Piecewise Linear Wavelet Collocation on Triangular Grids, Approximation of the Boundary Manifold and Quadrature

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Abstract

In this paper we consider a piecewise linear collocation method for the solution of a pseudo-differential equations of order $\mathbf{r} = 0, -1$ over a closed and smooth boundary manifold. The trial space is the space of all continuous and piecewise linear functions defined over a uniform triangular grid and the collocation points are the grid points. For the wavelet basis in the trial space we choose the three-point hierarchical basis together with a slight modification near the boundary points of the global patches of parametrization. We choose three, four, and six term linear combinations of Dirac delta functionals as wavelet basis in the space of test functionals. Though not all wavelets have vanishing moments, we derive the usual compression results, i.e. we prove that, for N degrees of freedom, the fully populated stiffness matrix of N^2 entries can be approximated by a sparse matrix with no more than $O(N[\log N]^{2.25})$ non-zero entries. The main topic of the present paper, however, is to show that the parametrization can be approximated by low order piecewise polynomial interpolation and that the integrals in the stiffness matrix can be computed by quadrature, where the quadrature rules are combinations of product integration applied to non analytic factors of the integrand and of high order Gauß rules applied to the analytic parts. The whole algorithm for the assembling of the matrix requires no more than $O(N[\log N]^{4.25})$ arithmetic operations, and the error of the collocation approximation, including the compression, the approximative parametrization, and the quadratures, is less than $O(N^{-1}[\log N]^2)$. Note that, in contrast to well-known algorithms by v.Petersdorff, Schwab, and Schneider, only a finite degree of smoothness is required.

1 Introduction

It is a well-known fact that usual finite element discretizations of linear integral equations (e.g. of boundary integral equations) lead to systems of linear equations with fully populated matrices. Thus, even an iterative solution method requires a huge number of arithmetic operations and a large storage capacity. In order to improve these finite element approaches for integral equations, several algorithms have been developed. One of these consists in employing wavelet bases of the finite element spaces. The basic idea goes back to Beylkin, Coifman, and Rokhlin [3], and has been thoroughly investigated by Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [13, 14, 33, 32, 31, 44] (cf. also the contributions by Alpert, Harten, Yad-Shalom, and the author [1, 22, 39]). In the present paper, we shall apply the wavelet technique to the piecewise linear collocation of two-dimensional boundary integral equations of order $\mathbf{r} = 0$ and $\mathbf{r} = -1$ corresponding to three-dimensional boundary value problems.

First we shall present a new simple biorthogonal wavelet basis (compare the definition of univariate biorthogonal wavelets by Cohen, Daubechies, and Feauveau [9]) of continuous piecewise linear functions defined over triangular grids. The grids will be supposed to be uniform refinements of a coarse initial triangulation, and the basis will be the system of three-point hierarchical basis functions, i.e. each basis function will be a linear combination of no more than three finite element functions defined over the corresponding level of a grid hierarchy. If the function is located in the interior of a triangular patch of the initial triangulation, then it will have two vanishing moments. If the basis function intersects the boundary of the coarse triangles corresponding to the initial triangulation, then no vanishing moment condition will be fulfilled. We shall prove that this basis is a Riesz basis in the Sobolev space of order s over the boundary manifold for $-0.5 < s < 1.5$ (compare the general approach by Dahmen [11]). In comparison to other bases of continuous wavelet functions our basis functions will have a rather small support, and we believe that this property is essential for the wavelet algorithm. Indeed, small supports lead to better compression rates, especially, for lower levels and to faster quadrature algorithms for the assembling of the stiffness matrix. Similar systems of hierarchical three-point functions have been analyzed before for the real plane and for manifolds by Junkherr, Stevenson, Lorentz, and Oswald [24, 46, 27]. For manifolds, however, the constructions are either more involved or the range of Sobolev orders for the Riesz property is smaller. In comparison to tensor product wavelets over rectangular partitions (cf. the almost analogous construction in [38]), we believe that triangular grids are easier to adapt to general geometries. Note, however, that the general construction of tensor product wavelets by Canuto, Dahmen, Schneider, Tabacco, and Urban [15, 16, 17, 5, 6, 7] offer interesting additional features, which seem to be useful, especially, for integral operators of different order and Galerkin discretizations. Piecewise linear and continuous wavelet functions over triangular grids have been constructed by Dahmen and Stevenson [18]. Note that, though these wavelets have larger supports, the corresponding wavelet transforms are fast and the Riesz property is satisfied for $-1.5 < s < 1.5$. A last alternative for the basis in the trial space is provided by discontinuous wavelet functions. These so called multiwavelets are easy to construct. They have been introduced by Alpert [1] and generalized to two-dimensional manifolds by v.Petersdorff, Schneider, and Schwab [30]. The corresponding spaces lead to larger systems of equations, and it seems to be an open question whether the increase in the degrees of freedom can be compensated by higher compression rates and better constants in the error estimates.

For the basis in the test space spanned by Dirac delta functionals, we shall take the usual test functionals which can be considered at as scaled versions of difference formulas (cf. the wavelet collocation methods by Dahmen, Prößdorf, Schneider, Harten, Yad-Shalom, and the author [14, 22, 39, 38, 40]). Applying the wavelet basis functions of the trial and test space, we shall obtain the well-known compression results for trial wavelets with vanishing moments due to Dahmen, v.Petersdorff, Prößdorf, Schneider, and Schwab [14, 33, 44]. The compression for trial functions without vanishing moments is the same as in [38] (cf. also the univariate analogue for the Galerkin method treated in [33, 4]). In particular, to compute an approximate collocation solution with optimal asymptotic order of convergence, it is sufficient to compute and store $O(N[\log N]^{1.75})$ entries of the fully populated $N \times N$ stiffness matrix. Here N stands for the number of degrees of freedom.

In general, the stiffness matrix cannot be computed exactly. This is the case, for instance, if the boundary manifold is given by a discrete set of points, only, or if no analytic formula is available to integrate the kernel and trial function. Therefore, we shall consider an algorithm for the approximation of the boundary surface and for the quadrature of the integrals. We emphasize that this is the most time consuming and the most difficult part of the wavelet method. To set up the stiffness matrix, we shall proceed as follows. Depending on the test functional, we shall define an appropriate partition of the supports of the trial basis functions. Over these subdomains we shall replace the parametrization of the boundary manifold by a quadratic or cubic interpolation. We shall assume that the kernel function is a finite sum of terms $(P,Q) \mapsto k(P,Q)p(P-Q)/|P-Q|^\alpha$, where $k(P,Q)$ is $2-\mathbf{r}$ times continuously differentiable and where $p(P-Q)$ is a polynomial with constant coefficients. For the part $k(P,Q)$ of the kernel function, we shall apply a low order product integration rule with the weight function chosen as the product of $Q \mapsto p(P-Q)/|P-Q|^\alpha$ times trial wavelet. The quadrature weights of the product rule, i.e., the integrals over the function $p(P-Q)/|P-Q|^\alpha$ times trial wavelet will be computed by Gauß rules of order less than $O(\log N)$. This way and using well-known ideas to treat singular integrals, we shall arrive at a fully discretized wavelet algorithm with $O(N[\log N]^{4.25})$ arithmetic operations to compute $O(N[\log N]^{2.25})$ entries of the stiffness matrix. Assuming that the collocation is stable, the asymptotic error of the exact collocation solution is known to be less than $O(N^{-(2-\mathbf{r})/2})$ which is optimal for piecewise linear trial spaces. The fully discrete wavelet algorithm will be shown to be stable, too, and to be convergent with an almost optimal error less than $O(N^{-(2-\mathbf{r})/2}[\log N]^2)$ for $\mathbf{r} = 0$ and less than $O(N^{-(2-\mathbf{r})/2}[\log N]^{1.625})$ for $\mathbf{r} = -1$.

