# DETERMINATION OF POINT WAVE SOURCES BY POINTWISE OBSERVATIONS : STABILITY AND RECONSTRUCTION

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*Dedicated to the memory of Professor Dr. Siegfried Prößdorf*

Abstract. We consider a wave equation with point source terms:

$$\left\{\begin{array}{ll} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + \lambda(t)\displaystyle\sum_{k=1}^{N} \alpha_k \delta(x-x_k), & 0<x<1,\, 0<t<T \\ u(x,0)=0, \quad u'(x,0)=0, \qquad 0<x<1 & \\ u(0,t)=u(1,t)=0, \qquad 0<t<T & \end{array}\right\}$$

where $\lambda \in C^1[0,T]$ is a known function such that $\lambda(0) \neq 0$, $\alpha_k \in \mathbb{R}$, $\delta(\cdot - x_k)$ is the Dirac delta function at $x_k$, $1 \le k \le N$. We discuss the inverse problem of determining point sources $\{N, \alpha_1, ..., \alpha_N, x_1, ...., x_N\}$ or $\{x_1, ...., x_N\}$ from observation data $u(\eta, t)$, $0 < t < T$ with given $\eta \in (0,1)$ and $T > 0$.

We prove uniqueness and stabilty in determining point sources in terms of the norm in $H^1(0,T)$ of observations. The uniqueness result requires that $\eta$ is an irrational number and $T \ge 1$, and our stability result further needs a-priori (but reasonable) informations of unknown $\{x_1, ..., x_N\}$. Moreover, we establish two schemes for reconstructing $\{x_1, ...., x_N\}$ which are stable against errors in $L^2(0,T)$.

## §1. Introduction.

In this paper, we discuss the following initial/boundary value problem for the wave equation :

$$(1.1)\qquad \left\{\begin{array}{ll} u''(x,t) = u_{xx}(x,t) + \lambda(t)\displaystyle\sum_{k=1}^{N} \alpha_k \delta(x-x_k), & 0<x<1,\, 0<t<T \\ u(x,0)=0, \quad u'(x,0)=0, \qquad 0<x<1 & \\ u(0,t)=u(1,t)=0, \qquad 0<t<T. & \end{array}\right\}$$

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Here we set $u'(x,t) = \frac{\partial u}{\partial t}(x,t)$, $u''(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t)$. Throughout this paper, $\lambda \in C^1[0,T]$ is known and we assume that $\lambda$ satisfies

$$\lambda(0) \neq 0, \tag{1.2}$$

$\alpha_k \in \mathbb{R}$, $1 \le k \le N$, $\delta(\cdot - x_k)$ is the Dirac delta function at $x_k$, that is,

$$< \delta(\cdot - x_k), \phi >= \phi(x_k) \quad \text{for } \phi \in C_0^\infty(0,1) \equiv \mathcal{D}(0,1).$$

Here and henceforth $< \cdot, \cdot >$ denotes the duality pairing between $\mathcal{D}'(0,1)$ and $\mathcal{D}(0,1)$, the dual of $\mathcal{D}(0,1)$.

We denote the dual of the Sobolev space $H_0^1(0,1)$ by $H^{-1}(0,1)$, identifying the dual of $L^2(0,1)$ with itself: $H_0^1(0,1) \subset L^2(0,1) \subset H^{-1}(0,1)$ (e.g. Lions and Magenes [16]). Henceforth $< \cdot, \cdot >_{H^{-1},H_0^1}$ denotes the duality paring between $H^{-1}(0,1)$ and $H_0^1(0,1)$. By the embedding theorem (e.g. Adams [1]) we have $H_0^1(0,1) \subset C[0,1]$ and so $\delta(\cdot - x_k) \in H^{-1}(0,1)$. Therefore

$$\sum_{k=1}^N \alpha_k \delta(\cdot - x_k) \in H^{-1}(0,1). \tag{1.3}$$

We can define the weak solution to (1.1) by the transposition method (e.g. Komornik [12], Lasiecka, Lions and Triggiani [13], Lions [15], Lions and Magenes [16]): We call $u = u(x,t)$ a weak solution to (1.1) if $u \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ and for any $(\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ we have

$$\begin{aligned} &< -u'(\cdot,t), \psi(\psi_0,\psi_1)(\cdot,t) >_{H^{-1},H_0^1} + (u(\cdot,t), \psi'(\psi_0,\psi_1)(\cdot,t))_{L^2(0,1)} \\ &+ \int_0^t \lambda(t) < \sum_{k=1}^N \alpha_k \delta(\cdot - x_k), \psi(\psi_0,\psi_1)(\cdot,t) >_{H^{-1},H_0^1} dt = 0, \quad 0 < t < T \end{aligned} \tag{1.4}$$

where $\psi(\psi_0,\psi_1) \in C([0,T]; H_0^1(0,1)) \cap C^1([0,T]; L^2(0,1))$ is the solution to

$$\left\{ \begin{array}{l} \psi''(x,t) = \psi_{xx}(x,t), \qquad 0 < x < 1,\ 0 < t < T \\ \psi(x,0) = \psi_0(x), \quad \psi'(x,0) = \psi_1(x), \qquad 0 < x < 1 \\ \psi(0,t) = \psi(1,t) = 0, \qquad 0 < t < T. \end{array} \right\} \tag{1.5}$$

For the existence of a unique $\psi(\psi_0,\psi_1)$, we can refer to [12], [16], for example. We set

$$P = \{N, \alpha_1, ..., \alpha_N, x_1, ..., x_N\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^N \times (0,1)^N.$$

Throughout this paper we assume that $x_1, ..., x_N$ in $P$ are mutually distinct. It is proved (e.g. [12], [13], [15]) that there actually exists a unique weak solution $u$ to (1.1), denoted by $u = u(P)(x,t)$, and

$$u(P) \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1)) \tag{1.6}$$

and there exists a constant $C_1 = C_1(T) > 0$ such that

$$\begin{aligned} &\|u(P)\|_{C([0,T];L^2(0,1))} + \|u(P)'\|_{C([0,T];H^{-1}(0,1))} \\ \le &C_1 \left\| \sum_{k=1}^{N} \alpha_k \delta(\cdot - x_k) \right\|_{H^{-1}(0,1)} \end{aligned} \tag{1.7}$$

for any $N \in \mathbb{N}$, $\alpha_k \in \mathbb{R}$, $x_k \in (0,1)$, $1 \le k \le N$. As is seen from Lemmata 3.1 and 3.2 in Section 3,

$$u(P) \in C_t^1([0,T]; L_x^2(0,1)) \cap C_x([0,1]; H_t^1(0,T)) \tag{1.8}$$

and especially,

$$u(P)(\eta, \cdot) \in H^1(0,T) \tag{1.9}$$

for an arbitrarily fixed $\eta \in (0,1)$.

The purpose of this paper is to discuss the

**Determination of point wave sources by pointwise observations.** Let $\eta \in (0,1)$ and $T > 0$ be given. Then we are required to determine

$$P = \{N, \alpha_1, ..., \alpha_N, x_1, ..., x_N\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^N \times (0,1)^N$$

from the pointwise observation

$$u(P)(\eta, t) \quad 0 < t < T.$$

More precisely, let us discuss the following three subjects for the inverse problem. Let $P = \{N, \alpha_1, ..., \alpha_N, x_1, ..., x_N\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^N \times (0,1)^N$ and $Q = \{M, \beta_1, ..., \beta_M, y_1, ..., y_M\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^M \times (0,1)^M$.

(I) (Uniqueness)

Does $u(P)(\eta, t) = u(Q)(\eta, t)$, $0 < t < T$ imply $P = Q$, namely,

$$M = N, \quad \alpha_k = \beta_k, \quad x_k = y_k, \quad 1 \le k \le N$$

after renumbering of $\{\beta_k, y_k\}_{1 \le k \le N}$ if necessary? We should determine conditions on an observation point $\eta$ and time length $T > 0$ guaranteeing the uniqueness.

(II) (Stability)

We estimate

$$\sum_{k=1}^{N} |\alpha_k - \beta_k| + \sum_{k=1}^{N} |x_k - y_k| \tag{1.10}$$

by an appropriate norm of $u(P)(\eta, \cdot) - u(Q)(\eta, \cdot)$ provided that $\eta \in (0,1)$ and $T > 0$ guarantee the uniqueness in (I).

(III) (Regularization)
We establish reconstruction schemes which are stable against $L^2$-errors of observation data.

As is seen from Theorem 2, for estimating the quantity in (1.10), it is necessary to take a stronger norm of observation errors than the norm of $L^2(0,T)$. For (III) we discuss reconstruction schemes on the basis of regularization by truncated singular value decomposition and regularization by discretization.

In the system (1.1), the $N$-point sources $\lambda(t)\sum_{k=1}^N \delta(x-x_k)$ with weights $\alpha_k$, $1 \le k \le N$, initiate the one dimensional vibration which is in the equilibrium at $t=0$. This system is related, for example, to a model of earthquakes (e.g. Aki and Richards [2]) although in such a model first of all we should consider a three dimensional Lamé system.

The system (1.1) can be rewritten in a general form:

$$
\text{(1.1')} \qquad \left\{
\begin{array}{ll}
u''(x,t) = u_{xx}(x,t) + \lambda(t)f(x), & 0<x<1,\ 0<t<T \\
u(x,0)=0, \quad u'(x,0)=0, & 0<x<1 \\
u(0,t)=u(1,t)=0, & 0<t<T.
\end{array}
\right\}
$$

In this paper, $f$ is assumed to be a linear combination of delta functions. On the other hand, as far as $f$ is an $L^2$-function, similar inverse problems are discussed in Yamamoto [21], and a detailed structure of the ill-posedness of the inverse problem is studied in Yamamoto [23]. In Yamamoto [24], an inverse problem similar to [21], is considered in the case where $f$ is an $L^2$-function and $\lambda$ depends also on $x$. For an inverse problem for the Lamé equation, we can refer to Grasselli and Yamamoto [10].

The remainder of this paper is composed of six sections and an appendix. In Section 2 we state main results for the uniqueness and the stability. In Section 3, we give preliminaries for the proof and in Sections 4 and 5 we prove the main results. In Section 6, we treat a simplified determination problem where $N=1$ and $\alpha_1=1$, and we prove a sharper result for the uniqueness and the stability. Finally in Section 7, we discuss two kinds of regularization methods under decomposition of the problem into a well-posed part and an ill-posed part.

Our technical keys in the uniqueness and stability are Duhamel's principle which reduces our inverse problem to the determination of initial values, a classical result by Ingham [11] concerning the non-harmonic Fourier analysis and a result on Diophantine approximation in number theory.

## §2. Main results.

We state our main results on uniqueness and stability. Let us remember that $u(P)$ is the weak solution to (1.1) with $P=\{N,\alpha_1,...,\alpha_N,x_1,...,x_N\} \in \mathbb{N}\times(\mathbb{R}^N\setminus\{0\})^N\times(0,1)^N$.

**Theorem 1.** *(Uniqueness) Let* $P=\{N,\alpha_1,...,\alpha_N,x_1,...,x_N\} \in \mathbb{N}\times(\mathbb{R}^N\setminus\{0\})^N\times(0,1)^N$ *and* $Q=\{M,\beta_1,...,\beta_M,y_1,...,y_M\}\in\mathbb{N}\times(\mathbb{R}\setminus\{0\})^M\times(0,1)^M$. *Let*

(2.1) $\qquad \eta$ *be an irrational number* *and*

$$T \geq 1. \tag{2.2}$$

*Then*

$$u(P)(\eta, t) = u(Q)(\eta, t), \qquad 0 < t < T \tag{2.3}$$

*implies*

$$P = Q, \tag{2.4}$$

*namely,*

$$M = N, \quad \beta_k = \alpha_k, \quad y_k = x_k \qquad 1 \leq k \leq N$$

*after renumbering* $(\beta_k, y_k)$, $1 \leq k \leq N$ *if necessary.*

For the uniqueness, the theorem requires that the observation time $T$ is greater than or equal to one, the travelling time for which the wave from one end $x = 0$ reaches another end $x = 1$. In this sense, the condition (2.2) is physically understandable.

