On a class of continuous coagulation–fragmentation equations

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Abstract. A model for the dynamics of a system of particles undergoing simultaneously coalescence and breakup is considered, each particle being assumed to be fully identified by its size. Existence of solutions to the corresponding evolution integral partial differential equation is shown for product-type coagulation kernels with a weak fragmentation. The failure of density conservation (or gelation) is also investigated in some particular cases.

1. Introduction

The coagulation-fragmentation equations are a model for the dynamics of cluster growth and describe the time evolution of a system of clusters under the combined effect of coagulation and fragmentation. Each cluster is identified by its size (or volume) which is assumed to be a positive real number in the model considered in this paper. From a physical point of view the basic mechanisms taken into account are the coalescence of two clusters to form a larger one and the breakage of clusters into smaller ones. It is also assumed that the rates of these reactions only depend on the sizes of the clusters involved in the reaction. Other effects (multiple-coagulation, spatial fluctuations, ...) are neglected. Examples of applications of these models arise in aerosol physics, polymer science and astronomy (see, e.g., [15] or the recent survey paper [2] and the references therein). Denoting by $c(x,t)$ the density of clusters of size x at time t, the continuous coagulation-fragmentation equations read [15]

$$\begin{aligned} c_t(x,t) \;=\; & \frac{1}{2}\int_0^x \phi(x-y,y)\; c(x-y,t)\; c(y,t)\; dy \\ & - c(x,t)\int_0^x \frac{y}{x}\;\psi(x,y)\; dy \\ & - c(x,t)\int_0^\infty \phi(x,y)\; c(y,t)\; dy \\ & + \int_x^\infty \psi(y,x)\; c(y,t)\; dy, \end{aligned} \tag{1.1}$$

$$c(x,0) = c_0(x), \tag{1.2}$$

where the size variable x ranges in $(0,+\infty)$, the time variable t ranges in $(0,+\infty)$ and c_t denotes the partial derivative of c with respect to time. Moreover, the reaction rates ϕ and ψ are non-negative functions which are called, respectively, the coagulation and fragmentation kernels. The first and fourth integrals in (1.1) account for the formation of clusters of size x due to coagulation of smaller clusters and fragmentation of larger ones, while the second and third integrals in (1.1) describe the loss of clusters of size x due to coalescence with other clusters and breakup.

The specific form of ϕ and ψ of course depends on the particular physical situation to be described. One interesting example of coagulation kernel is the so-called product kernel $\phi(x,y) = (xy)^\gamma$, $\gamma \geq 0$, and the main requirement we impose on the coagulation kernels to be considered in this paper is to have a product kernel as dominating part. More precisely, we assume that

$$\phi(x,y) = r(x)\ r(y) + \alpha(x,y), \quad (x,y) \in \mathbb{R}_+^2, \tag{1.3}$$

where the functions r and α satisfy

$$\begin{cases} r \in \mathcal{C}(\mathbb{R}_+), \quad \alpha \in \mathcal{C}(\mathbb{R}_+^2), \\ 0 \le \alpha(x,y) = \alpha(y,x) \le A\ r(x)\ r(y), \quad (x,y) \in [1,+\infty)^2, \end{cases} \tag{1.4}$$

for some positive real number A. Here and in the following we use the notations $\mathbb{R}_+ = [0,+\infty)$ and $\mathbb{R}_+^2 = \mathbb{R}_+ \times \mathbb{R}_+$.
As for the fragmentation kernel it is natural to assume that [15]

$$\psi \in \mathcal{C}(\mathbb{R}_+^2), \tag{1.5}$$

$$\psi(x,y) \ge 0, \quad (x,y) \in \mathbb{R}_+^2 \quad \text{and} \quad \psi(x,y) = 0 \quad \text{if} \quad y > x \ge 0. \tag{1.6}$$

We next require that the fragmentation is weaker than the coagulation, namely : there is a non-increasing and bounded function $\omega\ : \mathbb{R}_+ \to \mathbb{R}_+$ and a positive real number B such that

$$\lim_{x \to +\infty} \omega(x) = 0, \tag{1.7}$$

$$\int_0^x \psi(x,y)\ dy \le \omega(x)\ \max(x, r(x)), \quad x \in \mathbb{R}_+, \tag{1.8}$$

$$\psi(x,y) \le B\ \left(1 + \max(x, r(x))\right), \quad (x,y) \in \mathbb{R}_+^2. \tag{1.9}$$

We finally assume that the initial datum c_0 satisfies

$$c_0 \in X^+, \tag{1.10}$$

where X^+ is the positive cone of the Banach space

$$X = L^1(0,+\infty;(1+x)\ dx)$$

endowed with the norm $\|.\|$ defined by

$$\|u\| = \int_0^\infty (1+x)\ |u(x)|\ dx, \quad u \in X.$$

Thus,

$$X^+ = \{u \in X, \ \ u \ge 0 \text{ a.e.}\}.$$

We first investigate the existence of solutions to (1.1)-(1.2) for the class of kernels and initial data described above. Before stating our result, let us make precise the notion of solution to (1.1) to be used in the sequel.

Definition 1.1. Let $T \in (0,+\infty]$. A solution c to (1.1) on $[0,T)$ is a function $c\ : [0,T) \to X^+$ such that, for every $t \in (0,T)$, there holds

$$c \in \mathcal{C}([0,t]; L^1(0,+\infty)) \cap L^\infty(0,t;X), \tag{1.11}$$

$$\begin{cases} (x,y,s) \mapsto \phi(x,y)\ c(x,s)\ c(y,s) \in L^1((0,+\infty)^2 \times (0,t)), \\ (x,y,s) \mapsto \psi(y,x)\ c(y,s) \in L^1((0,+\infty)^2 \times (0,t)), \end{cases} \tag{1.12}$$

and for almost every $x \in \mathbb{R}_+$,

$$
\begin{aligned}
(1.13) \qquad c(x,t) &= c(x,0) \\
&+ \frac{1}{2} \int_0^t \int_0^x \phi(x-y,y)\ c(x-y,s)\ c(y,s)\ dyds \\
&- \int_0^t c(x,s) \int_0^x \frac{y}{x}\ \psi(x,y)\ dyds \\
&- \int_0^t c(x,s) \int_0^\infty \phi(x,y)\ c(y,s)\ dyds \\
&+ \int_0^t \int_x^\infty \psi(y,x)\ c(y,s)\ dyds.
\end{aligned}
$$

Existence of solutions to (1.1) in the sense of Definition 1.1 (the initial datum c_0 and thus the function c possibly enjoying additional regularity properties) has been the subject of several papers since the pioneering works of Melzak [15] and McLeod [14]. For bounded kernels ϕ and ψ existence of solutions to (1.1) is studied in [15, 1, 13], while the case of unbounded kernels has been investigated in [17, 4, 5] assuming the following growth condition on ϕ,

$$
(1.14) \qquad \phi(x,y) \leq K\ (1+x+y), \quad (x,y) \in \mathbb{R}_+^2.
$$

In the absence of fragmentation ($\psi \equiv 0$) global existence results are available assuming either (1.14) [9] or the weaker assumption [10]

$$
(1.15) \qquad \lim_{(x,y)\to+\infty} \frac{\phi(x,y)}{xy} = 0.
$$

Also the kernel $\phi(x,y) = xy$ has been considered in [14] (local existence) while explicit global solutions are constructed in [8] by means of the Laplace transform. Let us finally mention that under the assumption (1.15) existence of measure-valued solutions to the pure coagulation equations ($\psi \equiv 0$) and to the full coagulation-fragmentation equations (with suitable assumptions on ψ) has been obtained, respectively, in [16] and [7] by a probabilistic approach (see also [2]). The solutions obtained therein taking only their values in the space of measures thus satisfy (1.1) in a weaker sense than the one required by Definition 1.1.
Browsing on the aforementioned papers one sees that no existence results (in the sense of Definition 1.1) are available for (1.1) when $\psi \not\equiv 0$ and ϕ does not satisfy (1.14). Our first result is a step in that direction within the class of kernels described above.