Notice that alternative quadrature algorithms have been considered by Beylkin, Coifman, Rokhlin [3] for integral operators with smooth kernels and by v.Petersdorff, Schwab, and Schneider [33, 44] (cf. also the numerical implementation by Lage and Schwab [26]) for boundary integral operators with Green kernels over piecewise analytic boundaries. To our knowledge, the fully discrete algorithm of the present paper is the first which applies to boundary integral equations over surfaces with finite degree of smoothness. In fact, the required degree of smoothness for the geometry will be equal to the convergence order $2-\mathbf{r}$ increased by one, i.e., the same as for the conventional collocation algorithm. Moreover, beside the usual singular main part $p(P-Q)/|P-Q|^\alpha$ of Green kernels, the kernel function of the integral operator will be allowed to have an additional factor $k(P,Q)$ of finite smoothness degree $2-\mathbf{r}$. In the proof of corresponding error estimates, we shall show that the techniques developed for the compression algorithm apply to the analysis of the discretization as well. The only thing to do is to replace the decay properties in the matrix entries due to the vanishing moments of the trial functions and the norm estimates due to the smoothness of the solution by error estimates of the approximate parameter mappings and of the quadrature rules, respectively.

The powers of logarithms in the asymptotic convergence and complexity estimates are, of course, not optimal. Using the refined compression technique of Schneider [44], choosing wavelet basis functions with more vanishing moments, and applying higher order quadrature rules, the logarithmic powers can be dropped or, at least, their exponents can be reduced. Note, however, that the application of higher order moment conditions and quadratures requires additional smoothness assumptions. Furthermore, we believe that a simple algorithm like the one in the present paper is often more efficient than an asymptotically optimal method since the number of degrees of freedom does not tend to infinity in realistic numerical computations.

The plan of the paper is as follows. In Sect. 2 we shall describe the boundary manifold, the integral equation, and the conventional piecewise linear collocation method. We shall introduce the three-point hierarchical wavelet functions of the piecewise linear trial space, the test wavelet functionals, and the corresponding compression algorithm in Sect. 3. Sect. 4 will be devoted to the description of the interpolation of the parameter mappings and to the quadrature algorithm. All proofs will be deferred to Sects. 5 and 6. In particular, in Sect. 5 we shall prove the Riesz property of the wavelet basis, count the numbers of entries in the compressed matrix, and derive the compression estimates and preconditioners. Finally, the discretization including the approximation of the parametrizations and of the integration will be analyzed in Sect. 6.

2 The Piecewise Linear Collocation Method

2.1 The Manifold

We suppose that the integral equation to be solved is given on a closed boundary manifold $\Gamma \subset \mathbb{R}^3$ with finite degree of smoothness. More exactly, we assume that Γ is the union of m_Γ triangular patches Γ_m, i.e.

$$\begin{aligned} \Gamma &= \cup_{m=1}^{m_\Gamma} \Gamma_m, \quad \Gamma_m := \kappa_m(T), \\ T &:= \Big\{(s,t) \in \mathbb{R}^2 : \ 0 \le s \le 1, \ 0 \le t \le \min\{s, 1-s\}\Big\}. \end{aligned} \tag{2.1}$$

Here the κ_m denote parametrization mappings from the standard triangle T to the manifold Γ. We assume that the κ_m extend to mappings from a small neighbourhood of $T \subseteq \mathbb{R}^2$ to Γ and that these extensions are d_Γ times continuously differentiable. Here d_Γ is an integer which is assumed to be greater or equal to three when dealing with zero order operators and greater or equal to four when dealing with operators of order $\mathbf{r} = -1$. Further we suppose that the intersection of two patches Γ_m and $\Gamma_{m'}$ is either empty or a corner point for both patches or a whole side for Γ_m and $\Gamma_{m'}$. In the last case we assume that the representations

$$\begin{aligned} \Gamma_m \cap \Gamma_{m'} &= \Big\{\kappa_m\Big(c_1 + \lambda(c_2 - c_1)\Big) : \ 0 \le \lambda \le 1\Big\}, \\ \Gamma_m \cap \Gamma_{m'} &= \Big\{\kappa_{m'}\Big(c_1' + \lambda(c_2' - c_1')\Big) : \ 0 \le \lambda \le 1\Big\} \end{aligned}$$

satisfy the condition

$$\kappa_m\Big(c_1 + \lambda(c_2 - c_1)\Big) = \kappa_{m'}\Big(c_1' + \lambda(c_2' - c_1')\Big), \quad 0 \le \lambda \le 1. \tag{2.2}$$

Note that, for the numerical method, the parameter mappings κ_m need not to be given for all points of T. We shall use only the values of κ_m at the points of a uniform grid over the triangle T.

In the construction of the wavelet basis the numbering of the patches will play a crucial role since the basis functions will first be defined on Γ_1, then on Γ_2, and so on. To secure stability of the so constructed basis, we even need an assumption connected with the numbering. We suppose that, if the corner P of a patch Γ_m is contained in the union $\cup_{m'=1}^{m-1}\Gamma_{m'}$ of the preceding patches, then at least one of the sides of Γ_m ending at P is contained in $\cup_{m'=1}^{m-1}\Gamma_{m'}$. It is not hard to see that, for a boundary manifold Γ homeomorphic to the sphere and for any fixed triangulation, there always exists a numbering of the triangular patches which fulfills the assumption. However, the numbering assumption seems to be a severe topological restriction. It seems to us that, for boundaries homeomorphic to the torus a construction of similar basis systems is possible only if the triangular patches are combined with rectangular ones and if the piecewise linear functions over the triangular patches are combined with piecewise bilinear functions over the rectangular patches (cf. [38]).

To secure stability of the wavelet construction, we need a final assumption on the parametrizations. For any $m = 2, \dots, m_\Gamma - 1$, we suppose that, if one of the two "shorter" sides $\kappa_m(\{(s,s) : \ 0 \le s \le 0.5\})$ and $\kappa_m(\{(s,1-s) : \ 0.5 \le s \le 1\})$ is contained in $\cup_{m'=1}^{m-1}\Gamma_m$, then the other must also be contained in $\cup_{m'=1}^{m-1}\Gamma_m$. This last assumption can always be satisfied if the parameter mappings κ_m are replaced by a composition of κ_m with a suitable affine automorphism of T.

Since the manifold is at least thrice continuously differentiable, for each $Q \in \Gamma$, there exists a unit vector n_Q normal to Γ at Q and pointing into the exterior domain bounded by Γ. The Sobolev spaces $H^s(\Gamma)$ over Γ can be defined in the usual way. We define the space $H^s(\Gamma_m)$ over Γ_m as the image of the Sobolev space over T, i.e.

$$H^s(\Gamma_m) \ := \ \{f : \ f \circ \kappa_m \in H^s(T)\}.$$

Consequently, we get

$$\begin{aligned} H^s(\Gamma) \quad &= \quad \left\{ (f_m)_{m=1}^{m_\Gamma} \in \bigoplus_{m=1}^{m_\Gamma} H^s(\Gamma_m) : \ f_m|_{\Gamma_m \cap \Gamma_{m'}} = f_{m'}|_{\Gamma_m \cap \Gamma_{m'}} \right\}, \quad \frac{1}{2} < s < \frac{3}{2}, \\ H^s(\Gamma) \quad &= \quad \bigoplus_{m=1}^{m_\Gamma} H^s(\Gamma_m), \quad -\frac{1}{2} < s < \frac{1}{2}, \qquad (2.3) \\ \|f\|_{H^s(\Gamma)} \quad &\sim \quad \sqrt{\sum_{m=1}^{m_\Gamma} \|f|_{\Gamma_m}\|^2_{H^s(\Gamma_m)}}\,, \quad f \in H^s(\Gamma), \ -\frac{1}{2} < s < \frac{3}{2}. \end{aligned}$$

Finally, we note that the sphere can serve as a simple example for a boundary manifold fulfilling all assumptions. To get the corresponding parametrization mappings, we inscribe a tetrahedron and take the projections from the midpoint mapping the triangular faces of the tetrahedron onto triangular patches of the sphere. Composing these parametrizations with suitable affine mappings, we arrive at a representation (2.1) for the sphere. The numbering of these four parameter patches can be chosen arbitrarily.

