Nonlinear Equations in Non-Reflexive Banach Spaces and Strongly Nonlinear Differential Equations

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Abstract

In this paper, we study strongly nonlinear degenerate elliptic and parabolic equations of the form $F(x,u,Du,\ldots,D^{(2m-1)}u,Lu) = 0$ and $u_t = F(x,t,u,Du,\ldots,D^{(2m-1)}u,Lu)$, respectively, where L is a linear operator of the derivatives of highest (i.e., of $2m$–th) order. Under very weak restrictions on the growth of F with respect to the derivatives of u, existence results for weak solutions are proved. These existence results are based on general solvability results for nonlinear operator equations in Banach spaces which will be proved in this paper.

0 Introduction

In this paper, we study problems of the form

$$F(x,u,Du,\ldots,D^{(2m-1)}u,Lu) =: F(x,D^k u,Lu) = 0 \quad \text{in } G, \tag{0.1}$$

$$D^\beta u = 0 \quad \text{on } \partial G, \quad \text{for } |\beta| \le m-1, \tag{0.2}$$

and, respectively,

$$\frac{\partial u}{\partial t} + F(x,t,u,D^k u,Lu) = 0 \quad \text{in } Q, \tag{0.3}$$

$$D^\beta u = 0 \quad \text{on } \Gamma, \quad \text{for } |\beta| \le m-1, \tag{0.4}$$

$$u(0,x) = 0, \quad x \in G. \tag{0.5}$$

Here, $G \subset \mathbb{R}^n$, $n \ge 1$, denotes a bounded domain with sufficiently smooth boundary ∂G, and $Q := G \times [0,T]$, $\Gamma := \partial G \times [0,T]$. In addition, L denotes a linear differential operator of $2m$-th order $(m \in \mathbb{N})$, and $D^k u$ stands for the set of partial derivatives of the form $D^\alpha u$, $|\alpha| \le 2m$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex, $\alpha_i \in \mathbb{N}_0$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$; in this connection, $D^0 u := u$.

The solvability of problems of the form (0.1–2) and (0.3–5), respectively, has been studied before (see [6], [7], [9], [14], [16] and the references therein) under different conditions on F, namely under strong restrictions on the order of growth of the nonlinearity. In this paper we study cases when F depends in a linear way on the highest order derivatives of the unknown function u, which allows to investigate the problems under more general conditions for F (see also [16]).

The existence results for the problems under study are based on general solvability results for nonlinear operator equations and differential operator equations in Banach spaces which will be proved in this paper (see also [18], [19], [21]). More precisely, in Section 2 we will consider the equation

$$f(x) = y, \quad y \in Y, \tag{0.6}$$

where $f : \mathcal{D}(f) \subset X \to Y$ is some (nonlinear) operator acting between two Banach spaces X and Y, while in Section 3 we consider the Cauchy problem

$$\frac{dx}{dt} + f(t,x(t)) := x'(t) + f(t,x(t)) = y(t), \quad x(0) = 0, \tag{0.7}$$

where $f(t,\cdot)$ denotes some (nonlinear) operator acting from $L_{\Phi_0}(0,T;X)$ into $L_{\Phi_1}(0,T;Y)$. Here, Φ_0, Φ_1 are certain N-functions, and L_{Φ_0}, L_{Φ_1} denote the corresponding vector-valued Orlicz spaces. Recall (see [8]) that, given any N-function Φ, we have

$$L_\Phi(0,T;X) = \left\{u : (0,T) \to X \,\Big|\, \Big\| \|u(\cdot)\|_X \Big\|_\Phi < +\infty \right\}, \tag{0.8}$$

$$W^1_\Phi(0,T;X) = \left\{u \in L_\Phi(0,T;X) \,\Big|\, \frac{du}{dt} \in L_\Phi(0,T;X) \right\}, \tag{0.9}$$

where, for arbitrary v in the Orlicz space $L_\Phi(G)$ (for any domain $G \subset \mathbb{R}^k$, $k \in \mathbb{N}$) the Orlicz-norm $\|v\|_\Phi$ is defined by

$$\|v\|_\Phi := \sup \left\{ \Big| \int_G v(x)\, q(x)\, dx \Big| \;:\; q \in E_\Psi(G), \quad \int_G \Psi(q(x))\, dx \le 1 \right\} < +\infty. \tag{0.10}$$

Here, Ψ denotes the complementary N-function of Φ, and $E_\Phi(G)$ is the closure of the space of bounded functions in G with respect to the norm of $L_\Phi(G)$. Note that $(E_\Phi(G))^* = L_\Psi(G)$, if Φ and Ψ are complementary N-functions. Likewise, we denote by $E_\Phi(0,T;X)$ the closure of $L^\infty(0,T;X)$ with respect to the norm of $L_\Phi(0,T;X)$, and $W^1E_\Phi(0,T;X)$ is the closure of $W^{1,\infty}(0,T;X)$ with respect to the norm of $W^1_\Phi(0,T;X)$.

Since the domains of definition of the nonlinear operators are, in general, nonlinear sets, and since the solvability of (0.6) and (0.7) will be established via compactness methods (cf. [11]), we first study in Section 1 some nonlinear spaces and derive results concerning the (compact) imbedding between them. In Sections 4 and 5, we establish existence results for the problems (0.1–2) and (0.3–5), respectively, applying the general results of Sections 3 and 4. The final Section 6 brings examples; in particular, we study the equation

$$-\sum_{k=0}^{n} a_k\, e^{|D_k u|^{1+\beta}} \Delta u \;-\; a\, e^{|\Delta u|^2} \Delta u \;=\; h(x). \tag{0.11}$$

1 Some pn-spaces and imbedding theorems

Let two locally convex topological vector spaces X and Y and some, in general, nonlinear mapping $g : \mathcal{D}(g) \subset X \to Y$ be given. We introduce the notation

$$S_{gB} := \{x \in X \,|\, g(x) \in B\}, \tag{1.1}$$

if $(B, \|\cdot\|_B)$ is a Banach space such that $B \subset Y$. Clearly, to any Banach space $B \subset Y$ there exists a corresponding S_{gB}, characterized by B and g, and we have $S_{gB} \neq \emptyset$ if and only if $R(g) \cap B \neq \emptyset$, where $R(g)$ denotes the range of g in Y.

Definition 1.1 (see [18] or [15], [21])
A set $S \subset X$ is called *quasi-pseudonormed space* (or *qn - space*), if S is a topological space and if there exists a function $[\cdot]_S : S \to \mathbb{R}$ satisfying

(qn) $\qquad [x]_S \ge 0 \quad \forall\, x \in S, \qquad x = 0 \Rightarrow [x]_S = 0.$

If, in addition,

$$\textbf{(pn)} \qquad \begin{array}{lll} [x_1]_S \neq [x_2]_S & \Rightarrow x_1 \neq x_2\,, & x_1\,, x_2 \in S\,, \\ [x]_S = 0 & \Rightarrow x = 0\,, & \end{array}$$

holds, then S is called *pseudonormed space* (or *pn - space*). The function $[\,\cdot\,]_S$ will be called *q - norm* (or *p - norm*, respectively).

Next, let us denote

$$[x]_{S_{gB}} := \|g(x)\|_B \quad \forall\, x \in S_{gB}\,. \tag{1.2}$$

Obviously, if $R(g) \cap B \neq \emptyset$ and $g(0) = 0$, then S_{gB} is a qn-space in which the topology is defined by $[\,\cdot\,]_{S_{gB}}$, that is, it is defined by the topology of B by means of the mapping g, similarly as in Souslin topological spaces (cf. [3]).

Some properties of these spaces have been studied in [15], [17], [21], and the references therein; in this note, we will only consider some examples for these spaces which will be needed later.

Example 1.2 Let $\rho \geq 0$ and $\mu \geq 1$. We consider the spaces

$$\begin{aligned} \bar{S}_{1,\rho,\mu}(\Omega) &:= \Big\{ u \;\Big|\; [u]_{\bar{S}}^{\rho+\mu} := \sum_{i=1}^{n} \int_\Omega |u|^\rho\, |D_i\, u|^\mu\, dx < +\infty \Big\} \\ &= \Big\{ u \;\Big|\; D_i(|u|^{\rho/\mu}\, u) \in L_\mu(\Omega)\,,\ 1 \leq i \leq n \Big\}\,. \end{aligned} \tag{1.3}$$

Putting $g := (g_1, \ldots, g_n)$, where $g_i(u) := D_i(|u|^{\rho/\mu}\, u)$, $1 \leq i \leq n$, as well as $B := (L_\mu(\Omega))^n$, we easily see that

$$[u]_{\bar{S}} = \sum_{i=1}^{n} \|g_i(u)\|_{L_\mu}^{\mu/(\rho+\mu)}\,, \quad \text{for all } \rho \geq 0 \text{ and } \mu \geq 1\,. \tag{1.4}$$

Obviously, $\bar{S}_{1,\rho,\mu}(\Omega)$, $\rho \geq 0$, $\mu \geq 1$, is a quasi-pseudonormed space, and it holds

$$[\lambda\, u]_{\bar{S}} = |\lambda|\, [u]_{\bar{S}} \quad \forall\, \lambda \in \mathbb{R}\,,\ \forall\, u \in \bar{S}_{1,\rho,\mu}(\Omega)\,. \tag{1.5}$$

In addition to the spaces $\bar{S}_{1,\rho,\mu}(\Omega)$, we consider the spaces

$$S_{1,\rho,\mu}(\Omega) := \bar{S}_{1,\rho,\mu}(\Omega) \cap L_{\rho+\mu}(\Omega)\,. \tag{1.6}$$

Apparently,

$$S_{1,\rho,\mu}(\Omega) = \Big\{ u \;\Big|\; |u|^{\rho/\mu}\, u \in W_\mu^1(\Omega) \Big\}\,, \tag{1.7}$$

where $W_\mu^1(\Omega)$ denotes the standard Sobolev space.

For functions $u \in S_{1,\rho,\mu}(\Omega)$ we define

$$[u]_S := \|\, |u|^{\rho/\mu}\, u\, \|_{W_\mu^1}^{\mu/(\rho+\mu)}\,. \tag{1.8}$$

Clearly, $S_{1,\rho,\mu}(\Omega)$ is a pseudonormed space, and

$$[\lambda\, u]_S = |\lambda|\, [u]_S \quad \forall\, \lambda \in \mathbb{R}\,, \forall\, u \in S_{1,\rho,\mu}(\Omega)\,. \tag{1.9}$$

We now turn our attention to the general properties of the spaces S_{gB}. Since S_{gB} is at least a semi-metric space (cf. [18]), topological concepts like convergence with respect to the semi-distance (or distance), continuity, the imbedding of one qn-space into another, the compactness of such imbeddings, and so on, are defined. Moreover, concepts like separability, "reflexivity", completeness, $*$-completeness, of qn-spaces have been studied (cf. [15], [17]); these notions are analogously defined as the topology corresponding to the space B and the mapping g. In particular, the following results have been proved.

Proposition 1.3 *For the spaces S_{gB} defined in (1.1) the following holds true:*

(a) *If B is a separable Banach space and S_{gB} is a pn-space, then S_{gB} is separable. If S_{gB} is only a qn-space, and if, in addition, the inverse image $g^{-1}(y)$ of every $y \in R(g) \cap B$ is an at most countable set, then S_{gB} is separable.*

(b) *If B is a reflexive Banach space and the set $R(g) \cap B$ is weakly closed in B, then any bounded subset of S_{gB} is weakly compact in S_{gB} (this property of S_{gB} will be called "reflexivity" of S_{gB}).*

(c) *If $R(g) \cap B$ is closed (weakly closed) in B, then S_{gB} is a complete (weakly complete) qn-space.*

In the spaces S_{gB} often the following condition is satisfied.

$(\mathcal{N})$ There are $C \in (0, +\infty]$ and $\mu : \mathbb{R} \to \mathbb{R}_+$ such that

$$[\lambda\, x]_{S_{gB}} \leq \mu(\lambda)\; [x]_{S_{gB}}, \quad \forall\, |\lambda| < C,\ \forall\, x \in S_{gB}.$$

Hence, in many cases there exists a function ϕ, depending on g, such that the p-norm (q-norm) can be defined in the form

$$[x]_{S_{gB}} := \phi\left(\|g(x)\|_B\right), \tag{1.10}$$

in which case inequality in $(\mathcal{N})$ may be replaced by

$$[\lambda\, x]_{S_{gB}} \leq |\lambda|\; [x]_{S_{gB}}, \quad \forall\, |\lambda| < +\infty, \forall\, x \in S_{gB}. \tag{1.11}$$

For instance, in the spaces $\bar{S}_{1,\rho,\mu}(\Omega)$ and $S_{1,\rho,\mu}(\Omega)$ defined in Example 1.2, (1.11) holds even with equality. Note also, in particular, that in the case when g is a linear operator the set S_{gB} and the corresponding q-norm may coincide with the domain of definition of g and the graph-norm, respectively.

Definition 1.4 Let $g_0 : \mathcal{D}(g_0) \subset X \to Y$ and $g_1 : \mathcal{D}(g_1) \subset X \to Y$ denote two mappings.

(a) We write $g_0 \le g_1$ if and only if for any Banach space B with $B \subset Y$ the corresponding quasi-pseudonormed spaces $S_{g_0 B}$ and $S_{g_1 B}$ satisfy

$$S_{g_1 B} \subset S_{g_0 B}\,, \text{ where } R(g_i) \cap B \ne \emptyset \quad \text{for } i = 0,1\,. \tag{1.12}$$

(b) We say that $S_{g_1 B_1}$ is *continuously imbedded in* $S_{g_0 B_0}$ (for short: $S_{g_1 B_1} \hookrightarrow S_{g_0 B_0}$), if and only if there exists a continuous homogeneity function $\phi_{01} : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|g_0(x_1) - g_0(x_2)\|_{B_0} \le \phi_{01}\left(\|g_1(x_1) - g_1(x_2)\|_{B_1}\right) \quad \forall\, x_1, x_2 \in S_{g_1 B_1}\,. \tag{1.13}$$

The following results are easily established.

Proposition 1.5 *Let $S_{g_i B_i}$, $i = 0,1$, denote qn-spaces defined as in (1.1) such that the corresponding mappings g_0, g_1 satisfy $g_0 \le g_1$. Then it holds:*

(a) *If $B_1 \subset B_0$ then $S_{g_1 B_1} \subset S_{g_0 B_0}$.*

(b) *If B_1 is compactly imbedded in B_0 then the imbedding of $S_{g_1 B_1}$ in $S_{g_0 B_0}$ is compact.*

Next, we study classes of abstract mappings with values in qn-spaces, i.e. we consider sets of functions $x : [0,T] \to X$ satisfying $x(t) \in S_{gB}$ for a. e. $t \in (0,T)$ and investigate under which conditions such sets are compact.

To this end, let Φ_0, Φ_1, Φ denote N-functions, and let X_0 denote some Banach space such that $S_{gB} \subset X_0 \subset X$. We then define the classes of functions,

$$L_\Phi(0,T;S_{gB}) := \left\{ x : [0,T] \to X \;\middle|\; [x]_{L_\Phi(S_{gB})} := \left\| [x(\cdot)]_{S_{gB}} \right\|_{L_\Phi} < +\infty \right\}, \tag{1.14}$$

$$\begin{aligned} \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},X_0) \;:=\; & \Big\{ x : [0,T] \to X \;\Big|\; [x]_{\mathcal{P}_{\Phi_0\Phi_1}(S_{gB},X_0)} := [x]_{L_{\Phi_0}(S_{gB})} \\ & + \|\, x_t \,\|_{L_{\Phi_1}(X_0)} < +\infty \Big\}\,. \end{aligned} \tag{1.15}$$

Here, the subscript t stands for the derivative with respect to time, and $\|\cdot\|_{L_\Phi}$ denotes for N-functions Φ the norm in the corresponding Orlicz space $L_\Phi(0,T)$.

