\nabla u \in \mathfrak{L}^{2,\omega}(G_0;\mathbb{R}^N)$ and

$$(5.2) \qquad \begin{cases} \|\nabla u\|_{\mathfrak{L}^{2,\omega}(G_0;\mathbb{R}^N)} \leq \\ \leq c\left(\|f\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}}G)} + \|u\|_{W^{1,2}(G^\circ)}\right), \end{cases}$$

where the constant c in (5.2) depends only on G, G_0, N, ε and ω.

(iii) G is called admissible if it is G-admissible.

Remark 5.2 Let $G \subset \mathbb{R}^N$ be regular. The variational equation (5.1) is the weak formulation of the boundary value problem

$$
(5.3) \qquad \begin{cases} -\nabla \cdot (A\nabla u) + du &= -\nabla \cdot f + g \quad \text{in } G^\circ, \\ (A\nabla u)\cdot \nu &= f\cdot\nu + h \quad \text{on } \partial_{\mathcal{N}} G, \\ u &= 0 \quad \text{on } \partial_{\mathcal{D}} G. \end{cases}
$$

The Lax-Milgram Lemma yields that for all $(A,d) \in \mathfrak{A}(\varepsilon, G)$, $f \in L^2(G;\mathbb{R}^N)$, $g \in L^2(G)$ and $h \in L^2(\partial_{\mathcal{N}} G)$ there exists exactly one weak solution $u \in W_0^{1,2}(G)$ of the boundary value problem (5.3), and the linear map

$$(f,g,h) \in L^2(G;\mathbb{R}^N) \times L^2(G) \times L^2(\partial_{\mathcal{N}} G) \mapsto u \in W_0^{1,2}(G)$$

is continuous. Hence, G is admissible if and only if for each $0 < \varepsilon \le 1$ there exists an $\overline{\omega} > N-2$ such that for all $\omega < \overline{\omega}$, $(A,d) \in \mathfrak{A}(\varepsilon, G)$, $f \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(G)$ and $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)$ the weak solution $u \in W_0^{1,2}(G)$ to (5.3) belongs to $W_0^{1,2,\omega}(G)$, and the linear map

$$(f,g,h) \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N) \times \mathfrak{L}^{2,\omega-2}(G) \times \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G) \mapsto u \in W_0^{1,2,\omega}(G)$$

is continuous. In particular, if G is admissible, then for all weak solutions u to (5.3) it holds not only (5.2) with $G_0 = G$, but also

$$\|u\|_{W_0^{1,2,\omega}(G)} \le c\left(\|f\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)}\right)$$

(with another constant c, possibly).

It is well-known (cf., e.g., Troianiello [30, Theorem 2.19]) that bounded open subsets of $\mathbb{R}^N$ with smooth boundary are admissible. The aim of this section is to prove the following

Theorem 5.3 *Each regular subset $G \subset \mathbb{R}^N$ is admissible.*

In order to prove Theorem 5.3, we need some lemmas concerning localization and transformation properties of admissible sets and the admissibility of the sets E_1, E_2 and E_3.

Lemma 5.4 *Let $G \subset \mathbb{R}^N$ be regular, and suppose that for each $x \in \partial G$ there exist open neighborhoods U_0 and U of x in $\mathbb{R}^N$ with $U_0 \subset U$, such that $U_0 \cap G$ is $(U \cap G)$-admissible. Then G is admissible.*

Proof Because of the compactness of ∂G, there exist open subsets U_{0j}, U_j in $\mathbb{R}^N$ $(j = 1,\dots,n)$ such that

$$
(5.4) \qquad U_{0j} \subset U_j\,,\ \partial G \subset \bigcup_{j=1}^n U_{0j} \text{ and } U_{0j} \cap G \text{ is } (U_j \cap G)\text{-admissible.}
$$

Moreover, there exist open balls U_{0j}, U_j in G° $(j = n+1,\dots,n+m)$ such that

$$U_{0j} \subset U_j \text{ and } G \subset \bigcup_{j=1}^{n+m} U_{0j}.$$

Remark that each ball U_{0j} is U_j-admissible for $j = n+1, \ldots, n+m$.

Let $\{\alpha_1, \ldots, \alpha_{n+m}\} \subset C_0^\infty(\mathbb{R}^N)$ be a smooth partition of unity subordinate to the covering $\{U_{01}, \ldots, U_{0n+m}\}$ of G.

Now, take $(A, d) \in \mathfrak{A}(\varepsilon, G)$, $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(G)$, $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)$, and $u \in W_0^{1,2}(G)$ such that (5.1) holds. Then, we have for all $w \in W_0^{1,2}(U_j \cap G)$

$$
(5.5) \qquad \left\{
\begin{aligned}
&\int\limits_{U_j \cap G} (A\nabla(\alpha_j u) \cdot \nabla w + d\alpha_j uw) \, \mathrm{d}\lambda^N = \\
&= \int\limits_{G} (A\nabla u \cdot \nabla(\alpha_j w) + du\,\varphi_j w + A(u\nabla w - w\nabla u) \cdot \nabla\alpha_j) \, \mathrm{d}\lambda^N = \\
&= \int\limits_{G} (f \cdot \nabla(\alpha_j w) + g\alpha_j w) \, \mathrm{d}\lambda^N + \int\limits_{\partial_{\mathcal{N}} G} h\alpha_j w \, \mathrm{d}\lambda_{\partial G} + \\
&\qquad + \int\limits_{G} A(u\nabla w - w\nabla u) \cdot \nabla\alpha_j \, \mathrm{d}\lambda^N = \\
&= \int\limits_{U_j \cap G} (\alpha_j g + f \cdot \nabla\alpha_j - A\nabla u \cdot \nabla\alpha_j) \, w \, \mathrm{d}\lambda^N + \\
&\qquad + \int\limits_{U_j \cap G} (\alpha_j f + uA\nabla\alpha_j) \cdot \nabla w \, \mathrm{d}\lambda^N + \int\limits_{U_j \cap \partial_{\mathcal{N}} G} \alpha_j hw \, \mathrm{d}\lambda_{\partial G}.
\end{aligned}
\right.
$$

In order to apply the assumption (5.4), we use the multiplier properties (Theorem 2.3(ii) and Theorem 4.2), the continuous embedding $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,2}(G)$ (cf. Theorem 2.3(iv)) and the trace property $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,1}(\partial_{\mathcal{N}} G)$ (cf. Theorem 4.4). Thus, we have for $\mu = \min\{\omega, 2\}$

$$
(5.6) \qquad \left\{
\begin{array}{rcl}
\alpha_j f + uA\nabla\alpha_j & \in & \mathfrak{L}^{2,\mu}(U_j \cap G; \mathbb{R}^N), \\
\alpha_j g + f \cdot \nabla\alpha_j - A\nabla u \cdot \nabla\alpha_j & \in & \mathfrak{L}^{2,\mu-2}(U_j \cap G), \\
\alpha_j h & \in & \mathfrak{L}^{2,\mu-1}(U_j \cap \partial_{\mathcal{N}} G).
\end{array}
\right.
$$

Moreover, we have $\alpha_j u \in W_0^{1,2}(U_j \cap G)$ (here and later on we use the symbol $\alpha_j u$ for the restriction of the product $\alpha_j u$ to $U_j \cap G$, too). Hence, it follows from (5.4) and (5.5) that there exists an $N - 2 < \overline{\omega} < N$ such that, if $\mu < \overline{\omega}$, it holds $\nabla(\alpha_j u) \in \mathfrak{L}^{2,\mu}(U_{0j} \cap G; \mathbb{R}^N)$ and

$$
(5.7) \qquad \left\{
\begin{aligned}
&\|\nabla(\alpha_j u)\|_{\mathfrak{L}^{2,\mu}(U_{0j} \cap G; \mathbb{R}^N)} \le \\
&\le c \left(\|f\|_{\mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\mu-2}(G)} + \|h\|_{\mathfrak{L}^{2,\mu-1}(\partial_{\mathcal{N}} G)} + \|u\|_{W^{1,2}(G^\circ)} \right).
\end{aligned}
\right.
$$

Finally, the zero extension of the map $\nabla(\alpha_j u) \in \mathfrak{L}^{2,\mu}(U_{0j} \cap G; \mathbb{R}^N)$ is the map $\nabla(\alpha_j u) \in \mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)$, and, hence,

$$
\nabla u = \sum_{j=1}^{n+m} \nabla(\alpha_j u) \in \mathfrak{L}^{2,\mu}(G; \mathbb{R}^N)
$$

and

$$
(5.8)\qquad \begin{cases} \|\nabla u\|_{\mathfrak{L}^{2,\mu}(G;\mathbb{R}^N)} \le \sum_{j=1}^{n+m} \|\nabla(\alpha_j u)\|_{\mathfrak{L}^{2,\mu}(U_{0j}\cap G;\mathbb{R}^N)} \le \\ \le c\left(\|f\|_{\mathfrak{L}^{2,\mu}(G;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\mu-2}(G)} + \|h\|_{\mathfrak{L}^{2,\mu-1}(\partial_{\mathcal{N}}G)} + \|u\|_{W^{1,2}(G^\circ)}\right). \end{cases}
$$

Note, that the constants c in (5.7) and (5.8) do not depend on A, d, f, g, h and u.