2.2 The Integral Equation

Over Γ we consider a pseudo-differential operator A of order $\mathbf{r} = 0$ or $\mathbf{r} = -1$ mapping $H^{\mathbf{r}/2}$ into $H^{-\mathbf{r}/2}$. We suppose that A is an integral operator of the form $A = K$ for $\mathbf{r} = -1$ and $A = aI + K$ for $\mathbf{r} = 0$, where aI stands for the operator of multiplication by a function a which may be zero, and the integral operator K is defined by

$$Ku(P) \ := \ \int_\Gamma k(P, Q, n_Q) \frac{p(P - Q)}{|P - Q|^\alpha} u(Q) \, \mathrm{d}_Q \Gamma. \qquad (2.4)$$

The function p stands for a homogeneous polynomial of degree $\deg(p)$, the real number α is equal to $\mathbf{r} + 2 + \deg(p)$, and the kernel function k depends on the points $P, Q \in \Gamma$. This function need not to be a restriction to $\Gamma \times \Gamma$ of a function defined on the space $\mathbb{R}^3 \times \mathbb{R}^3$. It may depend for instance on the unit normals n_P and n_Q pointing into the exterior or on any different kind of differentiable vector field over Γ. To simplify the notation, we assume a special dependence and take $k = k(P, Q, n_Q)$ with k defined on at least a neighbourhood of $\{(P, Q, n) : \ P, Q \in \Gamma, \ n = n_Q\} \subset \Gamma \times \Gamma \times \mathbb{R}^3$. If $\mathbf{r} = 0$, then the integrand in (2.4) can be strongly singular and the integral is to be understood in the sense of a Cauchy principal value. To ensure the existence of this principal value, we assume that p is odd, i.e. $p(Q - P) = -p(P - Q)$. Note that in applications we often have a finite sum of integrals of the above type and additional terms of lower order. Only for simplicity of notation we restrict ourselves to the one term of (2.4).

For the operator A including the just defined integral operator K, we assume the continuity of the mapping

$$A: \ H^{s+\mathbf{r}}(\Gamma) \longrightarrow H^s(\Gamma) \tag{2.5}$$

with $s = 0$ and $s = 1.1$ (or $s = 1.1$ replaced by a different s with $1 < s < 1.5$) and the invertibility of (2.5) with $s = 0$. Further, we suppose a finite degree of smoothness, i.e. the function a is supposed to be twice continuously differentiable and the kernel k to be d_k times continuously differentiable. More precisely, for any d_k-th order derivative $\partial_P^{d_k}$ taken with respect to variable $P \in \Gamma$ and for any d_k-th order derivative $\partial_{Q,n}^{d_k}$ taken with respect to the variables $Q \in \Gamma$ and $n \in I\!R^3$, we require that $\partial_P^{d_k} \partial_{Q,n}^{d_k} k(P,Q,n_Q)$ is continuous. The degree of smoothness d_k is supposed to be greater or equal to two for $\mathbf{r} = 0$ and greater or equal to three for $\mathbf{r} = -1$. For an operator A which satisfies all these assumptions, we shall solve the operator equation $Au = v$ with known right-hand side v and unknown u. To get error estimates with optimal order, we finally assume $u \in H^2(\Gamma)$.

Let us consider some examples. For instance, single and double layer potential equations belong to our class of operator equations. Indeed, for the single layer case $A = A_s$ corresponding to Laplace's equation, the order $\mathbf{r}_s$ is -1, and

$$k_s(P,Q,n_Q) := \frac{1}{4\pi}, \quad p_s(P-Q) := 1, \quad \alpha_s = 1.$$

In case of the double layer operator $A = A_d$ we get the order $\mathbf{r}_d = 0$, and the multiplication function $a_d \equiv 0.5$ is constant. The integral operator K_d is the sum of three terms K_d^x, K_d^y, and K_d^z. The first term K_d^x is defined by

$$\begin{aligned} k_d^x(P,Q,n_Q) \ &= \ k_d^x\Big(P,Q,(n_Q^x,n_Q^y,n_Q^z)\Big) \ := \ -\frac{n_Q^x}{4\pi}, \quad \alpha_d \ := \ 3, \\ p_d^x(P-Q) \ &= \ p_d^x\Big((P^x-Q^x,P^y-Q^y,P^z-Q^z)\Big) \ := \ P^x - Q^x, \end{aligned}$$

and the second and third analogously by changing x to y and z, respectively. Note that the operator K_d without aI is a pseudo-differential operator of order -1. Boundary integral operators for the Stokes system or for Lamè's system can be represented in a similar fashion (cf. [28]).

To get a further example, we take the adjoint operator K_d^* and replace the normal vector field n_Q by an oblique field o_Q. We arrive at a strongly singular boundary integral operator $A = A_o$ which corresponds to the oblique derivative boundary value problem for Laplace's equation. In this case, $a_o := -0.5\, n_P \cdot o_P$ and $K_o = K_o^x + K_o^y + K_o^z$ with

$$\begin{aligned} k_o^x(P,Q,o_P) \ &= \ k_o^x\Big(P,Q,(o_P^x,o_P^y,o_P^z)\Big) \ := \ \frac{o_P^x}{4\pi}, \quad \alpha_o \ := \ 3, \\ p_o^x(P-Q) \ &= \ p_o^x\Big((P^x-Q^x,P^y-Q^y,P^z-Q^z)\Big) \ := \ P^x - Q^x. \end{aligned}$$

The definitions for the second and third kernels corresponding to K_o^y and K_o^z, respectively, are analogous.

2.3 Grid and Collocation Points

Let us introduce a hierarchy of uniform grids over the standard triangle T. For the step sizes 2^{-l}, $l = 0, \dots, L$, we set

$$\triangle_l^T \ := \ {}^1\!\triangle_l^T \cup {}^2\!\triangle_l^T,$$

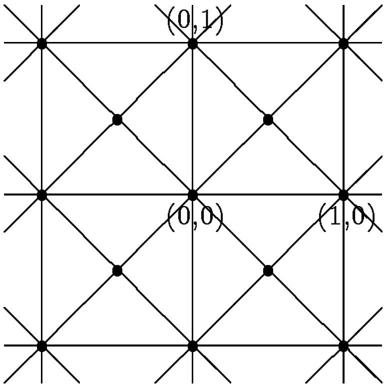

Figure 1: Grid $\triangle_0^{\mathbb{R}^2}$.

$$
\begin{aligned}
{}^1\!\triangle_l^T &:= \left\{(i2^{-l}, j2^{-l}) :\ 0 \le i \le 2^l,\ 0 \le j \le \min\{2^l - i, i\}\right\}, \\
{}^2\!\triangle_l^T &:= \left\{(2^{-l-1}, 2^{-l-1}) + (i2^{-l}, j2^{-l}) :\ 0 \le i < 2^l,\ 0 \le j < \min\{2^l - i, i+1\}\right\}
\end{aligned}
$$

and denote the grid points by $\tau = (s,t) \in \triangle_l^T$. The grid $\triangle_l^T$ is the restriction of the grid (cf. Figure 1)

$$
\triangle_l^{\mathbb{R}^2} := \left\{(i2^{-l}, j2^{-l}) :\ i,j \in \mathbb{Z}^2\right\} \cup \left\{(2^{-l-1}, 2^{-l-1}) + (i2^{-l}, j2^{-l}) :\ i,j \in \mathbb{Z}^2\right\}
$$

to the triangle T. Using the parametrizations, we arrive at a grid hierarchy on Γ.

$$
\triangle_l^\Gamma \ := \ \left\{\kappa_m(\tau) :\ m = 1, \ldots, m_\Gamma,\ \tau \in \triangle_l^T\right\}.
$$

Clearly, a grid point $P = \kappa_m(\tau)$ may have more than one representation. If P is in the interior of a side of the triangular patch Γ_m which is a common side with $\Gamma_{m'}$, then there are exactly two representations $P = \kappa_m(\tau)$ and $P = \kappa_{m'}(\tau')$. If P is a corner point of a patch, then there exist $k > 2$ representations $P = \kappa_{m_1}(\tau_1) = \kappa_{m_2}(\tau_2) = \ldots = \kappa_{m_k}(\tau_k)$. We introduce ${}^i\!\triangle_l^\Gamma$ as the set of those $P \in \triangle_l^\Gamma$ whose representation $P = \kappa_m(\tau)$ with the smallest m satisfies $\tau \in {}^i\!\triangle_l^T$, i.e.,

$$
{}^i\!\triangle_l^\Gamma \ := \ \cup_{m=1}^{m_\Gamma} \left\{\kappa_m(\tau) :\ \tau \in {}^i\!\triangle_l^T,\ \kappa_m(\tau) \notin \cup_{m'=1}^{m-1} \kappa_{m'}(\triangle_l^T)\right\},
$$

and arrive at $\triangle_l^\Gamma = {}^1\!\triangle_l^\Gamma \cup {}^2\!\triangle_l^\Gamma$. The points of $\triangle_l^\Gamma$ will be denoted by upper capital letters like P and Q.