In addition, we denote by $E_\Phi(0,T;S_{gB})$ the closure of $L^\infty(0,T;S_{gB})$ with respect to the metric of the space $L_\Phi(0,T;S_{gB})$, and $W^1E_\Phi(0,T;S_{gB})$ denotes the closure of $W^{1,\infty}(0,T;S_{gB})$ with respect to the metric of the space $W^1_\Phi(0,T;S_{gB})$.

It follows directly from the definition that $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},X_0) \subset L_\Phi(0,T;S_{gB})$ provided that $\Phi \le \Phi_0$. Moreover, the following assertions are easily verified.

Proposition 1.6 *Suppose that $S_{g_0 B_0}$ and $S_{g_1 B_1}$ are qn-spaces (or pn-spaces, respectively). Then also $L_\Phi(0,T;S_{g_i B_i})$ and $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_i B_i},X)$ $(i = 0,1)$ are qn-spaces (or pn-spaces, respectively). If, in addition,*

$$g_0 \le g_1\,, \quad B_1 \subset B_0\,, \quad \Phi_0' \le \Phi_0\,, \quad \Phi_1' \le \Phi_1\,, \quad \textit{and} \quad X_0 \subset X_0'\,, \tag{1.16}$$

then

$$L_{\Phi_0}(0,T;S_{g_1B_1}) \subset L_{\Phi_0'}(0,T;S_{g_0B_0})\,, \quad \textit{as well as} \tag{1.17}$$

$$\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0) \subset \mathcal{P}_{\Phi_0'\Phi_1'}(0,T;S_{g_0B_0},X_0')\,. \tag{1.18}$$

If any of the inclusions in (1.16) *is strict, then* (1.17) *and* (1.18) *are strict inclusions.*

In what follows, we always assume that the spaces $S_{g_i\,B_i}$, $i = 0,1$, are complete with respect to the corresponding topologies. We consider the following conditions.

(A1) g_1 is continuously differentiable as mapping from X into B_0, and there exist some constant $C > 0$ and some N-function Ψ_2 with $\Psi_2 \leq \Phi_1$ such that for the complementary N-function Φ_2 associated with Ψ_2 it holds

$$\Phi_2\left(\|g_1'(x)\|_{\mathcal{L}(X,B_0)}\right) \leq C\,[x]_{S_{g_1B_1}} \quad \forall\, x \in S_{g_1B_1}\,, \tag{1.19}$$

where $S_{g_1B_1} \hookrightarrow S_{g_0B_0}$.

(A2) g_1 is locally Lipschitz continuous in the following sense: to any pair $x_1, x_2 \in S_{g_1B_0}$ there exists some $k = k(x_1,x_2) > 0$ such that

$$\|g_1(x_1) - g_1(x_2)\|_{B_0} \leq k(x_1,x_2)\;\|x_1 - x_2\|_X \tag{1.20}$$

holds, where

$$\Phi_2(k(x_1,x_2)) \leq C\left([x_1]_{S_{g_1B_1}} + [x_2]_{S_{g_1B_1}}\right), \quad C > 0\,, \tag{1.21}$$

with the function Φ_2 defined in **(A1)**, and where $S_{g_1B_1} \hookrightarrow S_{g_0B_0}$.

(A3) It holds $B_1 \subset B_0 \subset B_2 \subset Y$, and there exist functions Ψ_3, $\Psi_4 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\Psi_3^{-1}(\tau) \geq c_0\,|\tau|$ and $\Psi_4(\tau) \geq c_1\,|\tau|$ for any $\tau \in \mathbb{R}_+$ with suitable positive constants c_0, c_1, such that for $g = g_1$ and $g = g_0$ it holds

$$\|g(x_1) - g(x_2)\|_{B_2} \leq C\left(\Psi_3\left([x_1]_{S_{gB_1}}\right) + \Psi_3\left([x_2]_{S_{gB_1}}\right)\right)\Psi_4\left(\|x_1 - x_2\|_{X_0}\right). \tag{1.22}$$

We have the following result.

Theorem 1.7 *Let the assumptions of Proposition 1.6 be satisfied, let* $g_0 \leq g_1$, *and assume that the imbedding* $B_1 \hookrightarrow B_0$ *is compact. If any of the assumptions* **(A1)**, **(A2)**, **(A3)** *is fulfilled then the imbedding* $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0) \hookrightarrow L_\Phi(0,T;S_{g_0B_0})$ *is compact for any* Φ *satisfying* $\Phi \leq \Phi_0$. *In particular, the imbedding* $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0) \hookrightarrow C^0(0,T;X_0)$ *is compact.*

Remark 1.8 If g_0 satisfies **(A1)** or **(A2)** then the assertion of Theorem 1.7 remains true even if the imbedding of $S_{g_1B_1}$ in $S_{g_0B_0}$ is not continuous, as follows from the proof given below.

The proof of Theorem 1.7 is based on the following lemmas.

Lemma 1.9 *Let* $g_0 \le g_1$ *, and let* M *be a bounded subset of* $L_\Phi(0,T;S_{g_1 B_1})$ *. In addition, suppose that* M *is equicontinuous in the sense that to every* $\epsilon > 0$ *there exists some* $\delta = \delta(\epsilon) > 0$ *such that for all* $|\tau| < \delta$ *and* $t > 0$ *with* $t + \tau \in [0,T]$ *it holds*

$$\|g_1(x(\cdot)) - g_1(x(\cdot + \tau))\|_{L_\Phi(0,t;B_0)} < \epsilon \quad \forall\, x \in M\,. \tag{1.23}$$

If the imbedding $B_1 \hookrightarrow B_0$ *is compact, then* M *is compact in* $L_\Phi(0,T;S_{g_1 B_1})$ *.*

Proof: Let $\epsilon > 0$ be arbitrary. The compactness of the imbedding $S_{g_1 B_1} \hookrightarrow S_{g_0 B_0}$ and the boundedness of M in $L_\Phi(0,T;S_{g_1 B_1})$ imply that for almost all $t \in [0,T]$ there exists a finite ϵ - net (which depends on t) of the set $\{x(t) \,|\, x \in M\}$ in $S_{g_1 B_0}$. Therefore, using the compactness $[0,T]$, applying the standard diagonalization procedure, and invoking (1.23), we can conclude the existence of a finite ϵ - net for M with respect to the topology of $L_\Phi(0,T;S_{g_1 B_0})$. Consequently, M is compact in $L_\Phi(0,T;S_{g_1 B_0})$. The assertion then follows from the continuity of the imbedding of $L_\Phi(0,T;S_{g_1 B_0})$ in $L_\Phi(0,T;S_{g_0 B_0})$. □

Lemma 1.10 *Let the assumptions of Theorem 1.7 be satisfied with* **(A1)**. *Then it holds for any* $x \in \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1 B_1},X_0)$

$$\begin{gathered}\|g_1(x(t+\delta)) - g_1(x(t))\|_{B_0} \le C \int_t^{t+\delta} \|g_1'(x(\tau))\|_{\mathcal{L}(X_0,Y)}\, \|x'(\tau)\|_{X_0}\, d\tau\,, \\ \textit{for all } t \in [0,T] \setminus E \textit{ and all } \delta > 0 \textit{ such that } t+\delta \in [0,T]\setminus E\,,\end{gathered} \tag{1.24}$$

with a suitable constant $C > 0$ *and some set* $E \subset [0,T]$ *of zero measure.*

Proof: The assertion follows directly from the usual Lipschitz inequality. □

Lemma 1.11

(a) *Let* $B_1 \subset B_0 \subset B_2$ *, where* $B_2 \subset Y$ *is some Banach space, and where the imbedding* $B_1 \hookrightarrow B_0$ *is compact. Moreover, let* $g_0 \le g_1$ *. Then to any* $\epsilon > 0$ *there is some* $K(\epsilon) > 0$ *such that for all* $x_1, x_2 \in S_{g_1 B_1}$ *it holds*

$$\begin{aligned}\|g_0(x_1) - g_0(x_2)\|_{B_0} \;&\le\; \epsilon \left(\phi_{01}\left([x_1]_{S_{g_1 B_1}}\right) + \phi_{01}\left([x_2]_{S_{g_1 B_1}}\right)\right) \\ &\qquad + K(\epsilon)\, \|g_0(x_1) - g_0(x_2)\|_{B_2}\,,\end{aligned} \tag{1.25}$$

where $\phi_{01} : \mathbb{R}_+ \to \mathbb{R}_+$ *is the homogeinity function introduced in Definition 1.4, (b) that corresponds to the mappings* g_0 *,* g_1 *and the spaces* B_0 *,* B_1 *.*

(b) *Let* $B_1 \subset B_0 \subset B_2$ *, where the imbedding* $B_1 \hookrightarrow B_2$ *is compact, and where* B_1 *is a reflexive Banach space. In addition, let* (1.25) *hold for any pair* x_1 *,* $x_2 \in S_{g_1 B_1}$ *. Then the imbedding* $S_{g_1 B_1} \hookrightarrow S_{g_0 B_0}$ *is compact. In particular, if* $S_{g_i B_i} = B_i$ *and* $g_0 = g_1 = id.$ *, then* $B_1 \hookrightarrow B_0$ *compactly.*

Remark 1.12 Lemma 1.11, (b) shows that under the assumptions of the lemma the fulfillment of (1.25) is both necessary and sufficient for the compactness of the imbedding $B_1 \hookrightarrow B_0$.

Proof of Lemma 1.11: The assertion of part **(a)** follows from a well-known compactness theorem (see [11]) and from the relation $S_{g_1 B_1} \hookrightarrow S_{g_0 B_0} \subset S_{g_0 B_2}$.

To verify the assertion of part **(b)**, we show that if M is a bounded subset of $S_{g_1 B_1}$ then M is relatively compact in $S_{g_0 B_0}$. To this end, let $M \subset S_{g_1 B_1} \subset S_{g_0 B_0}$ be bounded. Owing to the "reflexivity" of the space $S_{g_1 B_1}$, and since $S_{g_1 B_1} \hookrightarrow S_{g_1 B_2}$, we may select a weakly convergent sequence $\{x_m\} \subset M$. By the compactness assumptions, we may assume $\{x_m\}$ converges strongly in $S_{g_0 B_2}$.

It follows from inequality (1.25) that for any $\delta > 0$ there exist $m(\delta) > 0$, $k(\delta) > 0$ and m_1, $m_2 > m(\delta)$, such that

$$\begin{aligned} \|g_0(x_{m_1}) - g_0(x_{m_2})\|_{B_0} \;\leq\; & \delta \left(\phi_{01}\left([x_{m_1}]_{S_{g_1 B_1}}\right) + \phi_{01}\left([x_{m_2}]_{S_{g_1 B_1}}\right)\right) \\ & + K(\delta)\, \|g_0(x_{m_1}) - g_0(x_{m_2})\|_{B_2}\,. \end{aligned} \tag{1.26}$$

Consequently, the sequence $\{x_m\}$ is a Cauchy sequence, and hence strongly convergent in $S_{g_0 B_0}$. This concludes the proof of part **(b)** of the Lemma. □

Proof of Theorem 1.7: As a consequence of Lemma 1.9, it suffices to show that any bounded subset M of $\mathcal{P}_{\Phi_0 \Phi_1}(0, T; S_{g_1 B_1}, X_0)$ is equicontinuous in $L_{\Phi_0}(0, T; S_{g_0 B_0})$.

Suppose now that **(A1)** holds (the proof is analogous if either **(A2)** or **(A3)** are assumed to hold). It then follows from Lemma 1.10 that

$$\begin{aligned} & \|g_1(x(t+\delta)) - g_1(x(t))\|_{B_0} \\ \leq\; & C \int_t^{t+\delta} \|g_1'(x(\tau))\|_{\mathcal{L}(X, B_0)}\, \|x'(\tau)\|_{X_0}\, d\tau \\ \leq\; & C \Big[\int_t^{t+\delta} \Phi_2\Big(\|g_1'(x(\tau))\|_{\mathcal{L}(X,B_0)}\Big)\, d\tau + \int_t^{t+\delta} \Psi_2\Big(\|x'(\tau)\|_{X_0}\Big)\, d\tau \Big] \\ =\; & C \Big[\int_0^T \chi(\tau; [t, t+\delta])\, \Phi_2\Big(\|g_1'(x(\tau))\|_{\mathcal{L}(X,B_0)}\Big)\, d\tau \\ & \quad + \int_0^T \chi(\tau; [t, t+\delta])\, \Psi_2\Big(\|x'(\tau)\|_{X_0}\Big)\, d\tau \Big], \end{aligned} \tag{1.27}$$

where $\chi(\,\cdot\,; E)$ denotes the characteristic function of a set $E \subset [0, T]$.

Now observe that $\Phi_3 \circ \Psi_2 \leq \Phi_1$ and $\Psi_2 \ll \Phi_1$. Hence

$$\begin{aligned} & \|g_1(x(t)) - g_1(x(t+\delta))\|_{B_0} \\ \leq\; & C \left[\int_0^T \chi(\tau; [t, t+\delta])\, [x(\tau)]_{S_{g_1 B_1}}\, d\tau + \int_0^T \chi(\tau; [t, t+\delta])\, \Psi_2\Big(\|x'(\tau)\|_{X_0}\Big)\, d\tau \right] \\ \leq\; & C \Big[\|\chi(\,\cdot\,; [t, t+\delta])\|_{L_{\Psi_0}}\, [x(\cdot)]_{L_{\Psi_0}(S_{g_1 B_1})} \\ & \quad + \|\chi(\,\cdot\,; [t, t+\delta])\|_{L_{\Psi_3}}\, \Big\|\Psi_2\Big(\|x'(\cdot)\|_{X_0}\Big)\Big\|_{L_{\Phi_3}} \Big], \end{aligned} \tag{1.28}$$

where Ψ_3 is the N-function $\Psi_3 := \Phi_3^*$.

As M is bounded in $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0)$, there exists to every $\epsilon > 0$ some $\delta(\epsilon) > 0$ such that, with $\tilde{\Psi} := \sup\{\Psi_0\,,\,\Psi_3\}$,

$$\int_0^T \|g_1(x(t)) - g_1(x(t+\delta))\|_{B_0}\,dt \,\le\, C\left(\tilde{\Psi}^{-1}\left(\frac{1}{\delta}\right)\right)^{-1} \cdot [x]_{\mathcal{P}_{\Phi_0\Phi_1}(S_{g_1B_1},X_0)}\,. \qquad (1.29)$$

From this the equicontinuity of M immediately follows. Lemma 1.9 then implies the compactness of the imbedding $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0) \hookrightarrow L_{\Phi_0}(0,T;S_{g_1B_0})$, whence also the compactness of $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0) \hookrightarrow L_{\Phi}(0,T;S_{g_0B_0})$ follows if $\Phi \le \Phi_0$. The first part of the assertion is proved.

The second assertion can be verified as follows: proceeding as in the first part of the proof, we obtain an inequality of the form (1.28); then, we use a diagonalization procedure and the boundedness of the set M in the space $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0)$ to establish the validity of the second assertion. The details of the proof are analogous to the proof of an assertion of similar type which can be found in [3]. □

Remark 1.13 The spaces $S_{g_1B_1}$ or $S_{g_0B_0}$, but also $L_\Phi(0,T;S_{g_0B_0})$ and, in particular, $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{g_1B_1},X_0)$, may be linear spaces (see [5]); the N-functions Φ_0, Φ_1, Φ, Φ_2 may be increasing functions of power type (see, e.g., [5], [15], [16], [21]).