Theorem 1.2. *Assume that the coagulation and fragmentation kernels ϕ and ψ fulfill, respectively, (1.3)-(1.4) and (1.5)-(1.9). For every $c_0 \in X^+$ there exists at least one solution c to (1.1) on $[0,+\infty)$ with $c(0) = c_0$ satisfying*

$$
(1.16) \qquad \int_0^\infty x\ c(x,t)\ dx \leq \int_0^\infty x\ c_0(x)\ dx, \quad t \in \mathbb{R}_+.
$$

It is worth mentioning at this point that no growth condition is required on the function r in (1.4). Therefore, Theorem 1.2 also provides the existence of global solutions to the pure coagulation equations ($\psi \equiv 0$) when r increases superlinearly and when $\phi(x,y) = xy + \alpha(x,y)$, $\alpha \not\equiv 0$, both results being new to our knowledge.

Our next result deals with the large time behaviour of the total density ϱ of solutions c to (1.1)-(1.2) defined by

$$\varrho(t) = \int_0^\infty x\ c(x,t)\ dx, \quad t \in \mathbb{R}_+, \tag{1.17}$$

when $\psi \equiv 0$ and $r(x) \geq R\ x$, $x \in \mathbb{R}_+$, for some $R > 0$. It turns out that, in the coagulation-fragmentation process described by (1.1) there is neither sink nor source of clusters, so that the total density ϱ is expected to be constant through time evolution, i.e. $\varrho(t) = \varrho(0)$ for $t \in \mathbb{R}_+$. While this is true in some cases (e.g., when ϕ satisfies (1.14) [18, 5]) it is well-known that, if $\phi(x,y) = xy$ and $\psi \equiv 0$ there are explicit solutions for which this property fails to be true, a phenomenon known as gelation [8] . This picture is in fact valid for a wider class of coagulation kernels and initial data as the following result shows.

Theorem 1.3. *Assume that ϕ satisfies (1.3)-(1.4) and $\psi \equiv 0$. Assume further that there exists $R > 0$ such that*

$$r(x) \geq R\ x, \quad x \in \mathbb{R}_+. \tag{1.18}$$

Consider $c_0 \in X^+$, $c_0 \not\equiv 0$, and let c be a solution to (1.1) on $[0,+\infty)$ with initial datum c_0. Then

$$\varrho(t) := \int_0^\infty x\ c(x,t)\ dx \leq \frac{2^{1/2}\ |c_0|_{L^1}^{1/2}}{R}\ t^{-1/2}, \quad t \in (0,+\infty). \tag{1.19}$$

If c_0 satisfies in addition

$$I_q := \int_0^\infty x^{-q}\ c_0(x)\ dx < \infty \tag{1.20}$$

for some $q \in (0,+\infty)$, there holds

$$\varrho(t) \leq \varrho(0)\ \min\left\{1, \left(\frac{q+(q+2)t/T_*}{2(q+1)}\right)^{-(q+1)/(q+2)}\right\}, \tag{1.21}$$

where

$$T_* = \frac{2}{R^2}\ I_q^{1/(q+1)}\ \varrho(0)^{-(q+2)/(q+1)}. \tag{1.22}$$

Finally, if $c_0 \equiv 0$ on $(0,\delta)$ for some $\delta > 0$ we have

$$\varrho(t) \leq \varrho(0)\ \min\left\{1, \left(\frac{1+t/T_*}{2}\right)^{-1}\right\}, \tag{1.23}$$

where

$$T_* = \frac{2}{\delta\ R^2\ \rho(0)}. \tag{1.24}$$

It follows from Theorem 1.3 that the temporal decay of the total density ϱ depends strongly on the amount of clusters of very small size ($x \approx 0$) in the initial distribution, and the smaller this amount is, the faster is the decay of the total density ϱ. This fact has already been noticed in [8] when $\phi(x,y) = xy$ for some specific initial data for which explicit solutions are available. Theorem 1.3 thus provides an extension of the results of [8].

We finally investigate the possible occurrence of the gelation phenomenon in the coagulation-fragmentation equations, still assuming that ϕ fulfills (1.3)-(1.4) and (1.18). Indeed, as already mentioned above we expect from our assumptions (1.7)-(1.8) that the dynamics of the system of clusters will be dominated by coagulation which is the gelation-inducing mechanism. It is thus likely that gelation still takes place in the full coagulation-fragmentation equations under our assumptions. One partial result in that direction is the following.

Proposition 1.4. *Let ϕ and ψ be coagulation and fragmentation kernels satisfying, respectively, (1.3)-(1.4), (1.18) and (1.5)-(1.9) together with*

$$\int_0^x \psi(x,y)\ \left(1-\frac{y}{x}\right)\ dy \le \Gamma\ \min(1,x), \quad x\in\mathbb{R}_+, \tag{1.25}$$

for some $\Gamma>0$. Consider next $c_0\in X^+$ and denote by c a solution to (1.1) on $[0,+\infty)$ with initial datum c_0. If

$$\varrho(0) > \frac{2\Gamma}{R^2}, \tag{1.26}$$

then gelation occurs in a finite time, i.e.

$$T_{gel} = \inf\{t\in\mathbb{R}_+,\ \varrho(t)<\varrho(0)\} < \infty.$$

As far as we know only few results on the onset of gelation in the coagulation-fragmentation equations were available and only the case $\phi(x,y)=xy$ had been considered together with some special cases of fragmentation kernels ψ by formal arguments in [3] and [19]. Some fragmentation kernels considered in the above mentioned papers however do not fulfill (1.7)-(1.9). For the discrete coagulation-fragmentation equations a result in the spirit of Proposition 1.4 may be found in [11, Theorem 4] for a different (stochastic) notion of gelation.

2. Preliminaries

Let $(\xi_n)_{n\ge1}$ be a sequence of smooth cut-off functions such that $0\le\xi_n\le1$ and

$$\xi_n(x) = \begin{cases} 1 & \text{if} \quad 0\le x\le n, \\ 0 & \text{if} \quad x\ge n+1. \end{cases}$$

For $n\ge1$ we define a sequence of approximations of ϕ and ψ by

$$\phi_n(x,y) = \phi(x,y)\ \xi_n(x)\ \xi_n(y), \quad (x,y)\in\mathbb{R}_+^2, \tag{2.1}$$

$$\psi_n(x,y) = \psi(x,y)\ \xi_n(x), \quad (x,y)\in\mathbb{R}_+^2. \tag{2.2}$$

A straightforward consequence of (1.3)-(1.8) and the properties of ξ_n is the following result.

Lemma 2.1. *For each $n\ge1$ the functions ϕ_n and ψ_n are non-negative and bounded continuous functions on $\mathbb{R}_+^2$ and satisfy*

$$\phi_n(x,y) = r_n(x)\ r_n(y) + \alpha_n(x,y), \quad (x,y)\in\mathbb{R}_+^2, \tag{2.3}$$

$$0\le\alpha_n(x,y)\le A\ r_n(x)\ r_n(y), \quad (x,y)\in[1,+\infty)^2, \tag{2.4}$$

$$\int_0^x \psi_n(x,y)\ dy \le \omega(x)\ \max\left(x\xi_n(x), r_n(x)\right), \quad x\in\mathbb{R}_+, \tag{2.5}$$

where

$$r_n(x) = r(x)\ \xi_n(x), \quad \alpha_n(x,y) = \alpha(x,y)\ \xi_n(x)\ \xi_n(y). \tag{2.6}$$

We also consider a sequence of non-negative functions $(c_0^n)_{n\geq 1}$ in $\mathcal{D}(0,+\infty)$ such that

$$\lim_{n\to+\infty} \|c_0^n - c_0\| = 0. \tag{2.7}$$

Consequently,

$$C_0 := \sup_{n\geq 1} \|c_0^n\| < \infty. \tag{2.8}$$

Owing to Lemma 2.1 we may use the results of Melzak [15] to establish the existence of a solution to (1.1)-(1.2) with (ϕ, ψ, c_0) replaced by (ϕ_n, ψ_n, c_0^n). More precisely we have the following result.