To each grid $\triangle_l^\Gamma$ there corresponds a partition of Γ into triangular pieces. Indeed, let us introduce the sets of centroids

$$
\begin{aligned}
\square_0^{\mathbb{R}^2} &:= \left\{\left(\frac{1}{2}, \frac{1}{6}\right) + k,\ \left(\frac{1}{2}, \frac{5}{6}\right) + k,\ \left(\frac{1}{6}, \frac{1}{2}\right) + k,\ \left(\frac{5}{6}, \frac{1}{2}\right) + k :\ k \in \mathbb{Z}^2\right\}, \\
\square_l^{\mathbb{R}^2} &:= \left\{2^{-l}\tau :\ \tau \in \square_0^{\mathbb{R}^2}\right\}, \quad \square_l^T := T \cap \square_l^{\mathbb{R}^2}, \\
\square_l^\Gamma &:= \left\{\kappa_m(\tau) :\ \tau \in \square_l^T,\ m = 1, 2, \ldots, m_\Gamma\right\}.
\end{aligned}
$$

For each point $\tau \in \square_l^T$, there exist three uniquely defined neighbour points τ_1, τ_2, and τ_3 such that $\tau_1, \tau_2, \tau_3 \in \triangle_l^T$, that the triangle T_τ spanned by the three corners τ_1, τ_2, and τ_3 is of square measure $2^{-2l}/4$, and that τ is the centroid of T_τ. We arrive at the triangulation $\{T_\tau : \tau \in \Box_l^T\}$ of T. Note that, for $l' > l$, the centroids in $\Box_l^T$ are located at the boundaries of the smaller triangles $T_{\tau'}$ with $\tau' \in \Box_{l'}^T$. Hence there is a one to one correspondence between the triangles T_τ over several levels and the centroids in $\cup_{l=0}^{L}\Box_l^T$. Similarly to the triangulation over T, we define the triangulation $\{T_\tau : \tau \in \Box_l^{\mathbb{R}^2}\}$ of $\mathbb{R}^2$. For Γ and a point $Q = \kappa_m(\tau) \in \Box_l^\Gamma$, we set $\Gamma_Q := \{\kappa_m(\sigma) : \sigma \in T_\tau\}$ and arrive at the triangulation $\{\Gamma_Q : Q \in \Box_l^\Gamma\}$. Further, we denote the level l of the points $Q \in \Box_l^\Gamma$ by $l(Q)$. Notice that each partition triangle Γ_Q, $Q \in \Box_l^\Gamma$, of the generation l splits into four subtriangles of the generation $l+1$. We call Γ_Q the father of the four subtriangles and, for $Q \in \Box_l^\Gamma$, $l > 0$, we denote the father of Γ_Q by Γ_{Q^F}.

Beside the grids $\triangle_l^\Gamma$ we introduce the difference grids

$$\nabla_l^\Gamma \quad := \quad \begin{cases} \triangle_0^\Gamma & \text{if } l = -1 \\ \triangle_{l+1}^\Gamma \setminus \triangle_l^\Gamma & \text{if } l = 0, \dots, L-1, \end{cases}$$

and obtain $\triangle_L^\Gamma = \bigcup_{l=-1}^{L-1} \nabla_l^\Gamma$. For $P \in \triangle_L^\Gamma$, we denote the unique level l for which $P \in \nabla_l^\Gamma$ by $l(P)$. Analogously to ∇_l^Γ, we define the difference grids and the point levels over T and $\mathbb{R}^2$ and get $\triangle_L^T = \bigcup_{l=-1}^{L-1} \nabla_l^T$ as well as $\triangle_L^{\mathbb{R}^2} = \bigcup_{l=-1}^{L-1} \nabla_l^{\mathbb{R}^2}$. Finally, in accordance to the splitting $\triangle_l^T = {}^1\!\triangle_l^T \cup {}^2\!\triangle_l^T$, we introduce ${}^i\nabla_l^T = \nabla_l^T \cap {}^i\!\triangle_{l+1}^T$ for $i = 1, 2$ and get $\nabla_l^T = {}^1\nabla_l^T \cup {}^2\nabla_l^T$ as well as ${}^2\nabla_l^T = {}^2\!\triangle_{l+1}^T$. Similarly, we define ${}^i\nabla_l^{\mathbb{R}^2}$ and ${}^i\nabla_l^\Gamma$.

Now the set of collocation points will be the grid $\triangle_L^\Gamma$, i.e. the test functionals of the collocation scheme are the Dirac delta functionals δ_P with $P \in \triangle_L^\Gamma$. The test space Dir_L^Γ is the span of all these δ_P.

2.4 The Trial Functions

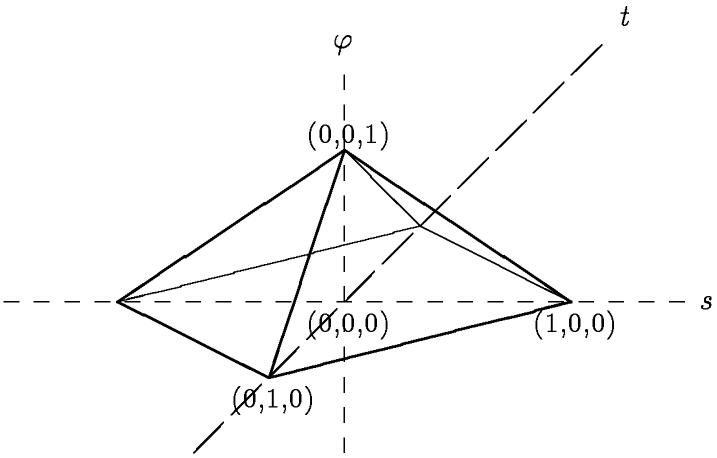

Figure 2: Hat function $(s,t) \mapsto {}^1\varphi(s,t)$.

To prepare the introduction of linear spaces, we first define two-dimensional hat functions for the grid $\triangle_0^{\mathbb{R}^2}$.

$$\begin{aligned} {}^1\varphi(s,t) &:= \max\Big\{0, 1 - \max\{|s-t|, |s+t|\}\Big\}, \\ {}^2\varphi(s,t) &:= \max\Big\{0, 1 - 2\max\{|s|, |t|\}\Big\}. \end{aligned}$$

Clearly, the function ${}^1\varphi$ and the function ${}^2\varphi$ shifted to the point $(0.5, 0.5)$ are piecewise linear functions subordinate to the triangulation $\{T_\tau : \tau \in \square_0^{\mathbb{R}^2}\}$ (cf. the grid in Figure 1, the graph of ${}^1\varphi$ in Figure 2, and the graph of ${}^2\varphi$ shifted to the point $(0.5, 0.5)$ in Figure 3). Note that ${}^2\varphi$ can be obtained from ${}^1\varphi$ by rotation with angle $\pi/4$ and by dilation with factor $\sqrt{2}$, i.e.,

$$ {}^2\varphi(s,t) := {}^1\varphi(s+t, s-t). $$

Now we get piecewise linear basis functions by dilating and shifting ${}^1\varphi$ and ${}^2\varphi$ to each grid point. More precisely, for each grid point on T, we set

$$ \varphi^l_\tau(\sigma) \;:=\; {}^i\varphi\left(2^l(\sigma-\tau)\right), \quad \tau \in {}^i\!\triangle^T_l. $$

With the help of the parametrizations we introduce the piecewise linear (with respect to the parametrization) hat functions over Γ. For each grid point $P \in \triangle^\Gamma_l$, we set

$$ \varphi^l_P(Q) \;:=\; \begin{cases} \varphi^l_\tau(\sigma) & \text{if there exist } m, \tau, \sigma \text{ s.t. } Q = \kappa_m(\sigma),\; P = \kappa_m(\tau) \\ 0 & \text{else.} \end{cases} \tag{2.6} $$