Now, we again apply assumption (5.4) and the variational equation (5.5). Then we get (5.6) with $\mu = \min\{\omega, 4\}$, and, hence, $\nabla u \in \mathfrak{L}^{2,\mu}(G;\mathbb{R}^N)$ and (5.8) for this new μ and a new constant $c > 0$ (with the same dependencies). Reiterating this procedure as often as necessary we obtain that ∇u belongs to $\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$, and it holds (5.8) for each $\mu = \omega < \overline{\omega}$. Therefore, G is admissible. □

Lemma 5.5 *Let G_0 and G be regular subsets of $\mathbb{R}^N$, $G_0 \subset G$, and let G_0 be G-admissible. Further, let U be an open neighborhood of $\overline{G}$ in $\mathbb{R}^N$ and Φ a Lipschitz transformation of U into $\mathbb{R}^N$. Then $\Phi(G_0)$ is $\Phi(G)$-admissible.*

Proof First of all, $H := \Phi(G)$ and $H_0 := \Phi(G_0)$ are regular sets of $\mathbb{R}^N$ (cf. Lemma 3.6), and it holds $\partial_{\mathcal{N}} H_0 = \Phi(\partial_{\mathcal{N}} G_0)$ and $\partial_{\mathcal{N}} H = \Phi(\partial_{\mathcal{N}} G)$.

Let $D\Phi(x)$ be the functional matrix of Φ in x. By $J\Phi(x) := \det D\Phi(x)$ we denote the Jacobian of Φ in x. Let $L \ge 1$ a common Lipschitz constant of both the transformations Φ and Φ^{-1}. Then, we have

$$
(5.9)\qquad \begin{cases} L^{-1}|\xi| \le |D\Phi(x)\xi| \le L\,|\xi| \quad \text{for all } \xi \in \mathbb{R}^N, \\ \quad L^{-N} \le |J\Phi(x)| \le L^N \end{cases}
$$

for λ^N-almost all $x \in G$.

Let $\Phi_{\mathcal{N}} : \partial_{\mathcal{N}} G \to \partial_{\mathcal{N}} H$ be the restriction of the Lipschitz transformation Φ on the Lipschitz hypersurface $\partial_{\mathcal{N}} G$. By $J\Phi_{\mathcal{N}}(x)$ we denote the Jacobian of $\Phi_{\mathcal{N}}$ in $x \in \partial_{\mathcal{N}} G$, which is defined by (4.4). Then, we have $L^{1-N} \le |J\Phi_{\mathcal{N}}(x)| \le L^{N-1}$ for $\lambda_{\partial G}$-almost all $x \in \partial_{\mathcal{N}} G$.

Now, take $(A, d) \in \mathfrak{A}(\varepsilon, H)$, $f \in \mathfrak{L}^{2,\omega}(H;\mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(H)$, $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} H)$ and $u \in W_0^{1,2}(H)$ such that

$$
(5.10)\qquad \int_H (A\nabla u \cdot \nabla v + duv)\,\mathrm{d}\lambda^N = \int_H (f \cdot \nabla v + gv)\,\mathrm{d}\lambda^N + \int_{\partial_{\mathcal{N}} H} hv\,\mathrm{d}\lambda_{\partial G}
$$

for all $v \in W_0^{1,2}(H)$. Because of Lemma 3.12 we have $u \circ \Phi \in W_0^{1,2}(G)$ and

$$
(5.11)\qquad \|u \circ \Phi\|_{W_0^{1,2}(G)} \le c\,\|u\|_{W_0^{1,2}(H)},
$$

where the constant c in (5.11) does not depend on u. Moreover, Theorem 2.1(ii) and Lemma 4.3 yield a constant $c > 0$ such that

$$
(5.12)\qquad \begin{cases} f \circ \Phi \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N), & \|f \circ \Phi\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} \le c\,\|f\|_{\mathfrak{L}^{2,\omega}(H;\mathbb{R}^N)}, \\ g \circ \Phi \in \mathfrak{L}^{2,\omega-2}(G), & \|g \circ \Phi\|_{\mathfrak{L}^{2,\omega-2}(G)} \le c\,\|g\|_{\mathfrak{L}^{2,\omega-2}(H)}, \\ h \circ \Phi_{\mathcal{N}} \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G), & \|h \circ \Phi_{\mathcal{N}}\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)} \le c\,\|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} H)}. \end{cases}
$$

Finally, from (5.9) it follows that the map

$$x \in G \mapsto \big(|J\Phi(x)|\, D\Phi(x)^{-1} A(\Phi(x))\, (D\Phi(x)^T)^{-1}, |J\Phi(x)|\, d(\Phi(x))\big) \in \mathfrak{S}_N \times \mathrm{I\!R}$$

belongs to $\mathfrak{A}(\delta, G)$ with $\delta = \varepsilon L^{-N-2}$. Therefore, the chain rule for derivatives, the transformation formulas for integrals and the variational equation (5.10) imply that for all $w \in W_0^{1,2}(G)$ we have

$$\text{(5.13)} \quad \left\{ \begin{aligned} &\int_G |J\Phi| \left((D\Phi^{-1})\,(A \circ \Phi)\,(D\Phi^{-1})^T\right) \nabla(u \circ \Phi) \cdot \nabla w \,\mathrm{d}\lambda^N + \\ &\qquad + \int_G |J\Phi|\,(d \circ \Phi) \cdot (u \circ \Phi) \cdot w \,\mathrm{d}\lambda^N = \\ &= \int_H \left(A \nabla u \cdot \nabla(w \circ \Phi^{-1}) + du \cdot (w \circ \Phi^{-1})\right) \mathrm{d}\lambda^N = \\ &= \int_H \left(f \cdot \nabla(w \circ \Phi^{-1}) + g \cdot (w \circ \Phi^{-1})\right) \mathrm{d}\lambda^N + \\ &\qquad + \int_{\partial_{\mathcal{N}} H} h \cdot (w \circ {\Phi_{\mathcal{N}}}^{-1})\,\mathrm{d}\lambda_{\partial H} = \\ &= \int_G |J\Phi| \left(D\Phi^{-1}(f \circ \Phi) \cdot \nabla w + (g \circ \Phi) \cdot w\right) \mathrm{d}\lambda^N + \\ &\qquad + \int_{\partial_{\mathcal{N}} G} |J\Phi_{\mathcal{N}}|\,(h \circ \Phi_{\mathcal{N}}) \cdot w \,\mathrm{d}\lambda_{\partial G}. \end{aligned} \right.$$

Now, Theorem 2.1 and Theorem 2.3(ii) yield that the maps

$$x \in G \mapsto |J\Phi(x)|\, D\Phi(x)^{-1}(f \circ \Phi)(x) \in \mathrm{I\!R}^N \quad \text{and} \quad x \in G \mapsto |J\Phi(x)|\,(g \circ \Phi)(x) \in \mathrm{I\!R}$$

belong to $\mathfrak{L}^{2,\omega}(G; \mathrm{I\!R}^N)$ and $\mathfrak{L}^{2,\omega-2}(G)$, respectively. Analogously, from the Lemma 4.2 and Lemma 4.3 it follows that the map

$$x \in \partial_{\mathcal{N}} G \mapsto |J\Phi_{\mathcal{N}}(x)|\,(h \circ \Phi_{\mathcal{N}})(x) \in \mathrm{I\!R}$$

belongs to $\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)$. Moreover, there is a constant $c > 0$ such that the norms of these maps can be estimated by

$$\begin{aligned} \big\| |J\Phi|\, D\Phi^{-1}\,(f \circ \Phi) \big\|_{\mathfrak{L}^{2,\omega}(G;\mathrm{I\!R}^N)} &\le c\, \|f\|_{\mathfrak{L}^{2,\omega}(H;\mathrm{I\!R}^N)}, \\ \big\| |J\Phi|\,(g \circ \Phi) \big\|_{\mathfrak{L}^{2,\omega-2}(G)} &\le c\, \|g\|_{\mathfrak{L}^{2,\omega-2}(H)}, \\ \big\| |J\Phi_{\mathcal{N}}|\,(h \circ \Phi_{\mathcal{N}}) \big\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)} &\le c\, \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} H)}. \end{aligned}$$

Hence, the assumption that G_0 is G-admissible, (5.12) and (5.13) imply, that there exists $N-2 < \overline{\omega} < N$ such that for all $\omega < \overline{\omega}$ we have $\nabla(u \circ \Phi) \in \mathfrak{L}^{2,\omega}(G_0; \mathrm{I\!R}^N)$ and

$$\text{(5.14)} \quad \left\{ \begin{aligned} &\|\nabla(u \circ \Phi)\|_{\mathfrak{L}^{2,\omega}(G_0;\mathrm{I\!R}^N)} \le \\ &\le c\left(\|f\|_{\mathfrak{L}^{2,\omega}(H;\mathrm{I\!R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(H)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} H)} + \|u\|_{W^{1,2}(H^\circ)}\right). \end{aligned} \right.$$

Applying again Theorem 2.1 and Theorem 2.3(ii), finally, there exists a constant $c > 0$ such that $\nabla u \in \mathfrak{L}^{2,\omega}(H_0;\mathbb{R}^N)$ and

$$
\text{(5.15)} \qquad \begin{cases} \|\nabla u\|_{\mathfrak{L}^{2,\omega}(H_0;\mathbb{R}^N)} \leq \\ \leq c\left(\|f\|_{\mathfrak{L}^{2,\omega}(H;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(H)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_N H)} + \|u\|_{W^{1,2}(H^\circ)}\right). \end{cases}
$$

Note, that the constants c in (5.14) and (5.15) does not depend on A, d, f, g, h and u. Hence, H_0 is H-admissible. □

Now we prove the main Theorem 5.3:

Let $x \in \partial G$. By Lemma 5.4 and Lemma 5.5, the proof is done, if we have found open neighborhoods U_0, U_1 and U_2 of x and a Lipschitz transformation Φ of U_2 into $\mathbb{R}^N$ such that

$$
\text{(5.16)} \qquad U_0 \subset U_1\,,\ \overline{U_1} \subset U_2,\ \text{and } \Phi(U_0 \cap G) \text{ is } \Phi(U_1 \cap G)\text{-admissible.}
$$

Let (Φ, U) be a chart of G in x and $\Phi(U \cap G) = E_k$ with $k = 1, 2$ or 3. If we take $0 < r < 1$, then it holds

$$
\begin{aligned} \Phi(\Phi^{-1}(B(0,r/4)) \cap G) &= E_k(0,r/4), \\ \Phi(\Phi^{-1}(B(0,r/2)) \cap G) &= E_k(0,r/2). \end{aligned}
$$

Hence, with the choice

$$
U_0 = \Phi^{-1}(B(0,r/4))\,,\ U_1 = \Phi^{-1}(B(0,r/2)) \text{ and } U_2 = \Phi^{-1}(B(0,r)),
$$

condition (5.16) is fulfilled if it is shown that for all $k \in \{1,2,3\}$ and $0 < r < 1$, the sets $E_k(0,r)$ are E_k-admissible.