Proposition 2.2. *For each $n \geq 1$ there is a unique function*

$$c^n \in \mathcal{C}(\mathbb{R}_+^2) \cap L^\infty(0,T; L^1(0,+\infty)), \quad T \in (0,+\infty),$$

such that, for every $(x,t) \in \mathbb{R}_+^2$ there holds

$$\begin{aligned} c^n(x,t) &= c_0^n(x) \\ &+ \frac{1}{2} \int_0^t \int_0^x \phi_n(x-y,y)\ c^n(x-y,s)\ c^n(y,s)\ dyds \\ &- \int_0^t c^n(x,s) \int_0^x \frac{y}{x}\ \psi_n(x,y)\ dyds \\ &- \int_0^t c^n(x,s) \int_0^\infty \phi_n(x,y)\ c^n(y,s)\ dyds \\ &+ \int_0^t \int_x^\infty \psi_n(y,x)\ c^n(y,s)\ dyds. \end{aligned} \tag{2.9}$$

Since the coagulation and fragmentation kernels ϕ_n and ψ_n are bounded and compactly supported in $\mathbb{R}_+^2$ we deduce from (2.9) the following useful identities.

Lemma 2.3. *Let g be a locally bounded function on $\mathbb{R}_+$ such that $g(x) \leq G\ (1+x)$, $x \in \mathbb{R}_+$, for some $G > 0$. For $n \geq 1$, $t \in (0,+\infty)$ and $s \in [0,t)$ there holds*

$$\begin{aligned} &\int_0^\infty g(x)\ \left(c^n(x,t) - c^n(x,s)\right)\ dx \\ &= \frac{1}{2} \int_s^t \int_0^\infty \int_0^\infty \phi_n(x,y)\ \tilde{g}(x,y)\ c^n(x,\sigma)\ c^n(y,\sigma)\ dxdyd\sigma \\ &+ \int_s^t \int_0^\infty c^n(x,\sigma) \int_0^x \psi_n(x,y)\ \left(g(y) - \frac{y}{x}\ g(x)\right)\ dy\ dxd\sigma, \end{aligned} \tag{2.10}$$

where

$$\tilde{g}(x,y) = g(x+y) - g(x) - g(y), \quad (x,y) \in \mathbb{R}_+^2, \tag{2.11}$$

$$\int_0^\infty x\ c^n(x,t)\ dx = \int_0^\infty x\ c_0^n(x)\ dx. \tag{2.12}$$

In fact (2.12) follows from (2.10) with $g(x) = x$.

We next use the special form (2.1) of the coagulation kernel to derive some estimates valid uniformly with respect to $n \geq 1$. In the following we denote by $(C_i)_{i\geq 1}$ any positive constant which depends only on ϕ, r, α, A, ψ, ω, B, c_0 and C_0 in (2.8). The dependence of the C_i's upon further parameters will be indicated explicitly.

Lemma 2.4. *For $M > 0$ and $n \geq M$ there holds*

$$\begin{aligned}(2.13)\qquad & \int_0^t \int_M^\infty \int_M^\infty \phi_n(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdyds \\ & \leq \frac{2C_0}{M} + 2 \int_0^t \int_M^\infty \left(\int_0^x \psi_n(x,y)\ dy \right)\ c^n(x,s)\ dxds.\end{aligned}$$

Proof. We take $g(x) = \min(x, M)$ in (2.10). As

$$\begin{aligned} &\tilde{g}(x,y) \leq 0 \quad \text{if} \quad x \in [0,M] \quad \text{or} \quad y \in [0,M], \\ &\tilde{g}(x,y) = -M \quad \text{if} \quad (x,y) \in [M,+\infty)^2, \end{aligned}$$

we obtain

$$\begin{aligned} & \int_0^\infty g(x)\ \left(c^n(x,t) - c_0^n(x)\right)\ dx \\ & \leq -\frac{M}{2} \int_0^t \int_M^\infty \int_M^\infty \phi_n(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdyds \\ & + \int_0^t \int_M^\infty c^n(x,s) \int_0^x \psi_n(x,y) \left(g(y) - \frac{My}{x} \right)\ dy\ dxds \\ & \leq -\frac{M}{2} \int_0^t \int_M^\infty \int_M^\infty \phi_n(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdyds \\ & + M \int_0^t \int_M^\infty c^n(x,s) \int_0^x \psi_n(x,y)\ dy\ dxds, \end{aligned}$$

hence (2.13). □

A simple consequence of Lemma 2.4 and (2.3)-(2.5) is the following result.

Lemma 2.5. *Let $T \in (0,+\infty)$. For $M > 0$, $t \in [0,T]$ and $n \geq M$ there holds*

$$(2.14)\qquad \int_0^t \left(\int_M^\infty r_n(x)\ c^n(x,s)\ dx \right)^2\ ds \leq C_1(T)\ \left(M^{-1} + \omega(M) \right).$$

Proof. We infer from Lemma 2.4, (2.12), (2.3), (2.5) and the properties of ω that

$$\begin{aligned} & \int_0^t \left(\int_M^\infty r_n(x)\ c^n(x,s)\ dx \right)^2\ ds \\ & \leq \frac{2C_0}{M} + 2\ \omega(M) \int_0^t \int_M^\infty \left(x\ \xi_n(x) + r_n(x)\right)\ c^n(x,s)\ dxds \\ & \leq C_1(T)\ \left(M^{-1} + \omega(M)\right) + 2\ \omega(M) \int_0^t \left(\int_M^\infty r_n(x)\ c^n(x,s)\ dx \right)\ ds \\ & \leq C_1(T)\ \left(M^{-1} + \omega(M)\right) + 2\ \omega(M)^2\ t \\ & + \frac{1}{2} \int_0^t \left(\int_M^\infty r_n(x)\ c^n(x,s)\ dx \right)^2\ ds, \end{aligned}$$

hence (2.14). □

Before going further we introduce the following notation : for $n \geq 1$, $a \in (1,+\infty]$, $\delta \in (0,+\infty)$ and $t \in \mathbb{R}_+$ we put

$$\mathcal{E}^n_{a,\delta}(t) = \sup \left\{ \begin{array}{l} \displaystyle\int_0^a \mathbf{1}_E(x)\ c^n(x,t)\ dx, \\ \\ E \ \text{ measurable subset of } \ \mathbb{R}_+ \ \text{ with } \ |E| \leq \delta \end{array} \right\}.$$

Here $\mathbf{1}_E$ denotes the characteristic function of E.
We may now proceed as in [17, Lemma 3.5] with some modifications to prove the following result.