Due to the assumptions on the parametrizations (cf. (2.2)) the basis functions are well defined. Note that if $P \in \triangle^\Gamma_l$ is in the interior of the parametrization patch Γ_m, then the support $\operatorname{supp}\varphi^l_P$ of φ^l_P is contained in Γ_m. If $P = \kappa_m(\tau) = \kappa_{m'}(\tau)$ is in the interior of a side, then $\operatorname{supp}\varphi^l_P \subseteq \Gamma_m \cup \Gamma_{m'}$. For corner points $P = \kappa_{m_1}(\tau_1) = \kappa_{m_2}(\tau_2) = \ldots = \kappa_{m_k}(\tau_k)$ of the triangular parametrization patches we get $\operatorname{supp}\varphi^l_P \subseteq \cup_{n=1}^k \Gamma_{m_n}$. We denote the span of the functions φ^l_P, $P \in \triangle^\Gamma_l$ by Lin^Γ_l. Obviously, this is the space of all continuous and piecewise linear functions over the partition $\{\Gamma_Q : Q \in \square^\Gamma_l\}$ corresponding to the grid $\triangle^\Gamma_l$, where linearity is understood with respect to the parametrization. The space Lin^Γ_L will be the set of trial functions for the collocation.

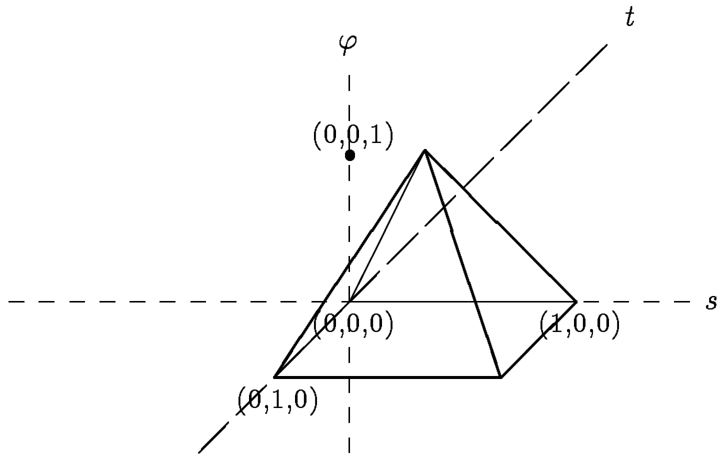

Figure 3: Hat function $(s,t) \mapsto {}^2\varphi(s-0.5, t-0.5)$.

2.5 The Collocation Scheme

Now the collocation method seeks an approximate solution u_L for the exact solution u of $Au = v$. This is sought in the trial space Lin^Γ_L by solving

$$ Au_L(P) = v(P), \quad P \in \triangle^\Gamma_L. \tag{2.7} $$

Using the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \xi_P \varphi_P^L$, the collocation equation can be written in form of a matrix equation $A_L\xi = \eta$, where we set

$$\xi := (\xi_P)_{P\in\triangle_L^\Gamma}, \quad \eta := (\eta_P)_{P\in\triangle_L^\Gamma}, \quad \eta_P := v(P).$$

The matrix of the linear system is the so called stiffness matrix given by

$$A_L := (a_{P',P})_{P',P\in\triangle_L^\Gamma}, \quad a_{P',P} := (A\varphi_P^L)(P').$$

Moreover, using the interpolation projection R_L defined by $R_L f := \sum_{P\in\triangle_L^\Gamma} f(P)\varphi_P^L$, the collocation can be treated as a projection equation of the form $R_L A u_L = R_L v$.

Throughout this paper we shall assume that the collocation method applied to the operator equation $Au = v$ is stable. For the exact definition of stability and some remarks we refer to Sect. 5.4. If the collocation is stable, if the exact solution u is in $H^2(\Gamma)$, and if $h \sim 2^{-L}$ denotes the step size of the discretization, then the approximate solution u_L satisfies the well-known optimal convergence estimates (cf. Sect. 5.4)

$$\|u - u_L\|_{L^2(\Gamma)} \leq Ch^2, \quad \mathbf{r} = 0, -1, \tag{2.8}$$

$$\|u - u_L\|_{H^{-1}(\Gamma)} \leq Ch^3, \quad \mathbf{r} = -1. \tag{2.9}$$

3 The Wavelet Algorithm

3.1 The Wavelet Basis of the Trial space

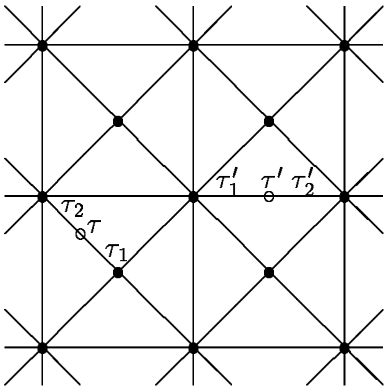

Figure 4: Neighbours τ_1 and τ_2.

Now we introduce a simple wavelet basis for the piecewise linear space. These functions have been considered first for the case of different grids in the plane $\mathbb{R}^2$ (cf. [24, 46, 27]) and are called three-point hierarchical basis functions. More precisely, for the plane and for any point $\tau \in \triangle_L^{\mathbb{R}^2}$, we set (cf. Figure 5 for the supports of such functions)

$$\psi_\tau := \begin{cases} \varphi_\tau^0 & \text{if } \tau \in \nabla_{-1}^{\mathbb{R}^2} \\ \varphi_\tau^{l+1} - \frac{1}{2}\left\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\right\} & \text{if } \tau \in {}^1\nabla_l^{\mathbb{R}^2} \text{ with } l = l(\tau) \in \{0,\dots,L-1\} \\ \varphi_\tau^{l+1} - \frac{1}{4}\left\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\right\} & \text{if } \tau \in {}^2\nabla_l^{\mathbb{R}^2} \text{ with } l = l(\tau) \in \{0,\dots,L-1\}. \end{cases} \tag{3.1}$$

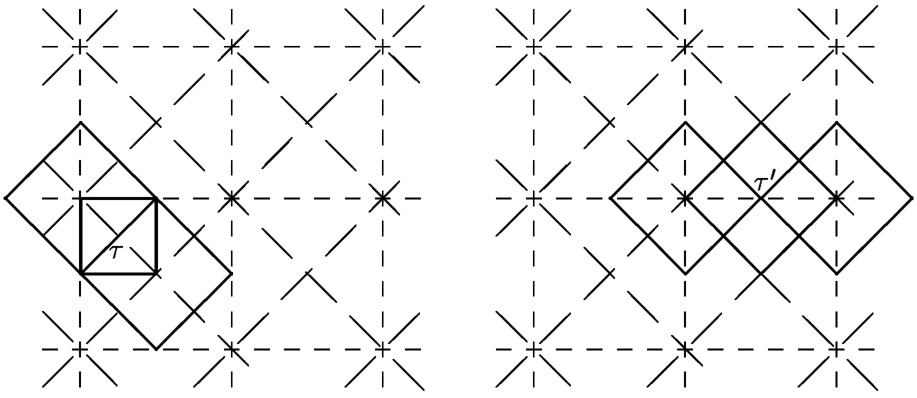

Figure 5: Supports of wavelets ψ_τ and $\psi_{\tau'}$.

Here τ_1 and τ_2 denote the uniquely defined neighbours of τ on $\triangle_{l+1}^{\mathbb{R}^2}$ (cf. Figure 4). Indeed any difference grid point $\tau \in {}^2\nabla_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$ has exactly two neighbour points τ_1 and τ_2 at minimal distance which belong to $\triangle_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$. Any difference grid point $\tau' \in {}^1\nabla_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$ has exactly two neighbour points τ_1' and τ_2' at minimal distance which belong to ${}^1\triangle_l^{\mathbb{R}^2} \subset \triangle_{l+1}^{\mathbb{R}^2}$. The functions ψ_τ with $\tau \in \nabla_l^{\mathbb{R}^2}$, $l = 0, \ldots, L-1$ have two vanishing moments, i.e. they are orthogonal to all constant and linear functions.

