RECONSTRUCTION OF SOURCE TERMS IN EVOLUTION EQUATIONS BY EXACT CONTROLLABILITY

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Abstract. For fixed $\rho = \rho(x,t)$, we consider the solution $u(f)$ to

$$u''(x,t) + Au(x,t) = f(x)\rho(x,t), \quad x \in \Omega,\ t > 0$$

$$u(x,0) = u'(x,0) = 0, \qquad x \in \Omega, \qquad B_j u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\ 1 \le j \le m,$$

where $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $\Omega \subset R^r$, $r \ge 1$ is a bounded domain with smooth boundary, A is a uniformly symmetric elliptic differential operator of order $2m$ with t-independent smooth coefficients, B_j, $1 \le j \le m$, are t-independent boundary differential operators such that the system $\{A, B_j\}_{1\le j\le m}$ is well-posed. Let $\{C_j\}_{1\le j\le m}$ be complementary boundary differential operators of $\{B_j\}_{1\le j\le m}$. We consider a multidimensional linear inverse problem : for given $\Gamma \subset \partial\Omega$, $T > 0$ and $n \in \{1, ..., m\}$, determine $f(x)$, $x \in \Omega$ from $C_j u(f)(x,t)$, $x \in \Gamma$, $0 < t < T$, $1 \le j \le n$.

By exact controllability based on the Hilbert uniqueness method, we reduce our inverse problem to an equation of the second kind which gives reconstruction of f. Moreover under extra regularity assumptions on ρ, we can prove that this equation is a Fredholm equation of the second kind. Our methodology is widely applicable to various equations in mathematical physics.

§1. Introduction.

We consider an initial - boundary value problem :

$$u''(x,t) + Au(x,t) = f(x)\rho(x,t), \qquad x \in \Omega,\ t > 0 \tag{1.1}$$

$$u(x,0) = u'(x,0) = 0, \qquad x \in \Omega \tag{1.2}$$

$$B_j u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\ 1 \le j \le m, \tag{1.3}$$

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where $u' = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial t^2}$, $\Omega \subset R^r$, $r \ge 1$ is a bounded domain with C^2- boundary, A is a uniformly symmetric elliptic differential operator of order $2m$ with t-independent smooth coefficients, B_j, $1 \le j \le m$, are boundary differential operators. More precisely, we set $x = (x_1, ..., x_r) \in R^r$, $\alpha = (\alpha_1, ..., \alpha_r) \in (N \cup \{0\})^r$, $|\alpha| = \alpha_1 + ... + \alpha_r$, $D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_r}\right)^{\alpha_r}$, and

$$(A\phi)(x) = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\alpha|} D_x^\alpha (a_{\alpha\beta}(x) D_x^\beta \phi)(x),$$

which $a_{\alpha\beta} = a_{\beta\alpha} \in C^\infty(\overline{\Omega})$ are real-valued for $|\alpha|$, $|\beta| \le m$, and we assume the uniform ellipticity : there exists a constant $M_0 > 0$ independent of $x \in \overline{\Omega}$ and $\xi \in R^r$ such that

$$M_0^{-1} |\xi|^{2m} \le \left| \sum_{|\alpha|,|\beta|=m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} \right| \le M_0 |\xi|^{2m}, \qquad x \in \overline{\Omega},\ \xi \in R^r,$$

where $\xi = (\xi_1, ..., \xi_r) \in R^r$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_r^{\alpha_r}$ with $\alpha = (\alpha_1, ..., \alpha_r)$, $|\xi|^2 = \xi_1^2 + ... + \xi_r^2$. Moreover we put

$$(B_j\psi)(x) = \sum_{|\alpha| \le m_j} b_{j\alpha}(x) D_x^\alpha \psi(x),$$

where $b_{j\alpha} \in C^\infty(\partial\Omega)$, $0 \le m_j < 2m$. Throughout this paper we assume that $\{B_j\}_{1 \le j \le m}$ is normal on $\partial\Omega$ (e.g. Lions and Magenes [17, Vol.I]) and that the system $\{A, B_j\}_{1 \le j \le m}$ is well-posed ([17, Vol.II]).

Henceforth let $\{C_j\}_{1 \le j \le m}$ be complementary boundary differential operators of $\{B_j\}_{1 \le j \le m}$, whose coefficients are t-independent and smooth in $x \in \partial\Omega$ ([17, Vol.I]).

In this paper, assuming that ρ is given while f is unknown to be determined from observations on a part of lateral boundary, we denote the weak solution to (1.1) - (1.3) by $u(f) = u(f)(x,t)$. For the weak solution, we can further refer to [17]. We discuss

Inverse Source Problem.