In order to show this, we have to prove some lemmas, again, and to use the following notation: For $x_0 \in \mathbb{R}^N$, $r > 0$ and $\delta \in \mathbb{R}$ we denote by

$$
B(x_0, r, \delta) := \{x \in B(x_0, r) : x_N - x_{0N} = \delta r\}
$$

the intersection of the hyperplane $\{x \in \mathbb{R}^N : x_N - x_{0N} = \delta r\}$ with the open ball $B(x_0, r)$.

Lemma 5.6 *For all $0 < \varepsilon \leq 1$ there exists an $\overline{\omega} > N - 2$ such that for all $\omega < \overline{\omega}$, $\delta \in \mathbb{R}$, $0 < \varrho \leq r < 1$ and $x_0 \in \mathbb{R}^N$ the following holds:*

Let $A \in L^\infty(B(x_0,r);\mathfrak{S}_N)$, $f \in L^2(B(x_0,r);\mathbb{R}^N)$, $g \in L^2(B(x_0,r))$, $h \in L^2(B(x_0,r,\delta))$ and $u \in W^{1,2}(B(x_0,r))$ satisfy

$$
\text{(5.17)} \qquad \int\limits_{B(x_0,r)} A\nabla u \cdot \nabla v \,\mathrm{d}\lambda^N = \int\limits_{B(x_0,r)} (f \cdot \nabla v + gv)\,\mathrm{d}\lambda^N + \int\limits_{B(x_0,r,\delta)} hv \,\mathrm{d}\lambda^{N-1}
$$

for all $v \in W_0^{1,2}(B(x_0,r))$, and let $A(x)\xi \cdot \xi \geq \varepsilon |\xi|^2$ for all $\xi \in \mathbb{R}^N$ and λ^N-almost all $x \in B(x_0,r)$. Then

$$
\text{(5.18)} \qquad \begin{cases} \|\nabla u\|^2_{L^2(B(x_0,\varrho);\mathbb{R}^N)} \leq c\Big\{\left(\dfrac{\varrho}{r}\right)^\omega \|\nabla u\|^2_{L^2(B(x_0,r);\mathbb{R}^N)} + \\ + \|f\|^2_{L^2(B(x_0,r);\mathbb{R}^N)} + r^2\|g\|^2_{L^2(B(x_0,r))} + r\|h\|^2_{L^2(B(x_0,r,\delta))}\Big\}, \end{cases}
$$

where the constant c in (5.18) depends only on N, ε and ω.

Proof Considering the transformation $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ defined as $\Phi(y) = x_0 + ry$, we get for all $v \in W_0^{1,2}(B(x_0, r))$

$$\int\limits_{B(x_0,r)} |v|^2 \,\mathrm{d}\lambda^N = r^N \int\limits_{B(0,1)} |v \circ \Phi|^2 \,\mathrm{d}\lambda^N,$$

$$\int\limits_{B(x_0,r,\delta)} |v|^2 \,\mathrm{d}\lambda^{N-1} = r^{N-1} \int\limits_{B(0,1,\delta)} |v \circ \Phi|^2 \,\mathrm{d}\lambda^{N-1}.$$

Due to Poincaré's inequality and to the classical trace theorem, there exists a constant c, depending on N only, such that for all such v we have

$$\int\limits_{B(0,1)} |v \circ \Phi|^2 \,\mathrm{d}\lambda^N \leq c \int\limits_{B(0,1)} |\nabla(v \circ \Phi)|^2 \,\mathrm{d}\lambda^N,$$

$$\int\limits_{B(0,1,\delta)} |v \circ \Phi|^2 \,\mathrm{d}\lambda^{N-1} \leq c \int\limits_{B(0,1)} |\nabla(v \circ \Phi)|^2 \,\mathrm{d}\lambda^N.$$

Applying the chain rule and the transformation formulas again it follows that

$$\text{(5.19)} \qquad \begin{cases} \displaystyle\int\limits_{B(x_0,r)} |v|^2 \,\mathrm{d}\lambda^N \leq c\, r^2 \int\limits_{B(x_0,r)} |\nabla v|^2 \,\mathrm{d}\lambda^N, \\ \displaystyle\int\limits_{B(x_0,r,\delta)} |v|^2 \,\mathrm{d}\lambda^{N-1} \leq c\, r \int\limits_{B(x_0,r)} |\nabla v|^2 \,\mathrm{d}\lambda^N. \end{cases}$$

Now, we consider the uniquely determined function $u_0 \in W_0^{1,2}(B(x_0, r))$ satisfying the variational equation (5.17). Taking $v = u_0$ as a test function in (5.17), we get

$$\varepsilon \,\|\nabla u_0\|^2_{L^2(B(x_0,r))} \leq \|f\|_{L^2(B(x_0,r);\mathbb{R}^N)} \|\nabla u_0\|_{L^2(B(x_0,r);\mathbb{R}^N)} + \\ + \|g\|_{L^2(B(x_0,r))} \|u_0\|_{L^2(B(x_0,r))} + \|h\|_{L^2(B(x_0,r,\delta))} \|u_0\|_{L^2(B(x_0,r,\delta))}.$$

Combining this with (5.19), we state

$$\text{(5.20)} \qquad \begin{cases} \|\nabla u_0\|^2_{L^2(B(x_0,r);\mathbb{R}^N)} \leq \\ \leq c\,(\|f\|^2_{L^2(B(x_0,r);\mathbb{R}^N)} + r^2 \|g\|^2_{L^2(B(x_0,r))} + r\|h\|^2_{L^2(B(x_0,r,\delta))}) \end{cases}$$

with a new constant c, depending on N and ε, only. Now, the function $w = u - u_0 \in W^{1,2}(B(x_0, r))$ satisfies

$$\int\limits_{B(x_0,r)} A\nabla w \cdot \nabla v \,\mathrm{d}\lambda^N = 0 \text{ for all } v \in W_0^{1,2}(B(x_0, r)).$$

But Lemma 5.6 is true in the case of vanishing f, g and h, see Troianiello [30, Lemma 2.15]. Hence, for $0 < \varrho \leq r < 1$ it holds

$$\|\nabla w\|^2_{L^2(B(x_0,\varrho);\mathbb{R}^N)} \leq c \left(\frac{\varrho}{r}\right)^{\omega} \|\nabla w\|^2_{L^2(B(x_0,r);\mathbb{R}^N)}$$

with a constant c, which depends only on N, ε and ω. Using $u = w + u_0$ and (5.20), we get (5.18). $\square$

For $0 < \varepsilon \leq 1$ let us denote by $\overline{\omega}(\varepsilon)$ the supremum of all $N-2 < \overline{\omega} < N$ such that the assertion of Lemma 5.6 is true. Obviously, $\overline{\omega}(\varepsilon)$ depends only on ε and the space dimension N, and the map $\varepsilon \mapsto \overline{\omega}(\varepsilon)$ is non-decreasing.

Lemma 5.7 *Let* $0 < \varepsilon \leq 1$, $\omega < \overline{\omega}(\varepsilon)$, $0 < R < 1$, $(A, d) \in \mathfrak{A}(\varepsilon, B(0,1))$, $f \in \mathfrak{L}^{2,\omega}(B(0,1); \mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(B(0,1))$, $h \in \mathfrak{L}^{2,\omega-1}(B(0,1,0))$ *and* $u \in W^{1,2}(B(0,1))$ *satisfy*

$$\int_{B(0,1)} (A\nabla u \cdot \nabla v + duv)\, \mathrm{d}\lambda^N = \int_{B(0,1)} (f \cdot \nabla v + gv)\, \mathrm{d}\lambda^N + \int_{B(0,1,0)} hv\, \mathrm{d}\lambda^{N-1} \tag{5.21}$$

for all $v \in W_0^{1,2}(B(0,1))$. *Then it holds that* $\nabla u|_{B(0,R)} \in \mathfrak{L}^{2,\omega}(B(0,R); \mathbb{R}^N)$ *and*

$$\left\{ \begin{array}{l} \|\nabla u\|^2_{\mathfrak{L}^{2,\omega}(B(0,R);\mathbb{R}^N)} \leq c \Big\{ \|u\|_{W^{1,2}(B(0,1))} + \\ + \|f\|_{\mathfrak{L}^{2,\omega}(B(0,1);\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(B(0,1))} + \|h\|_{\mathfrak{L}^{2,\omega-1}(B(0,1,0))} \Big\}, \end{array} \right. \tag{5.22}$$

where the constant c *in* (5.22) *depends only on* N, ε, ω *and* R.