The wavelet functions ψ_τ on the manifold Γ are slight modifications of (3.1). The definition is not very difficult. However, to motivate this definition, we shortly explain the construction:

- We start with the first parametrization patch Γ_1 and the definition of functions ψ_P such that $P \in \triangle_L^\Gamma \cap \Gamma_1$. First we restrict the functions ψ_τ from (3.1) to T. If these restrictions intersect the boundary of T, then we modify them adding restrictions of three-point basis functions $\psi_{\tau'}$ with τ' outside of T. The resulting basis functions $\psi_\tau^{\&}$ are restrictions of functions which are symmetric (even) with respect to the boundary of T. For $P = \kappa_1(\tau)$, we take the composition $\psi_P = \psi_\tau^{\&} \circ \kappa_1^{-1}$ to arrive at functions over the parametrization patch Γ_1. To get continuous trial functions over Γ, we extend the ψ_P with $P \in \nabla_l^\Gamma \cap \Gamma_1$, $l = -1, 0, \ldots, L-1$ from Γ_1 to Γ such that the extensions are piecewise linear on the partition $\{\Gamma_Q : Q \in \square_{l+1}^\Gamma\}$ corresponding to the grid $\triangle_{l+1}^\Gamma$ and vanish at all grid points from $\triangle_{l+1}^\Gamma \setminus \Gamma_1$.
- Next we define the functions ψ_P such that $P \in \triangle_L^\Gamma \cap \{\Gamma_2 \setminus \Gamma_1\}$. We start again with the restrictions of (3.1) to T. Since we have already basis functions over the boundary $\Gamma_1 \cap \Gamma_2$, we need basis functions on Γ_2 vanishing over $\Gamma_1 \cap \Gamma_2$, i.e. basis functions on T vanishing on the side S' for which $\kappa_2(S') = \Gamma_2 \cap \Gamma_1$. Therefore, we modify the functions on T such that they are restrictions of functions antisymmetric (odd) with respect to the side S' and symmetric (even) with respect to the sides S of T with $\kappa_2(S) \not\subset \Gamma_1$. Clearly all these functions vanish on S'. We take the composition with κ_2^{-1} to arrive at functions over the parametrization patch Γ_2 which vanish over $\Gamma_2 \cap \Gamma_1$. To get continuous trial functions, we extend these functions ψ_P with $P \in \nabla_l^\Gamma \cap \{\Gamma_2 \setminus \Gamma_1\}$, $l = -1, 0, \ldots, L-1$ from Γ_2 to Γ such that the extensions are piecewise linear on the partition $\{\Gamma_Q : Q \in \square_{l+1}^\Gamma\}$ corresponding to the grid $\triangle_{l+1}^\Gamma$ and vanish at all grid points from $\triangle_{l+1}^\Gamma \setminus \Gamma_2$.

- Analogously to the previous step, we define the functions ψ_P such that the point P is in $\triangle_L^\Gamma \cap \{\Gamma_3 \setminus (\Gamma_1 \cup \Gamma_2)\}$. Then we construct the functions ψ_P with point P in $\triangle_L^\Gamma \cap \{\Gamma_4 \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)\}$ and so on. Finally, we define ψ_P with point P in $\triangle_L^\Gamma \cap \{\Gamma_{m_\Gamma} \setminus \cup_{m=1}^{m_\Gamma - 1} \Gamma_m\}$.

For more details and the properties of the basis we refer to Sect. 5.1. The final definition of the three-point hierarchical wavelet functions over the manifold Γ is

$$\psi_P := \begin{cases} \varphi_P^0 & \text{if } P \in \nabla_{-1}^\Gamma \\ \varphi_P^{l+1} - \frac{1}{2}\left\{\varepsilon^{P,P_1}\varphi_{P_1}^{l+1} + \varepsilon^{P,P_2}\varphi_{P_2}^{l+1}\right\} & \text{if } P \in {}^1\nabla_l^\Gamma \text{ with } l \in \{0,\dots,L-1\} \\ \varphi_P^{l+1} - \frac{1}{4}\left\{\varepsilon^{P,P_1}\varphi_{P_1}^{l+1} + \varepsilon^{P,P_2}\varphi_{P_2}^{l+1}\right\} & \text{if } P \in {}^2\nabla_l^\Gamma \text{ with } l \in \{0,\dots,L-1\}, \end{cases} \tag{3.2}$$

where P_1 and P_2 are the uniquely defined neighbours on $\triangle_{l+1}^\Gamma$ of $P \in \nabla_l^\Gamma$, i.e. $P_1 = \kappa_m(\tau_1)$ and $P_2 = \kappa_m(\tau_2)$ if $P = \kappa_m(\tau)$ is the representation with the minimal $m \in \{1,\dots,m_\Gamma\}$ and if τ_1, τ_2 are the neighbours of τ. The coefficients $\varepsilon^{P,P'}$ are equal to one in almost all cases. Only if the point $P' = P_1, P_2$ is at the boundary of a parametrization patch, then a value $\varepsilon^{P,P'}$ different from one is needed. More precisely, the coefficients $\varepsilon^{P,P'}$ are given by (cf. Sect. 2.3 for the definition of ${}^i\triangle_L^\Gamma$)

$$\varepsilon^{P,P'} := \begin{cases} 1 & \text{if} \quad \text{there is a parametrization patch } \Gamma_m \text{ such that } P \text{ and } P' \text{ belong} \\ & \qquad \text{to the interior of the triangle } \Gamma_m \\ & \quad\ \text{or there exists a side } \Gamma_m \cap \Gamma_{m'} \text{ of a parametrization patch such} \\ & \qquad \text{that } P \text{ and } P' \text{ belong to the interior of the side } \Gamma_m \cap \Gamma_{m'} \\ 2 & \text{if} \quad \text{there exists a side } \Gamma_m \cap \Gamma_{m'} \text{ of a parametrization patch such} \\ & \qquad \text{that } m < m'\text{, that } P \text{ is an interior point of } \Gamma_m\text{, and that } P' \\ & \qquad \text{belongs to the interior of the side } \Gamma_m \cap \Gamma_{m'} \\ & \quad\ \text{or } P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch, } P' \in {}^2\triangle_0^\Gamma, \\ & \qquad \text{the point } P \text{ is an interior point of a side } \Gamma_{m_1} \cap \Gamma_{m_2}, \text{ and} \\ & \qquad m_1 < m_i,\ i = 2,\dots,k \\ 4 & \text{if} \quad P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch, } P' \in {}^1\triangle_0^\Gamma, \\ & \qquad \text{the point } P \text{ is an interior point of a side } \Gamma_{m_1} \cap \Gamma_{m_2}, \text{ and} \\ & \qquad m_1 < m_i,\ i = 2,\dots,k \\ & \quad\ \text{or } P' = \cap_{i=1}^k \Gamma_{m_i} \text{ is a corner of a parametrization patch, } P' \in {}^2\triangle_0^\Gamma, \\ & \qquad \text{the point } P \text{ is an interior point of the face } \Gamma_{m_1}, \text{ and} \\ & \qquad m_1 < m_i,\ i = 2,\dots,k \\ 0 & \text{else.} \end{cases} \tag{3.3}$$

Clearly, the support of ψ_P is contained in the union of all those Γ_m in which P or at least one of the neighbour points P_1 or P_2 is located. The basis $\{\psi_P :\ P \in \triangle_L^\Gamma\}$ spans the trial space Lin_L^Γ since the system is linearly independent (cf. (5.20)). Moreover, it represents a hierarchical basis, i.e.

$$\left\{\psi_P :\ P \in \triangle_L^\Gamma\right\} = \bigcup_{l=-1}^{L-1} \left\{\psi_P :\ P \in \nabla_l^\Gamma\right\},$$

$$Lin_0^\Gamma \subset Lin_1^\Gamma \subset \dots \subset Lin_L^\Gamma,$$

$$Lin_{l'}^\Gamma = \operatorname{span} \bigcup_{l=-1}^{l'-1} \left\{\psi_P :\ P \in \nabla_l^\Gamma\right\}.$$

The function ψ_P with $P \in \nabla_l^\Gamma,\ l = 0, \ldots, L-1$ and with supp ψ_P contained in the interior of only one parametrization patch has two vanishing moments, i.e. it is orthogonal to the set of all functions that are constant or linear with respect to the parametrization. Orthogonality means here orthogonality with respect to the L^2 scalar product in the parameter domain.