For given $\Gamma \subset \partial\Omega$, $T > 0$ and $n \in \{1, ..., m\}$, determine $f(x)$, $x \in \Omega$, from $C_j u(f)(x,t)$, $x \in \Gamma$, $0 < t < T$, $1 \le j \le n$.

In (1.1), the non-homogeneous term $f(x)\rho(x,t)$ is considered to cause actions such as vibrations, and the inverse source problem is significant in mathematical physics. Moreover when we discuss determination of spatially varying coefficients in A, we have to do with this type of inverse problem after subtraction or linearization (e.g. Lavrentiev, Romanov and Shishat·skiĭ[14], Romanov [22]). We notice that we want to determine f with a single boundary measurement.

In the case where $\rho = \rho(t)$ is independent of x, by means of Duhamel's principle (e.g. Rauch [21]), we can reduce the inverse problem to an observability problem, namely, determination of initial data. For the inverse problem in the case of x-indepednent $\rho = \rho(t)$, we can refer to Puel and Yamamoto [18], Yamamoto [24], [25], [26]. On the other hand, the inverse problem becomes more difficult for x-dependent ρ. Fot such a case, the method by Bukhgeim and Klibanov [3] is useful and their method is based on a weighted estimate called a Carleman estimate. For the uniqueness, we can refer to Bukhgeim and Klibanov [3], Iskakov [5], [6], [7], Khaĭdarov [9], Klibanov [10]. Moreover for similar inverse problems for Lamé systems and Maxwell's equations, we refer to Ikehata, Nakamura and Yamamoto [4], and Yamamoto [27], respectively. As for an inverse problem with many observations for a hyperbolic equation given by (1.1), we can refer to Rakesh and Symes [20]. For general references for these kinds of inverse problems, the readers can consult monographs : Isakov [8], Lavrentiev, Romanov and Shishat·skiĭ[14], Romanov [22].

Most of the papers above-mentioned mainly treat the uniqueness problem. For stability in determining functions in hyperbolic equations from a single boundary measurement, estimation of Hölder type has been proved (Khaĭdarov [9]. also see a remark (p.577) in [10]). Recently the author has established the best possible Lipschitz stability by combination of the Carleman estimate and the exact observability (Yamamoto [28]).

Reconstruction of f is practically important, but such discussions are very few (Bukhgeim [2]). The purpose of this paper is to reduce our inverse problem to an equation of the second kind by the exact controllability, which is a Fredholm equation of the second kind under a natural setting. Then our inverse problem is to solve the equation of the second kind. Further study for the equation will be made in a forthcoming paper.

This paper is composed of four sections. Section 2 is devoted to a brief explanation of the Hilbert Uniqueness Method. In Section 3, we state our main result. In Section 4, we prove the main result.

§2. Brief Explanation of the Hilbert Uniqueness Method.

We give a brief explanation of the Hilbert Uniqueness Method, according to Lions [16]. We refer also to Komornik [11], Lasiecka and Triggiani [13], Lions [15]. We set

$$\begin{aligned}\widetilde{F} &= \widetilde{F_1} \times \widetilde{F_2}\\ &=\{(\phi_1, \phi_2) \in C^\infty(\overline{\Omega})^2\,;\, B_j\phi_1 = 0 \text{ if the order of } B_j \text{ is less than } m\},\end{aligned}$$

and for $(\phi_1, \phi_2) \in \widetilde{F}$, we denote the solution to

$$w''(x,t) + Aw(x,t) = 0, \qquad x \in \Omega,\ 0 < t < T, \tag{2.1}$$

$$w(x,0) = \phi_1(x), \quad w'(x,0) = \phi_2(x), \qquad x \in \Omega \tag{2.2}$$

$$B_j w(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T,\ 1 \le j \le m \tag{2.3}$$

by $w(\phi_1, \phi_2) = w(\phi_1, \phi_2)(x,t)$. We pose

Assumption A (Unicity). *For a given measurable* $\Gamma \subset \partial\Omega$*, a finite* $T > 0$ *and* $n \in \{1, ..., m\}$*, if the solution* $w(\phi_1, \phi_2)$ *satisfies*

$$C_j w(x,t) = 0, \qquad x \in \Gamma,\ 0 < t < T,\ 1 \le j \le n$$

for $(\phi_1, \phi_2) \in \widetilde{F}$*, then* $w(\phi_1, \phi_2)(x,t) = 0$*,* $x \in \Omega$*,* $0 < t < T$ *follows.*