For the stability, we pose a strict condition:

$$\eta \text{ is an irrational algebraic number.} \tag{2.5}$$

Here $\eta \in (0, 1)$ is called an algebraic number if $\eta$ is a root of an algebraic equation with integer coefficients (e.g. Baker [3], [4]).

Moreover, for the statement of stability, we introduce a-priori informations for point sources: We assume

$$M = N, \quad \beta_k = \alpha_k \neq 0, \qquad 1 \leq k \leq N. \tag{2.6}$$

In other words, we exclusively discuss the estimation of point source locations. We number $\{x_1, ..., x_N\}$ and $\{y_1, ..., y_N\}$ as

$$x_1 < .... < x_N, \quad y_1 < .... < y_N. \tag{2.7}$$

As an a-priori assumption, we suppose that there is a small $\epsilon > 0$ such that

$$x_{i+1} - x_i > 3\epsilon, \qquad 1 \leq i \leq N - 1, \tag{2.8}$$

$$|x_i - y_i| < \frac{\epsilon}{3}, \qquad 1 \leq i \leq N \tag{2.9}$$

and

$$2\epsilon < x_1, x_N < 1 - 2\epsilon. \tag{2.10}$$

It is trivial that $\epsilon < \frac{1}{3N+1}$ and so we must assume that $\epsilon$ is smaller if $N$ is greater. The a-priori assumption (2.9) means that $\{x_1, ..., x_N\}$ and $\{y_1, ..., y_N\}$ are not very far from each other.

Now we are ready to state our stability result

**Theorem 2.** *(Conditional stability)*
*Let us assume (2.2), (2.5), (2.6) - (2.10). Then we have*

$$\sum_{i=1}^{N}|x_i - y_i| \le \frac{C}{\sqrt{\epsilon}}\|u(P)(\eta,\cdot) - u(Q)(\eta,\cdot)\|_{H^1(0,T)} \tag{2.11}$$

*where* $C = C(T, N, \alpha_1, ..., \alpha_N) > 0$ *is independent of* $x_k, y_k$, $1 \le k \le N$ *and* $\epsilon$.

The constant in our estimate (2.11) is bigger if $\epsilon$ is smaller. This means that the estimate (2.11) becomes worse although our a-priori information (2.9) is improved.

This theorem asserts stability under a-priori informations (2.6) - (2.10), and such stability is called conditional stability. In (2.11), the norm in the right hand side is finite by (1.9). If we take other a-priori informations, then the resulting stability conclusion may be changed.

It is well-known that the measure of algebraic numbers in $(0,1)$ is zero. Therefore the assumption (2.5) is very restrictive in choosing an observation point. Thus we should discuss the transcendental $\eta \in (0,1)$. However in the transcendental case, the rate of stability is very sensitive to the choice, as the Diophantine approximation suggests (e.g. Baker [3]), and the unified statement for the general transcendental $\eta$ is very difficult (e.g. Yamamoto [22]). In a special case of $M = N = 1$, we can obtain sharper results for the uniqueness and the stabilty. Such a special case is discussed in Section 6.

## §3. Preliminaries for the proof.

For the proofs of Theorems 1 and 2, in this section, we introduce operators and establish a representation formula of solutions by means of eigenfunctions and Duhamel's principle. Throughout this paper, all functions are assumed to be real-valued, and $L^2(a,b)$ and $H^s(a,b)$, $H^s_0(a,b)$ are the usual $L^2$-space and Sobolev spaces, respectively. Identifying the dual of $L^2(a,b)$ with itself, we denote the dual of $H^s_0(a,b)$ by $H^{-s}(a,b)$ (e.g. Lions and Magenes [16]).

We define an operator $A$ in $L^2(0,1)$ by

$$(Au)(x) = -\frac{d^2u}{dx^2}(x), \quad 0 < x < 1, \qquad \mathcal{D}(A) = H^2(0,1) \cap H^1_0(0,1). \tag{3.1}$$

Then we can define the fractional power $A^\alpha$ for any $\alpha \in \mathbb{R}$ (e.g Pazy [17]). For $\alpha > 0$, it follows that $A^{-\alpha}$ is bounded from $L^2(0,1)$ to itself and it is known (e.g. Fujiwara [9], Lions and Magenes [16]) that the completion of $L^2(0,1)$ by the norm $\|A^{-\alpha}u\|_{L^2(0,1)}$, is $H^{-2\alpha}(0,1)$. Furthermore there exists a constant $C_2 > 0$ such that

$$C_2^{-1}\|A^{\frac{1}{2}}u\|_{L^2(0,1)} \le \|u\|_{H^1_0(0,1)} \le C_2\|A^{\frac{1}{2}}u\|_{L^2(0,1)}, \quad u \in H^1_0(0,1). \tag{3.2}$$

For $P = \{N, \alpha_1, ..., \alpha_N, x_1, ...., x_N\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^N \times (0,1)^N$, we set

$$f(x) = \sum_{k=1}^{N} \alpha_k \delta(x - x_k) \tag{3.3}$$

for simplicity. Then we note

$$f \in H^{-1}(0,1) \tag{3.4}$$

by (1.3). For the original system (1.1), we consider

$$\left\{ \begin{array}{l} w''(x,t) = w_{xx}(x,t), \qquad 0 < x < 1,\, 0 < t < T \\ w(x,0) = 0, \quad w'(x,0) = f(x), \qquad 0 < x < 1 \\ w(0,t) = w(1,t) = 0, \qquad 0 < t < T. \end{array} \right\} \tag{3.5}$$

Then by the transposition method similar to (1.4), we see that there exists a unique weak solution $w = w(P) \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(0,1))$ to (3.5) (e.g. Komornik [12]). More precisely, $w = w(P)$ satisfies

$$\begin{aligned} &- < w'(\cdot,t), \psi(\cdot,t) >_{H^{-1},H_0^1} + (w(\cdot,t), \psi'(\cdot,t))_{L^2(0,1)} \\ &+ < f, \phi_0 >_{H^{-1},H_0^1} = 0, \qquad 0 < t < T \end{aligned} \tag{3.6}$$

for any solution $\psi = \psi(\phi_0, \phi_1)$ to (1.5).

Moreover, for $w(P), u(P) \in C([0,T]; L^2(0,1)) \cap C^1([0,T]; H^{-1}(0,1))$ we can apply Duhamel's principle (e.g. Rauch [19]) in a weak form.

**Lemma 3.1.** *Let*

$$\lambda \in C^1[0,T].$$

*Then*

$$u(P)(x,t) = \int_0^t \lambda(t-s) w(P)(x,s) ds, \quad 0 < x < 1,\, 0 < t < T. \tag{3.7}$$

In view of this lemma, it suffices to consider (3.5) in order to establish the representation formula of the solution $u(P)$.

Henceforth we set

$$\phi_k(x) = \sqrt{2} \sin k\pi x, \qquad 0 < x < 1,\, k \in \mathbb{N}. \tag{3.8}$$

**Lemma 3.2.** *We have*

$$w(P)(x,t) = \sum_{k=1}^{\infty} \frac{1}{k\pi} \left( \sum_{j=1}^{N} \alpha_j \phi_k(x_j) \right) \phi_k(x) \sin k\pi t, \tag{3.9}$$

*where the series is convergent in* $C([0,T]; L^2(0,1)) \cap C([0,1]; L^2(0,T))$.

**Proof of Lemma 3.2.** It is sufficient to prove the lemma in the case of $f(x) = \delta(x - x_1)$. Let us denote the right-hand side of (3.9) by $v = v(x,t)$. We can easily see that $v$ is convergent in $C([0,T]; L^2(0,1))$ and that $\sum_{k=1}^{\infty} \phi_k(x_1)\phi_k(x) \cos k\pi t$ is convergent in $C([0,T];H^{-1}(0,1))$. Therefore we have to verify that $v$ satisfies (3.6) for any $\psi_0 \in H_0^1(0,1)$ and $\psi_1 \in L^2(0,1)$. In terms of an eigenfunction expansion, we have

$$\psi(x,t) = \psi(\psi_0,\psi_1)(x,t)$$
$$= \sum_{k=1}^{\infty} (\psi_0,\phi_k)_{L^2(0,1)} \cos k\pi t \phi_k(x) + \sum_{k=1}^{\infty} \frac{(\psi_1,\phi_k)_{L^2(0,1)}}{k\pi} \sin k\pi t \phi_k(x).$$

By means of (3.1), we have

$$< \phi_k, \phi_l >_{H^{-1},H_0^1} = (A^{-\frac{1}{2}}\phi_k, A^{\frac{1}{2}}\phi_l)_{L^2(0,1)} = (\phi_k,\phi_l)_{L^2(0,1)} = \begin{cases} 1, & k = l \\ 0, & k \neq l, \end{cases}$$

so that by direct substitution we obtain

$$< -v'(\cdot,t), \psi(\cdot,t) >_{H^{-1},H_0^1} + (v(\cdot,t), \psi'(\cdot,t))_{L^2(0,1)}$$
$$= - \sum_{k=1}^{\infty} \phi_k(x_1)(\psi_0,\phi_k)_{L^2(0,1)}.$$

On the other hand, since

$$\psi_0(x) = \sum_{k=1}^{\infty} (\psi_0,\phi_k)_{L^2(0,1)} \phi_k(x), \quad 0 < x < 1$$

converges in $H_0^1(0,1)$, we obtain

$$\psi_0(x_1) = \sum_{k=1}^{\infty} (\psi_0,\phi_k)_{L^2(0,1)} \phi_k(x_1)$$

by the Sobolev embedding. Therefore we have

$$< -v'(\cdot,t), \psi(\cdot,t) >_{H^{-1},H_0^1} + (v(\cdot,t), \psi'(\cdot,t))_{L^2(0,1)}$$
$$= -\, \psi_0(x_1) = -< f, \psi_0 >_{H^{-1},H_0^1},$$

which implies (3.6).

Finally we have to prove that the series in (3.9) is convergent in $C([0,1];L^2(0,T))$. To this end, we show Lemma 3.3 which is a direct consequence of a classical result by Ingham [11].

**Lemma 3.3.**

*(1) For any $T > 0$, there exists a constant $C_3 = C_3(T) > 0$ such that*

$$\int_0^T \left| \sum_{k=1}^{\infty} a_k \sin k\pi t \right|^2 dt \le C_3 \sum_{k=1}^{\infty} a_k^2, \quad a_k \in \mathbb{R}. \tag{3.10}$$

*(2) Let*

$$T \ge 1. \tag{3.11}$$

*Then there exists a constant $C_4 = C_4(T) > 0$ such that*

$$\sum_{k=1}^{\infty} a_k^2 \le C_4 \int_0^T \left| \sum_{k=1}^{\infty} a_k \sin k\pi t \right|^2 dt. \tag{3.12}$$

**Proof of Lemma 3.3.** Setting

$$\lambda_k = k\pi, \quad k \in \mathbb{Z}$$

and

$$b_k = \begin{cases} \frac{a_k}{2\sqrt{-1}}, & k \ge 1 \\ -\frac{a_{-k}}{2\sqrt{-1}}, & k \le -1, \end{cases}$$

we see

$$\sum_{k=1}^{\infty} a_k \sin k\pi t = \sum_{k=-\infty, k\ne 0}^{\infty} b_k \exp(\sqrt{-1}\lambda_k t)$$

and

$$\int_{-T}^{T} \left| \sum_{k=-\infty,k\ne 0}^{\infty} b_k \exp(\sqrt{-1}\lambda_k t)\right|^2 dt = 2\int_0^T \left| \sum_{k=-\infty,k\ne 0}^{\infty} b_k \exp(\sqrt{-1}\lambda_k t)\right|^2 dt.$$

Thus direct application of Theorems 1 and 2 in Ingham [11] leads to Lemma 3.3.