2 Nonlinear equations in Banach spaces

Let X, Y be Banach spaces, S_{gB} a *qn*-space defined as in the previous section, and $f : S_{gB} \subset X \to Y$ a nonlinear mapping.

We will now study the solvability of functional equations of the form

$$\langle f(x), y^* \rangle = \langle y, y^* \rangle \quad \forall\, y^* \in M^* \subset Y^*\,, \qquad (2.1)$$

where $\langle \cdot\,,\,\cdot \rangle$ denotes the duality pairing for the pair (Y, Y^*). Here, Y^* is the dual space of Y (or, vice versa, Y is the dual of Y^*).

Definition 2.1 Let M^* be a subset of the dual space of Y, (of Y^*, respectively), and let $y \in M \subset Y$ be given. Then any $x \in S_{gB}$ satisfying (2.1) is called an M^*-*solution* to (2.1).

We consider the following conditions.

(B1) The mapping $f : S_{gB} \subset X \to Y$ is a *weakly compact (continuous)* mapping, i.e., whenever $x_m \to x_0$ weakly in S_{gB}, then there exists a subsequence $\{x_{m_k}\}$ of $\{x_m\}$ such that $f(x_{m_k}) \to f(x_0)$ weakly in Y as $m_k \nearrow \infty$.

(B2) It holds $B = B^{**}$, and there exist some separable topological vector space X_1 and a linear continuous operator $\varphi : X_1 \subset S_{gB} \to Y^*$ such that $\overline{\varphi(X_1)} \supset M^*$.

(B3) There exist some reflexive separable Banach space $Y_0^* \subset Y^*$ such that $M^* \subset Y_0^*$, a subspace $S_{g_1 B_1}$ of the qn-space S_{gB}, and some nonlinear mapping $\varphi : S_{g_1 B_1} \to Y^*$ such that the following conditions hold:

(i) $\overline{\varphi(S_{g_1 B_1})} \supset M^*$.

(ii) $\varphi(S_{g_1 B_1})$ contains a linear manifold which is dense in Y_0^*.

(iii) The inverse mapping φ^{-1} of φ is a weakly compact (continuous) mapping from Y_0^* into S_{gB} such that for every $y \in Y_0^*$ the image $\varphi^{-1}(y)$ is a closed and convex set (note that φ^{-1} is possibly set-valued so that the notion "weakly continuous" has to be understood in the appropriate sense, see, e.g. [2], [3]).

Theorem 2.2 *Suppose that S_{gB} is a weakly complete space, and let either the conditions* **(B1)**, **(B2)** *or* **(B1)**, **(B3)** *be satisfied. Furthermore, assume that a set $M \subset Y$ is given such that for each $y \in M$ there exists some $r = r(y) > 0$ such that*

$$\langle f(x), \varphi(x) \rangle \geq \langle y, \varphi(x) \rangle \quad \begin{cases} \forall\, x \in X_1\,, \quad [x]_{S_{gB}} \geq r & (\text{if } \mathbf{(B1)}, \mathbf{(B2)} \text{ hold}), \\ \forall\, x \in S_{g_1 B_1}\,, \quad [x]_{S_{gB}} \geq r & (\text{if } \mathbf{(B1)}, \mathbf{(B3)} \text{ hold}). \end{cases} \tag{2.2}$$

Then Eq. (2.1) *has an M^*-solution for any $y \in M$.*

Proof: Let **(B1)**, **(B2)** and the corresponding condition in (2.2) be satisfied. We argue by Galerkin approximation. To this end, let $\{x^k\}_{k\in\mathbb{N}}$ be a complete system in the (separable) space X_1. We then look for approximate solutions of the form $x_m = \sum_{k=1}^{m} c_m^k\, x^k$, where the unknown coefficient vector $c_m = (c_m^k)$ has to be determined from the system of algebraic equations

$$\Phi_k(c_m) := \langle f(x_m), \varphi(x^k) \rangle - \langle y, \varphi(x^k) \rangle = 0\,, \quad k = 1, \ldots, m\,. \tag{2.3}$$

Now observe that by **(B1)** the mapping $\Phi(c_m) := (\Phi_1(c_m), \ldots, \Phi_m(c_m))$ is continuous. From (2.2) it follows that for some $r > 0$ for all x_m with $[x_m]_{S_{gB}} \geq r$ the "acute angle"-condition is satisfied, i.e., on any sphere $S_{r_1}(0) \subset \mathbb{R}^m$, where $r_1 \geq r$, it holds

$$\sum_{k=1}^{m} \langle \Phi_k(c_m), c_m^k \rangle \geq 0 \quad \forall\, c_m \in \mathbb{R}^m\,, \quad \|c_m\| = r_1\,. \tag{2.4}$$

The solvability of (2.3) then follows from the well-known lemma "on the acute angle" which is equivalent to Brouwer's fixed-point theorem (see, e.g., [5], [11], [12]).

Hence, we obtain a sequence $\{x_m\} \subset S_{gB}$ of solutions to (2.3) which by construction is bounded in S_{gB}, i.e., we have $[x_m]_{S_{gB}} \leq r$, for all $m \in \mathbb{N}$. Invoking the "reflexivity" and the weak completeness of the space S_{gB}, we conclude the existence of a subsequence, again denoted $\{x_m\}$, and of some $x_0 \in S_{gB}$ such that $x_m \to x_0$ weakly in S_{gB}.

We now show that x_0 solves (2.1). In view of **(B2)**, it suffices to pass to the limit as $m \nearrow \infty$ in (2.3) for every fixed $k \in \mathbb{N}$. By **(B1)**, we may without loss of generality assume that $f(x_m) \to f(x_0)$ weakly in Y; hence passage to the limit yields that

$$\langle f(x_0), \varphi(x^k) \rangle = \langle y, \varphi(x^k) \rangle \quad \forall\, k \in \mathbb{N}. \tag{2.5}$$

Since $\{x^k\}_{k \in \mathbb{N}}$ is complete in X_1, we therefore have

$$\langle f(x_0), \varphi(x) \rangle = \langle y, \varphi(x) \rangle \quad \forall\, x \in X_1. \tag{2.6}$$

The assertion then follows from the fact that $\overline{\varphi(X_1)} \supset M^*$.

Let us now assume that the conditions **(B1)**, **(B3)** and the corresponding condition in (2.2) are fulfilled. We pick a complete system $\{y^k\}_{k \in \mathbb{N}}$ in Y_0^* and look for approximate solutions $x_m \in \varphi^{-1}(y_m^*)$ to Eq. (2.1), where $y_m^* := \sum_{k=1}^{m} c_m^k y^k \in Y_0^*$. The unknown coefficients c_m^k, $k = 1, \ldots, m$, have to be determined from the system of algebraic equations

$$\tilde{\Phi}_k(c_m) := \langle f(\varphi^{-1}(y_m^*)), y^k \rangle - \langle y, y^k \rangle = 0, \quad k = 1, \ldots, m. \tag{2.7}$$

The solvability of the system (2.7) again follows from the lemma of type "on the acute angle" that is based on Kakutani's fixed point theorem (see [2], [22], [23]), just in the same way as using Brouwer's fixed point theorem. Indeed, from **(B1)** and **(B3)** it follows that $\tilde{\Phi}_k$ is continuous (recall that f and φ^{-1} are weakly continuous mappings), while the "acute angle" - condition follows from **(B3)** and (2.2) just as in the previous case. Hence, arguing along the lines of the previous case, we obtain the existence of an M^* - solution also in this case. We may omit the details to the reader. Theorem 2.2 is proved. □

Remark 2.3 From the proof of Theorem 2.2 it follows that in the conditions **(B2)** and **(B3)**, respectively, the assumption $M^* \subset \overline{R(\varphi)}$ can be replaced by the following assumption:

There exists a space $Y_1^* \subset Y^*$ such that $R(\varphi) \subset Y_1^*$, $M^* \subset Y_1^*$, and $\overline{R(\varphi)} \supset Y_1^*$ (or $R(\varphi)$ contains a linear manifold which is everywhere dense in Y_1^*, respectively).

Next, we consider Eq. (2.1) under the assumption that f satisfies the following condition.

(B1') The mapping $f : S_{gB} \subset X \to Y^*$ is *weakly-star continuous*, that is, it satisfies condition **(B1)** with weak convergence replaced by weak-star convergence.

Theorem 2.4 *Let* **(B1')** *hold, let* S_{gB} *be weakly-star complete and let bounded subsets of* S_{gB} *be weakly-star compact in* S_{gB}. *Besides, suppose that either* **(B2)**

and the corresponding condition in (2.2) *or* **(B3)** *and the corresponding condition in* (2.2) *are satisfied with the corresponding modifications (i.e., after exchanging the roles of the spaces* Y *and* Y^*, *as well as replacing weak continuity by weak-star continuity). Then the problem*

$$\langle f(x), y\rangle = \langle y^*, y\rangle \quad \forall\, y \in M \subset Y \tag{2.8}$$

has for any $y^* \in M^*$ *a solution* $x \in S_{gB}$.

Proof: The proof is analogous to that of Theorem 2.2 and is therefore omitted here. □

From the above theorems we immediately get the following result (a similar result has been proved earlier in [5]).

Corollary 2.5 *Let* **(B1)** *and* **(B2)** *hold, and let* $M \subset Y$ *be such that* (2.2) *holds. Then to any* $y \in M$ *there exist* $x \in S_{gB}$ *and* $y_0 \in ker(\varphi^*) \cap Y$ *with* $f(x) = y + y_0$, *where* φ^* *denotes the adjoint of the linear continuous operator* φ.

We also conclude the following result which also applies when the nonlinear equations are considered in non-reflexive Banach spaces X and Y.

Corollary 2.6 (Solvability theorem) *Let the assumptions of Theorem 2.2 (or of Theorem 2.4, respectively) be satisfied, and let* M *(or* M^*, *respectively) denote the sets introduced there. In addition, suppose that* $R(\varphi)$ *contains a subset which is everywhere dense in* Y^* *(or in* Y, *respectively). Then the equation* $f(x) = y$ *(*$f(x) = y^*$, *respectively) is for any* $y \in M$ *(*$y^* \in M^*$, *respectively) solvable in* S_{gB}.

Moreover, we can conclude from the Theorems 2.2 and 2.4 a further result.

Theorem 2.7 *Let the assumptions of Theorem 2.2 (or of Theorem 2.4, respectively) be fulfilled, and let the spaces* X, Y, Y^*, S_{gB}, X_0, $S_{g_1B_1}$, *as well as the mappings* f, φ, *be defined as in Theorem 2.2. Besides, assume that the following condition is satisfied:*

(B4) *The operators* f *and* φ *induce a coercive pairing on* $S_{g_1B_1} \subset S_{gB}$ *in a generalized sense (*f *is coercive in a generalized sense), that is, there exists a continuous function* $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, *satisfying* $\Psi(\tau) \geq c\,|\tau|$ *on* $\mathbb{R}_+$ *for some* $c > 0$, *such that for* $x \in S_{g_1B_1}$ *it holds*

$$\frac{\langle f(x), \varphi(x)\rangle}{\Psi\left([x]_{S_{gB}}\right)} \nearrow +\infty\,, \quad \text{as } [x]_{S_{gB}} \nearrow \infty\,. \tag{2.9}$$

Then the equation $f(x) = y$ *is solvable in* S_{gB} *for every* $y \in Y$ *satisfying*

$$\sup \left\{ \frac{\langle y, \varphi(x) \rangle}{\Psi\left([x]_{S_{gB}}\right)} \;\Big|\; x \in S_{gB} \right\} < +\infty \,. \tag{2.10}$$

Proof: The proof of Theorem 2.7 follows from Theorem 2.2 (or from Theorem 2.4, respectively), where the only difference to Theorem 2.2 or Theorem 2.4 is given by the definition of the set M. □

From Theorem 2.7 we can easily conclude the following result.

Corollary 2.8 *Let the assumptions of Theorem 2.7 hold, and suppose that*

$$\frac{\|\varphi(x)\|_{Y_0^*}}{\Psi\left([x]_{S_{gB}}\right)} < +\infty \,, \quad \textit{whenever } [x]_{S_{gB}} \nearrow +\infty \,. \tag{2.11}$$

Then the equation $f(x) = y$ *is solvable for any* $y \in Y_0$ *, where* $Y_0 \subset Y$ *(or* $Y_0 \subset Y^*$ *, respectively).*

3 The Cauchy initial-value problem for operator differential equations in Banach spaces

Let X , Y , Y^* , B_0 , B_1 denote Banach spaces, and let $S_{gB} \subset X$ be a weakly complete "reflexive" (weakly-star complete bounded weakly-star compact) qn-space. We assume that either $(Y\,, Y^*)$ is a complementary pair of spaces (i.e. Y^* contains a bounded subspace for which Y is the dual space and vice versa, as, for example, in the case of Orlicz spaces or Orlicz-Sobolev spaces, see [8], [4], [7], [10]), or Y is the dual space of some Banach space Y_1.

We consider the evolution equation

$$\frac{dx(t)}{dt} + f(t\,, x(t)) = y(t) \,, \quad x(0) = 0 \,, \tag{3.1}$$

where $f(t\,, \cdot) : S_{gB} \to Y$ is for all $t \in [0, T]$ a nonlinear operator.

We make the following assumptions:

(C1) $f : \mathcal{P}_{\Phi_0 \Phi_1}(0, T; S_{gB}, Y) \to L_{\Phi_1}(0, T; Y)$ is a weakly-star continuous operator. Here, Φ_0 , Φ_1 are N-functions such that for some $q > 1$ we have

$$\Psi_0(\tau) \leq \Phi_1(\tau) \leq C|\tau|^q \,, \quad \Phi_1 \leq \Phi_0 \,. \tag{3.2}$$

(C2) There exists a mapping $\varphi : S_{g_1 B_1} \subset S_{gB} \to Y_1$ such that

$$\varphi \,:\, W^1 E_{\Phi_0}(0, T; S_{g_1 B_1}) \to W^1 E_{\Psi_1}(0, T; Y_1) \tag{3.3}$$

and such that one of the following conditions is satisfied:

(a) $\varphi : S_{g_1B_1} \to Y_1$ is a linear bounded operator that commutes with the derivative d/dt, such that $R(\varphi)$ contains a linear set that is everywhere dense in Y_1, and $S_{g_1B_1}$ is a separable locally convex topological vector space satisfying $S_{g_1B_1} \subset Y \cap S_{gB}$.

(b) $\varphi : S_{g_1B_1} \to Y_1$ is a nonlinear mapping of class C_1 having an inverse φ^{-1} which is weakly continuous from Y_1 in S_{gB} (from $W^1E_{\Psi_1}(0,T;Y_1)$ to $W^1E_{\Phi_0}(0,T;S_{gB})$). Furthermore, $R(\varphi)$ contains some everywhere in Y_1 dense linear set, and $\varphi(0) = 0$.