This is unicity in a Cauchy problem for $w'' + Aw = 0$, for which we refer to Bardos, Lebeau and Rauch [1] and Tataru [23] for example. On Assumption A, we can define a norm $\|(\phi_1, \phi_2)\|_F$ by

$$\|(\phi_1,\phi_2)\|_F \equiv \left(\|\phi_1\|_{F_1}^2 + \|\phi_2\|_{F_2}^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n} \|C_j w(\phi_1,\phi_2)\|^2_{L^2(\Gamma\times(0,T))}\right)^{\frac{1}{2}},$$

for any $(\phi_1, \phi_2) \in \widetilde{F}$, where $\|\eta\|_{L^2(\Gamma\times(0,T))} = \left(\int_\Gamma \int_0^T |\eta(x,t)|^2 dt dS_x\right)^{\frac{1}{2}}$. Let a Hilbert space $F \equiv F_1 \times F_2$ be the *completion* of $\widetilde{F}$ by the norm $\|\cdot\|_F$. Let $F' = F_1' \times F_2'$ be its dual. Throughout this paper, $\cdot'$ denotes the dual space and we identify the dual spaces $L^2(\Gamma \times (0,T))'$ of $L^2(\Gamma \times (0,T))$ and $L^2(\Omega)'$ of $L^2(\Omega)$ respectively with itself. The space F' is related to the exactly controllable set and the essence of the Hilbert Uniqueness Method is construction of the Hilbert space F'.

Next let us consider

$$\psi''(x,t) + A\psi(x,t) = 0, \quad x \in \Omega,\ 0 < t < T \tag{2.4}$$

$$\psi(x,T) = \psi'(x,T) = 0, \quad x \in \Omega \tag{2.5}$$

$$B_j\psi(x,t) = \begin{cases} v_j(x,t), & x \in \Gamma,\ 0 < t < T \ :\ 1 \le j \le n \\ 0, & x \in \partial\Omega \setminus \Gamma,\ 0 < t < T \ :\ 1 \le j \le n \\ 0, & x \in \partial\Omega,\ 0 < t < T \ :\ n+1 \le j \le m. \end{cases} \tag{2.6}$$

For the system (2.4) - (2.6) with a uniformly symmetric elliptic operator A of order $2m$, a general treatment (Theorem 4.1 (p.107 : Vol.II) in [17]) tells that for any $v = (v_1, ..., v_n) \in L^2(\Gamma \times (0,T))^n$, there exists a unique weak solution $\psi(v) \in H^{0,-1}(\Omega \times (0,T)) \equiv \left(H_0^1(0,T; L^2(\Omega)) \cap L^2(0,T; L^2(\Omega))\right)'$, where $H_0^1(0,T;L^2(\Omega)) = \{u \in H^1(0,T;L^2(\Omega))\,;\, u(\cdot,0) = u(\cdot,T) = 0\}$. Furthermore we refer to Theorems 6.1 and 6.2 (pp.118-119 : Vol.II) in [17], and especially for a wave equation, we also quote Lasiecka, Lions and Triggiani [12], Lions [16].

In applying a result (Theorem 0 below) on exact controllability, we however pose a stronger assumption for the regularity of $\psi(v)$.

Assumption B (Regularity in the control system). *For $v \in L^2(\Gamma \times (0,T))^n$, the weak solution $\psi(v)$ satisfies*

$$\psi(v) \in C^0([0,T]; F_2'), \quad \psi(v)' \in C^0([0,T]; F_1')$$

$$\|\psi(v)\|_{C^0([0,T];F_2')} \le M_1 \|v\|_{L^2(\Gamma\times(0,T))^n}$$

where $M_1 = M_1(\Omega, \Gamma, T) > 0$ is independent of v.

Example 1 : wave equation. ([11], [13], [16]) For an arbitrarily given $x_0 \in R^r$, we set

$$\Gamma_+(x_0) = \{x \in \partial\Omega; (x - x_0, \nu(x)) > 0\}$$
$$R_0 = R_0(x_0) = \sup_{x\in\partial\Omega} |x - x_0|, \tag{2.7}$$

where $\nu(x)$ is the outward unit normal to $\partial\Omega$ and $(\cdot,\cdot)$ is the inner product in R^r. We consider: $A = -\Delta$ (the Laplacian), $m = 1$,

$$B_1 u = u|_{\partial\Omega}, \qquad C_1 u = \frac{\partial u}{\partial n}_{|\Gamma}.$$

If

$$T > 2R_0$$

and a measurable set $\Gamma \subset \partial\Omega$ satisfies

$$\Gamma \supset \Gamma_+(x_0), \tag{2.8}$$

then

$$F_1 = H_0^1(\Omega), \qquad F_2 = L^2(\Omega), \tag{2.9}$$

and Assumptions A and B hold true.