Henceforth we denote a generic constant depending on $\lambda$ and $T$ by $C_5 = C_5(T)$. Now we return to the proof of Lemma 3.2. By (1) of Lemma 3.3, we see that for any $0 \le x \le 1$

$$\sup_{0\le x\le 1} \left\| \sum_{k=n}^{m} \frac{1}{k\pi}\phi_k(x_1)\phi_k(x)\sin k\pi t \right\|^2_{L^2(0,T)} \le C_5 \sup_{0\le x \le 1} \left( \sum_{k=n}^{m} \frac{1}{k^2\pi^2}\phi_k(x_1)^2\phi_k(x)^2 \right)$$
$$\le C_5 \sum_{k=n}^{m} \frac{1}{k^2\pi^2} \longrightarrow 0$$

as $m, n \longrightarrow \infty$. Thus the proof of Lemma 3.2 is complete.

By applying Lemma 3.2 in Lemma 3.1, we see that for any $\eta \in (0,1)$,

$$u(f)(\eta, t) = \int_0^t \lambda(t-s) w(f)(\eta, s) ds, \quad 0 < t < T. \tag{3.13}$$

Therefore by taking $t$-derivatives of both the sides of (3.13), we obtain

$$u(f)'(\eta,t) = \lambda(0)w(f)(\eta,t) + \int_0^t \lambda'(t-s)w(f)(\eta,s)ds, \quad 0 < t < T.$$

Since $\lambda(0) \ne 0$, this is a Volterra integral equation of the second kind, and we can uniquely solve it. Moreover,

$$\begin{aligned} &C_5^{-1}\|w(f)(\eta,\cdot)\|_{L^2(0,T)} \le \|u(f)(\eta,\cdot)\|_{H^1(0,T)} \\ &\le C_5 \|w(f)(\eta,\cdot)\|_{L^2(0,T)} \end{aligned} \tag{3.14}$$

holds for any $f = \sum_{j=1}^{N} \alpha_j \delta(\cdot - x_j)$. Thus for our inverse problem, it is sufficient to consider the following

**Reduced Inverse Problem.**

Let $w(P) = w(P)(x,t)$ be the weak solution to

$$\left\{ \begin{array}{l} w''(x,t) = w_{xx}(x,t), \qquad 0 < x < 1,\ 0 < t < T \\ w(x,0) = 0, \quad w'(x,0) = \sum_{k=1}^{N} \alpha_k \delta(x - x_k), \qquad 0 < x < 1 \\ w(0,t) = w(1,t) = 0, \qquad 0 < t < T \end{array} \right\} \tag{3.15}$$

where $P = \{N, \alpha_1, ..., \alpha_N, x_1, ..., x_N\}$. Let $\eta \in (0,1)$ be given, and let $P = \{N, \alpha_1, ..., \alpha_N, x_1, ..., x_N\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^N \times (0,1)^N$ and $Q = \{M, \beta_1, ..., \beta_M, y_1, ..., y_M\} \in \mathbb{N} \times (\mathbb{R} \setminus \{0\})^M \times (0,1)^M$.

(I) (Uniqueness)

Does

$$w(P)(\eta, t) = w(Q)(\eta, t), \quad 0 < t < T \tag{3.16}$$

imply

$$P = Q, \tag{3.17}$$

namely,

$$M = N, \quad \alpha_k = \beta_k, \quad x_k = y_k, \quad 1 \le k \le N? \tag{3.18}$$

(II) (Stability)

Can we estimate

$$\sum_{k=1}^{N} |\alpha_k - \beta_k| + \sum_{k=1}^{N} |x_k - y_k| \tag{3.19}$$

by an appropriate norm of $w(P)(\eta, \cdot) - w(Q)(\eta, \cdot)$?

## §4. Proof of Theorem 1.

Let $\eta \in (0,1)$ be irrational and let $T \ge 1$. We assume

$$w(P)(\eta, t) = w(Q)(\eta, t), \qquad 0 < t < T. \tag{4.1}$$

Then by Lemma 3.2, we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k\pi} < f, \phi_k >_{H^{-1}, H_0^1} \phi_k(\eta) \sin k\pi t = 0,$$

where $f = \sum_{j=1}^{N} \alpha_j \delta(\cdot - x_j) - \sum_{j=1}^{M} \beta_j \delta(\cdot - z_j)$. By $T \ge 1$, Lemma 3.3 (2) implies

$$< f, \phi_k >_{H^{-1}, H_0^1} \phi_k(\eta) = 0, \qquad k \in \mathbb{N}.$$

Since $\eta$ is irrational, $\phi_k(\eta) = \sqrt{2} \sin k\pi\eta \ne 0$, $k \in \mathbb{N}$, so that

$$< f, \phi_k >_{H^{-1}, H_0^1} = 0, \qquad k \in \mathbb{N}. \tag{4.2}$$

Since Span $\{\phi_k\}_{k \ge 1}$ is dense in $H_0^1(0,1)$, the equation (4.2) yields

$$< f, v >_{H^{-1}, H_0^1} = 0$$

for any $v \in H_0^1(0,1)$, namely, $f = 0$ in $H^{-1}(0,1)$. Therefore (4.1) implies $P = Q$.

**§5. Proof of Theorem 2.**

In this section, for the proof, setting

$$f = \sum_{k=1}^{N} \alpha_k \delta(\cdot - x_k) - \sum_{k=1}^{N} \alpha_k \delta(\cdot - y_k) \in H^{-1}(0,1), \tag{5.1}$$

we consider

$$\left\{ \begin{array}{ll} w''(x,t) = w_{xx}(x,t), & 0 < x < 1,\ 0 < t < T \\ w(x,0) = 0, \quad w'(x,0) = f(x), & 0 < x < 1 \\ w(0,t) = w(1,t) = 0, & 0 < t < T \end{array} \right\} \tag{5.2}$$

for $f \in H^{-1}(0,1)$. Then

**Lemma 5.1.** *Let us assume (2.1) and (2.2). Then there exists a constant $C_6 = C_6(T) > 0$ such that*

$$\|f\|_{H^{-2}(0,1)} \le C_6 \|w(f)(\eta, \cdot)\|_{L^2(0,T)}. \tag{5.3}$$

**Proof of Lemma 5.1.** Since $f \in H^{-1}(0,1)$, there exists a unique $F \in H_0^1(0,1)$ such that

$$AF = f. \tag{5.4}$$

We recall that $A$ is defined by (3.1). We set

$$W(F)(x,t) = \int_0^t w(f)(x,s)ds + F(x), \quad 0 < x < 1,\ 0 < t < T.$$

Since $A : L^2(0,1) \longrightarrow H^{-2}(0,1)$ is an isomorphism, we can take a constant $C_7 > 0$ independent of $f$ such that

$$\|f\|_{H^{-2}(0,1)} \le C_7 \|A^{-1} f\|_{L^2(0,1)} = C_7 \|F\|_{L^2(0,1)}.$$

Therefore it is sufficient to prove that

$$\|F\|_{L^2(0,1)} \le C_6 \|W(f)'(\eta, \cdot)\|_{L^2(0,T)}. \tag{5.5}$$

On the other hand, by the definition (3.6) of the weak solution, we can directly see that $W(F)$ is the weak solution to

$$\left\{ \begin{array}{ll} W''(x,t) = W_{xx}(x,t), & 0 < x < 1,\ 0 < t < T \\ W(x,0) = F(x), \quad W'(x,0) = 0, & 0 < x < 1 \\ W(0,t) = W(1,t) = 0, & 0 < t < T. \end{array} \right\} \tag{5.6}$$

Moreover by the eigenfunction expansion, we obtain

$$W(F)'(\eta,t)=\sum_{k=1}^{\infty}-k\pi(F,\phi_k)_{L^2(0,1)}\phi_k(\eta)\sin k\pi t \tag{5.7}$$

where the series is convergent in $L^2(0,T)$. In fact, since $F\in H_0^1(0,1)$, we have $k\pi(F,\phi_k)_{L^2(0,1)}=(F',\sqrt{2}\cos k\pi x)_{L^2(0,1)}$, $k\in\mathbb{N}$, by integration by parts. Since $\{\sqrt{2}\cos k\pi x\}_{k\in\mathbb{N}}$ is an orthonormal system and $F'\in L^2(0,1)$, we obtain

$$\sum_{k=1}^{\infty}k^2\pi^2(F,\phi_k)^2_{L^2(0,1)}\le\|F'\|^2_{L^2(0,1)}.$$

By applying Lemma 3.3 (1) and using this inequality and $|\phi_k(\eta)|\le\sqrt{2}$ for $k\in\mathbb{N}$, we see that the series in (5.7) is convergent in $L^2(0,1)$.

Let us complete the proof of Lemma 5.1. By Lemma 3.3 (2) and (5.7), we obtain

$$\sum_{k=1}^{\infty}(F,\phi_k)^2_{L^2(0,1)}(k\pi\phi_k(\eta))^2\le C_7\|W(F)'(\eta,\cdot)\|^2_{L^2(0,T)}. \tag{5.8}$$

On the other hand, by (2.5), Roth's theorem of Diophantine approximation (e.g. Baker [3], [4]) applies to obtain

$$\|k\eta\|\ge\frac{C_8}{k},\quad k\in\mathbb{N} \tag{5.9}$$

with a constant $C_8$ independent of $k\in\mathbb{N}$. Here and henceforth $\|k\eta\|$ denotes the distance between $k\eta$ and the nearest integer. Furthermore, for any $k\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that $k\eta=m+\|k\eta\|$ or $k\eta=m-\|k\eta\|$, so that

$$|\sin k\pi\eta|=|\sin\pi(m\pm\|k\eta\|)|=|\sin\pi\|k\eta\||.$$

Since $0\le\pi\|k\eta\|\le\frac{\pi}{2}$ by the definition of $\|k\eta\|$, we obtain

$$\frac{2}{\pi}\le\left|\frac{\sin\pi\|k\eta\|}{\pi\|k\eta\|}\right|=\frac{|\sin k\pi\eta|}{\pi\|k\eta\|},$$

namely,

$$|\sin k\pi\eta|\ge 2\|k\eta\|.$$

Combining this with (5.9), we obtain a constant $C_9>0$ independent of $k$, such that

$$|\sqrt{2}\sin k\pi\eta|\ge\frac{C_9}{k\pi},\quad k\in\mathbb{N}. \tag{5.10}$$

Substituting (5.10) into (5.8), we obtain

$$\sum_{k=1}^{\infty}(F,\phi_k)^2_{L^2(0,1)}\le C_7C_9^{-2}\|W(F)'(\eta,\cdot)\|^2_{L^2(0,T)},$$

implying the assertion of Lemma 5.1 by using the Parseval equality. Thus the proof of Lemma 5.1 is complete.