Lemma 2.6. *Let $T \in (0,+\infty)$ and $a \in (1,+\infty)$. For every $n \geq 1$, $t \in [0,T]$ and $\delta \in (0,+\infty)$ there holds*

$$\int_0^\infty c^n(x,t)\ dx \leq C_2(T), \tag{2.15}$$

$$\int_0^t \left(\int_0^\infty r_n(x)\ c^n(x,s)\ dx \right)^2\ ds \leq C_2(T), \tag{2.16}$$

$$\mathcal{E}^n_{a,\delta}(t) \leq C_3(a,T)\ \left(\mathcal{E}^n_{a,\delta}(0) + \delta \right). \tag{2.17}$$

Proof. We first take $g \equiv 1$ in (2.10) and use (2.8), (2.3), (2.4), (2.5) and (2.12) to obtain

$$\begin{aligned} &\int_0^\infty c^n(x,t)\ dx + \frac{1}{2}\ \int_0^t \left(\int_0^\infty r_n(x)\ c^n(x,s)\ dx \right)^2\ ds \\ &\leq C_0 + \int_0^t \int_0^\infty (x + r_n(x))\ \ c^n(x,s)\ dxds \\ &\leq C_2(T) + \frac{1}{4}\ \int_0^t \left(\int_0^\infty r_n(x)\ c^n(x,s)\ dx \right)^2\ ds, \end{aligned}$$

hence (2.15)-(2.16).
Next, let $a \in (1,+\infty)$, $\delta \in (0,+\infty)$ and consider a measurable subset E of $\mathbb{R}_+$ with $|E| \leq \delta$. Thanks to the non-negativity of ϕ_n, ψ_n and c^n it follows from (2.9) that

$$\begin{aligned} &\int_0^a \mathbf{1}_E(x)\ c^n(x,t)\ dx \leq \mathcal{E}^n_{a,\delta}(0) \\ &+\frac{1}{2}\ \int_0^t \int_0^a \int_0^x \mathbf{1}_E(x)\ \phi_n(x-y,y)\ c^n(x-y,s)\ c^n(y,s)\ dydxds \\ &+ \int_0^t \int_0^a \mathbf{1}_E(x)\ \int_x^\infty \psi_n(y,x)\ c^n(y,s)\ dydxds. \end{aligned}$$

The Fubini theorem then entails

$$\begin{aligned}&\int_0^a \mathbf{1}_E(x)\ c^n(x,t)\ dx \le \mathcal{E}^n_{a,\delta}(0)\\&+\int_0^t\int_0^a\int_0^a \mathbf{1}_{-y+E}(x)\ \phi_n(x,y)\ c^n(x,s)\ c^n(y,s)\ dydxds\\&+\int_0^t\int_0^a c^n(x,s)\ \int_0^x \mathbf{1}_E(y)\ \psi_n(x,y)\ dy\ dxds\\&+\int_0^t\int_a^\infty c^n(x,s)\ \int_0^x \mathbf{1}_E(y)\ \psi_n(x,y)\ dy\ dxds.\end{aligned}$$

The Lebesgue measure being invariant with respect to translation and ϕ_n, ψ_n being uniformly bounded on $[0,a]\times[0,a]$ with respect to $n\ge 1$ (the bound depending on a) we infer from the above estimate that

$$\begin{aligned}&\int_0^a \mathbf{1}_E(x)\ c^n(x,t)\ dx \le \mathcal{E}^n_{a,\delta}(0)\\&+C_4(a)\ \int_0^t \mathcal{E}^n_{a,\delta}(s)\ \int_0^a c^n(x,s)\ dx\ ds\\&+C_4(a)\ |E|\ \int_0^t\int_0^a c^n(x,s)\ dxds\\&+\int_0^t\int_a^\infty c^n(x,s)\ \int_0^x \mathbf{1}_E(y)\ \psi_n(x,y)\ dy\ dxds.\end{aligned}$$

It then follows from (2.12), (2.15), (2.16) and (1.9) that

$$\begin{aligned}\int_0^a \mathbf{1}_E(x)\ c^n(x,t)\ dx &\le \mathcal{E}^n_{a,\delta}(0)+C_5(a,T)\ \left(\int_0^t \mathcal{E}^n_{a,\delta}(s)\ ds+\delta\right)\\&+\ \int_0^t\int_a^\infty (1+x+r_n(x))\ c^n(x,s)\ |E|\ dxds\\&\le \mathcal{E}^n_{a,\delta}(0)+C_6(a,T)\ \left(\int_0^t \mathcal{E}^n_{a,\delta}(s)\ ds+\delta\right).\end{aligned}$$

The Gronwall lemma then yields (2.17). □

Lemma 2.7. *Let $T\in(0,+\infty)$ and $a\in(1,+\infty)$. For every $n\ge 1$, $t\in[0,T]$ and $s\in[0,t]$ there holds*

$$\int_0^a |c^n(x,t)-c^n(x,s)|\ dx \le C_7(a,T)\ (t-s)^{1/2}. \tag{2.18}$$

Proof. We take $g(x) = \mathbf{1}_{[0,a]}(x)\ sign\,(c^n(x,t) - c^n(x,s))$ in (2.10) and obtain

$$\begin{aligned}
&\int_0^a |c^n(x,t) - c^n(x,s)|\ dx \\
&\le \frac{3}{2} \int_s^t \int_0^a \int_0^a \phi_n(x,y)\ c^n(x,\sigma)\ c^n(y,\sigma)\ \ dxdyd\sigma \\
&+2 \int_s^t \int_0^a c^n(x,\sigma) \int_0^x \psi_n(x,y)\ dy\ dxd\sigma \\
&+ \int_s^t \int_a^\infty c^n(x,\sigma) \int_0^a \psi_n(x,y)\ dy\ dxd\sigma \\
&+ \int_s^t \int_0^a \int_a^\infty \phi_n(x,y)\ c^n(x,\sigma)\ c^n(y,\sigma)\ \ dxdyd\sigma
\end{aligned}$$

It then follows from (2.3), (2.4), (2.5), (2.12), (2.15) and (2.16) that

$$\begin{aligned}
&\int_0^a |c^n(x,t) - c^n(x,s)|\ dx \\
&\le C_8(a) \int_s^t \left(\sup_{\sigma\in[0,T]} |c^n(.,\sigma)|_{L^1}^2 + \int_0^a x\ c^n(x,\sigma)\ dx \right)\ d\sigma \\
&+ \int_s^t \int_a^\infty \omega(x)\ (x + r_n(x))\ c^n(x,\sigma)\ dxd\sigma \\
&+(1+A) \int_s^t \left(\int_0^a r_n(x)\ c^n(x,\sigma)\ dx \right) \left(\int_a^\infty r_n(x)\ c^n(x,\sigma)\ dx \right)\ d\sigma \\
&\le C_9(a,T)\ (t-s) + C_{10} \int_s^t \int_0^\infty (x + r_n(x))\ c^n(x,\sigma)\ dxd\sigma \\
&+(1+A)\ |r|_{L^\infty(0,a)} \int_s^t |c^n(.,\sigma)|_{L^1} \left(\int_0^\infty r_n(x)\ c^n(x,\sigma)\ dx \right)\ d\sigma \\
&\le C_{11}(a,T)\ \left((t-s) + (t-s)^{1/2} \right),
\end{aligned}$$

and the proof of Lemma 2.7 is complete. □

Owing to (2.12), Lemma 2.6 and Lemma 2.7 we may now prove a compactness result for the sequence (c^n).

Proposition 2.8. *For each $T \in (0,+\infty)$ the sequence (c^n) is relatively sequentially compact in $\mathcal{C}([0,T]; L^1(0,+\infty)_w)$.*

Here we have use the notation $\mathcal{C}([0,T]; Y_w)$ to denote the space of all weakly continuous functions from $[0,T]$ into the Banach space Y.