Example II: plate equation. (e.g. [11], [16]). Let $A = \Delta^2$, $m = 2$ and

$$B_1 u = u|_{\partial\Omega}, \quad B_2 u = \frac{\partial u}{\partial n}, \quad C_1 u = \Delta u_{|\Gamma}, \quad C_2 u = \frac{\partial \Delta u}{\partial n}_{|\Gamma}.$$

We set $n = 1$. If we choose Γ satisfying (2.8), then for any $T > 0$, $F_2 = L^2(\Omega)$ holds, and Assumptions A and B hold true.

By the Hilbert Uniqueness Method, we show boundary exact controllability:

Theorem 0. *(Théorème 3.2 (p.119) in [16]) On Assumptions A and B, for any $(\phi_1, \phi_2) \in F_2' \times F_1'$, there exists $v = (v_1, ..., v_n) \in L^2(\Gamma \times (0,T))^n$ such that the weak solution $\psi = \psi(v)$ to (2.4) - (2.6) satisfies*

$$\psi(v)(\cdot, 0) = \phi_1, \quad \psi(v)'(\cdot, 0) = \phi_2. \tag{2.10}$$

Moreover we can construct a map from (ϕ_1, ϕ_2) to v such that

$$\|v\|_{L^2(\Gamma\times(0,T))^n} \le M_1(\|\phi_1\|_{F_2'} + \|\phi_2\|_{F_1'}), \qquad (\phi_1, \phi_2) \in F_2' \times F_1',$$

where $M_1 = M_1(\Omega, \Gamma, T) > 0$ is independent of (ϕ_1, ϕ_2).

This theorem defines a bounded linear operator $g : F_2' \longrightarrow L^2(\Gamma \times (0,T))^n$ which maps $\phi_1 \in F_2'$ to $v \in L^2(\Gamma \times (0,T))^n$ realizing $\psi(v)(\cdot, 0) = \phi_1$ and $\psi(v)'(\cdot, 0) = 0$, and

$$\|g(\phi_1)\|_{L^2(\Gamma\times(0,T))^n} \le M_1 \|\phi_1\|_{F_2'}. \tag{2.11}$$

In (2.6), v_j, $1 \le j \le n$, are regarded as boundary controls which steer the system described by (2.4) - (2.5) to the equilibrium at time T starting from the initial state given by (ϕ_1, ϕ_2).

§3. Main result: reduction of the general inverse source problem to an equation of the second kind.

We discuss the initial - boundary value problem (1.1) - (1.3) with $\rho = \rho(x,t)$ satisfying

$$\left\| \int_0^T \rho'(\cdot, t)\psi(\cdot, t)dt \right\|_{F_2'} \le M_2 \|\psi\|_{C^0([0,T];F_2')}, \quad \psi \in C^0([0,T]; F_2') \tag{3.1}$$

$$\|f\rho(\cdot, 0)\|_{F_2} \le M_2 \|f\|_{F_2}, \quad f \in F_2 \tag{3.2}$$

$$\rho \in H^1(0,T; L^\infty(\Omega)) \tag{3.3}$$

$$\|f\rho'\|_{L^2(0,T;F_2)} \le M_2 \|f\|_{F_2}, \quad f \in F_2 \tag{3.4}$$

Here $M_2 > 0$ is independent of ψ and f. We always pose Assumptions A and B.

Remark. If we can characterize F_2, for example, as $F_2 = L^2(\Omega)$ (cf. Examples in Section 2), then the conditions (3.1) - (3.4) are equivalent to

$$\rho \in H^1(0,T; L^\infty(\Omega)), \quad \rho(\cdot, 0) \in L^\infty(\Omega). \tag{3.1'}$$

We recall that a linear operator $g : F_2' \longrightarrow L^2(\Gamma \times (0,T))^n$ is defined in Theorem 0 in Section 2 and satisfies (2.11). We define a linear operator S in F_2' by

$$(S\phi_1)(x) = \int_0^T \rho'(x,t)\psi(g(\phi_1))(x,t)dt, \qquad \phi_1 \in F_2'. \tag{3.5}$$

Then we are ready to state the main result:

Theorem. *Under Assumptions A and B, (3.1) - (3.4);*
(1) $S : F_2' \longrightarrow F_2'$ is a bounded linear operator.
(2) Let $v \in H^1(0,T;L^2(\Gamma))^n$. Then $f \in F_2$ satisfies

$$g^*\left(v' - (C_1u(f)', ..., C_nu(f)')\right) = 0 \tag{3.6}$$

if and only if $f \in F_2$ satisfies

$$\rho(\cdot,0)f + S^*f = g^*v'. \tag{3.7}$$

Here $S^* : F_2 \longrightarrow F_2$ is the adjoint of $S : F_2' \longrightarrow F_2'$, and g^* is the one of a bounded linear operator $g : F_2' \longrightarrow L^2(\Gamma \times (0,T))^n$. The operator equation (3.7) is our desired one of the second kind.