**Proof of Theorem 2.** By (3.14), it is sufficient to prove

$$
\begin{aligned}
|x_i - y_i| \le & \frac{C_5}{\sqrt{\epsilon}} \|w(P)(\eta,\cdot) - w(Q)(\eta,\cdot)\|_{L^2(0,T)} \\
\le & \frac{C_5}{\sqrt{\epsilon}} \|w(f)(\eta,\cdot)\|_{L^2(0,T)}.
\end{aligned} \tag{5.11}
$$

We use the notation (5.1). By the definition of $\|\cdot\|_{H^{-2}(0,1)}$, we have

$$
| < f, \mu >_{H^{-2},H_0^2} | \le \|f\|_{H^{-2}(0,1)} \|\mu\|_{H_0^2(0,1)},
$$

for $\mu \in H_0^2(0,1)$, where $< \cdot,\cdot >_{H^{-2},H_0^2}$ denotes the duality pairing between $H^{-2}(0,1)$ and $H_0^2(0,1)$. Therefore, applying Lemma 5.1, we obtain

$$
\left| \sum_{k=1}^{N} \alpha_k (\mu(x_k) - \mu(y_k)) \right| \le C_6 \|\mu\|_{H_0^2(0,1)} \|w(f)(\eta,\cdot)\|_{L^2(0,T)}, \ \mu \in H_0^2(0,1). \tag{5.12}
$$

Thus for the proof of (5.11), we have to choose suitable $\mu \in H_0^2(0,1)$ such that $\frac{d\mu}{dx}(x) > 0$ for $x \in (x_i - \frac{\epsilon}{3}, x_i + \frac{\epsilon}{3})$. To this end, for $1 \le i \le N$, we choose $\mu_i \in H_0^2(0,1)$ such that

$$
\mu_i(x) = \begin{cases} (x - (x_i - \epsilon))^2 (x - (x_i + 2\epsilon))^2 & \text{if } x_i - \epsilon \le x \le x_i + 2\epsilon \\ 0 & \text{otherwise.} \end{cases}
$$

Then by direct computations, we can obtain : there exists a constant $C_{10} > 0$ independent of $\epsilon > 0$ such that

$$
\inf_{x_i - \frac{\epsilon}{3} \le x \le x_i + \frac{\epsilon}{3}} \left| \frac{d\mu_i}{dx}(x) \right| \ge C_{10} \epsilon^3 \tag{5.13}
$$

and

$$
\|\mu_i\|_{H_0^2(0,1)} \le C_{11} \epsilon^{\frac{5}{2}}. \tag{5.14}
$$

Let us fix $1 \le i \le N$ and let us substitute $\mu = \mu_i$ into (5.12). By (2.8) - (2.10), we see that $\mu_i(x_j) = \mu_i(y_j) = 0$, $i \ne j$, so that

$$
|\alpha_i (\mu_i(x_i) - \mu_i(y_i))| \le C_6 \|\mu_i\|_{H_0^2(0,1)} \|w(f)(\eta,\cdot)\|_{L^2(0,T)}.
$$

Therefore by (2.9) and the mean value theorem, it follows from (5.13) and (5.14) that

$$
\begin{aligned}
|x_i - y_i| \le & C_6 \left( \inf_{x_i - \frac{\epsilon}{3} \le x \le x_i + \frac{\epsilon}{3}} \left| \frac{d\mu_i}{dx}(x) \right| \right)^{-1} \alpha_i^{-1} \|\mu_i\|_{H_0^2(0,1)} \|w(f)(\eta,\cdot)\|_{L^2(0,T)} \\
\le & C_6 C_{10}^{-1} C_{11} \alpha_i^{-1} \epsilon^{-3} \epsilon^{\frac{5}{2}} \|w(f)(\eta,\cdot)\|_{L^2(0,T)},
\end{aligned}
$$

which is (5.11). Thus the proof of Theorem 2 is complete.

**§6. Determination of a single point source.**

In this section, we consider a simple case of $N = 1$ and $\alpha_1 = 1$, that is,

$$\left\{ \begin{array}{l} u''(x,t) = u_{xx}(x,t) + \lambda(t)\delta(x - x_1), \qquad 0 < x < 1,\ 0 < t < T \\ u(x,0) = 0, \quad u'(x,0) = 0, \qquad 0 < x < 1 \\ u(0,t) = u(1,t) = 0, \qquad 0 < t < T \end{array} \right\} \tag{6.1}$$

where $x_1 \in (0,1)$ be an unknown source point. Let us denote the weak solution to (6.1) by $u(x_1) = u(x_1)(x,t)$. Then by Lemmata 3.2 and 3.3, we see

$$u(x_1)(\cdot,\cdot) \in C^1([0,T]; L^2(0,1)) \cap C([0,1]; H^1(0,T)). \tag{6.2}$$

Our simplified inverse problem consists in the determination of $x_1 \in (0,1)$ from $u(x_1)(\eta,t)$, $0 < t < T$ at a fixed observation point $\eta \in (0,1)$.

**Theorem 3.** *(Uniqueness)*
*Let*

$$\lambda(0) \ne 0 \tag{6.3}$$

*and*

$$T \ge 1. \tag{6.4}$$

*(1) If*

$$\eta \ne \frac{1}{2}, 0, 1, \tag{6.5}$$

*then*

$$u(x_1)(\eta,t) = u(y_1)(\eta,t), \qquad 0 < t < T \tag{6.6}$$

*implies* $x_1 = y_1$.
*(2) Let* $\eta = \frac{1}{2}$. *Then*

$$u(x_1)(\frac{1}{2},t) = u(y_1)(\frac{1}{2},t), \qquad 0 < t < T$$

*if and only if*

$$y_1 = x_1 \quad or \quad y_1 = 1 - x_1.$$

**Theorem 4.** *(Conditional stability)*
*Let us a-priori assume that*

$$|x_1 + y_1 - 1| \ge \epsilon \tag{6.7}$$

*for some* $\epsilon > 0$ *and let*

$$\eta \neq \frac{1}{2}, 0, 1.$$

*Then*

$$|x_1 - y_1| \le \frac{C}{\sin\frac{\pi\epsilon}{2}\sin\pi\eta}\|u(x_1)(\eta,\cdot) - u(y_1)(\eta,\cdot)\|_{H^1(0,T)} \tag{6.8}$$

*where* $C = C(T) > 0$ *is independent of* $\epsilon$, $\eta$ *and* $x_1$, $y_1$.

In Theorem 3, the choice of an observation point $\eta = \frac{1}{2}$ cannot distinguish $x_1$ from $1 - y_1$, but $u(x_1)(\frac{1}{2}, t)$ can be transformed to $u(1 - x_1)(\frac{1}{2}, t)$ by a change of independent variables $x \to 1 - x$. Thus by taking the symmetry with respect to $\eta = \frac{1}{2}$ into consideration, the system with a point source at $x_1$ and the one with a point source at $1 - x_1$ naturally give the same observation data at the mid point $\eta = \frac{1}{2}$.

In Theorem 4, the condition (6.7) is an a-priori information for the unknown $x_1$ and $y_1$. In particular, if we know that both $x_1$ and $y_1$ are in a half interval $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$, then (6.7) is satisfied. If $\epsilon \to 0$, then the estimate (6.8) becomes worse and is nonsense when $\epsilon = 0$. In this sense Theorem 4 shows conditional stability. On the other hand, the norm for data, the $H^1(0,T)$-norm, is consistent with the regularity (6.2).

For the proof, we consider

$$\left\{\begin{array}{ll} w''(x,t) = w_{xx}(x,t), & 0 < x < 1,\ 0 < t < T \\ w(x,0) = 0, \quad w'(x,0) = \delta(x - x_1), & 0 < x < 1 \\ w(0,t) = w(1,t) = 0, & 0 < t < T \end{array}\right\} \tag{6.9}$$

and we denote the weak solution to (6.9) by $w(x_1) = w(x_1)(x,t)$.

**Proof of Theorem 3.** By (3.14) and Lemma 3.2, it is sufficient to prove that $w(x_1)(\eta, t) = w(y_1)(\eta, t)$, $0 < t < T$, namely,

$$\sum_{k=1}^{\infty} \frac{1}{k\pi}\phi_k(\eta)(\phi_k(x_1) - \phi_k(y_1))\sin k\pi t = 0, \quad 0 < t < T \tag{6.10}$$

implies $x_1 = y_1$.

Since $T \ge 1$, we can apply Lemma 3.3 (2) and so the equality (6.10) is equivalent to $\sin k\pi\eta(\sin k\pi x_1 - \sin k\pi y_1) = 0$, namely,

$$\sin k\pi\eta\cos\frac{k\pi(x_1 + y_1)}{2}\sin\frac{k\pi(x_1 - y_1)}{2} = 0, \quad k \in \mathbb{N}. \tag{6.11}$$

By $0 < \eta < 1$, the equality (6.11) with $k = 1$, implies

$$\cos\frac{\pi(x_1 + y_1)}{2}\sin\frac{\pi(x_1 - y_1)}{2} = 0. \tag{6.12}$$

First, let $x_1 + y_1 \neq 1$. Then $\sin\frac{\pi(x_1-y_1)}{2} = 0$. By $-1 \le x_1 - y_1 \le 1$, this implies $x_1 = y_1$.
Second, let $x_1 + y_1 = 1$. Then equality (6.11) with $k = 2$ is

$$\sin 2\pi\eta \cos\pi(x_1+y_1)\sin\pi(x_1-y_1) = -\sin 2\pi\eta\sin\pi(x_1-y_1) = 0.$$

If $\eta \neq \frac{1}{2}, 0, 1$, then $\sin\pi(x_1 - y_1) = 0$, namely, $x_1 - y_1 \in \mathbb{Z}$. Since $0 < x_1, y_1 < 1$ and $x_1 + y_1 = 1$, we see that $x_1 - y_1 = 0$. Thus the proof of (1) of the theorem is complete.

Now let us complete the proof of (2). Let $\eta = \frac{1}{2}$. If $x_1 + y_1 \neq 1$, then from $w(x_1)(\frac{1}{2}, t) = w(y_1)(\frac{1}{2}, t)$, $0 < t < T$, we easily obtain $x_1 = y_1$ by (6.12). Therefore we see that $w(x_1)(\frac{1}{2}, t) = w(y_1)(\frac{1}{2}, t)$, $0 < t < T$ implies $y_1 = 1 - x_1$ or $y_1 = x_1$. The converse in (2) is straightforward. Thus the proof of (2) is complete.

**Proof of Theorem 4.** By Lemma 3.2 and $T \ge 1$ we can apply Lemma 3.3 (2) to

$$w(x_1)(\eta,t) - w(y_1)(\eta,t) = \sum_{k=1}^{\infty}\frac{1}{k\pi}(\phi_k(x_1)-\phi_k(y_1))\phi_k(\eta)\sin k\pi t, \quad 0<t<T,$$

so that

$$\begin{aligned}&\frac{1}{\pi^2}(\phi_1(x_1)-\phi_1(y_1))^2\phi_1(\eta)^2 \le \sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}(\phi_k(x_1)-\phi_k(y_1))^2\phi_k(\eta)^2\\ \le & C_4\|w(x_1)(\eta,\cdot)-w(y_1)(\eta,\cdot)\|^2_{L^2(0,T)}.\end{aligned}$$

Therefore by $\eta \neq \frac{1}{2}, 0, 1$, we have

$$\begin{aligned}&|\sin\pi x_1 - \sin\pi y_1| = 2\left|\cos\frac{\pi(x_1+y_1)}{2}\right|\left|\sin\frac{\pi(x_1-y_1)}{2}\right|\\ \le & \frac{\sqrt{C_4}\pi}{2\sin\pi\eta}\|w(x_1)(\eta,\cdot)-w(y_1)(\eta,\cdot)\|_{L^2(0,T)}.\end{aligned}$$

By (6.7) we obtain $\left|\cos\frac{\pi(x_1+y_1)}{2}\right| \ge \sin\frac{\pi\epsilon}{2}$ and by $0 < x_1 - y_1 < 1$, we have

$$\left|\sin\frac{\pi(x_1-y_1)}{2}\right| \ge \frac{2}{\pi}\frac{\pi}{2}|x_1-y_1|.$$

Therefore in terms of (3.14), our conclusion is straightforward.