(C3) $f(t,\cdot)$ and φ form a pair that is coercive in a generalized sense on the space $E_{\Phi_0}(0,T;S_{g_1B_1})$, i.e. there exist constants C, $\tilde{C} > 0$ and a continuous function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa \geq \Phi_0$ for $|\tau| > 1$ and

$$\int_0^T \langle f(t, x(t)), \varphi(x(t))\rangle \, dt \;\geq\; C\,\kappa\left([x]_{L_{\Phi_0}(S_{gB})}\right) - \tilde{C}. \tag{3.4}$$

(C4) There exist constants $0 \leq \mu < 1$, $\tilde{C}_1 \geq 0$, $\tilde{C}_2 \geq 0$, $C_2 \geq 0$, $C_1 > 0$ such that for any $x \in W^1E_{\Phi_0}(0,T;S_{g_1B_1})$ with $x(0) = 0$ and $\xi \in E_{\Phi_0}(0,T;S_{g_1B_1})$ the following inequalities hold:

$$\begin{aligned} \int_0^T \langle \xi, \varphi'(x)\xi\rangle(t)\, dt &\geq C_1 \left\|\xi\right\|^q_{L_{\Phi_1}(Y_1)}(1 + [x]^{\mu}_{L_{\Phi_0}(S_{gB})}) - \tilde{C}_1, \\ \int_0^T \langle x'(t), \varphi(x(t))\rangle\, dt &\geq C_2\left\|x\right\|^q_{L_{\Phi_1}(Y)} - \tilde{C}_2. \end{aligned} \tag{3.5}$$

Theorem 3.1 *Let the conditions* **(C1)** *to* **(C4)** *be fulfilled, and let the spaces* Y, Y_1, S_{gB} *satisfy the above conditions. Suppose that* $y \in L_{\Phi_1}(0,T;Y)$ *satisfies for some* $r \geq 0$ *the inequality*

$$\int_0^T \langle y, \varphi(x)\rangle(t)\, dt \leq \tilde{C}_0\,\kappa([x]_{L_{\Phi_0}(S_{gB})}) + \tilde{C}_1, \qquad \forall\, x \in E_{\Phi_0}(0,T;S_{g_1B_1}) \quad \text{with } [x]_{L_{\Phi_0}S_{gB}} \geq r, \tag{3.6}$$

where $\tilde{C}_0$, $\tilde{C}_1$ *are nonnegative constants satisfying* $C \geq \tilde{C}_0 + \varepsilon$ *for some* $\varepsilon > 0$. *Then* (3.1) *has a solution* $x \in \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y)$ *with* $x(0) = 0$ *in the sense of* $L_{\Phi_1}(0,T;Y)$, *that is, for any* $y^* \in E_{\Psi_1}(0,T;Y_1)$ *there holds*

$$\int_0^T \langle f(t,x(t)), y^*(t)\rangle\, dt + \int_0^T \langle x'(t), y^*(t)\rangle\, dt = \int_0^T \langle y(t), y^*(t)\rangle\, dt. \tag{3.7}$$

Proof: We apply the method of elliptic regularization and the results of the previous section. At first, we prove for $\epsilon > 0$ the solvability of the problem

$$\begin{cases} -\epsilon\, \dfrac{d^2x_\epsilon(t)}{dt^2} + \dfrac{dx_\epsilon(t)}{dt} + f(t, x_\epsilon(t)) = y(t), \quad t \in (0,T), \\ x_\epsilon(0) = 0, \quad x'_\epsilon(T) = 0. \end{cases} \tag{3.8}$$

To this end, we show that the conditions **(C1)** to **(C4)** imply that Theorem 2.4 can be applied to prove the solvability of the functional equality

$$\begin{aligned} & \epsilon \int_0^T \left\langle \frac{dx_\epsilon}{dt}\,,\, \frac{dy^*}{dt} \right\rangle (t)\, dt \,+\, \int_0^T \left\langle \frac{dx_\epsilon}{dt}\,,\, y^* \right\rangle (t)\, dt \,+\, \int_0^T \langle f(t, x_\epsilon(t))\,,\, y^*(t)\rangle\, dt \\ = \; & \int_0^T \langle y(t)\,,\, y^*(t)\rangle\, dt\,, \\ & \forall\; y^* \,\in\, W^1E_{\Psi_1}(0,T;Y_1)\,, \quad y^*(0) = y^*(T) = 0\,. \end{aligned} \tag{3.9}$$

Indeed, by **(C1)**, and since the operator defined by this equation depends linearly on x'_ϵ, we obtain that the latter is weakly-star continuous as a mapping from $\{x \in W^1L_{\Phi_0}(0,T;S_{gB}) \mid x(0) = x(T) = 0\}$ into $W^{-1}L_{\Phi_1}(0,T;Y)$. This yields that the first condition of Theorem 2.4 is fulfilled for equation (3.9).

From **(C2)** it follows that also the second condition of Theorem 2.4 is fulfilled, and the conditions **(C3)** and **(C4)** imply the generalized coercivity of the operator on the space $\{x \in W^1E_{\Phi_0}(0,T;S_{g_1B_1}) \mid x(0) = 0\}$.

Thus, applying Theorem 2.4, we obtain the solvability of (3.9), which in this case also means that (3.8) is solvable in the sense of the dual space of $\{y^* \in W^1E_{\Psi_1}(0,T;Y_1) \mid y^*(T) = y^*(0) = 0\}$, since $R(\varphi)$ is everywhere dense in the space $\{x \in W^1E_{\Psi_1}(0,T;Y_1) \mid x(0) = 0\}$.

It remains to show that this entails the weak-star solvability of the equation (3.8), i.e., we have to prove that if $x_\epsilon \in \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y)$ with $x_\epsilon(0) = 0$ is a solution to (3.9), it is under the assumptions of Theorem 3.1 also an $E_{\Psi_1}(0,T;Y_1)$-solution to (3.8).

To this end, let $x_\epsilon \in \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y)$, $x_\epsilon(0) = 0$, be an $E_{\Psi_1}(0,T;Y_1)$-solution for a given $y \in L_{\Phi_1}(0,T;Y)$. Using the general form of linear continuous functionals on the space $E_{\Phi_1}(0,T;Y_1)$, we get under the assumptions of Theorem 3.1 that $\epsilon \dfrac{d^2x_\epsilon}{dt^2} \in L_{\Phi_1}(0,T;Y)$ for all $\epsilon > 0$; indeed, we have

$$-\,\epsilon \int_0^T \left\langle \frac{d^2x_\epsilon}{dt^2}\,,\, y^* \right\rangle (t)\, dt \,=\, \int_0^T \left(\langle y\,,\, y^*\rangle \,-\, \left\langle \frac{dx_\epsilon}{dt} \,+\, f(t,x_\epsilon)\,,\, y^* \right\rangle \right) (t)\, dt\,, \tag{3.10}$$

with $y\,, x'_\epsilon\,,\, f(\cdot, x_\epsilon(\cdot)) \in L_{\Phi_1}(0,T;Y)$, so that also $\epsilon \dfrac{d^2x_\epsilon}{dt^2} \in L_{\Phi_1}(0,T;Y)$.
Consequently, $x'_\epsilon(T)$ is defined for every $\epsilon > 0$, and it follows immediately from the functional equation that $x'_\epsilon(T) = 0$. Thus, the solvability of (3.8) is proved.

Now, if we were able to show that we can pass to the limit as $\epsilon \searrow 0$ in (3.9), then it would be proved that (3.1) is solvable in a weak-star sense. For this purpose, the uniform boundedness of the set $\{x_\epsilon(t)\}$ in $\{x \in \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y) \mid x(0) = 0\}$ for $\epsilon \searrow 0$ has to be shown.

The functional equation yields

$$-\,\epsilon \int_0^T \left\langle \frac{d^2x_\epsilon}{dt^2}\,, \zeta \right\rangle \eta(t)\, dt \,+\, \int_0^T \left\langle \frac{dx_\epsilon}{dt}\,,\, \zeta \right\rangle \eta(t)\, dt \,=\, \int_0^T \langle y \,-\, f(t,x_\epsilon)\,,\, \zeta\rangle\, \eta(t)\, dt\,. \tag{3.11}$$

The function $x_\epsilon : [0,T] \to Y$ is a solution to the boundary value problem

$$- \epsilon \frac{d^2 x_\epsilon(t)}{dt^2} + \frac{dx_\epsilon(t)}{dt} = y_\epsilon(t) := y(t) - f(t, x_\epsilon(t)), \quad x_\epsilon(0) = 0, \quad x'_\epsilon(T) = 0. \tag{3.12}$$

From the assumptions on f, and the boundedness of $\{x_\epsilon\}_{\epsilon>0}$ in $L_{\Phi_0}(0,T;S_{gB})$, we obtain that $\{y_\epsilon\}_{\epsilon>0}$ is bounded in $L_{\Phi_1}(0,T;Y)$.

Now, solving (3.12) we get that

$$\frac{dx_\epsilon}{dt}(T-t) = \int_0^{T-t} y_\epsilon(T-\tau)\, \mathrm{e}^{-\frac{T-t-\tau}{\epsilon}}\, d\tau, \tag{3.13}$$

and since $\frac{1}{\epsilon}\int_0^\infty \exp(-\tau/\epsilon) d\tau = 1$, we get from Minkowski's inequality ([24]) or from Young's generalized inequality for convolution integrals ([13]) that $\{x'_\epsilon\}_{\epsilon>0}$ is bounded in the space $L_{\Phi_1}(0,T;Y)$. Thus, $\{x_\epsilon\}_{\epsilon>0}$ is a bounded subset of the space $\{x \in \mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y) \mid x(0) = 0\}$.

Since, for every $y^* \in W^1 E_{\Psi_1}(0,T;Y_1)$ the inequality

$$\left| \epsilon \int_0^T \left\langle \frac{dx_\epsilon}{dt}, \frac{dy^*}{dt} \right\rangle (t)\, dt \right| \le \epsilon \left\| \frac{dx_\epsilon}{dt} \right\|_{L_{\Phi_1}(Y)} \|y^*\|_{W^1_{\Psi_1}(0,T;Y_1)}$$
$$\le C\, \epsilon^{1-\frac{1}{q}} \|y^*\|_{W^1_{\Psi_1}(0,T;Y_1)} \tag{3.14}$$

holds for all $\epsilon > 0$, we may pass to the limit in (3.9) for $\epsilon \searrow 0$. It follows that the limit x of $\{x_\epsilon\}$ belongs to $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y) \cap \{x \mid x(0) = 0\}$ and satisfies the identity

$$\int_0^T \left\langle \frac{dx}{dt}, y^* \right\rangle (t)\, dt + \int_0^T \langle f(t,x(t)), y^*(t) \rangle\, dt = \int_0^T \langle y, y^* \rangle (t)\, dt, \tag{3.15}$$

for every $y^* \in W^1 E_{\Psi_1}(0,T;Y_1)$.

Finally, since $W^1 E_{\Psi_1}(0,T;Y_1)$ is dense in $E_{\Psi_1}(0,T;Y_1)$, we may conclude that x is a solution to problem (3.1). This concludes the proof of the assertion. □

Remark 3.2 It should be clear that if we change the assumptions correspondingly as in Section 2, then we obtain the solvability of (3.1) for weakly continuous operators $f(t,\cdot)$; for this case special results similar to Theorem 3.1 have already been proved in the earlier papers [5], [10], [17], [18], [19]. We would like to state as well that the proof given here is analogous to that of a similar theorem in [5].

From Theorem 3.1 it follows immediately

Corollary 3.3 *Let the assumptions of Theorem 3.1 be fulfilled, where*

$$\frac{\kappa\left([x]_{L_{\Phi_0}(S_{gB})}\right)}{\|\varphi(x)\|_{L_{\Psi_1}(Y_1)}} \to \infty, \quad \text{for } [x]_{S_{gB}} \to \infty.$$

Then the problem (3.1) *is* $E_{\Psi_1}(0,T;Y_1)$ *- solvable in* $\mathcal{P}_{\Phi_0\Phi_1}(0,T;S_{gB},Y) \cap \{x \mid x(0) = 0\}$ *for any* $y \in L_{\Phi_1}(0,T;Y)$.

4 Completely nonlinear differential equations of elliptic type

Let $G \subset \mathbb{R}^n$ $(n \geq 1)$ denote a bounded domain with a sufficiently smooth boundary ∂G. We consider the boundary value problem

$$A(u) := \begin{cases} F(x, u, Du, D^2u, \dots, D^{(2m-1)}u, Lu) = h(x), & x \in G, \\ D^\beta u|_{\partial G} = 0, & |\beta| \leq m - 1. \end{cases} \tag{4.1}$$

Here, denoting $W^{2m}_\Phi(G) := W^{2m}L_\Phi(G)$, we have $u \in W^{2m}_\Phi(G) \cap \overset{o}{W}{}^m_\Phi(G)$, and $h \in W^2_\Psi(G) \cap \overset{o}{W}{}^1_\Psi(G)$ and $F \equiv F_0 + F_1$ are given continuous functions, $\alpha = (\alpha_1 \dots, \alpha_n) \in \mathbb{N}^n_0$ is a multiindex, $|\alpha| := \sum_{i=1}^n \alpha_i$, $m \geq 1$, L a linear differential expression of order $2m$ that maps continuously from $W^{2m}_{p_0}(G) \cap \overset{o}{W}{}^m_{p_0}(G)$ into $L_p(G)$ for some $p_0 \geq p > 1$, and $D^k u := \{D^\alpha u \mid |\alpha| = k\}$.

We consider the following conditions:

(D1) The differential expression L generates a linear differential operator $L : W^{2m}_{p_0}(G) \cap \overset{o}{W}{}^m_{p_0}(G) \to L_p(G)$ with coefficients in $C^2(G)$, and there is some $C > 0$ such that

$$\|u\|_{W^2_{p_0}(G) \cap \overset{o}{W}{}^m_{p_0}(G)} \leq C \, \|L\,u\|_{L_p(G)}. \tag{4.2}$$

(In other words, L is a uniformly elliptic operator of order $2m$, see [24]).

(D2) $F_0(x, \xi, \eta) := F_0(x, \xi_0, \dots, \xi_\alpha, \dots, \eta)$ and $F_1(x, \xi, \eta)$ are continuously differentiable functions, i.e., $F_0, F_1 \in C^1$. Furthermore, it holds $F_0(x, \xi, 0) = 0$ and $F_1(x, \xi, \eta) = 0$ whenever $\eta = 0$ or $\xi_\alpha = 0$ for at least one α with $|\alpha| \leq m - 1$.

(This assumption and the fact that the equation is examined for $h \in \overset{o}{W}{}^1_\Psi(G)$ will imply that if the solution has additional smoothness, in a suitable sense, then it will follow from the equation that $Lu|_{\partial G} = 0$ holds.)

(D3) There exist functions φ, $\varphi_{0\alpha}$, $\varphi_{1\alpha} \geq 0$ for $|\alpha| \leq 2m-1$ that are N-functions or main parts of N-functions, and constants $C > 0$, $\tilde{C} > 0$ such that $\varphi(\tau) \geq \varphi_{0\alpha}(\tau)\, \varphi_{1\alpha}(\tau)$ for all $\tau \in \mathbb{R}_+$ and

$$C \left[\varphi(\eta) + \sum_{|\alpha| \leq 2m-1} \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha) \right] \leq F_{0\eta}(x, \xi, \eta)$$
$$\leq \tilde{C} \left[\varphi(\eta) + \sum_{|\alpha| \leq 2m-1} \varphi_{0\alpha}(\eta)\, \phi_{1\alpha}(\xi_\alpha) + 1 \right], \tag{4.3}$$

for any $\xi_\alpha, \eta \in \mathbb{R}$ and $x \in G$.

By Φ (resp. $\Phi_{1\alpha}$), we denote the N-function induced by φ (resp. by $\varphi_{1\alpha}$):

$$\Phi(\eta) := \int_0^{|\eta|} \tilde{\varphi}(\eta_1)\, d\eta_1 = \int_0^{|\eta|} \left[\int_0^{|\eta_1|} \varphi(\tau)\, d\tau \right] d\eta_1 . \qquad (*)$$

Ψ (resp. $\Psi_{1\alpha}$) denotes the N-function which is conjugate to Φ (resp. $\Phi_{1\alpha}$).