Corollary 1. *If f is a solution of our inverse problem, that is, $f \in F_2$ satisfies*

$$(C_1u(f), ..., C_nu(f)) = v \tag{3.6'}$$

for $v \in H^1(0,T;L^2(\Gamma))^n$, then f solves (3.7).

Remark. In general, $\mathcal{R}(g)$ is not dense in $L^2(\Gamma \times (0,T))^n$, so that g^* is not injective. Thus in Theorem, we can not replace (3.6) by (3.6').

Henceforth we assume

$$\rho(x,0) \neq 0, \qquad x \in \overline{\Omega}. \tag{3.8}$$

Then (3.7) is an equation of the second kind:

$$f + \frac{1}{\rho(\cdot,0)}S^*f = \frac{1}{\rho(\cdot,0)}g^*v'. \tag{3.9}$$

Moreover Corollary 1 asserts that it is sufficient to consider (3.9) for reconstructing f. For similar linear inverse problems with singular data such as Dirac delta functions in multidimensional cases and similar ones with smooth data in one-dimensional cases, we can reduce the problems to a Volterra equation of the second kind (e.g. Chapter 2 and Section 3 of Chapter 4 in [22]). However in multidimensional cases with not necessarily singular data, a general way for such reduction has not been published (cf. Bukhgeim [2]).

Here we do not give direct expression of S^*. In special cases, direct expression of S^* is not difficult. For example, in Example 1 in Section 2, let $r = 1$ (i.e., the spatial dimension is 1), $\Omega = (0,1)$, $\Gamma = \{0\}$ (one end point) and $T = 2$. Then we can construct the control operator $g : L^2(0,1) \longrightarrow L^2(0,2)$ by consideration of the dependency domain of the one-dimensional wave equation and D'Alembert's formula.

Next we have to study the unique solvability of the equation (3.9). First by the contraction mapping principle, we can readily see

Corollary 2. *Let*

$$\left\| \frac{\rho'(\cdot,\cdot)}{\rho(\cdot,0)} \right\|_{L^1(0,T;L^\infty(\Omega))} \tag{3.10}$$

be sufficiently small and let $v = (C_1u(f), ..., C_nu(f))$. *Then* f *is given as a unique solution of (3.9) by iteration.*

We consider a hyperbolic equation of the second order and we take $C_1u = \frac{\partial u}{\partial n}_{|\Gamma}$ as the boundary observation where the subboundary Γ satisfies (2.8):

$$u''(x,t) = \Delta u(x,t) - p(x)u(x,t) + f(x)\rho(x,t), \quad x \in \Omega,\ t > 0 \tag{3.11}$$

$$u(x,0) = u'(x,0) = 0, \qquad x \in \Omega \tag{3.12}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega, t > 0. \tag{3.13}$$

Moreover in addition to (3.1') we assume

$$p \in L^\infty(\Omega) \tag{3.14}$$

$$\rho, \frac{\rho}{\rho(\cdot,0)} \in H^2(0,T;L^\infty(\Omega)) \tag{3.15}$$

$$T > 2R_0 \tag{3.16}$$

where R_0 is given by (2.7). Then by the argument in the proof of Lemma 5.5 in Puel and Yamamoto [19], we can prove

Corollary 3. *Under the assumptions (3.14) - (3.16), the operator* $S^* : L^2(\Omega) \longrightarrow L^2(\Omega)$ *is compact. Therefore the equation (3.9) is a Fredholm equation of the second kind in* $L^2(\Omega)$.

In Corollary 3, for the unique solvability, it suffices to verify that $f + \frac{1}{\rho(\cdot,0)}S^*f = 0$ implies $f = 0$. This is equivalent to the uniqueness in some inverse problem and the method in Bukhgeim and Klibanov [3] may be helpful. In a forthcoming paper, we will treat details of the unique solvability.

§4. Proof of Theorem.