**Remark.** As is seen from the proof, we do not use (6.11) for all $k \in \mathbb{N}$. This suggests that our observation $w(x_1)(\eta, \cdot)$ can further determine more point sources like in Theorem 1.

**§7. Reconstruction of point sources from pointwise observations.**
In this section, we mainly discuss reconstruction of $x_1$, ...., $x_N$, the locations of the point sources provided that $N$ is given and $\alpha_1 = ... = \alpha_N = 1$. Henceforth we set

$$\Lambda = \{\{x_1, ...., x_N\}; 0 < x_1 < ... < x_N < 1\} \subset \mathbb{R}^N, \quad P = \{x_1, ..., x_N\} \in \Lambda.$$

Moreover we assume that

$$T \ge 1 \text{ and } \eta \text{ is irrational.} \tag{7.1}$$

Now we develop methods how $P$ can be reconstructed from observations $u(P)(\eta, t)$, $\eta$ fixed with $0 < \eta < 1$, of the considered system (1.1) in a stable way. The idea is to decompose the mapping $P \to u(P)(\eta, t)$ into a nonlinear well–posed part and a linear ill–posed part. We use the methods of Bruckner [5], [6] for the regularization of the ill–posed part and then give reconstruction formulas for the nonlinear part. Moreover, in the cases of $N = 1$ and $N = 2$, we can more explicitly give schemes.

As to given noisy data $u^\epsilon$ where $\epsilon > 0$ is a given noise level, let us consider two cases:
(i) $u^\epsilon \in L^2(0,T)$, $\|u - u^\epsilon\|_{L^2(0,T)} \le \epsilon$, $u = u(\eta, t)$.
(ii) $\underline{u}^\epsilon \in \mathbb{R}^n$, $\underline{u}^\epsilon = (u_1^\epsilon, ..., u_n^\epsilon)$, $|u(\eta, t_j) - u_j^\epsilon| \le \epsilon$, $j = 1, ..., n$, $\{t_j\}_{j=1}^n$ is an equidistant mesh on $[0, T]$.

We set

$$_0H^1(0,T) = \{u \in H^1(0,T); u(0) = 0\}. \tag{7.2}$$

From the remark following (3.13) in Section 3, it is clear that the mapping

$$\begin{aligned} &S : L^2(0,T) \to {}_0H^1(0,T) \\ &(Sw)(t) = \int_0^t \lambda(t-s)w(s)ds, \end{aligned} \tag{7.3}$$

where $\lambda \in C^1[0,T]$ satisfies $\lambda(0) \ne 0$, is an isomorphism from $L^2(0,T)$ onto ${}_0H^1(0,T)$, i.e., it is in both directions continuous and one-to-one. The inverse mapping $S^{-1}$ is obtained as the solution of a Volterra integral equation of the second kind. Moreover, if $w = w(P)(\eta, \cdot)$ is the observation of the system (3.5), then $u = Sw$ is the observation of (1.1) and vice versa.

The embedding $E : {}_0H^1(0,T) \longrightarrow L^2(0,T)$ is a compact operator. Let us define a map $\Theta : \Lambda \longrightarrow L^2(0,T)$ by

$$\Theta(\{x_1, ..., x_N\}) = w(P)(\eta, \cdot) \tag{7.4}$$

where $w(x_1, ..., x_N)(x,t) = w(P)(x,t)$ is the solution to (3.5) with $P = \{x_1, ..., x_N\}$ and $\alpha_1 = ... = \alpha_N = 1$. More explicitly, from Lemma 3.2, we see

$$w(P)(\eta, t) = \sum_{k=1}^{\infty} \frac{2}{k\pi} \left( \sum_{j=1}^{N} \sin k\pi x_j \right) \sin k\pi\eta \sin k\pi t, \quad 0 < t < T. \tag{7.5}$$

Then a decomposition scheme of the mappings is the following:

$$\mathbb{R}^N \quad \overset{\Theta}{\longrightarrow} \quad L^2(0,T) \quad \overset{S}{\longrightarrow} \quad {}_0H^1(0,T) \quad \overset{E}{\longrightarrow} \quad L^2(0,T)$$

$$P \quad \overset{\Theta^{-1}}{\longleftarrow} \quad w \quad \overset{S^{-1}}{\longleftarrow} \quad u \quad \overset{E^{-1}}{\longleftarrow} \quad u^\epsilon. \tag{7.6}$$

Here the upper diagram describes the mappings and spaces of the direct problem. The diagram below describes the inverse problem: Starting from noisy data $u^\epsilon$, over an approximation of the exact data $u$, evaluating an approximation of $w = S^{-1}u$, one finally has to reconstruct an approximation of $P = \Theta^{-1}w$. By (7.1), taking (7.6) and Lemma 3.3 (2) into consideration, we can prove that the operator $\Theta : \mathbb{R}^N \longrightarrow L^2(0,T)$ is one to one and $\Theta$ is continuous. Therefore, since $\Lambda$ is a relatively compact set in $\mathbb{R}^N$, a theorem in the general topology tells that the inverse $\Theta^{-1} : L^2(0,T) \longrightarrow \mathbb{R}^N$ is continuous. Moreover $S^{-1} : {}_0H^1(0,T) \longrightarrow L^2(0,T)$ is also continuous. The inverse $E^{-1}$ is not continuous from $L^2(0,T)$ to ${}_0H^1(0,T)$, so that the whole problem $E \circ S \circ \Theta$ is ill-posed, and is decomposed into a well–posed part $S \circ \Theta$ and an ill-posed part $E$.

As a first step of our reconstruction we will start from noisy data $u^\epsilon$ according to (i) or (ii) and construct new data $w^\epsilon$ as disturbed observations concerning the system (3.5). To this end we have to consider a regularization of the embedding

$${}_0H^1(0,T) \subset L^2(0,T).$$

We begin with the case (i) and apply the method in Bruckner [5]. Let us compute the singular values of the embedding ${}_0H^1(0,T) \subset L^2(0,T)$. Let us define a space

$$\widehat{H} = \{u \in H_0^1(0,2T); u(t) = u(2T-t), \quad 0 < t < T\}. \tag{7.7}$$

We equip ${}_0H^1(0,T)$ and $\widehat{H}$ with the scalar products and the norms of $H^1(0,T)$ and $H_0^1(0,2T)$, respectively:

$$\left\{ \begin{array}{ll} (u,v)_{{}_0H^1} = (u,v)_{L^2(0,T)} + \left(\dfrac{du}{dt}, \dfrac{dv}{dt}\right)_{L^2(0,T)}, & u,v \in {}_0H^1(0,T), \\ (U,V)_{\widehat{H}} = (U,V)_{L^2(0,2T)} + \left(\dfrac{dU}{dt}, \dfrac{dV}{dt}\right)_{L^2(0,2T)}, & U,V \in \widehat{H}. \end{array} \right\} \tag{7.8}$$

Furthermore let us define an extension operator $\gamma$ from ${}_0H^1(0,T) \to \widehat{H}$ by

$$(\gamma u)(t) = \left\{ \begin{array}{ll} u(t), & 0 \le t \le T \\ u(2T-t), & T < t < 2T. \end{array} \right. \tag{7.9}$$

By direct calculations, we see that

$$\frac{d(\gamma u)}{dt}(t) = \left\{ \begin{array}{ll} \frac{du}{dt}(t), & 0 < t < T \\ -\frac{du}{dt}(2T-t), & T < t < 2T \end{array} \right. \tag{7.10}$$

in the sense of $\mathcal{D}'(0,2T)$: the distributions in $(0,2T)$. Therefore $\gamma u \in \widehat{H}$ and

$$(\gamma u, \gamma v)_{\widehat{H}} = 2(u,v)_{{}_0H^1}, \quad u,v \in {}_0H^1(0,T). \tag{7.11}$$

Consequently by $\gamma$, the Hilbert spaces ${}_0H^1(0,T)$ and $\widehat{H}$ are isomorphic.

Let us set

$$L = \gamma^{-1}, \tag{7.12}$$

that is, $L$ is the restriction operator of functions on $(0,2T)$ to $(0,T)$. Then we have $L\widehat{H} = {}_0H^1(0,T)$. Moreover

$$g_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \cos \frac{(k-\frac{1}{2})\pi(t-T)}{T}, \quad 0<t<T,\ k \in \mathbb{N}, \tag{7.13}$$

is an orthonormal basis in $L^2(0,T)$. In fact, the orthonormality is straightforward. To prove the completeness, let us consider the eigenvalue problem $-\frac{d^2\phi}{dt^2}(t) = \lambda\phi(t)$, $0<t<T$ with $\phi(0) = \frac{d\phi}{dt}(T) = 0$. Then, as is easily checked,

$$\lambda_k = \frac{(k-\frac{1}{2})^2\pi^2}{T^2}, \qquad k \in \mathbb{N},$$

is the set of eigenvalues and $g_k$, $k \in \mathbb{N}$, is an eigenfunction for $\lambda_k$. Therefore by a result on the Sturm–Liouville problem (e.g. Levitan and Sargsjan [14]), we see that $\{g_k\}_{k\in\mathbb{N}}$ is complete in $L^2(0,T)$.

Let us set

$$\sigma_k = \left(1 + \frac{(k-\frac{1}{2})^2\pi^2}{T^2}\right)^{-\frac{1}{2}}, \quad k \in \mathbb{N}. \tag{7.14}$$

We can easily verify that

$$\Gamma_k(t) = \frac{1}{\sqrt{2}}\sigma_k(\gamma g_k)(t) = \frac{\sigma_k}{\sqrt{T}} \cos \frac{(k-\frac{1}{2})\pi(t-T)}{T}, \quad 0<t<2T,\ k \in \mathbb{N} \tag{7.15}$$

is an orthonormal basis in $\widehat{H}$. In fact, the orthonormality in $\widehat{H}$ is straightforward. For the completeness, we can proceed as follows. Let $v \in \widehat{H}$ satisfy $(v,\Gamma_k)_{\widehat{H}} = 0$, $k \in \mathbb{N}$. Then since $v$ is symmetric with respect to $t = T$, we have by partial integration

$$0 = (v,\Gamma_k)_{\widehat{H}} = 2\sigma_k^{-2}(Lv, L\Gamma_k)_{L^2(0,T)}, \quad k \in \mathbb{N},$$

namely, $(Lv, g_k)_{L^2(0,T)} = 0$, $k \in \mathbb{N}$. By the completeness of $\{g_k\}_{k\in\mathbb{N}}$, in $L^2(0,T)$, we can conclude that $v = 0$. Thus, $\{\Gamma_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $\widehat{H}$.

Thus

$$G_k(t) = \sqrt{2}L\Gamma_k = \sigma_k g_k(t), \quad 0<t<T, \quad k \in \mathbb{N} \tag{7.16}$$

is an orthonormal basis in ${}_0H^1(0,T)$.

Therefore the singular value decomposition of the compact embedding

$$E:\ {}_0H^1(0,T) \to L^2(0,T)$$

is

$$\{G_k, g_k, \sigma_k\}_{k\in\mathbb{N}}, \tag{7.17}$$

since $EG_k = \sigma_k g_k$ and

$$(G_j, E^*g_k)_{{}_0H^1} = (EG_j, g_k)_{L^2(0,T)} = \sigma_j\delta_{jk} = \sigma_k\delta_{jk} = \sigma_k(G_j, G_k)_{{}_0H^1} = (G_j, \sigma_k G_k)_{{}_0H^1},$$

that is, $E^*g_k = \sigma_k G_k$ holds by the completeness of $\{G_k\}_{k\in\mathbb{N}}$ in ${}_0H^1(0,T)$.