Proof. According to a variant of the Arzelà-Ascoli theorem (see, e.g., [20, Theorem 1.3.2]) we need only to check that the sequence (c^n) enjoys the following two properties :

(2.19) $\begin{cases} \text{The set } \{c^n(t), n \ge 1\} \text{ is weakly compact in } L^1(0,+\infty) \\ \text{for every } t \in [0,T]. \end{cases}$

(2.20) $\begin{cases} \text{The set } \{c^n, n \ge 1\} \text{ is weakly equicontinuous in} \\ L^1(0,+\infty) \text{ at every } t \in [0,T] \text{ (see [20, Definition 1.3.1])}. \end{cases}$

– Proof of (2.19). We fix $t \in [0,T]$. Let $\varepsilon \in (0,+\infty)$. By (2.8) and (2.12) we have

$$\int_M^\infty c^n(x,t)\, dx \le \frac{C_0}{M}, \quad M \in (0,+\infty). \tag{2.21}$$

We may therefore choose M_ε large enough such that

$$\int_{M_\varepsilon}^\infty c^n(x,t)\, dx \le \frac{\varepsilon}{2}, \quad n \ge 1. \tag{2.22}$$

Consider next $\delta \in (0,+\infty)$ and a measurable subset E of $(0,+\infty)$ with $|E| \le \delta$. Owing to (2.17) and (2.22) there holds

$$\begin{aligned} \int_E c^n(x,t)\, dx &\le \int_0^{M_\varepsilon} \mathbf{1}_E(x)\, c^n(x,t)\, dx + \frac{\varepsilon}{2}, \\ &\le C(M_\varepsilon,T)\, \left(\mathcal{E}^n_{+\infty,\delta}(0) + \delta\right) + \frac{\varepsilon}{2}. \end{aligned}$$

As (c_0^n) converges strongly to c_0 in $L^1(0,+\infty)$ by (2.7) we have

$$\lim_{\delta\to 0} \sup_{n\ge 1} \mathcal{E}^n_{+\infty,\delta}(0) = 0.$$

Consequently there is $\delta_\varepsilon > 0$ such that, if $|E| \le \delta_\varepsilon$,

$$\sup_{n\ge 1} \int_E c^n(x,t)\, dx \le \varepsilon. \tag{2.23}$$

Gathering (2.22) and (2.23) we deduce from the Dunford-Pettis theorem (see, e.g., [6]) that (2.19) holds true.

– Proof of (2.20). Let $\varepsilon \in (0,+\infty)$. As (2.21) holds uniformly with respect to $t \in [0,T]$ and $n \ge 1$ there is $a_\varepsilon \ge 1$ such that

$$\int_{a_\varepsilon}^\infty c^n(x,t)\, dx \le \frac{\varepsilon}{4}, \quad n \ge 1, \quad t \in [0,T]. \tag{2.24}$$

Let $t \in [0,T]$ and $s \in [0,t]$. By (2.18) and (2.24) we have

$$\begin{aligned} \int_0^\infty |c^n(x,t) - c^n(x,s)|\, dx &\le \int_0^{a_\varepsilon} |c^n(x,t) - c^n(x,s)|\, dx + \frac{\varepsilon}{2} \\ &\le C_7(a_\varepsilon,T)\, (t-s)^{1/2} + \frac{\varepsilon}{2} \\ &\le \varepsilon, \end{aligned} \tag{2.25}$$

provided

$$|t-s| \le \eta(\varepsilon,T) := \frac{\varepsilon^2}{4C_7(a_\varepsilon,T)^2}. \tag{2.26}$$

Therefore (c^n) is equicontinuous with respect to the strong topology of $L^1(0,+\infty)$ and thus also for the weak topology of $L^1(0,+\infty)$, hence (2.20).

We may then apply [20, Theorem 1.3.2] and obtain Proposition 2.8. □

The last result of this section states a continuity property of some bilinear integral operator with respect to the weak topology of $L^1 \times L^1$.

Lemma 2.9. *Consider $0 < a \le b < \infty$ and $H \in L^\infty((0,a)\times(0,b))$. We define a mapping Λ on $L^1(0,a)\times L^1(0,b)$ by*

$$\Lambda(u,v) = \int_0^a \int_0^b H(x,y)\ u(x)\ v(y)\ dxdy.$$

If (u_n) is a sequence in $L^1(0,a)$ converging weakly to u in $L^1(0,a)$ and (v_n) is a sequence in $L^1(0,b)$ converging weakly to v in $L^1(0,b)$ there holds

$$\lim_{n\to+\infty} \Lambda(u_n,v_n) = \Lambda(u,v).$$

The proof of Lemma 2.9 follows the lines of that of [17, Lemma 4.1] to which we refer.

3. Proof of Theorem 1.2

We are now in a position to prove Theorem 1.2. Indeed we infer from Proposition 2.8 that there is a subsequence of (c^n) (not relabeled) and a function

$$c \in \mathcal{C}([0,+\infty); L^1(0,+\infty)_w),$$

such that for each $T \in (0,+\infty)$ there holds

$$c_n \longrightarrow c \quad \text{in} \quad \mathcal{C}([0,T]; L^1(0,+\infty)_w). \tag{3.1}$$

As $c(.,t)$ is a weak limit of non-negative functions we deduce that $c(.,t) \ge 0$ a.e. in $(0,+\infty)$ for every $t \in \mathbb{R}_+$. We also claim that in fact,

$$c \in \mathcal{C}([0,+\infty); L^1(0,+\infty)). \tag{3.2}$$

Indeed, let $(t,s) \in \mathbb{R}_+^2$ and $\varepsilon \in (0,+\infty)$. Since $(c^n(t) - c^n(s))$ converges weakly to $c(t)-c(s)$ in $L^1(0,+\infty)$ we infer from (2.25) that

$$|c(t)-c(s)|_{L^1} \le \varepsilon,$$

as long as (t,s) fulfills (2.26), hence the claim (3.2).
Next, let $T \in (0,+\infty)$, $a \in (0,+\infty)$ and consider $M > a$. For $n \ge M$ it follows from (2.14) and the properties of ξ_n that

$$\int_0^T \left(\int_a^M r(x)\ c^n(x,s)\ dx \right)^2\ ds \le C_1(T)\ \left(a^{-1} + \omega(a)\right).$$

As $r\ \mathbf{1}_{[a,M]} \in L^\infty(0,+\infty)$ we infer from (3.1), (2.15) and the Lebesgue dominated convergence theorem that

$$\int_0^T \left(\int_a^M r(x)\ c(x,s)\ dx \right)^2\ ds \le C_1(T)\ \left(a^{-1} + \omega(a)\right).$$

As $M > a$ is arbitrary we finally obtain

$$\int_0^T \left(\int_a^\infty r(x)\ c(x,s)\ dx \right)^2\ ds \le C_1(T)\ \left(a^{-1} + \omega(a)\right). \tag{3.3}$$

In the same way we infer from (2.12), (2.15), (2.16) and (3.1) that

$$\int_0^T \left(\int_0^\infty r(x)\ c(x,s)\ dx \right)^2\ ds \le C_{12}(T), \tag{3.4}$$

$$\sup_{t\in[0,T]} \|c(t)\| \le C_{12}(T), \tag{3.5}$$

$$\int_0^\infty x\ c(x,t)\ dx \le \int_0^\infty x\ c_0(x)\ dx, \quad t\in[0,T]. \tag{3.6}$$

A first consequence of (1.3), (1.4), (1.8), (3.4), (3.5) and the Fubini theorem is that

$$\left\{ \begin{array}{l} (x,y,s)\mapsto \phi(x,y)\ c(x,s)\ c(y,s) \in L^1((0,+\infty)^2\times(0,T)), \\ \\ (x,y,s)\mapsto \psi(y,x)\ c(y,s) \in L^1((0,+\infty)^2\times(0,T)), \end{array} \right. \tag{3.7}$$

We now check that the function c is indeed a solution to (1.1)-(1.2) in the sense of Definition 1.1. For that purpose consider a function $g\in L^\infty(0,+\infty)$ with $|g|_{L^\infty}\le 1$ and $t\in(0,+\infty)$. Owing to (2.7) and (3.1) we have

$$\begin{aligned} \lim_{n\to+\infty} \int_0^\infty & (c^n(x,t)-c_0^n(x))\ g(x)\ dx \\ & = \int_0^\infty (c(x,t)-c_0(x))\ g(x)\ dx. \end{aligned} \tag{3.8}$$