Proof of (1). By Assumption B, (2.11) and (3.1) - (3.4),

$$\begin{aligned}\|S\phi_1\|_{F_2'} = &\left\| \int_0^T \rho'(\cdot,t)\psi(g(\phi_1))(\cdot,t)dt \right\|_{F_2'} \le M_2\|\psi(g(\phi_1))\|_{C^0([0,T];F_2')} \\ \le &M_1M_2\|g(\phi_1)\|_{L^2(\Gamma\times(0,T))^n} \le M_1^2M_2\|\phi_1\|_{F_2'},\end{aligned}$$

which implies the part of (1) of the theorem.

Proof of (2). Henceforth $< \cdot, \cdot >_{F_2', F_2}$ denotes the duality pairing between F_2' and F_2. First we show

Lemma 1 (duality equality). *Under Assumptions A and B, (3.1) - (3.4), for any* $v = (v_1, ..., v_n) \in L^2(\Gamma \times (0,T))^n$ *and* $f \in F_2$, *we have*

$$< \psi(v)(\cdot, 0), f\rho(\cdot, 0) >_{F_2', F_2} + \left\langle \int_0^T \rho'(\cdot, t)\psi(v)(\cdot, t)dt, f \right\rangle_{F_2', F_2}$$

$$= \sum_{j=1}^{n} \int_0^T \int_\Gamma C_j u(f)'(x,t) v_j(x,t) dS_x dt. \tag{4.1}$$

Proof of Lemma 1. First assuming that $v \in C_0^\infty(\Gamma \times (0,T))^n$ and $f \in C^\infty(\overline{\Omega})$, we see by Theorem 3.1 (pp.103 - 104 : Vol.II), Theorem 2.1 (pp.95 - 96 : Vol.II) and Theorem 8.2 (p.275 : Vol.I) in [17] that $\psi(v)$ and $u(f)$ are so regular that we can calculate

$$\int_0^T \left(\int_\Omega u(f)''(x,t)\psi(v)(x,t)dx \right) dt$$

by integration by parts, $\psi(v)(x,T) = \psi(v)'(x,T) = 0$ and $u(f)(x,0) = u(f)'(x,0) = 0$:

$$\int_0^T \left(\int_\Omega u(f)''(x,t)\psi(v)(x,t)dx \right) dt$$

$$= \int_\Omega \left([u(f)'(x,t)\psi(v)(x,t)]_{t=0}^{t=T} - \int_0^T u(f)'(x,t)\psi(v)'(x,t)dt \right) dx$$

$$= - \int_\Omega \left(\int_0^T u(f)'(x,t)\psi(v)'(x,t)dt \right) dx$$

$$= \int_\Omega \left(-[u(f)(x,t)\psi(v)'(x,t)]_{t=0}^{t=T} + \int_0^T u(f)(x,t)\psi(v)''(x,t)dt \right) dx$$

$$= \int_\Omega \left(\int_0^T u(f)(x,t)\psi(v)''(x,t)dt \right) dx.$$

Therefore using (1.1) and (2.4), we have

$$\int_0^T \left(\int_\Omega \psi(v)(x,t)Au(f)(x,t) - u(f)(x,t)A\psi(v)(x,t)dx \right) dt$$

$$= \int_\Omega f(x) \left(\int_0^T \rho(x,t)\psi(v)(x,t)dt \right) dx.$$

Applying the Green formula and taking into consideration the boundary conditions of $u(f)$ and $\psi(v)$, we see

$$\int_\Omega f(x) \left(\int_0^T \rho(x,t)\psi(v)(x,t)dt \right) dx = \sum_{j=1}^{n} \int_0^T \int_\Gamma C_j u(f)(x,t) v_j(x,t) dS_x dt \tag{4.2}$$

for $v \in C_0^\infty(\Gamma \times (0,T))^n$. Since $v \in C_0^\infty(\Gamma \times (0,T))^n$, the time derivative $\Psi = \psi(v)'$ satisfies

$$\Psi''(x,t) + A\Psi(x,t) = 0, \quad x \in \Omega,\ 0 < t < T$$

$$\Psi(x,T) = \Psi'(x,T) = 0, \quad x \in \Omega$$

$$B_j\Psi(x,t) = \begin{cases} v_j'(x,t), & x \in \Gamma,\ 0 < t < T \,:\, 1 \le j \le n \\ 0, & x \in \partial\Omega \setminus \Gamma,\ 0 < t < T \,:\, 1 \le j \le n \\ 0, & x \in \partial\Omega,\ 0 < t < T \,:\, n+1 \le j \le m. \end{cases}$$

Substituting Ψ into $\phi(v)$ in (4.2), we have

$$\int_\Omega f(x)\left(\int_0^T \rho(x,t)\psi(v)'(x,t)dt\right)dx = \sum_{j=1}^n \int_0^T \int_\Gamma C_ju(f)(x,t)v_j'(x,t)dS_xdt$$

Noting (3.2), (3.3) and the regularity of $\psi(v)$ and $C_ju(f)$, $1 \le j \le n$, and $v \in C_0^\infty(\Gamma \times (0,T))^n$, we apply integration by parts at the both sides to obtain (4.1) for any $v \in C_0^\infty(\Gamma \times (0,T))^n$ and $f \in C^\infty(\overline{\Omega})$.