We set

$$\|u\|_\chi^2 = \sum_{k=1}^{\infty} \sigma_k^{-2\chi} |(u, G_k)_{{}_0H^1}|^2, \qquad \chi \ge 0, \tag{7.18}$$

provided that the right-hand side is finite. We note that $\|u\|_0 = \|u\|_{{}_0H^1}$.

Then from [5], the following is known:

**Proposition 1.** *Let data $u^\epsilon$ and a real number $R$ satisfy*

$$\frac{\|u^\epsilon\|_{L^2(0,T)}}{\epsilon} > R > 1$$

*and let $R_\epsilon u^\epsilon \in {}_0H^1(0,T)$ be defined by*

$$R_\epsilon u^\epsilon = \sum_{k:\sigma_k > b} \sigma_k^{-1}(u^\epsilon, g_k)_{L^2(0,T)} G_k + \tau \sum_{k:\sigma_k = b} \sigma_k^{-1}(u^\epsilon, g_k)_{L^2(0,T)} G_k, \tag{7.19}$$

*where the singular value $b$ has the property*

$$\sum_{k:\sigma_k<b} |(u^\epsilon, g_k)_{L^2(0,T)}|^2 < (R\epsilon)^2 \le \sum_{k:\sigma_k\le b} |(u^\epsilon, g_k)_{L^2(0,T)}|^2$$

*and*

$$\tau = 1 - \left( \frac{(R\epsilon)^2 - \sum_{k:\sigma_k<b} |(u^\epsilon, g_k)_{L^2(0,T)}|^2}{\sum_{k:\sigma_k=b} |(u^\epsilon, g_k)_{L^2(0,T)}|^2} \right)^{\frac{1}{2}}. \tag{7.20}$$

*Then*

$$R_\epsilon u^\epsilon \to u \quad \text{in } {}_0H^1(0,T) \text{ as } \epsilon \to 0.$$

*If additionally for some $\chi > 0$, we have*

$$\|u\|_\chi^2 < \infty, \tag{7.21}$$

*then we obtain*

$$\|R_\epsilon u^\epsilon - u\|_{{}_0H^1} \le C_R \epsilon^{\frac{\chi}{\chi+1}} \|u\|_\chi^{\frac{1}{\chi+1}}, \tag{7.22}$$

*where $C_R$ is a constant which is independent of $\epsilon$ and $\|u\|_\chi$.*

**Remark.** Proposition 1 is a slight modification of the well-known truncated singular value decomposition method combined with an a-posteriori parameter selection procedure.

For applying (7.22) in Proposition 1, we have to verify (7.21) with some $\chi > 0$. In our inverse problem we actually prove

**Lemma 7.1.** *Let $P = \{N, \alpha_1, ...., \alpha_N, x_1, ..., x_N\}$ and $\eta \in (0,1)$. Then $u(P)(\eta, \cdot) \in H^{1+\nu}(0,T)$ and*

$$\|u(P)(\eta,\cdot)\|_\nu < \infty$$

*if $\nu < \frac{1}{2}$.*

The proof is technical and will be given in Appendix.

In view of Lemma 7.1, we apply Proposition 1 to our problem. Let us now define

$$w^\epsilon = S^{-1} R_\epsilon u^\epsilon. \tag{7.23}$$

Then for noisy data $u^\epsilon$, under the assumptions of Proposition 1, we see that

$$w^\epsilon \longrightarrow w \quad \text{in } L^2(0,T)$$

as $\epsilon \to 0$ and

$$\begin{aligned} \|w^\epsilon - w\|_{L^2(0,T)} &= \|S^{-1}R_\epsilon u^\epsilon - S^{-1}u\|_{L^2(0,T)} \\ &\le C\|R_\epsilon u^\epsilon - u\|_{{}_0H^1} = O\left(\epsilon^{\frac{\nu}{1+\nu}}\right) \end{aligned} \tag{7.24}$$

for $0 \le \nu < \frac{1}{2}$.

Next let us continue with the case (ii). Here we describe the solution of an approximation problem according to Bruckner [6]. Depending on the noise level $\epsilon$ and the time difference $d$ of consecutive observations $u_j^\epsilon$, $j = 1, ..., n$, where $d \cdot n = T$, we wish to construct functions $P(d,\epsilon) \in {}_0H^1(0,T)$ with the property $P(d(\epsilon),\epsilon) \to u$ in ${}_0H^1(0,T)$ as $\epsilon \to 0$.

Let us consider the Sobolev scale $\{H^\lambda(0,T)\}_{\lambda \ge 0}$ with norms $\|\cdot\|_{H^\lambda(0,T)}$ and finite dimensional spaces $Y_n$ of trial functions

$$Y_n \subset {}_0H^1(0,T), \quad Y_n \subset Y_{n+1},\ n \in \mathbb{N}, \quad \overline{\cup_{n\in\mathbb{N}} Y_n} = {}_0H^1(0,T), \tag{7.25}$$

and interpolation operators $K_n : \mathbb{R}^n \longrightarrow Y_n$, $n \in \mathbb{N}$, such that there is a unique interpolation function

$$K_n \underline{f} \in Y_n, \tag{7.26}$$

for every vector $\underline{f} = (f_1, ..., f_n) \in \mathbb{R}^n$ with the property

$$(K_n \underline{f})(t_j) = f_j, \quad j = 1, ..., n. \tag{7.27}$$

For $f \in C[0,T]$, we set $\underline{f} = (f(t_1), ..., f(t_n))$ and we define $J_n : C[0,T] \longrightarrow Y_n$ by

$$J_n f = K_n \underline{f} \tag{7.28}$$

and we assume

$$J_n y = y \qquad \text{for each } y \in Y_n.$$

Then $J_n$ is a projector of $H^1(0,T)$ onto $Y_n$. For the spaces $Y_n$ and the operators $J_n$ we suppose the following three properties.

*Approximation property:*

$$\|f - J_n f\|_{H^1(0,T)} \begin{cases} \longrightarrow 0 \quad (n \to \infty) & \text{if } f \in H^1(0,T) \\ \leq Cn^{-\lambda}\|f\|_{H^{1+\lambda}(0,T)} & \text{if } f \in H^{1+\lambda}(0,T),\ \lambda > 0. \end{cases} \tag{7.29}$$

*Inverse property:*

$$\|\psi\|_{H^1(0,T)} \leq C \cdot n\|\psi\|_{L^2(0,T)} \quad \text{for all } \psi \in Y_n. \tag{7.30}$$

*Finite property:*

$$\|\psi\|_{L^2(0,T)} \leq C \cdot \max_{1\leq j\leq n} |\psi(t_j)| \quad \text{for all } \psi \in Y_n. \tag{7.31}$$

Here and henceforth the letter $C > 0$ denotes some generic constant.

**Example for $Y_n$ and $J_n$ with (7.29) - (7.31).**
$Y_n$: the spaces of linear splines,
$J_n$: the linear interpolation operators.
More precisely, let $t_i = id$, $d = \frac{T}{n}$,

$$Y_n = \text{Span}\{B_j;\ j = 1, ..., n\}$$

where $B_j$ are the linear $B$–splines satisfying

$$B_j(0) = 0, \quad B_j(id) = 0 \quad \text{if } i \neq j, \quad B_i(id) = 1, \qquad i,j = 1, ..., n$$

and

$$J_n f = \sum_{j=1}^{n} f(jd) B_j,$$

the piecewise linear interpolation polynomials. Then

$$Y_{2^m} \subset Y_{2^{m+1}}, \qquad m \in \mathbb{N},$$

and

$$\overline{\cup_{m\in\mathbb{N}} Y_{2^m}} = {}_0H^1(0,T)$$

where the closure is understood in the sense of $H^1(0,T)$. As for details of the example, we can refer to Prößdorf and Silbermann [18].

The finite property (7.31) is immediate since

$$\|\psi\|_{L^2(0,T)} \leq C\|\psi\|_{C[0,T]}, \qquad \|\psi\|_{C[0,T]} \leq \max_{1\leq j\leq n} |\psi(jd)|$$

hold for $\psi \in Y_n$. The estimate (7.30) can be seen by straightforward calculations. Finally, the approximation property (7.29) is more involved. In the periodic case (i.e., if we consider functions on the torus $\mathbb{R}/\mathbb{Z}$), (7.29) can be looked up in Elschner and Schmidt [8] or Prößdorf and Silbermann [18]. Let $T = \frac{1}{2}$. Then, prolongating a function $f \in {}_0H^1(0,\frac{1}{2})$ to $\widetilde{f} \in {}_0H^1(\mathbb{R}/\mathbb{Z})$ (i.e. $\widetilde{f}(t) = f(t)$ if $0 \leq t \leq \frac{1}{2}$) in a suitable way and applying the periodical theory for $\tilde{f}$, the property (7.29) can be proven straightforwardly. Another way to prove (7.29) is developed in Schumaker [20].

From Bruckner [6] we obtain

**Proposition 2.** *In the case (ii), under the assumptions (7.29) - (7.31), we have*

$$\|u - K_n\underline{u}^\epsilon\|_{H^1(0,T)} \begin{cases} \to 0 \quad (n \to \infty) & \text{if } n = o(\epsilon^{-1}) \\ = O\left(\epsilon^{\frac{\lambda}{\lambda+1}}\right) & \text{if } n \sim \epsilon^{\frac{-1}{1+\lambda}} \text{ and } u \in H^{1+\lambda}(0,T). \end{cases}$$

**Remark.** Proposition 2 represents a special kind of regularization by discretization. The discretization parameter $n$ is the regularization parameter. See also Bruckner, Prößdorf and Vainikko [7].

In this case (ii) the new data are defined by

$$w^\epsilon = S^{-1}K_{n(\epsilon)}\underline{u}^\epsilon \tag{7.32}$$

where we set

$$n(\epsilon) = \epsilon^{\frac{-1}{1+\nu}}$$

where $0 < \nu < \frac{1}{2}$. Then by Proposition 2 and Lemma 7.1, we have

$$\|w^\epsilon - w\|_{L^2(0,T)} = O\left(\epsilon^{\frac{\nu}{\nu+1}}\right) \tag{7.33}$$

as $\epsilon \longrightarrow 0$.

Based on these considerations we propose the following steps of stable reconstruction. Here we exclusively discuss the case (i) because we can similarly implement in the case (ii). Let noisy data $u^\epsilon$, $\epsilon > 0$ be given with the property :

(i) $u^\epsilon \in L^2(0,T)$ and $\|u - u^\epsilon\|_{L^2(0,T)} \leq \epsilon$, $u = u(t)$ is the unknown exact data.

**Step 1.**

Construction of $R_\epsilon u^\epsilon \in {}_0H^1(0,T)$ by the evaluation (7.19).

**Step 2.**

Construction of $w^\epsilon = S^{-1}R_\epsilon u^\epsilon \in L^2(0,T)$ by solving the second kind Volterra integral equation

$$\frac{d}{dt}(R_\epsilon u^\epsilon)(t) = \lambda(0)w^\epsilon(t) + \int_0^t \frac{d\lambda}{dt}(t-s)w^\epsilon(s)ds.$$

We know from Proposition 1 that

$$\|w^\epsilon - w(P)(\eta,\cdot)\|_{L^2(0,T)} \le C\cdot \epsilon^{\frac{\nu}{1+\nu}}, \tag{7.34}$$

where $0<\nu\le\frac{1}{2}$ and $w = S^{-1}u$.