Proof Let $\{r_k\}_{k \in \mathbb{N}}$ be the following decreasing sequence

$$r_k := R + 2^{-k}(1-R).$$

Because of (5.21), for all $0 < r \leq 4^{-N} \min\{R, 1-R\}$, $x_0 \in \overline{B(0, r_1)}$ and $v \in W_0^{1,2}(B(x_0, r))$ we have

$$\int_{B(x_0,r)} A\nabla u \cdot \nabla v\, \mathrm{d}\lambda^N = \int_{B(x_0,r)} (f \cdot \nabla v + (g - du)v)\, \mathrm{d}\lambda^N + \int_{B(x_0,r,-x_{0N}/r)} hv\, \mathrm{d}\lambda^{N-1}.$$

Hence, Lemma 5.6 yields that for $\mu < \overline{\omega} < \overline{\omega}(\varepsilon)$ and $0 < \varrho \leq r \leq 4^{-N} \min\{R, 1-R\}$ we have (5.18) with $\delta = -x_{0N}/r$ and therefore

$$\begin{aligned} &\|\nabla u\|^2_{L^2(B(x_0,\varrho);\mathbb{R}^N)} \leq c \Big\{ \Big(\frac{\varrho}{r}\Big)^{\overline{\omega}} \|\nabla u\|^2_{L^2(B(x_0,r);\mathbb{R}^N)} + \|u\|^2_{L^2(B(x_0,r))} + \\ &+ r^{\mu} \Big(\|f\|^2_{\mathfrak{L}^{2,\mu}(B(0,1);\mathbb{R}^N)} + \|g\|^2_{\mathfrak{L}^{2,\mu-2}(B(0,1))} + \|h\|^2_{\mathfrak{L}^{2,\mu-1}(B(0,1,0))} \Big) \Big\}. \end{aligned}$$

Set for $\mu \leq \omega$

$$\begin{aligned} &\kappa_\mu(f, g, h, u) := \|u\|^2_{W^{1,2}(B(0,1))} + \\ &+ \|f\|^2_{\mathfrak{L}^{2,\mu}(B(0,1),\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\mu-2}(B(0,1))} + \|h\|_{\mathfrak{L}^{2,\mu-1}(B(0,1,0))}, \end{aligned}$$

and let $\omega < \overline{\omega}$. Because of the continuous embedding $W^{1,2}(B(0,1)) \hookrightarrow \mathfrak{L}^{2,2}(B(0,1))$ we can find a constant c depending only on μ, ε, R and N such that

$$\|u\|^2_{\mathfrak{L}^{2,\mu}(B(0,1))} \leq c\, \kappa_\mu(f, g, h, u) \quad \text{for } \mu = \min\{\omega, 2\}.$$

This yields for such μ

$$\|\nabla u\|^2_{L^2(B(x_0,\varrho);\mathrm{I\!R}^N)} \le c\Big\{ \Big(\frac{\varrho}{r}\Big)^{\overline{\omega}} \|\nabla u\|^2_{L^2(B(x_0,r);\mathrm{I\!R}^N)} + r^\mu \kappa_\mu(f,g,h,u)\Big\}.$$

Now we apply a fundamental lemma of Campanato (cf. [7, Lemma 1.1], [13, Section 3.2]) and obtain for $0 < \varrho \le r \le 4^{-N}\min\{R, 1-R\}$ and $\mu = \min\{\omega, 2\}$

$$\|\nabla u\|^2_{L^2(B(x_0,\varrho);\mathrm{I\!R}^N)} \le c\Big\{ \Big(\frac{\varrho}{r}\Big)^{\mu} \|\nabla u\|^2_{L^2(B(x_0,r);\mathrm{I\!R}^N)} + \varrho^\mu \kappa_\mu(f,g,h,u)\Big\},$$

where the constant c depends only on μ, ε, R and N. Dividing by ϱ^μ, and specifying $r = 4^{-N}\min\{R, 1-R\}$, we get $\nabla u|_{B(0,r_1)} \in \mathfrak{L}^{2,\mu}(B(0,r_1);\mathrm{I\!R}^N)$ and

$$\|\nabla u\|^2_{\mathfrak{L}^{2,\mu}(B(0,r_1);\mathrm{I\!R}^N)} \le c\,\kappa_\mu(f,g,h,u) \quad \text{for } \mu = \min\{\omega, 2\}.$$

Hence, again Theorem 2.3(iv) yields $u|_{B(0,r_1)} \in \mathfrak{L}^{2,\mu}(B(0,r_1);\mathrm{I\!R}^N)$ and

$$\|u\|^2_{\mathfrak{L}^{2,\mu}(B(0,r_1))} \le c\,\kappa_\mu(f,g,h,u) \quad \text{for } \mu = \min\{\omega, 4\}.$$

Repeating the same arguments as above we can prove

$$u|_{B(0,r_2)} \in \mathfrak{L}^{2,\mu}(B(0,r_2)) \quad \text{for } \mu = \min\{\omega, 6\}$$

with a corresponding norm estimate. After less than N steps of this iteration we arrive at $\mu = \omega$ and the claim of the lemma, because it holds $R < r_k \le 1$ and $4^{-N}\min\{R, 1-R\} < r_k - r_{k-1}$ for all $k = 0, 1, \ldots, N$. □

Lemma 5.8 *Let $0 < r < 1$. Then $E_1(0,r)$ is E_1-admissible, and $E_2(0,r)$ is E_2-admissible.*

Proof Remembering the notation $E_0 := B(0,1)$, for $k \in \{1,2\}$ and $u \in L^2(E_1)$ we define $S_k u \in L^2(E_0)$ by

$$(S_k u)(x) := \begin{cases} u(x) & \text{for } x \in E_k, \\ (-1)^k u(\hat{x}, -x_N) & \text{for } x = (\hat{x}, x_N) \in E_0 \setminus E_k. \end{cases}$$

Thus, $S_1 u$ and $S_2 u$ are the extensions of u to E_0 by anti-reflection and by reflection, respectively. It is well-known that $u \in W_0^{1,2}(E_k)$ if and only if $S_k u \in W_0^{1,2}(E_0)$ and in this case

$$\|S_k u\|_{W^{1,2}(E_0)} = \sqrt{2}\,\|u\|_{W^{1,2}(E_1)}.$$

Moreover, for $0 \le \omega < N$ we have $u \in \mathfrak{L}^{2,\omega}(E_1)$ if and only if $S_k u \in \mathfrak{L}^{2,\omega}(E_0)$ and, in this case it holds

$$\|S_k u\|_{\mathfrak{L}^{2,\omega}(E_0)} \le \sqrt{2}\,\|u\|_{\mathfrak{L}^{2,\omega}(E_1)} \le \sqrt{2}\,\|S_k u\|_{\mathfrak{L}^{2,\omega}(E_0)},$$

cf. Troianiello [30, Lemma 1.16 and Remark after the proof of Theorem 1.17].

Now, we extend elements $f \in L^2(E_1;\mathrm{I\!R}^N)$ to $R_k f \in L^2(E_0;\mathrm{I\!R}^N)$ and elements $(A,d) \in \mathfrak{A}(\varepsilon, E_1)$ to $(R_k A, R_k d) \in \mathfrak{A}(\varepsilon, E_0)$ by

$$\begin{cases} (R_k f)_j(\hat{x}, x_N) := (-1)^k f_j(\hat{x}, -x_N) & \text{for } j < N, \\ (R_k f)_N(\hat{x}, x_N) := (-1)^{k+1} f_N(\hat{x}, -x_N), & \end{cases}$$

$$\begin{cases} (R_kA)_{ij}(\hat{x},x_N) & := & A_{ij}(\hat{x},-x_N) & \text{for } i,j<N \text{ or } i=j=N, \\ (R_kA)_{ij}(\hat{x},x_N) & := & -A_{ij}(\hat{x},-x_N) & \text{otherwise,} \end{cases}$$

$$(R_kd)(\hat{x},x_N) := d(\hat{x},-x_N)$$

for $(\hat{x},x_N) \in E_0 \backslash E_k$. Then, we get $S_k(du) = (R_kd)(S_ku)$, $R_k(Af) = (R_kA)(R_kf)$ and $R_k(\nabla u) = \nabla(S_ku)$ for all $(A,d) \in \mathfrak{A}(\varepsilon, E_1)$, $f \in L^2(E_1;\mathrm{I\!R}^N)$ and $u \in W^{1,2}(E_1)$.

Finally, for $k \in \{1,2\}$, $v \in W_0^{1,2}(E_0)$ and $h \in L^2(B(0,1,0))$ we define $T_kv \in W_0^{1,2}(E_k)$ by

$$(T_kv)(\hat{x},x_N) := v(\hat{x},x_N) + (-1)^k v(\hat{x},-x_N) \quad \text{for } (\hat{x},x_N) \in E_k$$

and $T_1h = 0$, $T_2h = 2h$, respectively. The functions T_1v and T_2v are the restrictions of the antisymmetric and symmetric part of $2v$ to E_k, respectively, and we have

$$\int\limits_{E_0} ((R_kf)\cdot\nabla v + (S_kg)v)\,\mathrm{d}\lambda^N + \int\limits_{E_k\backslash E_1} (T_kh)v\,\mathrm{d}\lambda^{N-1} =$$

$$= \int\limits_{E_1} (f\cdot\nabla(T_kv) + g(T_kv))\,\mathrm{d}\lambda^N + \int\limits_{E_k\backslash E_1} h(T_kv)\,\mathrm{d}\lambda^{N-1}$$

for all $f \in L^2(E_1;\mathrm{I\!R}^N)$, $g \in L^2(E_1)$, $h \in L^2(B(0,1,0))$ and $v \in W_0^{1,2}(E_0)$.