3.2 The Wavelet Basis of the Test space

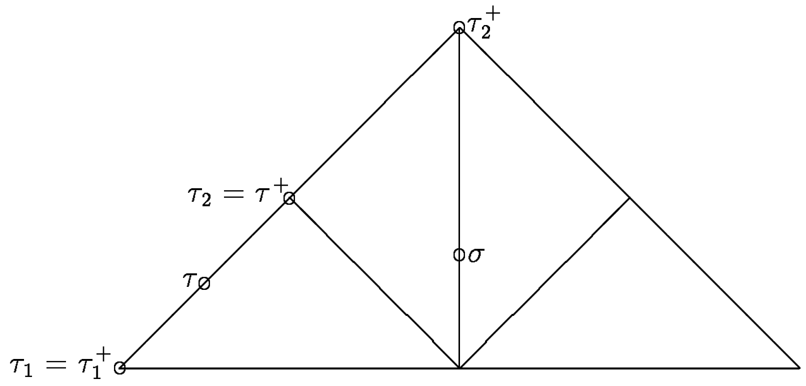

Figure 6: First case, point τ at the boundary of T_σ.

Let us retain the definition of neighbour points $P_1, P_2 \in \triangle_l^\Gamma$ of $P \in \nabla_l^\Gamma,\ l = 0, \ldots, L-1$ from the last subsection, and recall that δ_P stands for the Dirac delta functional at point P. With this notation, we introduce the functionals

$$\vartheta_P := \begin{cases} \delta_P & \text{if } P \in \nabla_{-1}^\Gamma \\ \delta_P - \frac{1}{2}\left\{\delta_{P_1} + \delta_{P_2}\right\} & \text{if } P \in \nabla_l^\Gamma \text{ with } l = l(P) \in \{0, \ldots, L-1\}. \end{cases} \tag{3.4}$$

Clearly, the support supp ϑ_P is contained in Γ_m if P belongs to Γ_m. In particular, supp ϑ_P is on the side of a parametrization patch if P is on this side. If P is a corner of a parametrization patch, then supp $\vartheta_P = \{P\}$. The set $\{\vartheta_P :\ P \in \triangle_L^\Gamma\}$ is a hierarchical basis of the test space Dir_L^Γ (cf. the Sects. 2.3 and 5.2). For any $P \in \nabla_l^\Gamma,\ l = 0, \ldots, L-1$, the functional ϑ_P has two vanishing moments, i.e. it vanishes over the set of all functions that are constant or linear with respect to the parametrization. To simplify the notation, some times we shall write $f(\vartheta_P)$ for $\vartheta_P(f)$.

The basis $\{\vartheta_P\}$ will be suitable for the collocation applied to operators of order $\mathbf{r} = 0$. For $\mathbf{r} = -1$, a basis with more vanishing moments is needed (cf. [14, 44]). This wavelet basis $\{\vartheta_P^+ :\ P \in \triangle_L^\Gamma\}$ is given by

$$\vartheta_P^+ := \begin{cases} \delta_P & \text{if } P \in \nabla_{-1}^\Gamma \\ \vartheta_P & \text{if } P \in \nabla_l^\Gamma \text{ with } l = l(P) \in \{0, 1\} \\ \vartheta_P - \frac{1}{4}\vartheta_{P^+} & \text{if } P \in \nabla_l^\Gamma \text{ with } l = l(P) \in \{2, \ldots, L-1\}. \end{cases} \tag{3.5}$$

Here P^+ is defined as follows. We assume that $P = \kappa_m(\tau)$ with $\tau \in \nabla_l^T$ and that τ is in the closed triangle T_σ with $\sigma \in \Box_{l-1}^T$ (cf. the notation of Sect. 2.3 and recall that T_σ is a partition triangle of the level $l-1$ partition defined by its centroid σ). We distinguish three cases. If τ is at the boundary of T_σ (cf. Figure 6), then we choose τ^+ to be the midpoint of that side of T_σ at which τ is located, and we set $P^+ := \kappa_m(\tau^+)$. If τ is not at the boundary and not at the symmetry axis of T_σ (cf. Figure 7), then we choose τ^+ to be the midpoint of that side of T_σ which is parallel to the straight line segment $\tau_1\tau_2$ defined by the two neighbours τ_1, τ_2 of τ from the grid $\triangle_l^T$. Again, we set $P^+ := \kappa_m(\tau^+)$. Finally, if τ is not at the boundary but at the symmetry axis of T_σ (cf. Figure 8), then we choose a neighbour triangle $T_{\sigma'}$ of T_σ which has a small side in common with T_σ. Clearly, the hypotenuse of $T_{\sigma'}$ is parallel to the straight line segment $\tau_1\tau_2$ defined by the two neighbours of τ from the grid $\triangle_l^T$. We choose τ^+ to be the midpoint of the hypotenuse of $T_{\sigma'}$ and set $P^+ := \kappa_m(\tau^+)$. Note that, if τ_1, τ_2 and τ_1^+, τ_2^+ denote the neighbour points of τ and τ^+, respectively, then the straight lines through τ, τ_1, τ_2 and through $\tau^+, \tau_1^+, \tau_2^+$ are parallel in all three cases. In accordance with (3.5), we get

$$\vartheta_P^+ := \delta_{\kappa_m(\tau)} - \frac{1}{2}\left\{\delta_{\kappa_m(\tau_1)} + \delta_{\kappa_m(\tau_1)}\right\} - \frac{1}{4}\delta_{\kappa_m(\tau^+)} + \frac{1}{8}\left\{\delta_{\kappa_m(\tau_1^+)} + \delta_{\kappa_m(\tau_1^+)}\right\},$$

for $P \in \nabla_l^\Gamma$ with $l \geq 2$. The set $\{\vartheta_P^+ : P \in \triangle_L^\Gamma\}$ is a hierarchical basis of Dir_L^Γ, too (cf. Sect. 5.2). For any $P \in \nabla_l^\Gamma$, $l = 2, \ldots, L-1$, the functional ϑ_P^+ has three vanishing moments, i.e. it vanishes over all polynomials of total degree less than three.

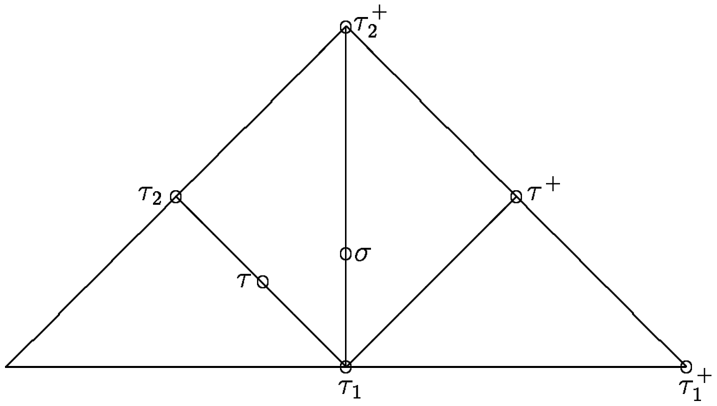

Figure 7: Second case, point τ not at the boundary of T_σ.

3.3 Wavelet Transforms

For the trial space Lin_L^Γ we have two different systems of basis functions $\{\varphi_P^L\}$ and $\{\psi_P\}$ at our disposal. We denote the basis transform by $\mathcal{T}_A$ (lower index A stands for ansatz), i.e. the matrix $\mathcal{T}_A$ maps the coefficient vector $\xi^L := (\xi_P^L)_{P\in\triangle_L^\Gamma}$ of the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \xi_P^L \varphi_P^L$ into the coefficient vector $\beta := (\beta_P)_{P\in\triangle_L^\Gamma}$ of the representation $u_L = \sum_{P\in\triangle_L^\Gamma} \beta_P \psi_P$. This transform can be determined by a pyramid type algorithm which is called fast wavelet transform.

To describe this, we write $\beta = (\beta^{-1}, \beta^0, \ldots, \beta^{L-1})$ for $\beta^l = (\beta_P^l)_{P\in\nabla_l^\Gamma} := (\beta_P)_{P\in\nabla_l^\Gamma}$ and introduce the auxiliary coefficient vectors $\xi^l := (\xi_P^l)_{P\in\triangle_l^\Gamma}$ by $\sum_{P\in\triangle_l^\Gamma} \xi_P^l \varphi_P^l = \sum_{P\in\triangle_l^\Gamma} \beta_P \psi_P$.