**Step 3.**

We reconstruct a quasi-inverse to $\Theta$ from $w^\epsilon \in L^2(0,T)$ satisfying (7.34) as follows. For simplicity, we further a-priori assume

$$0 < x_j < \frac{1}{2}, \qquad 1\le j\le N. \tag{7.35}$$

We solve

$$\sum_{j=1}^N \sin(2m-1)\pi x_j^\epsilon = \frac{(2m-1)\pi}{\sin(2m-1)\pi\eta}\int_0^1 w^\epsilon(t)\sin(2m-1)\pi t dt, \qquad m=1,....,N \tag{7.36}$$

with respect to $x_1^\epsilon,....,x_N^\epsilon \in (0,\frac{1}{2})$. Henceforth we number $x_1^\epsilon,....,x_N^\epsilon$ as

$$0 < x_1^\epsilon \le \cdots \le x_N^\epsilon < \frac{1}{2}.$$

Here we note that (7.1) implies that $\sin(2m-1)\pi\eta \ne 0$ for $1\le m\le N$.

The system (7.36) of trigonometric equations is uniquely solvable for given $w^\epsilon$.

In fact, we can prove that $\sin(2m-1)\pi x_j^\epsilon = P_m(\sin \pi x_j^\epsilon)$ where $P_m$ is a polynomial of order $2m-1$ and the coefficients of even orders vanish, so that the system (7.36) is equivalent to

$$\left\{\begin{array}{l} \sum_{j=1}^N \sin \pi x_j^\epsilon = q_1(a_1^\epsilon,....,a_N^\epsilon) \\ \sum_{j=1}^N (\sin \pi x_j^\epsilon)^3 = q_2(a_1^\epsilon,....,a_N^\epsilon) \\ \vdots \\ \sum_{j=1}^N (\sin \pi x_j^\epsilon)^{2N-1} = q_N(a_1^\epsilon,....,a_N^\epsilon), \end{array}\right\} \tag{7.37}$$

where we set

$$a_m^\epsilon = \frac{(2m-1)\pi}{\sin(2m-1)\pi\eta}\int_0^1 w^\epsilon(t)\sin(2m-1)\pi t dt, \quad 1\le m\le N$$

and $q_1, ..., q_N$ are polynomials of $a_1^\epsilon, ...., a_N^\epsilon$. Setting

$$\sin \pi x_j^\epsilon = \alpha_j^\epsilon, \qquad 1 \le j \le N, \tag{7.38}$$

we can reduce the roots of (7.37) to zeros $\alpha_1^\epsilon, ...., \alpha_N^\epsilon$ of $N$ symmetric polynomials. Then it is sufficient to solve (7.38) with respect to $0 < x_j^\epsilon < \frac{1}{2}$, $1 \le j \le N$. Then $\{x_1^\epsilon, ...., x_N^\epsilon\}$ is our desired approximation for locations of point sources.

In fact, by (7.5) we take scalar products in $L^2(0,1)$ of $w(P)(\eta, \cdot)$ with $\sin(2m-1)\pi t$, $1 \le m \le N$, so that

$$\begin{aligned} \sum_{j=1}^N \sin(2m-1)\pi x_j =& \frac{(2m-1)\pi}{\sin(2m-1)\pi\eta} \int_0^1 w(P)(\eta, t) \sin(2m-1)\pi t dt \\ \equiv& a_m, \qquad m = 1, ...., N. \end{aligned} \tag{7.39}$$

Then, noting that $T \ge 1$ from (7.1), by (7.34) we see that

$$|a_m - a_m^\epsilon| \le C\epsilon^{\frac{\nu}{1+\nu}}, \qquad 1 \le m \le N.$$

Therefore in a way similar to the reduction of (7.37) to zeros of $N$ symmetric polynomials, we see that

$$\begin{aligned} &\left|\sum_{j=1}^N (\sin \pi x_j^\epsilon - \sin \pi x_j)\right|, \quad \left|\sum_{j=1}^N \left\{(\sin \pi x_j^\epsilon)^3 - (\sin \pi x_j)^3\right\}\right|, ....., \\ &\left|\sum_{j=1}^N \left\{(\sin \pi x_j^\epsilon)^{2N-1} - (\sin \pi x_j)^{2N-1}\right\}\right| \le C\epsilon^{\frac{\nu}{1+\nu}}, \end{aligned} \tag{7.40}$$

so that

$$|x_j^\epsilon - x_j| \le C\epsilon^{\frac{\nu}{1+\nu}}, \qquad 1 \le j \le N$$

will follow under the extra assumtption (7.35).

**Test case in Step 3.** We take the case of $N = 2$ and clarify the effectiveness of the above process. By a formula: $\sin 3\theta = 3 \sin\theta - 4 \sin^3\theta$, we can rewrite (7.37) as

$$\sin \pi x_1^\epsilon + \sin \pi x_2^\epsilon = a_1^\epsilon \quad \text{and} \quad (\sin \pi x_1^\epsilon)^3 + (\sin \pi x_2^\epsilon)^3 = \frac{3a_1^\epsilon - a_2^\epsilon}{4}.$$

Therefore

$$\sin \pi x_1^\epsilon + \sin \pi x_2^\epsilon = a_1^\epsilon$$

and

$$\sin \pi x_1^\epsilon \times \sin \pi x_2^\epsilon = \frac{4(a_1^\epsilon)^3 - 3a_1^\epsilon + a_2^\epsilon}{12 a_1^\epsilon},$$

that is, we obtain $\sin \pi x_1^\epsilon$ and $\sin \pi x_2^\epsilon$ as the roots of the quadratic equation

$$y^2 - a_1^\epsilon y + \frac{4(a_1^\epsilon)^3 - 3a_1^\epsilon + a_2^\epsilon}{12a_1^\epsilon} = 0. \tag{7.41}$$

Similarly we see that $\sin \pi x_1$ and $\sin \pi x_2$ are the roots of

$$y^2 - a_1 y + \frac{4a_1^3 - 3a_1 + a_2}{12a_1} = 0. \tag{7.42}$$

The estimate (7.34) in Step 2 guarantees

$$|a_1 - a_1^\epsilon|, \quad |a_2 - a_2^\epsilon| \le C\epsilon^{\frac{\nu}{1+\nu}}.$$

From (7.41) and (7.42), noting that $0 < x_1, x_2 < \frac{1}{2}$, we can conclude that

$$|\sin \pi x_1^\epsilon - \sin \pi x_1|, \quad |\sin \pi x_2^\epsilon - \sin \pi x_2| \le C\epsilon^{\frac{\nu}{1+\nu}}$$

for small $\epsilon > 0$. Again by $0 < x_1^\epsilon, x_2^\epsilon, x_1, x_2 < \frac{1}{2}$, we see that

$$|x_1^\epsilon - x_1|, \quad |x_2^\epsilon - x_2| \le C\epsilon^{\frac{\nu}{1+\nu}}$$

where $C > 0$ depends on $x_1$ and $x_2$.

**Appendix. Proof of Lemma 7.1.**
It is sufficient to prove the lemma in the case of $N = 1$ and $\alpha_1 = 1$. Henceforth $C > 0$ denotes a generic constant depending on $T$ and $\nu$.

First we show

**Lemma A.1.** *Let $0 \le \nu < \frac{1}{2}$ and let $\{a_k\}_{k\in\mathbb{N}}$ satisfy $\sup_{k\in\mathbb{N}} |a_k| \le M$. Then*

$$\left\| \sum_{k=1}^\infty \frac{a_k}{k} \sin k\pi t \right\|_{H^\nu(0,T)}^2 \le \left( \frac{C}{2} M^2 \pi^{2\nu} \right)^{\frac{1}{2}} \left( \sum_{k=1}^\infty k^{2\nu-2} \right)^{\frac{1}{2}} < \infty.$$

**Proof of Lemma A.1.** Without loss of generality, we may assume that $T = 2$. In fact, $T < 2$ is straightforward from the case of $T = 2$ and if $T > 2$, then we can choose $n_0 \in \mathbb{N}$ such that $T \le 2n_0$. Then

$$\left\| \sum_{k=1}^\infty \frac{a_k}{k} \sin k\pi t \right\|_{H^\nu(0,T)}^2 \le \left\| \sum_{k=1}^\infty \frac{a_k}{k} \sin k\pi t \right\|_{H^\nu(0,2n_0)}^2 \le n_0 \left\| \sum_{k=1}^\infty \frac{a_k}{k} \sin k\pi t \right\|_{H^\nu(0,2)}^2$$

by the periodicity of $\sin k\pi t$. Here we also recall the definition of the norm in $H^\nu(0, n_0)$ (e.g. Adams [1]).

Let us define an operator $A_0$ in $L^2(0,2)$ by

$$(-A_0 u)(t) = \frac{d^2u}{dt^2}(t), \quad 0 < t < 2$$

with $\mathcal{D}(A_0) = H^2(0,2) \cap H_0^1(0,2)$. Then the spectrum $\sigma(A_0)$ consists only of eigenvalues $\{\frac{k^2\pi^2}{4}\}_{k\in\mathbb{N}}$ and $\sin\frac{k\pi t}{2}$ is an eigenfunction for $\frac{k^2\pi^2}{4}$, and it is known that $\{\sin\frac{k\pi t}{2}\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(0,2)$. Then $A_0^{\frac{\nu}{2}}$ is well-defined and

$$\|A_0^{\frac{\nu}{2}} u\|^2_{L^2(0,2)} = \left(\frac{\pi}{2}\right)^{2\nu} \sum_{k=1}^{\infty} k^{2\nu} |(u, \sin\frac{k\pi t}{2})_{L^2(0,2)}|^2 < \infty$$

for $u \in \mathcal{D}(A_0^{\frac{\nu}{2}})$. On the other hand, $\mathcal{D}(A_0^{\frac{\nu}{2}}) = H^\nu(0,2)$ and

$$\|u\|^2_{H^\nu(0,2)} \le C\|A_0^{\frac{\nu}{2}} u\|^2_{L^2(0,2)}, \qquad 0 \le \nu < \frac{1}{2}$$

(e.g. Fujiwara [9]). Therefore

$$\begin{aligned}
&\left\| \sum_{k=1}^{\infty} \frac{a_k}{k} \sin k\pi t \right\|^2_{H^\nu(0,2)} \\
\le & C\left(\frac{\pi}{2}\right)^{2\nu} \sum_{k=1}^{\infty} k^{2\nu} \left| \left( \sum_{m=1}^{\infty} \frac{a_m}{m} \sin m\pi t, \sin\frac{k\pi t}{2} \right)_{L^2(0,2)} \right|^2 \\
= & C\pi^{2\nu} \sum_{l=1}^{\infty} l^{2\nu-2} |a_l|^2 \le CM^2\pi^{2\nu} \sum_{l=1}^{\infty} l^{2\nu-2} < \infty
\end{aligned}$$

by $0 \le \nu < \frac{1}{2}$. Thus the proof of Lemma A.1 is complete.

Henceforth we mainly consider

$$w(x_1)(\eta, t) = \sum_{k=1}^{\infty} \frac{2}{k\pi} \sin k\pi x_1 \sin k\pi\eta \sin k\pi t.$$

Then

(1) $$u = u(x_1)(\eta, \cdot) = S(w(x_1)(\eta, \cdot)).$$

By Lemma A.1 we see

(2) $$w(x_1)(\eta, \cdot) \in H^\nu(0, T)$$

for $0 \le \nu < \frac{1}{2}$.