Now, consider $(A,d) \in \mathfrak{A}(\varepsilon,E_1)$, $f \in \mathfrak{L}^{2,\omega}(E_1;\mathrm{I\!R}^N)$ and $g \in \mathfrak{L}^{2,\omega-2}(E_1)$, $h \in \mathfrak{L}^{2,\omega-1}(B(0,1,0))$ and take $u \in W_0^{1,2}(E_k)$ such that for all $v \in W_0^{1,2}(E_k)$ we have

$$\int\limits_{E_1} (A\nabla u\cdot\nabla v + duv)\,\mathrm{d}\lambda^N = \int\limits_{E_1} (f\cdot\nabla v + gv)\,\mathrm{d}\lambda^N + \int\limits_{E_k\backslash E_1} hv\,\mathrm{d}\lambda^{N-1}$$

Then, for all $w \in W_0^{1,2}(E_0)$ it follows that

$$\int\limits_{E_0} ((R_kA)\nabla(S_ku)\cdot\nabla w + (R_kd)(S_ku)w)\,\mathrm{d}\lambda^N =$$

$$= \int\limits_{E_1} (A\nabla u\cdot\nabla(T_kw) + du(T_kw))\,\mathrm{d}\lambda^N =$$

$$= \int\limits_{E_1} (f\cdot\nabla(T_kw) + g(T_kw))\,\mathrm{d}\lambda^N + \int\limits_{E_k\backslash E_1} h(T_kw)\,\mathrm{d}\lambda^{N-1} =$$

$$= \int\limits_{E_0} ((R_kf)\cdot\nabla w + (S_kg)w)\,\mathrm{d}\lambda^N + \int\limits_{E_k\backslash E_1} (T_kh)w\,\mathrm{d}\lambda^{N-1}.$$

Hence, Lemma 5.7 yields the following: For $0 < r < 1$ and $0 \leq \omega < \overline{\omega}(\varepsilon)$ we have

$$\nabla(S_ku)|_{B(0,r)} = R_k(\nabla u)|_{B(0,r)} \in \mathfrak{L}^{2,\omega}(B(0,r);\mathrm{I\!R}^N)$$

and

$$\begin{aligned} &\|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_1(0,r);\mathrm{I\!R}^N)} \leq \|\nabla(S_ku)\|_{\mathfrak{L}^{2,\omega}(B(0,r);\mathrm{I\!R}^N)} \leq \\ &\leq c_1(\|R_kf\|_{\mathfrak{L}^{2,\omega}(E_0;\mathrm{I\!R}^N)} + \|S_kg\|_{\mathfrak{L}^{2,\omega-2}(E_0)} + \|T_kh\|_{\mathfrak{L}^{2,\omega-1}(B(0,1,0))} + \|S_ku\|_{W^{1,2}(E_0)}). \end{aligned}$$

Therefore,

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(E_1(0,r);\mathbb{R}^N)} \leq$$
$$\leq c_2(\|f\|_{\mathfrak{L}^{2,\omega}(E_1;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(E_1)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(B(0,1,0))} + \|u\|_{W^{1,2}(E_1)}),$$

where the constants c_1 and c_2 do not depend on A, d, f, g, h and u. □

Remark 5.9 From Lemma 5.4 to Lemma 5.8 it follows that Theorem 5.3 is proved if G is open (i.e. $\partial G = \partial_{\mathcal{D}} G$) or if G is closed (i.e. $\partial G = \partial_{\mathcal{N}} G$), because in that cases there exists an atlas of ∂G, consisting of charts of type 1 and 2, only.

In order to finish the proof of Theorem 5.3, it suffices to prove the following

Lemma 5.10 E_3 *is admissible.*

Proof Again, denote $E_0 := B(0,1)$, and let $\overline{E}_0$ be its closure. Further, denote $E_5 := \{x \in \overline{E}_0 : x_N < 0\}$.

There exists a Lipschitz transformation of $\mathbb{R}^N$ onto $\mathbb{R}^N$ which maps E_3 onto E_5. Hence, because of Lemma 5.5, it is sufficient to show that E_5 is admissible.

In order to do this, we define suitable extensions to $\overline{E_0}$ of $(A,d) \in \mathfrak{A}(\varepsilon, E_5)$, $f \in \mathfrak{L}^{2,\omega}(E_5;\mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(E_5)$ and $u \in W_0^{1,2}(E_5)$ as in the proof of Lemma 5.8. Further, we extend $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} E_5)$ to $\partial_{\mathcal{N}} \overline{E_0}$ by $S_5 h(x) := -h(\hat{x}, -x_N)$ for almost all $x = (\hat{x}, -x_N) \in \partial_{\mathcal{N}} E_5$. After that we proceed as in the proof of Lemma 5.8, using the fact that $\overline{E}_0$ is admissible (according to Remark 5.9 above). □

6 Solution Regularity and Fredholm Property

In this section G is a fixed regular subset of $\mathbb{R}^N$. Again, by $\langle\cdot,\cdot\rangle_G$ and J_G we denote the dual pairing and the duality map of $W_0^{1,2}(G)$, respectively (cf. (3.15)). By definition (cf. (3.16)), J_G is an isometric isomorphism from $W_0^{1,2,\omega}(G)$ onto $W^{-1,2,\omega}(G)$.

For $0 < \varepsilon \leq 1$ let $\overline{\omega}(\varepsilon, G)$ be the supremum of all $0 < \overline{\omega} < N$ such that for all $\omega < \overline{\omega}$ the following is true: For each $(A,d) \in \mathfrak{A}(\varepsilon, G)$, $f \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(G)$, $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)$ and $u \in W_0^{1,2}(G)$ with

$$\int_G (A\nabla u \cdot \nabla v + duv)\, \mathrm{d}\lambda^N = \int_G (f \cdot \nabla v + gv)\, \mathrm{d}\lambda^N + \int_{\partial_{\mathcal{N}} G} hv \,\mathrm{d}\lambda_{\partial G}$$

for all $v \in W_0^{1,2}(G)$, it holds $\nabla u \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$ and

$$\|\nabla u\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} \leq c\left(\|f\|_{\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)} + \|g\|_{\mathfrak{L}^{2,\omega-2}(G)} + \|h\|_{\mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)}\right),$$

where the constant c depends only on G, N, ε and ω. Because of Theorem 5.3 we have

$$N - 2 < \overline{\omega}(\varepsilon, G) \leq \overline{\omega}(1, G).$$

Let $F : L^2(G;\mathbb{R}^N) \times L^2(G) \times L^2(\partial_{\mathcal{N}} G) \mapsto W^{-1,2}(G)$ be the linear bounded operator which is defined by

$$\langle F(f,g,h), v\rangle_G := \int\limits_G (f \cdot \nabla v + gv)\, \mathrm{d}\lambda^N + \int\limits_{\partial_{\mathcal{N}} G} hv\, \mathrm{d}\lambda_{\partial G} \text{ for all } v \in W_0^{1,2}(G).$$

Remark 6.1 Let $\omega < \overline{\omega}(1,G)$. Then Theorem 5.3 yields that F is a bounded operator from $\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N) \times \mathfrak{L}^{2,\omega-2}(G) \times \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)$ onto $W^{-1,2,\omega}(G)$. In other words: A functional $\phi \in W^{-1,2}(G)$ belongs to $W^{-1,2,\omega}(G)$ if and only if there exist $f \in \mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(G)$ and $h \in \mathfrak{L}^{2,\omega-1}(\partial_{\mathcal{N}} G)$ such that $\phi = F(f,g,h)$.

In Griepentrog [17] will be derived other, more direct criteria for a functional $\phi \in W^{-1,2}(G)$ to belong to $W^{-1,2,\omega}(G)$ and corresponding expressions for a norm in $W^{-1,2,\omega}(G)$ which is equivalent to (3.16). In Rakotoson [24, 25] analogous characterizations are given for elements of $W^{-1,2,\omega}_{\mathrm{loc}}(G)$.

For $A \in L^\infty(G;\mathfrak{S}_N)$, $b, c \in L^\infty(G;\mathbb{R}^N)$, $d \in L^\infty(G)$ and $e \in L^\infty(\partial_{\mathcal{N}} G)$ we denote by $L(A,b,c,d,e)$ the linear bounded operator from $W_0^{1,2}(G)$ into $W^{-1,2}(G)$ which is defined by

$$\langle L(A,b,c,d,e)u, v\rangle_G := \int\limits_G ((A\nabla u + bu)\cdot \nabla v + (c \cdot \nabla u + du)v)\, \mathrm{d}\lambda^N + \int\limits_{\partial_{\mathcal{N}} G} euv\, \mathrm{d}\lambda_{\partial G}$$

for $u, v \in W_0^{1,2}(G)$. Furthermore, let $L_\omega(A,b,c,d,e)$ be the restriction of $L(A,b,c,d,e)$ to $W_0^{1,2,\omega}(G)$.