Next consider $a\in(1,+\infty)$. For $n\ge1$ and $s\in(0,t)$ we put

$$\begin{aligned} K_{1,n}(a,s) &= \int_0^a \int_0^a \phi_n(x,y)\ \tilde{g}(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdy, \\ K_{2,n}(a,s) &= 2\int_0^a \int_a^\infty \phi_n(x,y)\ \tilde{g}(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdy, \\ K_{3,n}(a,s) &= \int_a^\infty \int_a^\infty \phi_n(x,y)\ \tilde{g}(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdy, \\ F_{1,n}(a,s) &= \int_0^a c^n(x,s) \int_0^x \psi_n(x,y)\ \left(g(y)-\frac{y}{x}\ g(x)\right)\ dy\ dx, \\ F_{2,n}(a,s) &= \int_a^\infty c^n(x,s) \int_0^x \psi_n(x,y)\ \left(g(y)-\frac{y}{x}\ g(x)\right)\ dy\ dx, \end{aligned}$$

and

$$\begin{aligned} K_1(a,s) &= \int_0^a \int_0^a \phi(x,y)\ \tilde{g}(x,y)\ c(x,s)\ c(y,s)\ dxdy, \\ K_2(a,s) &= 2\int_0^a \int_a^\infty \phi(x,y)\ \tilde{g}(x,y)\ c(x,s)\ c(y,s)\ dxdy, \\ K_3(a,s) &= \int_a^\infty \int_a^\infty \phi(x,y)\ \tilde{g}(x,y)\ c(x,s)\ c(y,s)\ dxdy, \\ F_1(a,s) &= \int_0^a c(x,s) \int_0^x \psi(x,y)\ \left(g(y)-\frac{y}{x}\ g(x)\right)\ dy\ dx, \\ F_2(a,s) &= \int_a^\infty c(x,s) \int_0^x \psi(x,y)\ \left(g(y)-\frac{y}{x}\ g(x)\right)\ dy\ dx, \end{aligned}$$

where $\tilde{g}$ is defined by (2.11).
For $n \geq a$ we have $\phi_n \equiv \phi$ in $[0,a] \times [0,a]$ and it follows from Lemma 2.9 and (3.1) that for each $s \in (0,t)$ there holds

$$\lim_{n \to +\infty} K_{1,n}(a,s) = K_1(a,s).$$

The above inequality, (2.15) and the Lebesgue dominated convergence theorem then entail

$$\lim_{n \to +\infty} \int_0^t K_{1,n}(a,s)\ ds = \int_0^t K_1(a,s)\ ds. \tag{3.9}$$

It next follows from (2.3), (2.4), (2.14) and (2.16) that

$$\begin{aligned} &\int_0^t |K_{2,n}(a,s) + K_{3,n}(a,s)|\ ds \\ &\leq 9\ (1+A)\ \int_0^t \left(\int_0^\infty r_n(x)\ c^n(x,s)\ dx \right)\ \left(\int_a^\infty r_n(x)\ c^n(x,s)\ dx \right)\ ds \end{aligned}$$

$$\int_0^t |K_{2,n}(a,s) + K_{3,n}(a,s)|\ ds \leq C_{13}(t)\ \left(a^{-1} + \omega(a)\right). \tag{3.10}$$

Similarly it follows from (1.3), (1.4), (3.3) and (3.4) that

$$\int_0^t |K_2(a,s) + K_3(a,s)|\ ds \leq C_{14}(t)\ \left(a^{-1} + \omega(a)\right). \tag{3.11}$$

Therefore by (3.9)-(3.11) we have

$$\limsup_{n \to +\infty} \left| \int_0^t \sum_{i=1}^3 \left(K_{i,n}(a,s) - K_i(a,s)\right)\ ds \right| \leq C_{15}(t)\ \left(a^{-1} + \omega(a)\right).$$

At this point notice that $K_{1,n}(a,s) + K_{2,n}(a,s) + K_{3,n}(a,s)$ and $K_1(a,s) + K_2(a,s) + K_3(a,s)$ do not depend on $a \in (1,+\infty)$. The above inequality being valid for every $a \in (1,+\infty)$ we finally obtain, thanks to (1.7)

$$\begin{aligned} \lim_{n \to +\infty} &\int_0^t \int_0^\infty \int_0^\infty \phi_n(x,y)\ \tilde{g}(x,y)\ c^n(x,s)\ c^n(y,s)\ dxdyds \\ &= \int_0^t \int_0^\infty \int_0^\infty \phi(x,y)\ \tilde{g}(x,y)\ c(x,s)\ c(y,s)\ dxdyds. \end{aligned} \tag{3.12}$$

Next, for $n \geq a$ we have $\psi_n \equiv \psi$ in $[0,a] \times [0,a]$ and

$$\left| \int_0^x \psi(x,y)\ \left(g(y) - \frac{y}{x}\ g(x)\right)\ dy \right| \leq 2\ |\psi|_{L^\infty((0,a)\times(0,a))} \quad \text{a.e. in} \quad (0,a).$$

It then follows from (3.1) that

$$\lim_{n \to +\infty} \int_0^t F_{1,n}(a,s)\ ds = \int_0^t F_1(a,s)\ ds. \tag{3.13}$$

We also infer from (2.5), (2.12) and (2.16) that

$$\begin{aligned} \int_0^t |F_{2,n}(a,s)|\ ds \quad &\leq \quad 2\ \int_0^t \int_a^\infty c^n(x,s)\ \int_0^x \psi_n(x,y)\ dy\ dxds \\ &\leq 2\ \omega(a)\ \int_0^t \int_0^\infty (x + r_n(x))\ c^n(x,s)\ dxds \end{aligned}$$

$$
\int_0^t |F_{2,n}(a,s)| \ ds \le C_{16}(t)\ \omega(a). \tag{3.14}
$$

Proceeding in a similar way we deduce from (1.8), (3.4) and (3.5) that

$$
\int_0^t |F_2(a,s)| \ ds \le C_{17}(t)\ \omega(a). \tag{3.15}
$$

Combining (3.13)-(3.15) yields

$$
\limsup_{n\to+\infty} \left| \int_0^t \sum_{i=1}^2 (F_{i,n}(a,s) - F_i(a,s)) \ ds \right| \le C_{18}(t)\ \omega(a).
$$

As $F_{1,n}(a,s) + F_{2,n}(a,s)$ and $F_1(a,s) + F_2(a,s)$ do not depend on $a \in (1,+\infty)$ we may let $a \to +\infty$ in the above inequality and obtain

$$
\begin{aligned}
&\lim_{n\to+\infty} \int_0^t \int_0^\infty c^n(x,s) \ \int_0^x \psi_n(x,y) \ \left(g(y) - \frac{y}{x}\ g(x)\right) \ dy \ dxds \\
&= \int_0^t \int_0^\infty c(x,s) \ \int_0^x \psi(x,y) \ \left(g(y) - \frac{y}{x}\ g(x)\right) \ dy \ dxds.
\end{aligned} \tag{3.16}
$$

We now let $n \to +\infty$ in (2.10) and use (3.8), (3.12) and (3.16) to obtain that c satisfies

$$
\begin{aligned}
&\int_0^\infty g(x) \ (c(x,t) - c_0(x)) \ dx \\
&= \frac{1}{2} \int_0^t \int_0^\infty \int_0^\infty \phi(x,y)\ \tilde{g}(x,y)\ c(x,s)\ c(y,s)\ dxdyds \\
&+ \int_0^t \int_0^\infty c(x,s) \ \int_0^x \psi(x,y) \ \left(g(y) - \frac{y}{x}\ g(x)\right) \ dy \ dxds,
\end{aligned} \tag{3.17}
$$

the function $\tilde{g}$ still being defined by (2.11). But on account of (3.7) the Fubini theorem allows us to rewrite the first term of the right-hand side of (3.17) as

$$
\begin{aligned}
\int_0^\infty g(x) \ \int_0^t \Bigg(\frac{1}{2} \int_0^x & \phi(x-y,y)\ c(x-y,s)\ c(y,s)\ dy \\
& - c(x,s) \ \int_0^\infty \phi(x,y)\ c(y,s)\ dy \Bigg) \ dsdx
\end{aligned}
$$

and the second term on the right-hand side of (3.17) as

$$
\begin{aligned}
\int_0^\infty g(x) \ \int_0^t \Bigg(\int_x^\infty & \psi(y,x)\ c(y,s)\ dy \\
& - c(x,s) \ \int_0^x \frac{y}{x}\ \psi(x,y)\ dy \Bigg) \ dsdx.
\end{aligned}
$$