Next we can easily prove

$$S \in \mathcal{B}(L^2(0,T), H^1(0,T)) \cap \mathcal{B}(H^1(0,T), H^2(0,T)).$$

Here $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded linear operators from a Banach space $X$ to another Banach space $Y$. By the interpolation theorem (e.g. Lions and Magenes [16]), we see that

$$S \in \mathcal{B}(H^\nu(0,T), H^{\nu+1}(0,T)) \tag{3}$$

with $0 \le \nu < \frac{1}{2}$. Consequently (1) - (3) yield

$$u \in H^{\nu+1}(0,T), \qquad u(0) = 0. \tag{4}$$

We set

$$h_k(t) = \frac{\sqrt{2}}{\sqrt{T}} \sin \frac{(k-\frac{1}{2})\pi(t-T)}{T}, \quad 0 < t < T,\ k \in \mathbb{N}. \tag{5}$$

Define an operator $A_1$ in $L^2(0,T)$ by

$$\begin{aligned} &(-A_1 u)(t) = \frac{d^2u}{dt^2}(t), \qquad 0 < t < T \\ &\mathcal{D}(A_1) = \{u \in H^2(0,T); \frac{du}{dt}(0) = u(T) = 0\}. \end{aligned} \tag{6}$$

Then we can prove

**Lemma A.2.** *$\mathcal{D}(A_1^{\frac{\nu}{2}}) = H^\nu(0,T)$ for $0 \le \nu < \frac{1}{2}$ and there exists a constant $C > 0$ such that*

$$(-A_1)^{\frac{\nu}{2}} u = \sum_{k=1}^\infty \left( \frac{(k-\frac{1}{2})^2\pi^2}{T^2} \right)^{\frac{\nu}{2}} (u, h_k)_{L^2(0,T)} h_k \tag{7}$$

*and*

$$\sum_{k=1}^\infty \left( \frac{(k-\frac{1}{2})\pi}{T} \right)^{2\nu} |(u,h_k)_{L^2(0,T)}|^2 \le C \|u\|^2_{H^\nu(0,T)} \tag{8}$$

*for all $u \in H^\nu(0,T)$.*

**Proof of Lemma A.2.** Henceforth we set

$$\mu_k = (k - \frac{1}{2})\frac{\pi}{T}, \quad \lambda_k = \frac{k\pi}{T}, \qquad k \in \mathbb{N} \tag{9}$$

and

$$x_k(t) = \frac{1}{\sqrt{T}} \cos \mu_k t, \quad y_k(t) = \frac{1}{\sqrt{T}} \sin \lambda_k t, \qquad -T \le t \le T,\ k \in \mathbb{N}. \tag{10}$$

In Span$\{h_k\}_{k\in\mathbb{N}}$, we define a scalar product and a norm by

$$(u,v)_{X_\nu} = \sum_{k=1}^\infty \mu_k^{2\nu} (u,h_k)_{L^2(0,T)} \overline{(v,h_k)_{L^2(0,T)}}$$

and

$$\|u\|_{X_\nu}^2 = (u,u)_{X_\nu}$$

for $u, v \in$ Span $\{h_k\}_{k\in\mathbb{N}}$. The completion of Span$\{h_k\}_{k\in\mathbb{N}}$ in the norm $\|\cdot\|_{X_\nu}$ is denoted by $X_\nu$. Moreover we define an operator $-A_2$ in $L^2(-T,T)$ by

$$(-A_2u)(t) = \frac{d^2u}{dt^2}(t), \qquad -T < t < T$$

$$\mathcal{D}(A_2) = H^2(-T,T) \cap H_0^1(-T,T). \tag{11}$$

Then we can define fractional powers and

$$A_2^{\frac{\nu}{2}} u = \sum_{k=1}^{\infty} \mu_k^\nu (u, x_k)_{L^2(-T,T)} x_k + \sum_{k=1}^{\infty} \lambda_k^\nu (u, y_k)_{L^2(-T,T)} y_k. \tag{12}$$

Moreover we can easily see that $\{x_k, y_k\}_{k\in\mathbb{N}}$ is the set of the eigenfunctions of the operator $A_2$, and $\{x_k, y_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(-T,T)$. Now we denote

$$x_k = (-1)^k \frac{1}{\sqrt{2}} h_k, \qquad k \in \mathbb{N}. \tag{13}$$

Therefore by (12), we see that

$$\|A_2^{\frac{\nu}{2}} u\|_{L^2(-T,T)}^2 = \sum_{k=1}^{\infty} (\mu_k^{2\nu} |(u, x_k)_{L^2(-T,T)}|^2 + \lambda_k^{2\nu} |(u, y_k)_{L^2(-T,T)}|^2). \tag{14}$$

On the other hand, by Fujiwara [9], we have

$$\mathcal{D}(A_2^{\frac{\nu}{2}}) = H^\nu(-T,T), \qquad 0 \le \nu < \frac{1}{2} \tag{15}$$

and

$$C^{-1}\|u\|_{H^\nu(-T,T)} \le \|A_2^{\frac{\nu}{2}} u\|_{L^2(-T,T)} \le C\|u\|_{H^\nu(-T,T)}, \quad u \in \mathcal{D}(A_2^{\frac{\nu}{2}}). \tag{16}$$

Therefore we obtain

$$\begin{aligned} C^{-1} \sum_{k=1}^{\infty} (\mu_k^{2\nu} |(u, x_k)_{L^2(-T,T)}|^2 + \lambda_k^{2\nu} |(u, y_k)_{L^2(-T,T)}|^2) &\le \|u\|_{H^\nu(-T,T)}^2 \\ \le C \sum_{k=1}^{\infty} (\mu_k^{2\nu} |(u, x_k)_{L^2(-T,T)}|^2 + \lambda_k^{2\nu} |(u, y_k)_{L^2(-T,T)}|^2), &\quad u \in H^\nu(-T,T). \end{aligned} \tag{17}$$

Finally we define an isomorphism $K$ from $H^\nu(0,T)$ onto a closed subspace of $H^\nu(-T,T)$ :

$$(Ku)(t) = \begin{cases} u(t), & 0 \le t \le T \\ u(-t), & -T \le t < 0. \end{cases}$$

That is, $Ku$ is an even extension of the function in $(0,T)$ to one in $(-T,T)$. Then we see : $u \in X_\nu$ if and only if $Ku \in H^\nu(-T,T)$ and

$$C^{-1}\|u\|_{X_\nu} \le \|Ku\|_{H^\nu(-T,T)} \le C\|u\|_{X_\nu}, \qquad u \in H^\nu(-T,T). \tag{18}$$

In fact, since $Ku$ is an even function, we have $(u, y_k)_{L^2(-T,T)} = 0$, $k \in \mathbb{N}$, so that

$$\|Ku\|_{H^\nu(-T,T)}^2 \le C \sum_{k=1}^{\infty} \mu_k^{2\nu} |(Ku, x_k)_{L^2(-T,T)}|^2 \le C \sum_{k=1}^{\infty} \mu_k^{2\nu} |(u, h_k)_{L^2(0,T)}|^2 = C\|u\|_{X_\nu}^2$$

by (13) and (17). Similarly the reverse inequality can be proved. Thus we see (18). Further we need

**Lemma A.3.**

$$\|u\|_{H^\nu(0,T)} \le \|Ku\|_{H^\nu(-T,T)} \le 2\|u\|_{H^\nu(0,T)}, \quad u \in H^\nu(0,T).$$

**Proof of Lemma A.3.** The case $\nu = 0$ is readily verified. Let $0 < \nu < \frac{1}{2}$. Then we have

$$\|Ku\|^2_{H^\nu(-T,T)} = \|Ku\|^2_{L^2(-T,T)} + |Ku|^2_{H^\nu(-T,T)}$$

and

$$\|u\|^2_{H^\nu(0,T)} = \|u\|^2_{L^2(0,T)} + |u|^2_{H^\nu(0,T)},$$

with

$$|u|^2_{H^\nu(-T,T)} = \int_{-T}^{T}\int_{-T}^{T} \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\nu}} dt\, ds$$

and

$$|u|^2_{H^\nu(0,T)} = \int_0^T\int_0^T \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\nu}} dt\, ds$$

(e.g. Adams [1]). Therefore $\|u\|_{H^\nu(0,T)} \le \|Ku\|_{H^\nu(-T,T)}$ is straightforward. For the second inequality, since $\|Ku\|_{L^2(-T,T)} = \sqrt{2}\|u\|_{L^2(0,T)}$, it is sufficient to prove

$$|Ku|_{H^\nu(-T,T)} \le 2|u|_{H^\nu(0,T)}.$$

Noting the definition of $Ku$, we have

$$\begin{aligned}
&|Ku|^2_{H^\nu(-T,T)} \\
=&\left(\int_0^T\int_0^T + \int_{-T}^0\int_0^T + \int_0^T\int_{-T}^0 + \int_{-T}^0\int_{-T}^0\right)\left(\frac{|Ku(t)-Ku(s)|^2}{|t-s|^{1+2\nu}}\right) dt\, ds \\
=&2|u|^2_{H^\nu(0,T)} + 2\int_0^T\int_0^T \frac{|u(t)-u(s)|^2}{|t+s|^{1+2\nu}} dt\, ds.
\end{aligned}$$

Here we obtain

$$\begin{aligned}
&\int_0^T\int_0^T \frac{|u(t)-u(s)|^2}{|t+s|^{1+2\nu}} dt\, ds = \int_0^T\left(\int_0^T \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\nu}}\frac{|t-s|^{1+2\nu}}{|t+s|^{1+2\nu}} dt\right) ds \\
\le&\int_0^T\left(\int_0^T \frac{|u(t)-u(s)|^2}{|t-s|^{1+2\nu}} dt\right) ds,
\end{aligned}$$

so that

$$|Ku|^2_{H^\nu(-T,T)} \le 4|u|^2_{H^\nu(0,T)}.$$

Thus the proof of Lemma A.3 is complete.

Now we proceed to completing the proof of Lemma A.2. By (18) and Lemma A.3, we obtain

$$\|u\|_{X_\nu} \le C\|u\|_{H^\nu(0,T)},$$

namely,

$$\sum_{k=1}^{\infty} \mu_k^{2\nu} |(u, h_k)_{L^2(0,T)}|^2 \le C\|u\|_{H^\nu(0,T)}^2,$$

which completes the proof of Lemma A.2.

We are ready to completing the proof of Lemma 7.1. We recall (4). By (7.7), (7.13), (7.16) and (5) we have

$$\sigma_k^{-2\nu} |(u, G_k)_{{}_0H^1}|^2 \le Ck^{2\nu}\sigma_k^2 \left| (u, g_k)_{L^2(0,T)} - \left( \frac{du}{dt}, \frac{(k-\frac{1}{2})\pi}{T} h_k \right)_{L^2(0,T)} \right|^2$$

$$\le Ck^{2\nu}k^{-2} \left( |(u, g_k)_{L^2(0,T)}|^2 + k^2 \left| \left( \frac{du}{dt}, h_k \right)_{L^2(0,T)} \right|^2 \right).$$

Since $u \in L^2(0,T)$ and $\{g_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(0,T)$, noting that $2\nu - 2 < 0$, we see

$$\sum_{k=1}^{\infty} Ck^{2\nu}k^{-2} |(u, g_k)_{L^2(0,T)}|^2 \le C \sum_{k=1}^{\infty} |(u, g_k)_{L^2(0,T)}|^2 \le C\|u\|_{L^2(0,T)}^2.$$

Finally, since (4) implies $\frac{du}{dt} \in H^\nu(0,T)$, so that the inequality (8) yields

$$\sum_{k=1}^{\infty} Ck^{2\nu}k^{-2}k^2 \left| \left( \frac{du}{dt}, h_k \right)_{L^2(0,T)} \right|^2 = \sum_{k=1}^{\infty} Ck^{2\nu} \left| \left( \frac{du}{dt}, h_k \right)_{L^2(0,T)} \right|^2 < \infty.$$

Thus we obtain

$$\|u\|_\nu^2 = \sum_{k=1}^{\infty} \sigma_k^{-2\nu} |(u, G_k)_{{}_0H^1}|^2$$

$$\le C \sum_{k=1}^{\infty} k^{2\nu}k^{-2} |(u, g_k)_{L^2(0,T)}|^2 + C \sum_{k=1}^{\infty} k^{2\nu} \left| \left( \frac{du}{dt}, h_k \right)_{L^2(0,T)} \right|^2 < \infty$$

for $0 \le \nu < \frac{1}{2}$.

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