(D4) There exists a $p \geq 2$ such that $\Phi(\tau) \geq c\,|\tau|^p$, and $W_p^1(G) \subset L_M(G)$ for an N-function M satisfying $M \geq \Phi$.

(We remark that if Φ satisfies a Δ_3- or a Δ^2-condition, then the existence of some $p \geq 2$ satisfying **(D4)** becomes obvious, because in this case there is a p such that $p > n$ and $\Phi \geq c\,|\tau|^p$ ([8]). Consequently, $W_p^1(G) \subset L_M(G)$ holds for all M (see [4]).)

(D5) There exist convex functions $\varphi_0, \varphi_1, \varphi_2, \varphi_{j\alpha} \geq 0$, $j = 0, 1, \ldots, 7$, $|\alpha| \leq 2m - 1$, and constants $C_j \geq 0$, $j = 1, \ldots, 5$, such that the following inequalities hold.

(I)
$$\sum_{i=1}^{n} \left[\left| F_{0\,x_i}(x,\xi,\eta) \right| + \sum_{|\alpha| \leq 2m-2} \left| F_{0\,\xi_\alpha}(x,\xi,\eta) \right| \left| \xi_{\bar{\alpha}_i} \right| \right]$$
$$\leq C_1 \left\{ |\tilde{\varphi}(\eta)| + \sum_{|\alpha| \leq 2m-1} |\tilde{\varphi}_{0\alpha}(\eta)|\ \varphi_{1\alpha}(\xi_\alpha) \right\},$$

where $\bar{\alpha}_i := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n)$.

(II) $$\left| F_{0\,\xi_\beta}(x,\xi,\eta) \right| \leq C_2 \left\{ \varphi_0(\eta) + \sum_{|\alpha| \leq 2m-1} \varphi_{2\alpha}(\eta)\, \varphi_{3\alpha}(\xi_\alpha) \right\}, \quad |\beta| = 2m-1 .$$

(III) $$\left| F_{1\eta}(x,\xi,\eta) \right| \leq C_3 \left\{ \varphi(\eta) + \sum_{|\alpha| \leq 2m-1} \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha) \right\}, \quad \text{with } C_3 < \frac{C}{3} .$$

(IV) $$\left| F_{1\xi_\beta}(x,\xi,\eta) \right| \leq C_4 \left\{ \varphi_2(\eta) + \sum_{|\alpha| \leq 2m-1} \varphi_{6\alpha}(\eta)\, \varphi_{7\alpha}(\xi_\alpha) \right\}, \quad |\beta| = 2m-1 .$$

(V)
$$\sum_{i=1}^{n} \left\{ \left| F_{1x_i}(x,\xi,\eta) \right| + \sum_{|\alpha| \leq 2m-2} \left| F_{1\xi_\alpha}(x,\xi,\eta) \right| \left| \xi_{\bar{\alpha}_i} \right| \right\}$$
$$\leq C_5 \left\{ \varphi_1(\eta) + \sum_{|\alpha| \leq 2m-1} \varphi_{4\alpha}(\eta)\, \varphi_{5\alpha}(\xi_\alpha) \right\},$$

where $\bar{\alpha}_i = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n)$.

In addition,

$$\begin{gathered}
\varphi_0(\eta) \leq \Phi^{\frac{1}{q}}(\eta)|\eta|^{-1}, \quad \varphi_1(\eta) < |\tilde{\phi}(\eta)|, \quad \varphi_2(\eta) < \Phi^{\frac{1}{q}}(\eta)|\eta|^{-1}, \\
\varphi_{2\alpha}(\eta) \leq \Phi_{0\alpha}^{\frac{1}{q}}(\eta)|\eta|^{-1}, \quad \varphi_{3\alpha}(\xi_\alpha) \leq \varphi_{1\alpha}^{\frac{1}{q}}(\xi_\alpha), \quad \varphi_{4\alpha}(\eta) \leq |\tilde{\phi}_{0\alpha}(\eta)|, \\
\varphi_{5\alpha}(\xi_\alpha) \leq \varphi_{1\alpha}(\xi_\alpha), \quad \varphi_{6\alpha}(\eta) < \Phi_{0\alpha}^{\frac{1}{q}}(\eta)|\eta|^{-1}, \quad \varphi_{7\alpha}(\xi_\alpha) < \varphi_{1\alpha}^{\frac{1}{q}}(\xi_\alpha), \\
\forall\ \xi_\alpha, \eta \in \mathbb{R}, \quad |\alpha| \leq 2m-1, \quad q = p' .
\end{gathered} \qquad (4.4)$$

We introduce the following space of measurable functions

$$\mathcal{H}^L(G) \;:=\; \Big\{u \;\Big|\; \sum_{i=1}^{n}\int_G \Big[\varphi(L\,u) + \sum_{|\alpha|\le 2m-1} \varphi_{0\alpha}(L\,u)\;\varphi_{1\alpha}(D^\alpha u)\Big]|D_i L\,u|^2\,dx + \int_G \Phi(L\,u)\,dx \;<\; +\infty\Big\}. \tag{4.5}$$

Theorem 4.1 *Let* **(D1)** *to* **(D5)** *be fulfilled. Then* (4.1) *is solvable almost everywhere in* G *for any* $h \in W^2_\Psi(G) \cap \overset{o}{W}{}^1_\Psi(G)$ *and the solution* u *belongs to the space* $\mathcal{H}^L(G) \cap \{u \mid D^\alpha u|_{\partial G} = 0\,, \quad |\alpha| \le m-1\} := \overset{o}{\mathcal{H}}{}^L(G)\,.$

In the case when F_0, F_1 have additional smoothness, condition **(D5)** concerning the growth of the nonlinearities can be relaxed somewhat. We consider the following assumption:

(D6) There exist convex functions $\varphi_3\,,\ \varphi_{11\alpha}\,,\ \varphi_{j\,\alpha} \ge 0\,,\ j = 8, 9, 10\,,\ |\alpha| \le 2m-1\,,$ and constants $C_j\,,\ \tilde{C}_j \ge 0\,,\ j = 6, \ldots, 10\,,$ such that with $\bar{\alpha}_i := (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n)\,,\ \bar{\beta}_i := (\beta_1, \ldots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \ldots, \beta_n)$ the following inequalities are fulfilled:

$$\sum_{|\alpha|\le 2m-2}\sum_{i=1}^{n}\Big\{\Big|F_{0\,x_i\xi_\alpha}(x,\xi,\eta)\Big|\;\Big|\xi_{\bar{\alpha}_i}\Big| + \sum_{|\beta|\le 2m-2}\Big|F_{0\xi_\alpha\xi_\beta}(x,\xi,\eta)\Big|\;\Big|\xi_{\bar{\alpha}_i}\Big|\;\Big|\xi_{\bar{\beta}_i}\Big|\Big\}$$

$$\le\; C_6\Big\{\Big|\tilde{\varphi}(\eta)\Big| + \sum_{|\alpha|\le 2m-1}\Big[\Big|\tilde{\varphi}_{0\alpha}(\eta)\Big|\;\varphi_{1\alpha}(\xi_\alpha) + \varphi_{8\alpha}(\xi_\alpha)\Big]\Big\} + \tilde{C}_6\,,$$

$$\sum_{i=1}^{n}\Big\{\Big|F_{0\,x_i^2}(x,\xi,\eta)\Big| + \sum_{|\alpha|\le 2m-2}\Big|F_{0\,\xi_\alpha}(x,\xi,\eta)\Big|\;|\eta|\Big\}$$

$$\le\; C_7\Big\{\varphi_3(\eta) + \sum_{|\alpha|\le 2m-1}\varphi_{9\alpha}(\eta)\;\varphi_{10\alpha}(\xi_\alpha)\Big\} + \tilde{C}_7\,,$$

$$\sum_{i=1}^{n}\sum_{|\alpha|=2m-1}\Big\{\Big|F_{0\,x_i\xi_\alpha}(x,\xi,\eta)\Big| + \sum_{|\beta|\le 2m-2}\Big|F_{0\,\xi_\beta\xi_\alpha}(x,\xi,\eta)\Big|\;\Big|\xi_{\bar{\beta}_i}\Big|\Big\}\,|\eta|$$

$$\le\; C_8\Big\{\varphi_0(\eta) + \sum_{|\alpha|\le 2m-1}\Big[\varphi_{2\alpha}(\eta)\;\;\varphi_{3\alpha}(\xi_\alpha) + \varphi_{11\alpha}(\xi_\alpha)\Big]\Big\} + \tilde{C}_8\,,$$

$$\sum_{i=1}^{n}\Big\{\Big|F_{0\,x_i\eta}(x,\xi,\eta)\Big| + \sum_{|\beta|\le 2m-2}\Big|F_{0\,\xi_\beta\eta}(x,\xi,\eta)\Big|\;\Big|\xi_{\bar{\beta}_i}\Big|\Big\}\,|\eta|$$

$$\le\; C_9\Big\{|\tilde{\varphi}(\eta)| + \sum_{|\alpha|\le 2m-1}\varphi_{4\alpha}(\eta)\;\varphi_{5\alpha}(\xi_\alpha)\Big\}\,,$$

$$\left|F_{0\,\xi_\alpha}(x,\xi,\eta)\right| \le C_{10}\left\{\varphi_0(\eta) + \sum_{|\beta|\le 2m-1} \varphi_{2\beta}(\eta)\ \varphi_{3\beta}(\xi_\alpha)\right\}, \quad |\alpha| = 2m-1 .$$

Moreover, the functions φ_0 , $\varphi_{2\alpha}$, $\varphi_{3\alpha}$, $\varphi_{4\alpha}$, $\varphi_{5\alpha}$ satisfy the conditions stated in (4.4), and it holds

$$\begin{gathered}\varphi_{8\alpha}(\xi_\alpha) < |\tilde\varphi(\xi_\alpha)|\,, \quad \varphi_3(\eta) < \varphi_0(\eta)\,, \quad \varphi_{3\alpha}(\eta) < \varphi_{2\alpha}(\eta)\,, \\ \varphi_{10\alpha}(\xi_\alpha) < \varphi_{3\alpha}(\xi_\alpha)\,, \quad \varphi_{11\alpha}(\xi_\alpha) < \Phi^{\frac{1}{q}}(\xi_\alpha)\,, \quad |\alpha| \le 2m-1\,. \end{gathered} \tag{4.6}$$

In addition, the corresponding derivatives of the function F_1 satisfy inequalities of the same kind, and $F_{1\eta}$, $F_{1\xi_{2m-1}}$ satisfy the inequalities from condition **(D5)**, where the convex functions occurring on the right-hand sides of these inequalities are estimated from above by strict inequalities as in (4.4).

Theorem 4.2 *Let* $F = F(x,\xi,\eta)$ *and* L *satisfy* **(D1)** *to* **(D4)**. *Besides, let the partial derivatives* F_{x_i} , F_{ξ_α} , $|\alpha| \le 2m-2$, *be of class* C^1 *and satisfy* **(D6)**. *Then* (4.1) *is solvable almost everywhere in* G *for every* $h \in W^2_\Psi(G) \cap \overset{o}{W}{}^1_\Psi(G)$ *, and the solution* u *belongs to* $\mathcal{H}^L(G)$.

Remark 4.3 If F does not enjoy the smoothness necessary for the application of Theorem 4.2 but one of the functions F_0 or F_1 does, then we use for one of the functions **(D5)** and for the other one **(D6)**, and the assertion of the theorem still remains valid.

Proof of Theorems 4.1 and 4.2: We apply Theorem 2.7. To this end, we have to examine the properties of the spaces from (4.5) first.

We consider the spaces

$$S_{1,\varphi,2}(G) := \left\{u \,\Big|\, [u]_S := \Phi^{-1}\left(\int_G \varphi(u) \sum_{i=1}^n \left|D_i u\right|^2 dx\right) + \|u\|_{L_\Phi} < \infty\right\}, \tag{4.7}$$

$$S_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G) := \left\{u \,\Big|\, [u]_S := \tilde\Phi^{-1}\left(\int_G \sum_{j=1}^n \varphi_{0\alpha}(u)\, \varphi_{1\alpha_j}(D_j u) \sum_{i=1}^n \left|D_i u\right|^2 dx\right) < \infty\right\}, \tag{4.8}$$

where φ , $\varphi_{0\alpha}$, $\varphi_{1\alpha} \ge 0$, $|\alpha| = k$, are convex functions satisfying certain conditions which are, for example, of the type of condition **(D3)**.

We remark that these spaces have been considered before under different assumptions for the functions φ , $\varphi_{0\alpha}$, $\varphi_{1\alpha}$ (e.g., for the case $\tilde\varphi(u) := |u|^\rho u$, $\varphi_1 := Id.$ in the papers [5], [7], [18], and for more general cases in [19] and others). Moreover, the spaces (4.7) and (4.8) are qn-spaces (see [18]); consequently all results of Section 1 are valid for these spaces.

Furthermore, we apply the definition of the Sobolev spaces of the form $L^m_p(G)$ and choose the spaces S_{gB} as basic spaces instead of $L_p(G)$. Thus, we obtain the following classes of spaces

$$S^k_{1,\varphi,2}(G) := \Big\{ u \,\Big|\, D^\alpha u \in S_{1,\varphi,2}(G)\,,\ |\alpha| = k \Big\}\,,$$
$$M^L_\Phi(G) := \Big\{ u \,\Big|\, L\,u \in L_\Phi(G) \Big\}\,,$$
$$S^k_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G) := \Big\{ u \,\Big|\, D^\alpha u \in S_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G)\,,\ |\alpha| = k \Big\}\,. \tag{4.9}$$

These spaces could, by analogy, be called nonlinear Sobolev spaces or Orlicz-Sobolev spaces depending on which spaces are chosen as basis, $S_{1,\varphi,2}(G)$, $S_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G)$ or $S_{1,p,2}(G)$.

In what follows, we will consider spaces of the form

$$\underset{o}{S}{}^L_{1,\varphi,2}(G) := \Big\{ u \,\Big|\, Lu \in S_{1,\varphi,2}(G)\,,\ D^\alpha u\Big|_{\partial G} = 0\,,\ |\alpha| \le m-1 \Big\}\,, \tag{4.10}$$

$$S^{(L)}_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G) := \left\{ u \,\Big|\, \sum_{|\alpha|\le 2m-1} \int_G \varphi_{1\alpha}(D^\alpha u)\,\varphi_{0\alpha}(Lu) \sum_{i=1}^n \Big|D_i Lu\Big|^2\,dx < +\infty \right\}\,, \tag{4.11}$$

which are directly connected to the equations examined by us. For these spaces, we have to prove imbedding results that are based on the following inequalities.

Proposition 4.4 *Let $m \in \mathbb{N}$ and let φ, $\varphi_{0\alpha}$, $\varphi_{1\alpha}$, $|\alpha| \le 2m-1$, be nonnegative convex functions with $\varphi_{0\alpha}(\tau)\,\varphi_{1\alpha}(\tau) \le \varphi(\tau)$, for $\tau \in \mathbb{R}$. In addition, let $\varphi = \Phi''$, $\varphi_{0\alpha} = \Phi''_{0\alpha}$, where Φ, $\Phi_{0\alpha}$, $\varphi_{1\alpha}$ are N-functions defined as in (*) in* **(D3)**. *Then the following inequalities are valid for any ξ_α, η, $\eta_i \in \mathbb{R}$:*

$$|\tilde\varphi(\eta)|\,|\eta_i| \le \epsilon\,\varphi(\eta)\,\eta_i^2 + C(\epsilon)\,\Phi(\eta)\,, \quad \text{where } \tilde\phi := \Phi'\,, \tag{4.12}$$

$$\varphi_{1\alpha}(\xi_\alpha)\ |\tilde\varphi_{0\alpha}(\eta)|\ |\eta_i| \le \epsilon_1\,\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,\eta_i^2 + C(\epsilon_1)\,\Phi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,,$$
$$\text{for } \alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}_0^n\,, \quad |\alpha| \le 2m-1\,. \tag{4.13}$$

(Here, and in the sequel, ϵ and ϵ_i, $i \in \mathbb{N}$, denote positive constants, and $C(\epsilon)$, $C(\epsilon_i)$ denote positive constants depending on ϵ, ϵ_i.)