Therefore (3.17) reads

$$\begin{aligned}
&\int_0^\infty g(x)\ (c(x,t)-c_0(x))\ dx \\
&= \int_0^\infty g(x)\ \int_0^t \left(\frac{1}{2}\ \int_0^x \phi(x-y,y)\ c(x-y,s)\ c(y,s)\ dy \right. \\
&\qquad\qquad - c(x,s)\ \int_0^x \frac{y}{x}\ \psi(x,y)\ dy \\
&\qquad\qquad - c(x,s)\ \int_0^\infty \phi(x,y)\ c(y,s)\ dy \\
&\qquad\qquad \left. + \int_x^\infty \psi(y,x)\ c(y,s)\ dy \right)\ dsdx.
\end{aligned}$$

This equality being valid for every $g \in L^\infty(0,+\infty)$ we have shown that c fulfills Definition 1.1 (iii). Recalling (3.5), (3.6) and (3.7) we see that c is a solution to (1.1) on $[0,+\infty)$ in the sense of Definition 1.1 with $c(0)=c_0$ and satisfying (1.16). The proof of Theorem 1.2 is therefore complete.

Remark 3.1. If $r(x) \le C\ (1+x)^\gamma$ for some $\gamma \in [0,1/2)$ and $\psi(x,y) = F(y,x-y)$ for some symmetric and continuous function F satisfying

$$F(x,y) \le C\ (1+x+y)^\beta, \quad \beta \in [0,1),$$

Theorem 1.2 follows from [17, Theorem 4.2]. Our result thus improves [17, Theorem 4.2] along the direction of coagulation kernels growing faster than $(1+x)^{1/2}(1+y)^{1/2}$.

4. Gelation in the pure coagulation model

Throughout this section we assume that $\psi \equiv 0$ and that ϕ satisfies (1.3)-(1.4) and (1.18). Also c_0 is a function in X^+ and we denote by c a solution to (1.1) on $[0,+\infty)$ with initial datum c_0 (recall that such a solution exists by Theorem 1.2). We then put

$$\varrho(t) = \int_0^\infty x\ c(x,t)\ dx, \quad t \in \mathbb{R}_+. \tag{4.1}$$

From Definition 1.1 we deduce the following identity.

Lemma 4.1. *Let $g \in L^\infty(0,+\infty)$. For $t \in (0,+\infty)$ and $s \in [0,t)$ there holds*

$$\begin{aligned}
&\int_0^\infty g(x)\ (c(x,t)-c(x,s))\ dx \\
&= \frac{1}{2}\ \int_s^t \int_0^\infty \int_0^\infty \phi(x,y)\ \tilde{g}(x,y)\ c(x,\sigma)\ c(y,\sigma)\ dxdyd\sigma,
\end{aligned} \tag{4.2}$$

where $\tilde{g}$ is defined by (2.11).

As a consequence of Lemma 4.1 we obtain that ϱ is a non-increasing function.

Lemma 4.2. *For $t \in (0,+\infty)$ and $s \in [0,t)$ there holds*

$$\varrho(t) \le \varrho(s). \tag{4.3}$$

Next let $w\ : (0,+\infty) \to \mathbb{R}_+$ be a non-negative and non-increasing function such that

$$w(x+y) \le w(x) + w(y), \quad (x,y) \in (0,+\infty)^2. \tag{4.4}$$

Then, if c_0 enjoys the additional integrability property

$$\int_0^\infty w(x)\ c_0(x)\ dx < \infty, \tag{4.5}$$

so does $c(.,t)$ and

$$\int_0^\infty w(x)\ c(x,t)\ dx \le \int_0^\infty w(x)\ c(x,s)\ dx. \tag{4.6}$$

Proof. Let $M \in (0,+\infty)$ and take $g(x) = x\ \mathbf{1}_{[0,M]}(x)$ in (4.2). As

$$\tilde{g}(x,y) \le x + y - x - y = 0 \quad \text{if} \quad (x,y) \in [0,M] \times [0,M],$$
$$\tilde{g}(x,y) \le -g(x) - g(y) \le 0 \quad \text{if} \quad x \ge M \quad \text{or} \quad y \ge M,$$

we obtain

$$\int_0^M x\ c(x,t)\ dx \le \int_0^M x\ c(x,s)\ dx.$$

The above inequality and Definition 1.1 (i) then entail (4.3) by letting $M \to +\infty$.
Next the function w being as in Lemma 4.2 we define

$$w_\varepsilon(x) = \min\left(w(\varepsilon), w(x)\right), \quad x \in (0,+\infty),$$

for $\varepsilon \in (0,1)$. By (4.4) the function w_ε is a non-negative and non-increasing bounded function in $(0,+\infty)$ and

$$w_\varepsilon(x+y) \le w_\varepsilon(x) + w_\varepsilon(y), \quad (x,y) \in (0,+\infty)^2. \tag{4.7}$$

Also for each $x \in (0,+\infty)$ there holds

$$\lim_{\varepsilon \to 0} w_\varepsilon(x) = w(x).$$

We may then take $g \equiv w_\varepsilon$ in (4.2) and obtain, thanks to (4.7),

$$\int_0^\infty w_\varepsilon(x)\ c(x,t)\ dx \le \int_0^\infty w_\varepsilon(x)\ c(x,s)\ dx. \tag{4.8}$$

We first take $s = 0$ in (4.8) and let $\varepsilon \to 0$. The monotone convergence theorem and (4.5) entail

$$\int_0^\infty w(x)\ c(x,t)\ dx < \infty.$$

We may then let $\varepsilon \to 0$ in (4.8) and obtain (4.6). □

After this preparation we are ready to prove Theorem 1.3. Let $s \in (0,+\infty)$ and $t \in (s,+\infty)$. We take $g \equiv 1$ in (4.2) and use (1.18) to obtain

$$\int_s^t |\varrho(\sigma)|^2\ d\sigma \le \frac{2}{R^2} \int_0^\infty c(x,s)\ dx. \tag{4.9}$$

– Proof of (1.19). It follows from (4.3) and (4.9) with $s = 0$ that

$$t\ \varrho(t)^2 \le \frac{2}{R^2} \int_0^\infty c_0(x)\ dx,$$

hence (1.19).