Lemma 6.2 *Let $\omega < \overline{\omega}(1,G)$. Then $L_\omega(A,b,c,d,e)$ maps $W_0^{1,2,\omega}(G)$ continuously into $W^{-1,2,\omega}(G)$. Moreover, it depends continuously in the sense of the operator norm in $\mathcal{L}(W_0^{1,2,\omega}(G); W^{-1,2,\omega}(G))$ on A, b, c, d and e.*

Proof Let $A \in L^\infty(G;\mathfrak{S}_N)$, $b, c \in L^\infty(G;\mathbb{R}^N)$, $d \in L^\infty(G)$, $e \in L^\infty(\partial_{\mathcal{N}} G)$ and $u \in W_0^{1,2,\omega}(G)$. Because of the Theorem 2.3(ii) $A\nabla u + bu$ belongs to $\mathfrak{L}^{2,\omega}(G;\mathbb{R}^N)$ and depends bilinearly and continuously on (A,b) and u. Analogously, $c \cdot \nabla u + du$ belongs to $\mathfrak{L}^{2,\omega-2}(G)$ and depends bilinearly and continuously on (c,d) and u. Finally, because of Theorem 4.4 and of Lemma 4.2, the product eu belongs to $\mathfrak{L}^{2,\omega+1}(\partial_{\mathcal{N}} G)$ and depends bilinearly and continuously on e and u. Hence,

$$L(A,b,c,d,e)u = F(A\nabla u + bu, c \cdot \nabla u + du, eu)$$

belongs to $W^{-1,2,\omega}(G)$ and depends bilinearly and continuously on (A,b,c,d,e) and u, cf. Remark 6.1. $\square$

The main result of this section is the following

Theorem 6.3 *Let $A \in L^\infty(G;\mathfrak{S}_N)$ and $0 < \varepsilon \le 1$ such that for λ^N-almost all $x \in G$ it holds*

$$\varepsilon\,|\xi|^2 \le A(x)\xi \cdot \xi \le \frac{1}{\varepsilon}\,|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^N,$$

and let $\omega < \overline{\omega}(\varepsilon, G)$. Then the following is true:

(i) *If $L(A,b,c,d,e)u \in W^{-1,2,\omega}(G)$ for a certain $u \in W_0^{1,2}(G)$, then $u \in W_0^{1,2,\omega}(G)$.*

(ii) *$L_\omega(A,b,c,d,e)$ is a Fredholm operator (index zero) from $W_0^{1,2,\omega}(G)$ into $W^{-1,2,\omega}(G)$.*

Proof (i) We have $L(A,b,c,d,e)u = J_G w$ for a certain $w \in W_0^{1,2,\omega}(G)$. Therefore it holds

$$L(A,0,0,1,0)u = F(\nabla w - bu, w - c\cdot\nabla u - (d-1)u, -eu). \tag{6.1}$$

Now we proceed as in the proof of Lemma 5.4. We use the multiplier properties Theorem 2.3(ii) and Theorem 4.2, the continuous embedding $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,2}(G)$ and the trace property $W_0^{1,2}(G) \hookrightarrow \mathfrak{L}^{2,1}(\partial_{\mathcal{N}}G)$, and get for $\mu = \min\{\omega, 2\}$

$$\left\{\begin{array}{rcl} \nabla w - bu & \in & \mathfrak{L}^{2,\mu}(G;\mathbb{R}^N), \\ w - c\cdot\nabla u - (d-1)u & \in & \mathfrak{L}^{2,\mu-2}(G), \\ -eu & \in & \mathfrak{L}^{2,\mu-1}(\partial_{\mathcal{N}}G). \end{array}\right. \tag{6.2}$$

If $\mu < \overline{\omega}(\varepsilon,G)$, then Theorem 5.3 yields that $u \in W_0^{1,2,\mu}(G)$. Hence, we get (6.2) with $\mu = \min\{\omega,4\}$, and so on.

(ii) First we show that that $L(A,b,c,d,e)$ is a Fredholm operator from $W_0^{1,2}(G)$ into $W^{-1,2}(G)$. We have

$$L(A,b,c,d,e) = L(A,0,0,1,0) + L(0,b,c,d-1,e).$$

Because of the Lax-Milgram Lemma, $L(A,0,0,1,0)$ is an isomorphism from $W_0^{1,2}(G)$ onto $W^{-1,2}(G)$. But $L(0,b,c,d-1,e)$ is completely continuous from $W_0^{1,2}(G)$ into $W^{-1,2}(G)$ because of Lemma 3.9 and Theorem 4.4, so the claim is proved.

Now let us prove assertion (ii) of the theorem. We have, because of the claim above,

$$W^{-1,2}(G) = \operatorname{im} L(A,b,c,d,e) \oplus \ker(L(A,c,b,d,e)\circ J_G^{-1}) \tag{6.3}$$

and $\dim\ker L(A,b,c,d,e) = \dim\ker L(A,c,b,d,e) < \infty$. Here we used that the operator $L(A,c,b,d,e)$ is the adjoint to the operator $L(A,b,c,d,e)$. Assertion (i) implies

$$\begin{array}{rcl} \ker L(A,b,c,d,e) & = & \ker L_\omega(A,b,c,d,e), \\ \ker L(A,c,b,d,e) & = & \ker L_\omega(A,c,b,d,e), \end{array}$$

and, hence, $\dim\ker L_\omega(A,b,c,d,e) = \dim\ker L_\omega(A,c,b,d,e) < \infty$. Further, from assertion (i) follows

$$W^{-1,2,\omega}(G) \cap \operatorname{im} L(A,b,c,d,e) = \operatorname{im} L_\omega(A,b,c,d,e).$$

Therefore, (6.3) yields

$$W^{-1,2,\omega}(G) = \operatorname{im} L_\omega(A,b,c,d,e) \oplus \ker(L_\omega(A,c,b,d,e)\circ J_G^{-1}).$$

It remains to show that $\operatorname{im} L_\omega(A,b,c,d,e)$ is closed in $W^{-1,2,\omega}(G)$. Thus, let $L_\omega(A,b,c,d,e)u_j \to \phi$ in $W^{-1,2,\omega}(G)$ for $j \to \infty$ (with $u_j \in W_0^{1,2,\omega}(G)$). Then there exists an $u \in W_0^{1,2}(G)$ such that $L(A,b,c,d,e)u = \phi$, because $\operatorname{im} L(A,b,c,d,e)$ is closed in $W^{-1,2}(G)$. Now assertion (i) works again. We get $u \in W_0^{1,2,\omega}(G)$ and, hence, $\phi \in \operatorname{im} L_\omega(A,b,c,d,e)$. $\square$

Corollary 6.4 *Let $\mathfrak{I}(\varepsilon,G)$ be the set of all (A,b,c,d,e) such that $L(A,b,c,d,e)$ is injective and that it holds*

$$\varepsilon\,|\xi|^2 < \operatorname*{ess\,inf}_{x\in G}\,(A(x)\xi\cdot\xi) \quad\text{and}\quad \operatorname*{ess\,sup}_{x\in G}\,(A(x)\xi\cdot\xi) < \frac{1}{\varepsilon}\,|\xi|^2 \tag{6.4}$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$*. Then the following is true:*

(i) *The set* $\mathfrak{I}(\varepsilon, G)$ *is open in* $L^\infty(G; \mathfrak{S}_N) \times L^\infty(G; \mathbb{R}^N)^2 \times L^\infty(G) \times L^\infty(\partial_\mathcal{N} G)$*.*

(ii) *Let* $\omega < \overline{\omega}(\varepsilon, G)$*. Then* $L(A,b,c,d,e)^{-1}F(f,g,h)$ *depends, in the norm of the space* $W_0^{1,2,\omega}(G)$*, analytically on* $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, G)$*,* $f \in \mathfrak{L}^{2,\omega}(G; \mathbb{R}^N)$*,* $g \in \mathfrak{L}^{2,\omega-2}(G)$ *and* $h \in \mathfrak{L}^{2,\omega-1}(\partial_\mathcal{N} G)$*.*

Proof (i) Let $L(A,b,c,d,e)$ be injective. Then, because of its Fredholm property, it is an isomorphism from $W_0^{1,2}(G)$ onto $W^{-1,2}(G)$. But the set of isomorphisms is open in $\mathcal{L}(W_0^{1,2}(G); W^{-1,2}(G))$. Hence, if $(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e})$ is close to (A,b,c,d,e) in $L^\infty(G; \mathfrak{S}_N) \times L^\infty(G; \mathbb{R}^N)^2 \times L^\infty(G) \times L^\infty(\partial_\mathcal{N} G)$, then $L(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e})$ is injective, too. Moreover, (6.4) is an open condition in $L^\infty(G; \mathfrak{S}_N)$.

(ii) $L(A,b,c,d,e)$ and $F(f,g,h)$ depend linearly and continuously (in the norms of $\mathcal{L}(W_0^{1,2,\omega}(G), W^{-1,2,\omega}(G))$ and $W^{-1,2,\omega}(G)$, respectively) and, hence, analytically on (A,b,c,d,e) and (f,g,h), respectively. Therefore, $L(A,b,c,d,e)^{-1}F(f,g,h)$ depends analytically (in the norm of $W_0^{1,2,\omega}(G)$) on (A,b,c,d,e,f,g,h), too. Remark that the assumptions $\omega < \overline{\omega}(\varepsilon, G)$ and $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, G)$ imply that $L(A,b,c,d,e)$ is an isomorphism from $W_0^{1,2,\omega}(G)$ onto $W^{-1,2,\omega}(G)$ (because of Theorem 6.3 (ii)). $\square$

Remark 6.5 Consider the boundary value problem (1.1) with $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, \Omega \cup \Gamma)$, and suppose $\Omega \cup \Gamma$ to be regular. Then Theorem 2.3 (iii) and (iv) and Corollary 6.4 imply that, for all $N-2 < \omega < \overline{\omega}(\varepsilon, \Omega \cup \Gamma)$, the weak solution to (1.1) depends, in the norm of the Hölder space $C^{0,\alpha}(\overline{\Omega})$ with

$$\alpha = 1 - \frac{N-\omega}{2},$$

analytically on $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, \Omega \cup \Gamma)$, $f \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$, $g \in \mathfrak{L}^{2,\omega-2}(\Omega)$ and $h \in \mathfrak{L}^{2,\omega-1}(\Gamma)$.

Now, consider (1.1) with right hand sides $f \in L^p(\Omega; \mathbb{R}^N)$ with $p > N$ and $g = 0$ and $h = 0$. Then, because of the continuous embeddings $L^p(\Omega; \mathbb{R}^N) \hookrightarrow \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N)$ (cf. Theorem 2.1(i)) and $W_0^{1,2,\omega}(\Omega \cup \Gamma) \hookrightarrow C^{0,1-(N-\omega)/2}(\overline{\Omega})$ with

$$\omega = \min\left\{\overline{\omega}(\varepsilon, \Omega \cup \Gamma), N - \frac{2N}{p}\right\},$$

the weak solution to (1.1) depends, in the norm of $C^{0,\alpha}(\overline{\Omega})$, analytically on $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, \Omega \cup \Gamma)$ and $f \in L^p(\Omega; \mathbb{R}^N)$.