Proposition 4.5 *Let φ_0, $\varphi_{2\alpha}$, $\varphi_{3\alpha} \ge 0$, $|\alpha| \le 2m-1$, be convex functions with*

$$\varphi_0(\eta) \le \Phi^{\frac{1}{q}}(\eta)\,|\eta|^{-1}\,, \quad \varphi_{2\alpha}(\eta) \le \Phi^{\frac{1}{q}}_{0\alpha}(\eta)\,|\eta|^{-1}\,, \quad \varphi_{3\alpha}(\tau) \le \varphi^{\frac{1}{q}}_{1\alpha}(\tau)\,,$$

where Φ, $\varphi_{1\alpha}$, $\varphi_{0\alpha}$ are N-functions just like in Proposition 4.4. Then, the following inequalities hold for any η, ξ_α, $\xi_{\bar\alpha}$, $\eta_i \in \mathbb{R}$ with $p \ge 2$, $q = p'$:

$$\varphi_0(\eta)\,\Big|\xi_{\bar\alpha}\Big|\,\Big|\eta_i\Big| \le \epsilon_1\,\varphi(\eta)\,\eta_i^2 + \epsilon_2\,|\xi_{\bar\alpha}|^p + \epsilon_3\,\Phi(\eta) + C(\epsilon_1,\epsilon_2,\epsilon_3)\,. \tag{4.14}$$

$$\varphi_{2\alpha}(\eta)\,\varphi_{3\alpha}(\xi_\alpha)\,|\xi_{\bar\alpha}|\,|\eta_i| \le \epsilon_1\,\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,\eta_i^2 + \delta_{\bar\alpha}\,\epsilon_2\,|\xi_{\bar\alpha}|^p + \epsilon_3\,\Phi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)$$
$$+\,\epsilon_4\,\Phi(\xi_\alpha) + C(\epsilon_1,\epsilon_2,\epsilon_3\,\epsilon_4)\,, \quad \text{where } \delta_{\bar\alpha} = \begin{cases} 1, & |\bar\alpha| = 2m\,, \\ 0 & |\bar\alpha| \le 2m-1\,. \end{cases} \tag{4.15}$$

Proposition 4.6 *Let* φ_1 , $\varphi_{4\alpha}$, $\varphi_{5\alpha}$, $|\alpha| \le 2m-1$ *, be nonnegative convex functions with*

$$\varphi_1(\eta) \le |\tilde{\varphi}(\eta)|\,, \quad \varphi_{4\alpha}(\eta) \le |\tilde{\varphi}_{0\alpha}(\eta)|\,, \quad \varphi_{5\alpha}(\tau) \le \varphi_{1\alpha}(\tau)\,,$$

where $|\tilde{\varphi}|$, $|\tilde{\varphi}_{0\alpha}|$, $\varphi_{1\alpha}$ *are* N *-functions as in Proposition 4.4. Then the following inequalities hold for any* ξ_α , η , $\eta_i \in \mathbb{R}$ *:* ,

$$\varphi_1(\eta)|\eta_i| \;\le\; \epsilon_1\, \varphi(\eta)\, \eta_i^2 \;+\; \epsilon_2\, \Phi(\eta) \;+\; C(\epsilon_1, \epsilon_2)\,, \tag{4.16}$$

$$\varphi_{4\alpha}(\eta)\, \varphi_{5\alpha}(\xi_\alpha)\, |\eta_i| \;\le\; \epsilon_1\, \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha)\, \eta_i^2 \;+\; \epsilon_2\, \Phi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha) \;+\; C(\epsilon_1, \epsilon_2)\,. \tag{4.17}$$

Proposition 4.7 *Let* φ_2 , $\varphi_{6\alpha}$, $\varphi_{7\alpha}$, $|\alpha| \le 2m-1$ *, be nonnegative convex functions with*

$$\varphi_2(\eta) < \Phi^{\frac{1}{q}}(\eta)\, |\eta|^{-1}\,, \quad \varphi_{6\alpha}(\eta) < \Phi_{0\alpha}^{\frac{1}{q}}(\eta)\, |\eta|^{-1}\,, \quad \varphi_{7\alpha}(\xi_\alpha) \le \varphi_{1\alpha}^{\frac{1}{q}}(\xi_\alpha)\,,$$

where Φ , $\Phi_{0\alpha}$, $\varphi_{1\alpha}$ *are* N *-functions as in the preceding propositions. Then, with*

$$\delta_{\bar{\alpha}} \;=\; \begin{cases} 1 & |\bar{\alpha}| \;=\; 2m\,, \\ 0 & |\bar{\alpha}| \;\le\; 2m \;-\; 1\,, \quad |\alpha| \;\le\; 2m - 1\,, \end{cases} \tag{4.18}$$

the following inequalities hold for any ξ_α , η , $\xi_{\bar{\alpha}}$, $\eta_i \in \mathbb{R}$:

$$\varphi_2(\eta)\, |\xi_{\bar{\alpha}}|\, |\eta_i| \;\le\; \epsilon_1\, \varphi(\eta)\, |\eta_i|^2 \;+\; \epsilon_2\, \delta_{\bar{\alpha}}\, |\xi_{\bar{\alpha}}|^p \;+\; \epsilon_3\, \Phi(\eta) \;+\; \epsilon_4\, \Phi(\xi_\alpha) \;+\; C(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)\,, \tag{4.19}$$

$$\begin{aligned} \varphi_{6\alpha}(\eta)\, \varphi_{7\alpha}(\xi_\alpha)\, |\xi_{\bar{\alpha}}|\; |\eta_i| \;\;\le\;\; & \epsilon_1\, \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha)\, |\eta_i|^2 \;+\; \epsilon_2\, \delta_{\bar{\alpha}}\, |\xi_{\bar{\alpha}}|^p \;+\; \epsilon_3\, \Phi(\xi_\alpha) \\ & + \epsilon_4\, \Phi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha) \;+\; C(\epsilon)\,. \end{aligned} \tag{4.20}$$

Proposition 4.8 *Let* Φ , $\Phi_{0\alpha}$, $\varphi_{1\alpha}$, $|\alpha| \le 2m-2$ *, be* N *-functions defined as in* (*) *in* **(D3)** *by nonnegative functions* φ , $\varphi_{0\alpha}$ *that are at least main parts of certain* N *-functions. Then the following inequalities hold for any* ξ_α , η , $\eta_i \in \mathbb{R}^1$:

$$\begin{aligned} \varphi(\tau) \;&\ge\; \varphi_{0\alpha}(\tau)\, \varphi_{1\alpha}(\tau)\,, \quad \forall\, \tau \,\in\, \mathbb{R}\,, & (4.21) \\ \varphi(\eta)\, |\eta_i|\, |\zeta| \;&\le\; C_1\, \varphi(\eta)\, |\eta_i|^2 \;+\; C_2\, \Phi(\eta) \;+\; C_3\, \Phi(\zeta)\,, & (4.22) \\ \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha)\, |\eta_i|\, |\zeta| \;&\le\; C_1\, \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha)\, \eta_i^2 \;+\; C_2\, \Phi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha) \\ &\quad + C_3\, \Phi(\zeta) + C_4\, \Phi(\xi_\alpha) + C_5\,, & (4.23) \\ |\tilde{\varphi}_{0\alpha}(\eta)|\; |\varphi_{1\alpha}'(\xi_\alpha)|\; |\xi_{\bar{\alpha}}|\; |\zeta| \;&\le\; C_1\, \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha)\, \eta^2 \;+\; C_2\, \Phi(\eta) \;+\; C_3\, \Phi(\xi_\alpha) \\ &\quad + C_4\, \Phi(\xi_{\bar{\alpha}}) \;+\; C_5\, \Phi(\zeta) \;+\; C_6\,, & (4.24) \\ |\tilde{\varphi}_{1\alpha}(\eta)|\; \varphi_{1\alpha}(\xi_\alpha)\; |\zeta| \;&\le\; C_1\, \varphi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha)\, \eta^2 \;+\; C_2\, \Phi_{0\alpha}(\eta)\, \varphi_{1\alpha}(\xi_\alpha) \\ &\quad + C_3\, \Phi(\zeta) \;+\; C_4\, \Phi(\xi_\alpha) \;+\; C_5\,. & (4.25) \end{aligned}$$

Proofs of Propositions 4.4 to 4.8: The proofs of these propositions are derived from Young's inequality (Fenchel-Moreau) with a small parameter, applying the properties of N - functions and results from [19]. It constitutes no major difficulty, therefore we will not demonstrate the details, here. We confine ourselves to demonstrate just one of the inequalities in Proposition 4.5 and in Proposition 4.8.

In the case of inequality (4.14) in Proposition 4.5, we have, using Theorem 1 in [19], $|\bar{\alpha}| = 2m$, and

$$\begin{aligned} \varphi_0(\eta)\,|\xi_{\bar{\alpha}}|\,|\eta_i| &\le \Phi^{\frac{1}{q}}(\eta)\,|\eta_i|\,|\eta|^{-1}\,|\xi_{\bar{\alpha}}| \\ &\le (\Phi(\eta))^{\frac{2-q}{2q}}\,\varphi^{1/2}(\eta)\,|\xi_{\bar{\alpha}}|\,|\eta_i| \\ &\le \epsilon_1\,\varphi(\eta)\eta_i^2 + C(\epsilon_1)\,\Phi^{\frac{2-q}{q}}(\eta)|\xi_{\bar{\alpha}}|^2 \\ &\le \epsilon_1\,\varphi(\eta)\,\eta_i^2 + \epsilon_2\,|\xi_{\bar{\alpha}}|^p + \epsilon_3\,\Phi(\eta) + C(\epsilon_1,\epsilon_2,\epsilon_3)\,, \end{aligned} \tag{4.26}$$

and inequality (4.23) in Proposition 4.8 follows from

$$\begin{aligned} &\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,|\eta_i|\,|\zeta| \\ \le\;& C_1\,\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,\eta_i^2 + C_2\,\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,\zeta^2 \\ \le\;& C_1\,\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,\eta_i^2 + C_2\,[|\tilde{\varphi}_{0\alpha}(\eta)|\,|\zeta| + \tilde{\varphi}_{0\alpha}(\zeta)\,\zeta]\,\varphi_{1\alpha}(\xi_\alpha) \\ \le\;& C_1\,\varphi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha)\,\eta_i^2 + C_2\,\Phi_{0\alpha}(\eta)\,\varphi_{1\alpha}(\xi_\alpha) + C_3\,\Phi(\zeta) + C_4\,\Phi(\xi_\alpha) + C_5\,. \end{aligned} \tag{4.27}$$

□

Proposition 4.8 leads to

Corollary 4.9 *Let* φ, $\varphi_{0\alpha}$, $\varphi_{1\alpha}$, Φ, $\Phi_{0\alpha}$, $|\alpha| \le 2m-1$, *be functions satisfying the assumptions of Proposition* 4.8, *and let* Ψ *be the conjugate Young function to* Φ. *Then the following implications hold.*

$$u \in \mathop{S}_{\circ}{}^{L}_{1,\varphi,2}(G) \Longrightarrow \tilde{\varphi}(Lu) \in W^1_\Psi(G)\,, \tag{4.28}$$

$$u \in \overset{o}{M}{}^{L}_{\Phi}(G) \cap S^{(L)}_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G) \Longrightarrow \varphi_{0\alpha}(Lu)\,\varphi_{1\alpha}(D^\alpha u) \in W^1_\Psi(G)\,, \quad |\alpha| \le 2m-1\,, \tag{4.29}$$

where

$$\overset{o}{M}{}^{L}_{\Phi}(G) := M^L_\Phi(G) \cap \left\{ u \,\middle|\, D^\beta u\big|_{\partial G} = 0\,, |\beta| \le m-1 \right\}. \tag{4.30}$$

We introduce two further denotations. We put

$$S^L_{1,\varphi,2}(G) := S_{\tilde{\varphi}\circ L, W^1_\Psi(G)}\,, \quad S^{(L)}_{1,\varphi_0,\varphi_1,2}(G) := S_{\bar{\varphi}_0, S_{\bar{\varphi}_1, W^1_{\Psi_0}(G)}}\,,$$

$$\text{whenever } \bar{\varphi}_0 = \bar{\varphi}_0(\varphi_{0\alpha},\varphi_{1\alpha})\,,\ \bar{\varphi}_1 = \bar{\varphi}_1(\varphi_0,\varphi_1) \text{ are given functions.} \tag{4.31}$$

We then have the following result.

Lemma 4.10 *Let the functions* φ, $\varphi_{0\alpha}$, $\varphi_{1\alpha}$, Φ, $\Phi_{0\alpha}$, Ψ, $|\alpha| \le 2m-1$, *satisfy the assumptions of Proposition 4.8 and suppose that*

$$\varphi_{0\alpha}(\tau)\ \varphi_{1\alpha}(\tau) \le \varphi(\tau)\,, \quad \Phi(\tau) \ge |\tau|^p\,, \quad \forall\ \tau\,.$$

Then the following imbeddings are compact:

$$\underset{\circ}{\mathcal{H}}^L(G) := \underset{\circ}{S}{}^L_{1,\varphi,2}(G) \cap \Big(\bigcap_{\alpha} \underset{\circ}{S}{}^{(L)}_{1,\varphi_{0\alpha},\varphi_{1\alpha},2}(G) \Big) \subset \underset{\circ}{S}{}_{\tilde{\varphi}\circ L, W^1_\Psi(G)} \hookrightarrow \overset{o}{M}{}^{L}_{\Phi}(G)\,, \tag{4.32}$$

$$\underset{\circ}{\mathcal{H}}^L(G) \subset \underset{\circ}{S}{}_{\tilde{\varphi}\circ L, W^1_\Psi(G)} \hookrightarrow W^{2m}_p(G) \cap \overset{o}{W}{}^{m}_{p}(G) \hookrightarrow W^{2m-1}_{\Phi}(G)\,. \tag{4.33}$$

Proof. It follows from the imbedding theorems for Orlicz–Sobolev spaces (see [4]) that the imbedding $W^1_\Psi(G) \hookrightarrow L_\Psi(G)$ is compact. Using the results of Section 1, and invoking the conditions **(D1)** and **(D4)**, we find that the imbeddings

$$\underset{\circ}{S}{}_{\tilde{\varphi}\circ L, W^1_\Psi(G)} \hookrightarrow \overset{o}{M}{}^{L}_{\Phi}(G) \subset \overset{o}{M}{}^{L}_{p}(G)\,;\ \overset{o}{M}{}^{L}_{p}(G) \cong W^{2m}_{p_0}(G) \cap \overset{o}{W}{}^{m}_{p_0}(G) \tag{4.34}$$

are compact. Consequently, the imbedding $\underset{\circ}{\mathcal{H}}^L(G) \hookrightarrow W^{2m}_{p_0}(G)$ is compact, too, because under the assumptions of the lemma we have that $\|u\|_{W^{2m}_{p_0}(G)\cap \overset{o}{W}{}^{m}_{p_0}(G)}$ is equivalent to $\|Lu\|_{L_p(G)}$. □

5 Proof of the existence theorem for (4.1)

Since the proof of Theorem 4.2 is similar to that of Theorem 4.1, only a little lengthier, we only prove the latter here. To this end, we show that the operator A generated by (4.1) is E_Φ-weakly continuous, and coercive in a generalized sense. The result will then be a consequence of Theorem 2.7. The following lemmas yield its applicability.