– Proof of (1.21). Here the function c_0 enjoys the additional property (1.20). Clearly the function $w(x) = x^{-q}$ satisfies the assumptions of Lemma 4.2. Consequently,

$$\int_0^\infty x^{-q}\ c(x,s)\ dx \leq I_q = \int_0^\infty x^{-q}\ c_0(x)\ dx,$$

which yields, together with the Hölder inequality,

$$\int_0^\infty c(x,s)\ ds \leq \varrho(s)^{q/(q+1)}\ \left(\int_0^\infty x^{-q}\ c(x,s)\ dx\right)^{1/(q+1)}$$

$$\int_0^\infty c(x,s)\ ds \leq \varrho(s)^{q/(q+1)}\ I_q^{1/(q+1)}. \tag{4.10}$$

We then obtain from (4.9) and (4.10) that

$$\int_s^t |\varrho(\sigma)|^2\ d\sigma \leq \frac{2}{R^2}\ I_q^{1/(q+1)}\ \varrho(s)^{q/(q+1)}.$$

As the above inequality is valid for every $t > s$ we finally obtain

$$\int_s^\infty |\varrho(\sigma)|^2\ d\sigma \leq \frac{2}{R^2}\ I_q^{1/(q+1)}\ \varrho(s)^{q/(q+1)}, \quad s \in \mathbb{R}_+. \tag{4.11}$$

Introducing

$$E(s) = \varrho(s)^{q/(q+1)}, \quad m = \frac{q+2}{q} \quad \text{and} \quad \frac{1}{\kappa} = \frac{2}{R^2}\ I_q^{1/(q+1)},$$

(4.11) reads

$$\int_s^\infty E(\sigma)^{m+1}\ d\sigma \leq \frac{1}{\kappa}\ E(s), \quad s \in \mathbb{R}_+,$$

and (1.21) follows from [12, Theorem 9.1].

– Proof of (1.23). We take $w = \mathbf{1}_{[0,\delta]}$ in Lemma 4.2. Since $c_0 \equiv 0$ on $(0,\delta)$ we obtain

$$\int_0^\delta c(x,\sigma)\ dx = 0$$

for every $\sigma \in \mathbb{R}_+$, hence

$$c(x,\sigma) = 0 \ \text{ a.e. in } \ (0,\delta), \quad \sigma \in \mathbb{R}_+. \tag{4.12}$$

Recalling (4.9) it follows from (4.12) that

$$\int_s^t |\varrho(\sigma)|^2\ d\sigma \leq \frac{2}{R^2\ \delta}\ \varrho(s).$$

The above inequality being valid for every $t > s$ we have in fact

$$\int_s^\infty |\varrho(\sigma)|^2\ d\sigma \leq \frac{2}{R^2\ \delta}\ \varrho(s), \quad s \in \mathbb{R}_+.$$

We then use once more [12, Theorem 9.1] to conclude that (1.23) holds true.

Remark 4.3. Theorem 1.3 gives some upper bound on the gelation time T_{gel} defined by

$$T_{gel} = \inf\{t \in \mathbb{R}_+,\ \varrho(t) < \varrho(0)\} < \infty.$$

Indeed it follows from (1.19) that

$$T_{gel} \leq \frac{2\ |c_0|_{L^1}}{R^2\ \varrho(0)^2}.$$

This upper bound is however not optimal [8].

5. Gelation in the coagulation-fragmentation model

Let ϕ and ψ be coagulation and fragmentation kernels satisfying, respectively, (1.3)-(1.4), (1.18) and (1.5)-(1.9), (1.25). We next consider $c_0 \in X^+$ and denote by c a solution to (1.1) on $[0,+\infty)$ with initial datum c_0. Similarly as in the previous section we deduce from Definition 1.1 the following identity.

Lemma 5.1. *Let $g \in L^\infty(0,+\infty)$. For $t \in (0,+\infty)$ and $s \in [0,t)$ there holds*

$$\begin{aligned}(5.1)\qquad &\int_0^\infty g(x)\ \left(c(x,t)-c(x,s)\right)\ dx \\ &= \frac{1}{2}\int_s^t \int_0^\infty \int_0^\infty \phi(x,y)\ \tilde{g}(x,y)\ c(x,\sigma)\ c(y,\sigma)\ dxdyd\sigma \\ &+ \int_s^t \int_0^\infty c(x,\sigma)\ \int_0^x \psi(x,y)\ \left(g(y)-\frac{y}{x}\ g(x)\right)\ dy\ dxd\sigma,\end{aligned}$$

where $\tilde{g}$ is defined by (2.11).

Putting

$$(5.2)\qquad \varrho(t) = \int_0^\infty x\ c(x,t)\ dx, \quad t \in \mathbb{R}_+,$$

we obtain the following estimate on ϱ.

Lemma 5.2. *For $t \in (0,+\infty)$ and $s \in [0,t)$ there holds*

$$(5.3)\qquad \int_s^t |\varrho(\sigma)|^2\ d\sigma \le \frac{2}{R^2}\ |c(s)|_{L^1} + \frac{2\ \Gamma}{R^2}\ \int_s^t \varrho(\sigma)\ d\sigma,$$

$$(5.4)\qquad \varrho(t) \le \varrho(s).$$

Proof. We take $g \equiv 1$ in (5.1) and obtain

$$\begin{aligned}&\frac{1}{2}\int_s^t \int_0^\infty \int_0^\infty \phi(x,y)\ c(x,\sigma)\ c(y,\sigma)\ dxdyd\sigma \\ &\le \int_0^\infty c(x,s)\ dx + \int_s^t \int_0^\infty c(x,\sigma)\ \int_0^x \psi(x,y)\ \left(1-\frac{y}{x}\right)\ dy\ dxd\sigma.\end{aligned}$$

Then (5.3) follows from (1.3), (1.4), (1.18), (1.25) and the above inequality.
Next, let $M \in (0,+\infty)$ and take $g(x) = \min(x,M)$ in (5.1). As

$$\begin{aligned}\tilde{g}(x,y) &\le x+y-x-y = 0 \quad \text{if} \quad (x,y) \in [0,M]\times[0,M], \\ \tilde{g}(x,y) &\le M - g(x) - g(y) \le 0 \quad \text{if} \quad x \ge M \quad \text{or} \quad y \ge M,\end{aligned}$$

it follows from (1.25) that

$$\begin{aligned}
&\int_0^\infty \min(x,M)\ (c(x,t)-c(x,s))\ dx \\
&\le \int_s^t \int_M^\infty c(x,\sigma)\ \int_0^x \psi(x,y)\ \left(g(y)-\frac{M\ y}{x}\right)\ dy\ dxd\sigma \\
&\le M\ \int_s^t \int_M^\infty c(x,\sigma)\ \int_0^x \psi(x,y)\ \left(1-\frac{y}{x}\right)\ dy\ dxd\sigma \\
&\le \Gamma\ \int_s^t \int_M^\infty x\ c(x,\sigma)\ dxd\sigma.
\end{aligned}$$

As $c \in L^\infty(0,t;X)$ we may let $M \to +\infty$ in the above inequality and obtain (5.4). □

We are now in a position to prove Proposition 1.4. For $t \in \mathbb{R}_+$ we put

$$M(t) = \int_0^t \varrho(s)\ ds.$$

Let $t \in (0,+\infty)$. On the one hand it follows from the Jensen inequality that

$$M(t)^2 \le t\ \int_0^t |\varrho(s)|^2\ ds. \tag{5.5}$$

On the other hand (5.3) entails

$$\int_0^t |\varrho(s)|^2\ ds \le \frac{2}{R^2}\ |c_0|_{L^1} + \frac{2\ \Gamma}{R^2}\ M(t). \tag{5.6}$$

Combining (5.5) and (5.6) yields

$$M(t)^2 - \frac{2\ \Gamma\ t}{R^2}\ M(t) - \frac{2\ t}{R^2}\ |c_0|_{L^1} \le 0,$$

hence

$$M(t) \le \frac{\Gamma\ t}{R^2}\ \left(1+\left(1+\frac{2\ R^2}{\Gamma^2\ t}\ |c_0|_{L^1}\right)^{1/2}\right).$$

Recalling that ϱ is non-increasing by (5.4) we obtain

$$\varrho(t) \le \frac{\Gamma}{R^2}\ \left(1+\left(1+\frac{2\ R^2}{\Gamma^2\ t}\ |c_0|_{L^1}\right)^{1/2}\right). \tag{5.7}$$

As the limit as $t \to +\infty$ of the right-hand side of (5.7) is $2\ \Gamma/R^2$ Proposition 1.4 follows from (1.26) and (5.7).

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