Analogously, suppose $f = 0$ and $g = 0$ and $h \in L^{p-1}(\Gamma)$ with $p > N$ and $p \geq 3$. Then, because of the continuous embeddings $L^{p-1}(\Gamma) \hookrightarrow \mathfrak{L}^{2,\omega-1}(\Gamma)$ and, furthermore, $W_0^{1,2,\omega}(\Omega \cup \Gamma) \hookrightarrow C^{0,1-(N-\omega)/2}(\overline{\Omega})$ with

$$\omega = \min\left\{\overline{\omega}(\varepsilon, \Omega \cup \Gamma), N - \frac{2(N-1)}{p-1}\right\},$$

the weak solution to (1.1) depends, in the norm of $C^{0,\alpha}(\overline{\Omega})$, analytically on $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, \Omega \cup \Gamma)$ and $h \in L^{p-1}(\Gamma)$.

Finally, suppose $f = 0$ and $h = 0$ and $g \in L^{p/2}(\Omega)$ with $p > N$ and $p \geq 4$. Then, because of the continuous embeddings $L^{p/2}(\Omega) \hookrightarrow \mathfrak{L}^{2,\omega-2}(\Omega)$ and, furthermore, $W_0^{1,2,\omega}(\Omega \cup \Gamma) \hookrightarrow C^{0,1-(N-\omega)/2}(\overline{\Omega})$ with

$$\omega = \min\left\{\overline{\omega}(\varepsilon, \Omega \cup \Gamma), N + 2 - \frac{4N}{p}\right\},$$

the weak solution to (1.1) depends, in the norm of $C^{0,\alpha}(\overline{\Omega})$, analytically on $(A,b,c,d,e) \in \mathfrak{I}(\varepsilon, \Omega \cup \Gamma)$ and $g \in L^{p/2}(\Omega)$.

Remark that our approach does not cover the case $N = 3$ and $g \in L^{p/2}(\Omega)$ with $3 < p < 4$. Nevertheless, in GRIEPENTROG [17] these problems will be solved by using a more direct criterion for a functional to belong to the space $W^{-1,2,\omega}(\Omega \cup \Gamma)$. In fact, there will be shown, that Corollary 6.4 remains true under weaker assumptions on f, g and h, namely,

$$f \in \mathfrak{L}^{2,\omega}(\Omega; \mathbb{R}^N), g \in \mathfrak{L}^{\frac{2N}{N+2}, \frac{\omega N}{N+2}}(\Omega) \text{ and } h \in \mathfrak{L}^{\frac{2(N-1)}{N}, \frac{\omega(N-1)}{N}}(\Gamma).$$

7 Generalizations to Elliptic Systems

In this last section we consider weak solutions to boundary value problems for linear elliptic systems of the type

$$(7.1) \qquad \begin{cases} -\nabla \cdot (A_{ij}\nabla u_j + b_{ij}u_j) + c_{ij} \cdot \nabla u_j + d_{ij}u_j &= -\nabla \cdot f_i + g_i \quad \text{in } \Omega, \\ (A_{ij}\nabla u_j + b_{ij}u_j) \cdot \nu + e_{ij}u_j &= f_i \cdot \nu + h_i \quad \text{on } \Gamma_i, \\ u_i &= 0 \quad \text{on } \partial\Omega \setminus \Gamma_i, \end{cases}$$

In (7.1) and in the sequel (if no other settling, as in (7.2) below, is prescribed), the summation over repeated subscripts is understood, and free subscripts vary from 1 to n. Further, $\Omega \subset \mathbb{R}^N$ is a bounded domain, Γ_i are subsets of $\partial\Omega$, $\nu : \partial\Omega \to \mathbb{R}^N$ is the unit outward normal vector field on $\partial\Omega$, and $A_{ij} \in L^\infty(\Omega; \mathfrak{M}_N)$, $b_{ij}, c_{ij} \in L^\infty(\Omega; \mathbb{R}^N)$, $d_{ij} \in L^\infty(\Omega)$ and $e_{ij} \in L^\infty(\Gamma_i)$. We assume that

$$\text{the sets } G_i := \Omega \cup \Gamma_i \text{ are regular.}$$

Moreover, it is supposed that there exists an $\varepsilon \in (0,1]$ such that for all $\xi \in \mathbb{R}^N$ and λ^N-almost all $x \in \Omega$ we have

$$(7.2) \qquad \varepsilon\,|\xi|^2 \le A_{ii}(x)\xi \cdot \xi \text{ and } A_{ii}(x) \in \mathfrak{S}_N \text{ (no summation over } i)$$

and

$$(7.3) \qquad \|A_{ij}\|_{L^\infty(\Omega;\mathfrak{M}_N)} \le \frac{1}{\varepsilon}.$$

The results of this section follow from the results of the previous Section 6 in a straightforward way. We will formulate them not in the language of operators, like in Theorem 6.3, but in the language of weak solutions to (7.1).

A weak solution to (7.1) is, by definition, a tuple $u = (u_1, \dots, u_n)$ such that $u_i \in W_0^{1,2}(G_i)$ and

$$\int_\Omega ((A_{ij}\nabla u_j + b_{ij}u_j) \cdot \nabla v + (c_{ij} \cdot \nabla u_j + d_{ij}u_j)v)\,\mathrm{d}\lambda^N + \int_{\Gamma_i} e_{ij}u_j v\,\mathrm{d}\lambda_{\partial\Omega} =$$

$$= \int_\Omega (f_i \cdot \nabla v + g_i v)\,\mathrm{d}\lambda^N + \int_{\Gamma_i} h_i v\,\mathrm{d}\lambda_{\partial\Omega} \quad \text{for all } v \in W_0^{1,2}(G_i).$$

Theorem 7.1 *For all $\omega < \min\{\overline{\omega}(\varepsilon, G_i) : 1 \le i \le n\}$ there exists a $\delta > 0$ such that, if*

$$(7.4) \qquad \|A_{ij}\|_{L^\infty(\Omega;\mathfrak{M}_N)} \le \delta \quad \text{for } i > j,$$

the following holds:

(i) *If u is a weak solution to* (7.1) *with $f_i \in \mathfrak{L}^{2,\omega}(\Omega, \mathbb{R}^N)$, $g_i \in \mathfrak{L}^{2,\omega-2}(\Omega)$ and $h_i \in \mathfrak{L}^{2,\omega-1}(\Gamma_i)$, then $u \in W^{1,2,\omega}(\Omega)^n$.*

(ii) *Suppose that $u = 0$ is the only weak solution to* (7.1) *with $f_i = 0$, $g_i = 0$ and $h_i = 0$, i.e. to the homogeneous system, corresponding to* (7.1). *Then, for arbitrary $f_i \in \mathfrak{L}^{2,\omega}(\Omega, \mathbb{R}^N)$, $g_i \in \mathfrak{L}^{2,\omega-2}(\Omega)$ and $h_i \in \mathfrak{L}^{2,\omega-1}(\Gamma_i)$, there exists exactly one weak solution to* (7.1), *and this solution depends analytically (in the sense of $W^{1,2,\omega}(\Omega)^n$) on the coefficients A_{ij}, b_{ij}, c_{ij}, d_{ij} and e_{ij} (in the sense of the corresponding L^∞-spaces) and on the right hand sides f_i, g_i and h_i (in the sense of the corresponding Campanato spaces).*

Proof For

$$\begin{aligned}
A &= [A_{ij}]_{i,j=1}^n \in L^\infty(\Omega; \mathfrak{M}_N)^{n\times n}, \\
b &= [b_{ij}]_{i,j=1}^n, \;\; c \;=\; [c_{ij}]_{i,j=1}^n \in L^\infty(\Omega; \mathbb{R}^N)^{n\times n}, \\
d &= [d_{ij}]_{i,j=1}^n \in L^\infty(\Omega)^{n\times n}, \\
e &= [e_{ij}]_{i,j=1}^n \in L^\infty(\Gamma_1)^n \times \cdots \times L^\infty(\Gamma_n)^n
\end{aligned}$$

we denote by

$$L(A,b,c,d,e) : W_0^{1,2}(G_1) \times \cdots \times W_0^{1,2}(G_n) \to W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$$

the linear bounded operator, the i-th component of which is defined by

$$\begin{aligned}
&\langle [L(A,b,c,d,e)u]_i, v\rangle_{G_i} := \\
&:= \int_\Omega ((A_{ij}\nabla u_j + b_{ij}u_j)\cdot \nabla v + (c_{ij}\cdot \nabla u_j + d_{ij}u_j)v)\,\mathrm{d}\lambda^N + \int_{\Gamma_i} e_{ij}u_j v\,\mathrm{d}\lambda_{\Gamma_i}
\end{aligned}$$

for all $v \in W_0^{1,2}(G_i)$ (summation over j, but not over i). Further, for

$$\begin{aligned}
f &= [f_i]_{i=1}^n \in L^2(\Omega; \mathbb{R}^N)^n, \\
g &= [g_i]_{i=1}^n \in L^2(\Omega)^n, \\
h &= [h_i]_{i=1}^n \in L^2(\Gamma_1) \times \cdots \times L^2(\Gamma_n)
\end{aligned}$$

we define $F(f,g,h) \in W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$ by

$$\langle [F(f,g,h)]_i, v\rangle_{G_i} := \int_\Omega (f_i \cdot \nabla v + g_i v)\;\mathrm{d}\lambda^N + \int_{\Gamma_i} h_i v\,\mathrm{d}\lambda_{\Gamma_i} \;\text{ for all } v \in W_0^{1,2}(G_i).$$