Lemma 5.1 *Suppose the assumptions of Theorem* 4.1 *are satisfied, and let the operator* $K : X_0 := W^{2m+2}_{p_0}(G) \cap \overset{o}{W}{}^{m}_{p_0}(G) \cap \{u | Lu|_{\partial G} = 0\} \to L_p(G)$ *be defined as* $K\,u := -\Delta Lu + k\,Lu$. *Then the pair* (A, K) *is a coercive pair in a generalized sense, that is, for any* $u \in X_0$ *it holds*

$$\begin{aligned} \langle A(u)\,,\,K(u)\rangle &:= \int_G \Big[F_0(x, D^\alpha u, Lu) + F_1(x, D^\alpha u, Lu)\Big](-\,\Delta Lu + k\,Lu)\,dx \\ \geq\ C\Bigg\{ &\int_G \varphi(Lu) \sum_{i=1}^n \Big|D_i Lu\Big|^2 dx + \int_G \sum_{|\alpha|\leq 2m-1} \varphi_{0\alpha}(Lu)\,\varphi_{1\alpha}(D^\alpha u) \sum_{i=1}^n \Big|D_i Lu\Big|^2 dx \\ &+ \int_G \Big[\varphi(Lu) + \sum_{|\alpha|\leq 2m-1} \varphi_{0\alpha}(Lu)\,\varphi_{1\alpha}(D^\alpha u)\Big](Lu)^2\,dx \Bigg\} - \tilde{C}\,, \end{aligned} \tag{5.1}$$

where $C > 0\,,\ \tilde{C} \geq 0$ are some constants.

Proof: In view of **(D2)**, we have

$$\langle A(u)\,,\,K\,u\rangle = k \int_G \Big[F_0(x, D^\alpha u, Lu) + F_1(x, D^\alpha u, Lu)\Big] Lu\,dx$$

$$+ \sum_{i=1}^{n} \int_G \Big[F_{0\eta}(x, D^\alpha u, Lu) \; + \; F_{1\eta}(x, D^\alpha u, Lu)\Big] \Big|D_i Lu\Big|^2 \, dx$$

$$+ \quad \sum_{i=1}^{n} \int_G \left[F_{0x_i}(x, D^\alpha u, Lu) \; + \sum_{|\beta| \le 2m-1} F_{0\xi_\beta}(x, D^\alpha u, Lu) D^{\bar{\beta}_i} u\right] D_i Lu \, dx$$

$$+ \quad \sum_{i=1}^{n} \int_G \left[F_{1x_i}(x, D^\alpha u, Lu) + \sum_{|\beta| \le 2m-1} F_{1\xi_\beta}(x, D^\alpha u, Lu) D_i D^\beta u\right] D_i Lu \, dx. \quad (5.2)$$

Applying now the conditions **(D3)** and **(D5)** of Theorem 4.1, we obtain

$$\langle A(u)\,,\, K\, u\rangle \;\ge\; C' \int_G \left[\varphi(Lu) \; + \sum_{|\beta| \le 2m-1} \varphi_{0\beta}(Lu)\, \varphi_{1\beta}(D^\beta u)\right] \sum_{i=1}^{n} \Big|D_i Lu\Big|^2 \, dx$$

$$+ \, k_1 \int_G \left[\Big|\tilde{\varphi}(Lu)\, Lu \; + \sum_{|\beta| \le 2m-1} \tilde{\varphi}_{0\beta}(Lu)\, Lu\, \varphi_{1\beta}(D^\beta u)\right] \, dx$$

$$- \, C_1 \int_G \left\{\Big|\tilde{\varphi}(Lu)\Big| \; + \sum_{|\beta| \le 2m-1} \Big|\tilde{\varphi}_{0\beta}(Lu)\Big| \varphi_{1\beta}(D^\beta u)\right\} \sum_{i=1}^{n} |D_i Lu| \, dx$$

$$- \, C_2 \int_G \left[\varphi_0(Lu) \; + \sum_{|\beta| \le 2m-1} \varphi_{2\beta}(Lu)\, \varphi_{3\beta}(D^\beta u)\right] \sum_{i=1}^{n} |D_i D^\beta u| |D_i Lu| \, dx$$

$$- \, C_4 \int_G \left[\varphi_2(Lu) \; + \sum_{|\beta| \le 2m-1} \varphi_{6\beta}(Lu)\, \varphi_{7\beta}(D^\beta u)\right] \sum_{i=1}^{n} \Big|D_i D^\beta u\Big| \Big|D_i Lu\Big| \, dx$$

$$- \, C_5 \int_G \left[\Big|\varphi_1(Lu)\Big| \; + \sum_{|\beta| \le 2m-1} \varphi_{4\beta}(Lu)\, \varphi_{5\beta}(D^\beta u)\right] \sum_{i=1}^{n} \Big|D_i Lu\Big| \, dx \; - \; C_1 \, . \quad (5.3)$$

Here, we take into consideration that the following inequality holds under the conditions **(D2)**, **(D3)**, and **(D5)**.

$$\langle A(u)\,,\, k\, Lu\rangle \;\ge\; \tilde{C}_1 \int_G \left[\tilde{\varphi}(Lu) Lu \; + \sum_{|\alpha| \le 2m-1} \tilde{\varphi}_{0\alpha}(Lu)\, Lu\, \varphi_{1\alpha}(D^\alpha u)\right] \, dx \; - \; \tilde{C}_2 \, . \quad (5.4)$$

Furthermore, applying Propositions 4.4 to 4.7 to those summands in (5.3) that have no well-defined sign, we obtain

$$\langle A(u)\,,\, K\, u\rangle \;\ge\; \tilde{k} \int_G \left[\tilde{\varphi}(Lu)\, Lu \; + \sum_{|\alpha| \le 2m-1} \tilde{\varphi}_{0\alpha}(Lu)\, L\, u\, \varphi_{1\alpha}(D^\alpha u)\right] \, dx$$

$$+ \, \bar{C} \int_G \left[\varphi(Lu) \; + \sum_{|\alpha| \le 2m-1} \varphi_{0\alpha}(Lu)\, \varphi_{1\alpha}(D^\alpha u)\right] \sum_{i=1}^{n} \Big|D_i Lu\Big|^2 \, dx$$

$$- \, \epsilon_1 \sum_{|\beta| \le 2m-1} \int_G \Phi(D^\beta u) \, dx \; - \; \epsilon_2 \int_G \Phi(Lu) \, dx \; - \; \epsilon_3 \sum_{|\alpha| = 2m} \int_G \Big|D^\alpha u\Big|^{p_0} \, dx$$

$$- \, \epsilon_4 \sum_{|\alpha| \le 2m-1} \int_G \Phi_{0\alpha}(Lu)\, \varphi_{1\alpha}(D^\alpha u) \, dx \; - \; \bar{C}_2 \, . \quad (5.5)$$

From this, using the inequality (4.2) in condition **(D1)** and Proposition 4.4, and choosing k sufficiently large, we derive that (5.1) holds. Thus, the generalized coercivity is proved. □

Next, we will prove that A is E_Φ-weakly continuous from $\underset{o}{\mathcal{H}}^L(G)$ into $L_\Psi(G)$. In fact, we show a stronger result.

Lemma 5.2 *Under the assumptions of Theorem* 4.1, *the operator* $A : \underset{o}{\mathcal{H}}^L(G) \to L_\Psi(G)$ *is* E_Φ *- weakly continuous, and it is even fully continuous.*

Proof: Let $\{u_r\} \subset \underset{o}{\mathcal{H}}^L(G)$ be an E_Φ-weakly convergent sequence in $\underset{o}{\mathcal{H}}^L(G)$, i.e. $u_r \to u_0 \in \underset{o}{S}^L_{1,\varphi,2}(\overset{o}{G})$, E_Φ-weakly in $\underset{o}{\mathcal{H}}^L(G)$, but then also in $\underset{o}{S}^L_{1,\varphi,2}(\overset{o}{G})$.

Lemma 4.1 yields then that a subsequence can be chosen, again denoted $\{u_r\}$, that converges strongly in $\underset{o}{M}^L_\Phi(G)$, i.e. $Lu_r \to Lu_0$ in $L_\Phi(G)$, and also $Lu_r \to Lu_0$ in $L_p(G)$.

Hence, by **(D4)** and Lemma 4.1,

$$D^\alpha u_r \to D^\alpha u_0 , \quad |\alpha| \le 2m - 1 , \quad Lu_r \to Lu_0 \text{ in } L_\Phi(G) . \tag{5.6}$$

On the other hand, we get from the assumptions on the functions F_0, F_1 that F_0, F_1 map continuously from $\underset{o}{\mathcal{H}}^L(G)$ into $L_\Psi(G)$. Besides, it follows from the assumptions of the theorem that $F : \overset{o}{M}{}^L_\Phi(G) \to L_\Psi(G)$ is continuous. We also get that

$$F(\,\cdot\,, D^\alpha u_r, Lu_r) \to \chi \in L_\Psi(G) , \quad E_\Phi\text{-weakly in } L_\Psi(G) , \tag{5.7}$$

possibly after choosing a subsequence, and by virtue of (5.6) it follows that

$$D^\alpha u_r \to D^\alpha u_0 , \quad |\alpha| \le 2m - 1 , \quad Lu_r \to Lu_0 , \quad \text{almost everywhere in } G . \tag{5.8}$$

Hence, $\chi(x) = F(x, D^\alpha u_0, Lu_0)$, and

$$F(\,\cdot\,, u_r, D^\alpha u_r, Lu_r) \to F(\,\cdot\,, u_0, D^\alpha u_0, Lu_0) \quad \text{almost everywhere in } G . \tag{5.9}$$

Thus, we finally have shown that

$$F(\,\cdot\,, D^\alpha u_r, Lu_r) \to F(\,\cdot\,, D^\alpha u_0, Lu_0) \quad \text{in } L_\Psi(G) . \tag{5.10}$$

This proves the full continuity of A from $\underset{o}{\mathcal{H}}^L(G)$ into $L_\Psi(G)$ and thus, Lemma 5.2 is proved. □

Proof of Theorem 4.1: We have $R(K) = K(W^{2m+2}_{p_0}(G) \cap \overset{o}{W}{}^m_{p_0}(G) \cap \{u \mid Lu|_{\partial G} = 0\})$, and thus $R(K)$ contains a linear subset which is everywhere dense in $E_\Phi(G)$. Moreover, we have, for all $h \in W^2_\Psi(G) \cap \overset{o}{W}{}^1_\Psi(G)$,

$$\left|\langle h\,,\,K\,u\rangle\right| = \left|\int_G h(-\,\Delta Lu + k\,Lu)\,dx\right| = \left|\int_G (-\,\Delta h + k\,h)Lu\,dx\right|$$

$$
\begin{aligned}
&\le \left[\|\Delta h\|_{L_\Psi(G)} + k\,\|h\|_{L_\Psi(G)}\right]\|Lu\|_{L_\Phi(G)} \\
&\le C(\epsilon)\,\Psi\left(\|h\|_{W^2_\Psi(G)}\right) + \epsilon \int_G \Phi(Lu)\,dx + C\,. \qquad (5.11)
\end{aligned}
$$

Therefore, all assumptions of Theorem 2.7 are satisfied, and consequently, (4.1) is solvable in the sense of Definition 2.1 for any $h \in W^2_\Psi(G) \cap \overset{o}{W}{}^1_\Psi(G)$. In addition, we can infer from Lemma 5.2 that the solvability is in the sense of almost everywhere in G, and that $u \in \underset{o}{\mathcal{H}}{}^L(G)$. This concludes the proof of Theorem 4.1. □

6 Parabolic equations

We consider the initial-boundary value problem

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + F_0(x,t,D^\alpha u, Lu) + F_1(x,t,D^\alpha u, Lu) = h(x,t)\,,\ (x,t) \in Q\,, \\
u(x,0) = 0\,,\ D^\beta u\Big|_{\partial G\times[0,T]} = 0\,,\ |\beta| \le m-1\,,\ \Gamma = \partial G \times [0,T]\,,
\end{cases} \tag{6.1}
$$

for $u \in W^1_\Psi(0,T;L_\Psi(G)) \cap L_\Phi(0,T;W^{2m}_\Phi(G)\cap \overset{o}{W}{}^m_\Phi(G))$, $h \in L_\Psi(0,T;W^2_\Psi(G)) \cap \overset{o}{W}{}^1_\Psi(G))$, where $Q = G\times[0,T]$, $D^\alpha u = (u, D\,u, \ldots, D^{(2m-1)}u)$, and L is a linear differential expression of $2m$-th order just like in Section 4 ($D^k u = \{D^\alpha u \big|\, |\alpha| = k\}$).

We assume the following conditions to be fulfilled.

(D7) The operator L generated by the differential expression L satisfies **(D1)** and is a positive, selfadjoint operator commuting with $\partial/\partial t$. In addition, it holds with $L = L_0^2$, $L_0 = L_0^*$, for any u, $v \in L_{p_0}(0,T;W^{2m+2}_{p_0}(G)\cap \overset{o}{W}{}^m_{p_0}(G)) \cap \{u \mid Lu|_\Gamma = 0\}$,

$$
-\int_Q (\Delta Lu)v\,dx\,dt = \int_Q \sum_{i=1}^n D_i\,v\,d_i\,Lu\,dx\,dt = \int_Q \sum_{i=1}^n (D_i L_0 u)(D_i L_0 v)\,dx\,dt\,. \tag{6.2}
$$

(Note that from this condition it follows that L is a differential expression with constant coefficients.)

(D8) The functions $F_0 = F_0(x,t,\xi,\eta)$, $F_1 = F_1(x,t,\xi,\eta)$ satisfy a Carathéodory condition with respect to the variables $(x,t) \in Q$, $\xi \in \mathbb{R}^{\overline{m}}$ (where $\overline{m} = n^{2m-1}+1$) and $\eta \in \mathbb{R}$.

We introduce the spaces

$$
\begin{aligned}
\underset{o}{\mathcal{P}}_{1,\varphi,2,\Psi}(Q) &:= W^1_\Psi(0,T;L_\Psi(G)) \cap L_\Phi(0,T;\overset{o}{S}{}^L_{1,\varphi,2}(G)) \cap \{u \mid u(x,0) = 0\}\,, \\
\underset{o}{\mathcal{P}}{}^L(Q) &:= \underset{o}{\mathcal{P}}_{1,\varphi,2,\Psi}(Q) \cap L_\Psi(0,T;\underset{o}{\mathcal{H}}{}^L(G))\,,
\end{aligned} \tag{6.3}
$$

where $\underset{\circ}{\mathcal{H}}^L(G)$, $\overset{o\,L}{S}_{1,\varphi,2}(G)$ are the spaces introduced in Section 4. Consequently, these spaces are also qn-spaces and have the corresponding properties.

Theorem 6.1 *Let* **(D7)** *and* **(D8)** *hold, and assume that the conditions* **(D2)** *to* **(D5)** *on the functions* $F_0 = F_0(x,t,\xi,\eta)$, $F_1 = F_1(x,t,\xi,\eta)$ *are satisfied uniformly in* t. *Then* (6.1)*is for any* $h \in L_\Psi(0,T; W^2_\Psi(G) \cap \overset{o\,1}{W}_\Psi(G))$ *solvable (more precisely:* E_Φ*-solvable) in* $\underset{\circ}{\mathcal{P}}^L(Q)$ *(and a.e. in* Q*), and* $\partial u/\partial t$ *belongs to* $L_\Psi(0,T; W^1_\Psi(G))$.