Next for $v \in C_0^\infty(\Gamma \times (0,T))^n$ and $f \in F_2$, we prove (4.1). For this, we show

Lemma 2. *Under Assumption A and (3.2) - (3.4), we have*

$$\|C_ju(f)'\|_{L^2(\Gamma\times(0,T))} \le M_3\|f\|_{F_2}, \quad f \in F_2,\ 1 \le j \le n,$$

where $M_3 > 0$ is independent of $f \in F_2$.

Sketch of Proof of Lemma 2. First $U = u(f)'$ satisfies

$$U''(x,t) + AU(x,t) = f(x)\rho'(x,t), \qquad x \in \Omega,\ 0 < t < T$$

$$U(x,0) = 0, \quad U'(x,0) = f(x)\rho(x,0), \qquad x \in \Omega$$

$$B_jU(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T,\ 1 \le j \le m.$$

Let V be the solution to

$$V''(x,t) + AV(x,t) = f(x)\rho'(x,t), \qquad x \in \Omega,\ 0 < t < T$$

$$V(x,0) = V'(x,0) = 0, \qquad x \in \Omega$$

$$B_jV(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t < T,\ 1 \le j \le m.$$

Then we have $U = w(0, f\rho(\cdot,0)) + V$. Here we recall that $w(0, f\rho(\cdot,0))$ is given by (2.1) - (2.3). On the other hand, by Duhamel's principle (e.g. [21]), we obtain

$$V(x,t) = \int_0^t W(x,t;s)ds, \qquad x \in \Omega,\ 0 < t < T,$$

where $W = W(x,t;s)$ satisfies

$$W''(x,t;s) + AW(x,t;s) = 0, \qquad x \in \Omega,\ s < t$$

$$W(x,s;s) = 0, \quad W'(x,s;s) = f(x)\rho'(x,s), \qquad x \in \Omega$$

$$B_j W(x,t;s) = 0, \qquad x \in \partial\Omega,\ s < t,\ 1 \le j \le m.$$

Therefore we have

$$C_j u(f)' = C_j U = C_j w(0, f\rho(\cdot,0)) + \int_0^t C_j W(x,t;s)ds, \quad 1 \le j \le n,$$

so that

$$\begin{aligned}\left(\sum_{j=1}^n \|C_j u(f)'\|^2_{L^2(\Gamma\times(0,T))}\right)^{\frac12} \le& \left(2\sum_{j=1}^n \|C_j w(0, f\rho(\cdot,0))\|^2_{L^2(\Gamma\times(0,T))}\right)^{\frac12} \\ &+ \left(2\sum_{j=1}^n \int_0^T \int_\Gamma \left|\int_0^t C_j W(x,t;s)ds\right|^2 dS_x dt\right)^{\frac12}.\end{aligned} \tag{4.3}$$

On the other hand, by the definition of F_2, we see

$$\left(\sum_{j=1}^n \|C_j w(0, f\rho(\cdot,0))\|^2_{L^2(\Gamma\times(0,T))}\right)^{\frac12} = \|f\rho(\cdot,0)\|_{F_2} \tag{4.4}$$

and

$$\begin{aligned}&\sum_{j=1}^n \int_s^T \int_\Gamma |C_j W(x,t;s)|^2 dS_x dt \le \sum_{j=1}^n \int_s^{s+T} \int_\Gamma |C_j W(x,t;s)|^2 dS_x dt \\ &= \|f\rho'(\cdot,s)\|^2_{F_2}.\end{aligned} \tag{4.5}$$

Therefore by (3.4), (4.5), Schwarz's inequality and change of orders of integrations, we have

$$\begin{aligned}&\sum_{j=1}^n \int_0^T \int_\Gamma \left|\int_0^t C_j W(x,t;s)ds\right|^2 dS_x dt \\ \le& T\sum_{j=1}^n \int_0^T \left(\int_\Gamma \int_0^t |C_j W(x,t;s)|^2 ds dS_x\right) dt \\ =& T\int_0^T \sum_{j=1}^n \left(\int_s^T \int_\Gamma |C_j W(x,t;s)|^2 dS_x dt\right) ds \le T\int_0^T \|f\rho'(\cdot,s)\|^2_{F_2} ds \\ \le& M_2^2 T \|f\|^2_{F_2}.\end{aligned} \tag{4.6}$$