Obviously, u is a weak solution to (7.1) if and only if

$$L(A,b,c,d,e)u = F(f,g,h). \tag{7.5}$$

As in the previous section (see Remark 6.1 and Lemma 6.2) one shows that the operator F maps the spaces $\mathfrak{L}^{2,\omega}(\Omega;\mathbb{R}^N)^n \times \mathfrak{L}^{2,\omega-2}(\Omega)^n \times \mathfrak{L}^{2,\omega-1}(\Gamma_1) \times \cdots \times \mathfrak{L}^{2,\omega-1}(\Gamma_n)$ into the spaces $W^{-1,2,\omega}(G_1)\times\cdots\times W^{-1,2,\omega}(G_n)$ and the operator $L(A,b,c,d,e)$ maps $W_0^{1,2,\omega}(G_1)\times\cdots\times W_0^{1,2,\omega}(G_n)$ into $W^{-1,2,\omega}(G_1)\times\cdots\times W^{-1,2,\omega}(G_n)$, respectively. Moreover, by the same argument the functional $F(f,g,h)$ depends continuously (in the sense of $W^{-1,2,\omega}(G_1)\times\cdots\times W^{-1,2,\omega}(G_n)$)

on $f \in \mathfrak{L}^{2,\omega}(\Omega;\mathbb{R}^N)^n$, $g \in \mathfrak{L}^{2,\omega-2}(\Omega)^n$ and $h \in \mathfrak{L}^{2,\omega-1}(\Gamma_1) \times \cdots \times \mathfrak{L}^{2,\omega-1}(\Gamma_n)$, and $L(A,b,c,d,e)$ depends continuously (in the sense of $\mathcal{L}(W_0^{1,2,\omega}(G_1) \times \cdots \times W_0^{1,2,\omega}(G_n); W^{-1,2,\omega}(G_1) \times \cdots \times W^{-1,2,\omega}(G_n)))$ on A, b, c, d and e.

Let I_n be the unit $n \times n$-matrix.

In a first step we show that for all $\varepsilon \in (0,1)$ and all $\omega < \min\{\overline{\omega}(\varepsilon, G_i) : 1 \le i \le n\}$ there exists a $\delta > 0$ such that, if (7.3), (7.2) and (7.4) are satisfied, the operator $L(A,0,0,I_n,0)$ is an isomorphism from $W_0^{1,2,\omega}(G_1) \times \cdots \times W_0^{1,2,\omega}(G_n)$ onto $W^{-1,2,\omega}(G_1) \times \cdots \times W^{-1,2,\omega}(G_n)$. For that it suffices to show that for all $\varepsilon \in (0,1)$ and all $\omega < \min\{\overline{\omega}(\varepsilon, G_i) : 1 \le i \le n\}$ the following holds: If (7.3), (7.2) and

$$A_{ij} = 0 \quad \text{for } i > j \tag{7.6}$$

are satisfied, then $L(A,0,0,I_n,0)$ is bijective from $W_0^{1,2,\omega}(G_1) \times \cdots \times W_0^{1,2,\omega}(G_n)$ onto $W^{-1,2,\omega}(G_1) \times \cdots \times W^{-1,2,\omega}(G_n)$, and there exists a constant c, which depends on ε and ω only, such that

$$\|L(A,0,0,I_n,0)^{-1}\|_{\mathcal{L}(W^{-1,2,\omega}(G_1)\times\cdots\times W^{-1,2,\omega}(G_n);W_0^{1,2,\omega}(G_1)\times\cdots\times W_0^{1,2,\omega}(G_n))} \le c.$$

Thus, take $\omega < \min\{\overline{\omega}(\varepsilon, G_i) : 1 \le i \le n\}$ and, furthermore, $(\phi_1, \cdots, \phi_n) \in W^{-1,2,\omega}(G_1) \times \cdots \times W^{-1,2,\omega}(G_n)$, suppose (7.3), (7.2) and (7.6) to be satisfied, and consider the equation

$$L(A,0,0,I_n,0)(u_1, \cdots, u_n) = (\phi_1, \cdots, \phi_n). \tag{7.7}$$

This equation is equivalent to a system of n variational equations, the last one of which is

$$\int_\Omega (A_{nn}\nabla u_n \cdot \nabla v + u_n v)\, \mathrm{d}\lambda^N = \langle \phi_n, v\rangle_{G_n} \quad \text{for all } v \in W_0^{1,2}(G_n). \tag{7.8}$$

Because of Theorem 6.3, there exists exactly one $u_n \in W_0^{1,2,\omega}(G_n)$ which satisfies (7.8), and

$$\|u_n\|_{W_0^{1,2,\omega}(G_n)} \le c\,\|\phi_n\|_{W^{-1,2,\omega}(G_n)},$$

where the constant c depends only on ε and ω. The next to the last variational equation of the system equivalent to (7.7) is

$$\left\{\begin{aligned} &\int_\Omega ((A_{n-1n-1}\nabla u_{n-1} + A_{n-1n}\nabla u_n)\cdot\nabla v + u_{n-1}v)\, \mathrm{d}\lambda^N = \\ &= \langle \phi_{n-1}, v\rangle_{G_{n-1}} \quad \text{for all } v \in W_0^{1,2}(G_{n-1}). \end{aligned}\right. \tag{7.9}$$

Because of Theorem 6.3 again, there exists exactly one $u_{n-1} \in W_0^{1,2,\omega}(G_{n-1})$ such that u_n and u_{n-1} which satisfy (7.8) and (7.9), and

$$\|u_{n-1}\|_{W_0^{1,2,\omega}(G_{n-1})} \le c\left(\|\phi_n\|_{W^{-1,2,\omega}(G_n)} + \|\phi_{n-1}\|_{W^{-1,2,\omega}(G_{n-1})}\right)$$

with a new constant c, which depends only on ε and ω again. Here we used assumption (7.3). Continuing this procedure we get finally the claim of the first step.

In the second step let us prove assertion (i) of the theorem. Thus, take $\omega < \min\{\overline{\omega}(\varepsilon, G_i) : 1 \le i \le n\}$, $f_i \in \mathfrak{L}^{2,\omega}(\Omega, \mathbb{R}^N)$, $g_i \in \mathfrak{L}^{2,\omega-2}(\Omega)$ and $h_i \in \mathfrak{L}^{2,\omega-1}(\Gamma_i)$, suppose (7.3), (7.2) and (7.4) (with the δ from the first step), and let u be a solution to (7.5). Then

$$L(A,0,0,I_n,0)u = F(f - bu, g - c\cdot\nabla u - (d - I_n)u, h - eu).$$

Here, for $u \in W^{1,2}(\Omega)^n$, we denote by bu, $c \cdot \nabla u$, du and eu the elements of $L^2(\Omega; \mathbb{R}^N)^n$, $L^2(\Omega)^n$ and $L^2(\Gamma_1) \times \cdots \times L^2(\Gamma_n)$ with components $b_{ij}u_j$, $c_{ij} \cdot \nabla u_j$, $d_{ij}u_j$ and $e_{ij}u_j$, respectively. Now we proceed as in the proof of Theorem 6.3 (i) (cf. (6.1)). Using the isomorphism property from step one, we get the desired result.

Finally, in the third step let us prove assertion (ii) of the theorem. We have to show that $L(A, b, c, d, e)$ is a Fredholm operator from $W_0^{1,2,\omega}(G_1) \times \cdots \times W_0^{1,2,\omega}(G_n)$ into $W^{-1,2,\omega}(G_1) \times \cdots \times W^{-1,2,\omega}(G_n)$. For $\omega = 0$ this is true, because we have

$$L(A, b, c, d, e) = L(A, 0, 0, I_n, 0) + L(0, b, c, d - I_n, e),$$

$L(A, 0, 0, I_n, 0)$ is an isomorphism from $W_0^{1,2}(G_1) \times \cdots \times W_0^{1,2}(G_n)$ onto the space $W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$ (step one), and $L(0, b, c, d - I_n, e)$ is completely continuous from $W_0^{1,2}(G_1) \times \cdots \times W_0^{1,2}(G_n)$ into $W^{-1,2}(G_1) \times \cdots \times W^{-1,2}(G_n)$ (because of the completely continuous embeddings $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ and $W_0^{1,2}(G_j) \hookrightarrow L^2(\partial_{\mathcal{N}} G_j)$). Now we can proceed as in the proof of Theorem 6.3 (ii), using the just proved assertion (i). Here we have to use the fact that the adjoint to the operator $L(A, b, c, d, e)$ is an operator of the type $L(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e})$, where (because of (7.4))

$$\|\tilde{A}_{ij}\|_{L^\infty(\Omega; \mathfrak{M}_N)} \leq \delta \quad \text{for } i < j,$$

and that for such close to triangular operators the assertion (i) holds true, too. $\square$

Remark 7.2 In this section we did not suppose any ellipticity condition apart from (7.2). Especially, we did not assume the differential operator in (7.1) to be strongly elliptic, i.e. we did not suppose that there exists a constant $c > 0$ such that

$$(A_{ij}(x)\xi \cdot \xi)v_i v_j \geq c\,|\xi|^2 |v|^2$$

for all $\xi \in \mathbb{R}^N$ and all $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and λ^N-almost all $x \in \Omega$.

But, if the constant δ in (7.4) is sufficiently small, then this differential operator is elliptic in the sense of Agmon, Douglas and Nirenberg [1] (see also [32, Chapter 9.2]), i.e.

$$\det\,[A_{ij}(x)\xi \cdot \xi]_{i,j=1}^n \neq 0 \tag{7.10}$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and λ^N-almost all $x \in \Omega$. Moreover, in this case the differential operator is even normally elliptic in the sense of Amann [2], i.e. all eigenvalues of the matrix in (7.10) have positive real parts (because this matrix is close to a triangular one with positive diagonal elements).

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