Remark 6.2 The assertion of Theorem 6.1 remains valid when the assumptions needed for the validity of Theorem 4.1 are replaced by those needed for Theorem 4.2.

Proof of Theorem 6.1: We apply Theorem 3.1. At first, it is not difficult to verify that the operator $A(\,\cdot\,,t)$ generated by the elliptic part of problem (6.1) and the operator K defined in Section 5 satisfy the conditions **(C2)**, **(C3)** of Theorem 3.1, by Lemma 5.1. Moreover, using **(D7)**, we obtain the validity of **(C4)** from the inequalities

$$\int_Q \frac{\partial u}{\partial t}\left(-\Delta L\frac{\partial u}{\partial t} + k\,L\frac{\partial u}{\partial t}\right)dx\,dt$$

$$= \int_Q \sum_{i=1}^n \left(D_i\,L_0\frac{\partial u}{\partial t}\right)^2 dx\,dt + \int_Q k\left(L_0\frac{\partial u}{\partial t}\right)^2 dx\,dt \geq \tilde{C}\left\|\frac{\partial L_0 u}{\partial t}\right\|^2_{L_2(0,T;W^1_2(G))}, \quad (6.4)$$

$$\int_0^t\int_G \frac{\partial u}{\partial t}\left(-\Delta Lu + k\,Lu\right)dx\,dt \geq \tilde{C}_1\|L_0u(\,\cdot\,,t)\|^2_{W^1_2(G)}, \quad t\in[0,T]. \qquad (6.5)$$

The E_Φ-weak continuity of the operator A is proved as in Lemma 5.2 with the help of a compactness theorem following from Theorem 1.7 in Section 1. It is not difficult to see that the space $\underset{\circ}{\mathcal{P}}^L_{1,\varphi,2,\Psi}(Q)$ satisfies the assumptions of Theorem 1.7. From Theorem 1.7. follows therefore the compactness of the imbedding

$$\underset{\circ}{\mathcal{P}}^L_{1,\varphi,2,\Psi}(Q) \hookrightarrow L_\Phi(0,T;\underset{\circ}{M}^L_\Phi(G)), \qquad (6.6)$$

and, consequently, the compactness of the imbedding

$$\underset{\circ}{\mathcal{P}}^L(Q) \hookrightarrow L_\Phi(0,T;\underset{\circ}{M}^L_\Phi(G)). \qquad (6.7)$$

But this means that, if $u_r \to u_0$ E_Φ-weakly in $\underset{\circ}{\mathcal{P}}^L(Q)$, $r\nearrow\infty$, then we have for some subsequence, again denoted $\{u_r\}$, that

$$D^\alpha u_r \to D^\alpha u_0, \quad |\alpha| \leq 2m-1, \quad Lu_r \to Lu_0, \quad \text{in } L_\Phi(Q). \qquad (6.8)$$

To continue, we now use the same argumentation as in the proof of Lemma 5.2 to obtain the E_Φ-weak continuity of the operator $A : \underset{\circ}{\mathcal{P}}^L(Q) \to L_\Psi(Q)$.

Thus, all assumptions of Theorem 3.1 are fulfilled for (6.1), whence the existence result follows.

In order to complete the proof of Theorem 6.1, it remains to remark that (6.1) can be written in the form

$$\frac{\partial u}{\partial t} = h(x,t) - F(x,t,D^\alpha u, Lu), \tag{6.9}$$

and that, under the assumptions of the theorem, the right-hand side belongs to $L_\Psi(0,T;W^1_\Psi(G))$. Then, also $\partial u/\partial t \in L_\Psi(0,T;W^1_\Psi(G))$, and Theorem 6.1 is completely proved. □

Remark 6.3 If $F_1(x,t,\xi,\eta) \equiv F_1(x,t)$, then also $F_1 \in L_\Psi(0,T;W^2_\Psi(G)\cap \overset{o}{W}{}^1_\Psi(G))$.

7 Examples

Let $G \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with a sufficiently smooth boundary ∂G.

1. We consider the boundary value problem:

$$\left\{ \begin{array}{l} -\left[\sum_{k=0}^{n} a_k \left|D_k u\right|^{\rho_k} + a\left(|\Delta u|^\rho \ln(1+|\Delta u|^2) + |\Delta u|^{\tilde\rho_0}\right)\right] \Delta u \\ + \sum\limits_{|\alpha|+\beta_\alpha \leq \rho+1} b_\alpha(x)(\Delta u)^{\beta_\alpha} \prod\limits_{k=0}^{n} (D_k u)^{\alpha_k} = h(x),\ u\Big|_{\partial G} = 0 \end{array} \right. . \tag{7.1}$$

Theorem 4.1 yields for this equation the following existence result.

Theorem 7.1 *Let* $\rho \geq \rho_k \geq \rho_0 \geq 2$, $|\alpha| + \beta_\alpha \leq \rho+1$, $\rho > \tilde\rho_0 \geq 0$, $\alpha_k = 0$ *or* $\alpha_k \geq 1$, $a_k \geq 0$, $k = 0,\ldots,n$, $a > 0$, $b_\alpha \in C^1$, $(\alpha_0,\ldots,\alpha_n) := \alpha$, $\beta_\alpha \geq 0$. *Additionally, let one of the following conditions be fulfilled.*

(i) $\beta_\alpha = 0$, $\exists k_0 : 0 \leq k_0 \leq n$, $\alpha_{k_0} \geq \frac{\rho_{k_0}}{2} + 1$, $a_{k_0} > 0$, $|\alpha| < \rho+1$.

(ii) $\beta_\alpha = 0$, *if* $\alpha_k = 0$, $1 \leq k \leq n$, $\alpha_0 < \rho+1$, *then* $\alpha_0 = 0$ *or* $\alpha \geq 2$ *and* $b_\alpha \in W^2_{q_1}(G)\cap \overset{o}{W}{}^1_{q_1}(G)$, $q_1 > \dfrac{q+2}{\rho+1-\alpha_0}$.

(iii) $\beta_\alpha \geq 1$, *then* $\sum_{k=0}^{n} (\text{sign } a_k)\dfrac{\alpha_k}{\rho_k} + \dfrac{\beta_\alpha - 1}{\bar\beta} = 1$, $\tilde\rho_0 \leq \bar\beta \leq \rho$,

$$\sum_{|\alpha|+\beta_\alpha\leq\rho+1} 3|b_\alpha|\beta_\alpha \leq a\,\bar\beta, \qquad \sum_{|\alpha|+\beta_\alpha\leq\rho+1} 2|b_\alpha|\alpha_k \leq a_k\,\rho_k,\ 0 \leq k \leq n.$$

Then, problem (7.1) *is solvable for any* $h \in W^2_\Psi(G)\cap \overset{o}{W}{}^1_\Psi(G)$ *almost everywhere in* G, *and the solution* u *belongs to* $\underset{o}{S}{}^{\Delta}_{1,\varphi,2}(G) \cap \underset{o}{S}{}^{\Delta}_{1,\tilde\rho_0,2}(G) \cap \underset{o}{S}{}^{(\Delta)}_{3,\rho_0,2}(G) \cap S^{1(\Delta)}_{2,\bar\rho,2}(G)$.

Due to [1], the Laplacian Δ satisfies all conditions imposed on L. If one of the conditions of the theorem is satisfied, then the last summand in (7.1) is a term of low order that satisfies the assumptions of Theorem 4.1 under the conditions **(i)** and **(iii)**, or the assumptions of Theorem 4.2 under the condition **(ii)** made for $F_1(x, D^\alpha u, Lu)$.

Here, $\underset{o}{\mathcal{H}}^L(G)$ is specified by the following spaces (see [17], [19]):

$$\underset{o}{S}^{\Delta}_{1,\varphi,2}(G) := \left\{ u \,\middle|\, [u]_S := \Phi^{-1}\left(\int_G |\Delta u|^\rho \ln\left(1 + |\Delta u|^2\right) \sum_{i=1}^n |D_i \Delta u|^2 \, dx \right) + \Phi^{-1}\left(\int_G \ln\left(1 + |\Delta u|^2\right) \cdot |\Delta u|^{\rho+2} \, dx \right) < +\infty \right\}, \tag{7.2}$$

$$\underset{o}{S}^{\Delta}_{1,\tilde{\rho}_0,2}(G) := \left\{ u \,\middle|\, [u]_S^{\tilde{\rho}_0+2} := \int_G |\Delta u|^{\tilde{\rho}_0} \sum_{i=1}^n |D_i \Delta u|^2 \, dx + \int_G |\Delta u|^{\tilde{\rho}_0+2} \, dx < +\infty \,,\ u\Big|_{\partial G} = 0 \right\}, \tag{7.3}$$

$$S^{(\Delta)}_{3,\rho_0,2}(G) := \left\{ u \,\middle|\, [u]_S^{\rho_0+2} := \int_G |u|^{\rho_0} \sum_{i=1}^n |D_i \Delta u|^2 \, dx < +\infty \right\}, \tag{7.4}$$

$$S^{1(\Delta)}_{2,\bar{\rho},2}(G) := \left\{ u \,\middle|\, [u]_S := \left[\sum_{k=1}^n \left(\int_G |D_k u|^{\rho_k} \sum_{i=1}^n |D_i \Delta u|^2 \, dx \right)^{\frac{2}{(\rho_k+2)}} \right]^{\frac{1}{2}} < +\infty \right\}. \tag{7.5}$$

2. We consider the boundary value problem

$$- \left(a_0 \, e^{|u|^{1+\alpha}} + \sum_{i=1}^n a_i \, e^{|D_i u|^{1+\beta}} + a \, e^{(\Delta u)^2} - b \right) \Delta u = h(x) \,, \quad x \in G \,,$$
$$u\Big|_{\partial G} = 0 \,, \quad 0 < \alpha \leq \beta < 1 \,. \tag{7.6}$$

The following existence result for problem (7.6) is a consequence of Theorem 4.1.

Theorem 7.2 *Let* $0 \leq b \leq a + \sum_{k=0}^n a_k - 1$, *and* $a > 1$, $a_k \geq 0$, $0 < \alpha \leq \beta < 1$. *Then, problem* (7.6) *is solvable for any* $h \in W^2_{\Psi}(G) \cap \overset{o}{W}{}^1_{\Psi}(G)$ *almost everywhere in* G , *and the solution is contained in* $\underset{o}{S}^{\Delta}_{1,\varphi,2}(G) \cap S^{(\Delta)}_{3,\varphi_0,2}(G) \cap S^{1(\Delta)}_{2,\bar{\varphi},2}(G) := \underset{o}{\mathcal{H}}^{\Delta}(G)$, *with* $\varphi(\tau) := e^{\tau^2}$, $\Phi(\eta) := \int_0^{|\eta|} d\eta_1 \int_0^{|\eta_1|} \varphi(\tau) \, d\tau$, $\varphi_0(\tau) = e^{\tau^{1+\alpha}} - 1$, $\varphi_k(\tau) := e^{\tau^{1+\beta}} - 1$, $\Psi = \Psi^*$.

We now state the fundamental inequalities necessary for proving the generalized coercivity of the operator A .

Proposition 7.3 *Let* $0 < \alpha \leq \beta < 1$, $\alpha \leq \nu \leq 1$, $\beta \leq \mu \leq 1$. *Then, the following inequalities hold for any* $u \in C^3(G) \cap C^1_0(G)$.

$$\int_G e^{|u|^{1+\alpha}} |u|^\alpha |D_j u| \, |\Delta u| \, |D_j \Delta u| \, dx$$

$$\leq \epsilon \int_G e^{|u|^{1+\alpha}} |D_j \Delta u|^2\, dx + \epsilon_1 \int_G \left[e^{|D_j u|^{1+\beta}} + e^{|\Delta u|^2} \right] (\Delta u)^2\, dx$$
$$+ C_0(\epsilon, \epsilon_1) \int_G e^{|u|^{(1+\nu)}}\, dx + C(\epsilon, \epsilon_1)\,, \tag{7.7}$$

$$\int_G e^{|D_j u|^{1+\beta}} |D_j u|^{\beta} |D_i D_j u|\, |\Delta u|\, |D_i \Delta u|\, dx$$
$$\leq \epsilon \int_G e^{|D_j u|^{1+\beta}} |D_j \Delta u|^2\, dx + \epsilon_1 \int_G \left[e^{|D_j u|^{1+\mu}} + e^{|\Delta u|^2} |\Delta u|^2 \right] dx$$
$$+ \epsilon_2 \int_G |D_i D_j u|^p\, dx + C(\epsilon, \epsilon_1, \epsilon_2)\,. \tag{7.8}$$

Proof: We have, with suitable $p_1, p_2 > 1$,

$$\int_G e^{|u|^{1+\alpha}} |u|^{\alpha} |D_i u|\, |D_i \Delta u|\, |\Delta u|\, dx$$
$$\leq \epsilon \int_G e^{|u|^{1+\alpha}} |D_i \Delta u|^2\, dx + C(\epsilon) \int_G e^{|u|^{1+\alpha}} |u|^{2\alpha} |D_i u|^2 |\Delta u|^2\, dx$$
$$\leq \epsilon \int_G e^{|u|^{1+\alpha}} |D_i \Delta u|^2\, dx + \epsilon_1 \int_G \left(e^{|D_i u|^{1+\beta}} - 1 \right) |\Delta u|^2\, dx$$
$$+ C(\epsilon, \epsilon_1) \int_G e^{p_1 |u|^{1+\alpha}} |u|^{2\alpha p_2} |\Delta u|^2\, dx$$
$$\leq \epsilon \int_G e^{|u|^{1+\alpha}} |D_i \Delta u|^2\, dx + \epsilon_1 \int_G \left[e^{|D_i u|^{1+\beta}} + e^{|\Delta u|^2} \right] (\Delta u)^2\, dx$$
$$+ C_1(\epsilon, \epsilon_1) \int_G e^{|u|^{1+\nu}}\, dx + C_2(\epsilon, \epsilon_1)\,, \tag{7.9}$$

and, for the other inequality,

$$\int_G e^{|D_j u|^{1+\beta}} |D_j u|^{\beta} |D_i D_j u|\, |\Delta u|\, |D_i \Delta u|\, dx$$
$$\leq \epsilon \int_G e^{|D_j u|^{1+\beta}} |D_i \Delta u|^2\, dx + C(\epsilon) \int_G e^{|D_j u|^{1+\beta}} |D_j u|^{2\beta} |D_i D_j u|^2 |\Delta u|^2\, dx$$
$$\leq \epsilon \int_G e^{|D_j u|^{1+\beta}} |D_i \Delta u|^2\, dx + \epsilon_1 \int_G |D_i D_j u|^p\, dx$$
$$+ C_1(\epsilon, \epsilon_1) \int_G e^{q |D_j u|^{1+\beta}} |D_j u|^{2q\beta} |\Delta u|^{2q}\, dx$$
$$\leq \epsilon_1 \int_G |D_i D_j u|^p\, dx + \epsilon \int_G e^{|D_j u|^{1+\beta}} |D_i \Delta u|^2\, dx + \epsilon_2 \int_G e^{|\Delta u|^2} (\Delta u)^2\, dx$$
$$+ C_1(\epsilon, \epsilon_1, \epsilon_2) \int_G e^{|D_j u|^{1+\mu}}\, dx + C_2(\epsilon, \epsilon_1, \epsilon_2)\,. \tag{7.10}$$

□

Remark 7.3 Parabolic equations having an elliptic part as in Examples 1 and 2 can be treated in exactly the same way.

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