Applying (4.4) and (4.6) in (4.3), by (3.2) we obtain

$$\left(\sum_{j=1}^{n} \|C_j u(f)'\|^2_{L^2(\Gamma\times(0,T))}\right)^{\frac{1}{2}} \le \sqrt{2}\|f\rho(\cdot,0)\|_{F_2} + \sqrt{2T}M_2\|f\|_{F_2}$$
$$\le \sqrt{2}(M_2 + M_2\sqrt{T})\|f\|_{F_2}.$$

Thus the proof of Lemma 2 is complete.

Now we proceed to the proof of (4.1) for $v \in C_0^\infty(\Gamma\times(0,T))^n$ and $f \in F_2$. By the definition of F_2, there exist $f_l \in C^\infty(\overline{\Omega})$ such that $\|f_l - f\|_{F_2} \longrightarrow 0$ as $l \longrightarrow \infty$. By Lemma 2 we see

$$\|(C_j u(f_l) - C_j u(f))'\|_{L^2(\Gamma\times(0,T))} \longrightarrow 0, \qquad 1 \le j \le n$$

as $l \longrightarrow \infty$. Consequently, since (4.1) holds for f_l, by (3.2), we can make l tend to ∞, so that we obtain (4.1) for any $v \in C_0^\infty(\Gamma\times(0,T))^n$ and $f \in F_2$.

Finally let $v \in L^2(\Gamma\times(0,T))^n$. Since $C_0^\infty(\Gamma\times(0,T))^n$ is dense in $L^2(\Gamma\times(0,T))^n$, there exist $v_l \in C_0^\infty(\Gamma\times(0,T))^n$, $l \ge 1$, such that

$$\|v_l - v\|_{L^2(\Gamma\times(0,T))^n} \longrightarrow 0$$

as $l \longrightarrow \infty$. By Assumption B, we have

$$\|\psi(v_l) - \psi(v)\|_{C([0,T];F_2')} \longrightarrow 0, \tag{4.7}$$

as $l \longrightarrow \infty$. On the other hand, since (4.1) holds for $f \in F_2$ and $v_l = (v_1^{(l)}, ..., v_n^{(l)}) \in C_0^\infty(\Gamma\times(0,T))^n$, by (3.1) and (4.7) we can make l tend to ∞ in (4.1) with $v = v_l$. Therefore we see (4.1) for any $v \in L^2(\Gamma\times(0,T))^n$ and $f \in F_2$. Thus the proof of Lemma 1 is complete.

Now we proceed to completing the proof of Theorem. For any $\phi_1 \in F_2'$, we apply Theorem 0 in Section 2 to obtain $g(\phi_1) \equiv (g(\phi_1)_1, ..., g(\phi_1)_n) \in L^2(\Gamma\times(0,T))^n$ such that $\psi(g(\phi_1))(\cdot,0) = \phi_1$ and $\psi(g(\phi_1))'(\cdot,0) = 0$. Substituting $v = g(\phi_1)$ in (4.1) and noting (3.5), we can obtain

$$< \phi_1, \rho(\cdot,0)f >_{F_2',F_2} + < S\phi_1, f >_{F_2',F_2}$$
$$= \sum_{j=1}^{n}\int_0^T\int_\Gamma g(\phi_1)_j(x,t)C_j u(f)'(x,t)dS_x dt, \quad \phi_1 \in F_2',\ f \in F_2,$$

that is,

$$< \phi_1, \rho(\cdot,0)f + S^*f >_{F_2',F_2} = < \phi_1, g^*(C_1u(f)', ..., C_nu(f)') >_{F_2',F_2},$$
$$\phi_1 \in F_2',\ f \in F_2. \tag{4.8}$$

Let us complete the proof of the part (2). First let us assume (3.6). Then by (4.8), we have $< \phi_1, \rho(\cdot,0)f + S^*f >_{F_2',F_2} = < \phi_1, g^*v' >_{F_2',F_2}$ for all $\phi_1 \in F_2'$, which is (3.7).

Second assume (3.7). Then by (4.8) we obtain

$$< \phi_1, g^*v' >_{F_2',F_2} = < \phi_1, g^*(C_1u(f)', ..., C_nu(f)') >_{F_2',F_2}$$

for any $\phi_1 \in F_2'$, which implies (3.6).

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