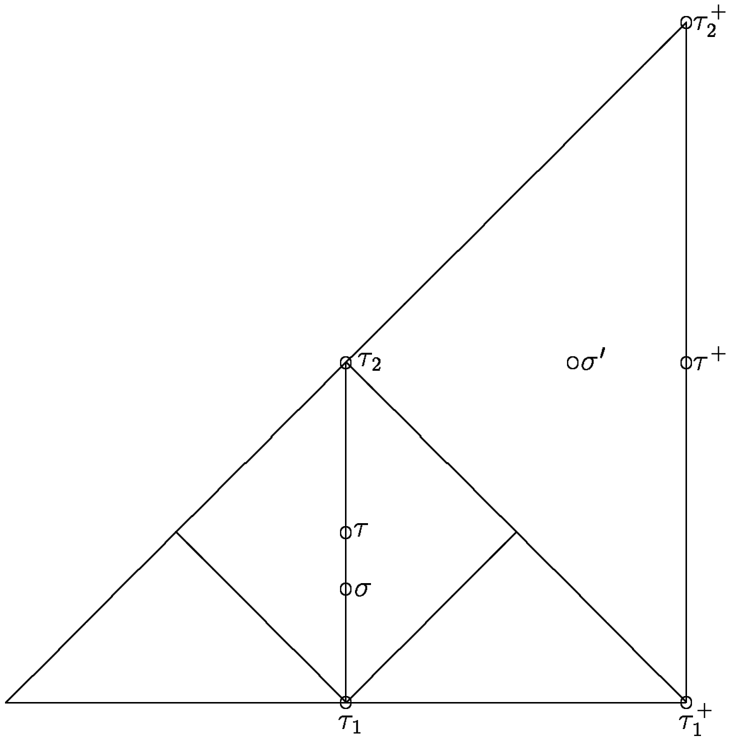

Figure 8: Third case, point τ not at the boundary of T_σ.

Now the algorithm for $\mathcal{T}_A$ looks as follows.

Wavelet Transform $\mathcal{T}_A$

$$
\begin{array}{l}
\text{initial value } \xi^l \text{ is given for } l = L \\
\text{do } l = L, L-1 \ldots, 1 \\
\quad \text{use the splitting } Lin_l^\Gamma = Lin_{l-1}^\Gamma \oplus \text{span}\,\{\psi_P :\ P \in \nabla_{l-1}^\Gamma\} \text{ to compute} \\
\quad \xi^{l-1} \text{ and } \beta^{l-1} \text{ from } \sum_{P\in\triangle_l^\Gamma} \xi_P^l \varphi_P^l = \sum_{P\in\triangle_{l-1}^\Gamma} \xi_P^{l-1}\varphi_P^{l-1} + \sum_{P\in\nabla_{l-1}^\Gamma} \beta_P^{l-1}\psi_P \\
\text{enddo} \\
\text{set } \beta^{-1} := \xi^0 \\
\text{form } \beta = (\beta^{-1}, \beta^0, \ldots, \beta^{L-1})
\end{array}
\tag{3.6}
$$

Similarly, the inverse transform $\mathcal{T}_A^{-1}$ can be realized by:

Wavelet Transform $\mathcal{T}_A^{-1}$

$$
\begin{array}{l}
\text{initial values } \beta^l \text{ are given for } l = -1, 0, \ldots, L-1 \\
\text{set } \xi^0 = \beta^{-1} \\
\text{do } l = 1, 2, \ldots, L \\
\quad \text{use the splitting } Lin_l^\Gamma = Lin_{l-1}^\Gamma \oplus \text{span}\,\{\psi_P :\ P \in \nabla_{l-1}^\Gamma\} \text{ to compute} \\
\quad \xi^l \text{ from } \sum_{P\in\triangle_l^\Gamma} \xi_P^l \varphi_P^l = \sum_{P\in\triangle_{l-1}^\Gamma} \xi_P^{l-1}\varphi_P^{l-1} + \sum_{P\in\nabla_{l-1}^\Gamma} \beta_P^{l-1}\psi_P \\
\text{enddo}
\end{array}
\tag{3.7}
$$

For the implementation of the inner part in the do loop of (3.7), we substitute the two scale relations (cf. (3.2) and (3.3))

$$
\varphi_P^{l-1} \quad = \sum_{P'\in\triangle_l^\Gamma:\, P'\in\operatorname{supp}\varphi_P^{l-1}} \varphi_P^{l-1}(P')\varphi_{P'}^l, \tag{3.8}
$$

$$\psi_P = \sum_{P'\in\triangle_l^\Gamma} d_{P',P}\varphi_{P'}^l, \quad P\in\nabla_{l-1}^\Gamma, \tag{3.9}$$

$$d_{P',P} := \begin{cases} 1 & \text{if } P'=P \\ -\frac{1}{2}\varepsilon^{P,P'} & \text{if } P'\in\{P_1,P_2\} \text{ and } P\in{}^1\nabla_l^\Gamma \\ -\frac{1}{4}\varepsilon^{P,P'} & \text{if } P'\in\{P_1,P_2\} \text{ and } P\in{}^2\nabla_l^\Gamma \end{cases} \tag{3.10}$$

into the splitting equation $\sum_{P\in\triangle_l^\Gamma}\xi_P^l\varphi_P^l = \sum_{P\in\triangle_{l-1}^\Gamma}\xi_P^{l-1}\varphi_P^{l-1}+\sum_{P\in\nabla_{l-1}^\Gamma}\beta_P^{l-1}\psi_P$ and compare the coefficients of the φ_P^l. This yields the representation $\xi^l = M_1\xi^{l-1}+M_2\beta^{l-1}$ with the sparse matrices

$$M_1 = \left(\varphi_P^{l-1}(P')\right)_{P'\in\triangle_l^\Gamma,P\in\triangle_{l-1}^\Gamma}, \quad M_2 = \left(d_{P',P}\right)_{P'\in\triangle_l^\Gamma,P\in\nabla_{l-1}^\Gamma}.$$

There exists a small constant dependent only on the geometry of Γ such that the number of non-zero entries in each column of M_1 and M_2 is less than this number. Hence, the multiplication by M_1 and M_2 requires only $O(2^{2l})$ arithmetic operations, and $O(2^{2L})$ operations are sufficient for the whole algorithm (3.7). For the algorithm (3.6), equation $\xi^l = M_1\xi^{l-1}+M_2\beta^{l-1}$ is to be solved for the unknowns ξ^{l-1} and β^{l-1}. If this is done by an appropriate iterative solver, then the whole algorithm (3.6) requires no more than $O(2^{2L})$, too.

Analogously to the trial space, we have two different bases in the test space. By $\mathcal{T}_T$ (lower index T stands for test space) we denote the linear transform which maps the vector $\gamma = (\gamma_P)_{P\in\triangle_L^\Gamma} := (\vartheta_P(f))_{P\in\triangle_L^\Gamma}$ of functionals applied to a function f into the vector of function values $\eta = (\eta_P)_{P\in\triangle_L^\Gamma} := (\delta_P(f))_{P\in\triangle_L^\Gamma} = (f(P))_{P\in\triangle_L^\Gamma}$. Again, the transform can be realized by a fast wavelet algorithm. We write $\gamma = (\gamma^{-1},\gamma^0,\dots,\gamma^{L-1})$ for $\gamma^l := (\gamma_P)_{P\in\nabla_l^\Gamma}$ and introduce the auxiliary coefficient vectors $\eta^l = (\eta_P^l)_{P\in\triangle_l^\Gamma} := (\eta_P)_{P\in\triangle_l^\Gamma}$. Now we arrive at the following algorithm.

Wavelet Transform $\mathcal{T}_T$

initial values γ^l are given for $l=-1,\dots,L-1$
set $\eta^0 = \gamma^{-1}$
do $l = 1,2,\dots,L$
  compute η^l : (3.11)
  if $P\in\triangle_{l-1}^\Gamma$ then $\eta_P^l = \eta_P^{l-1}$
  if $P\in\nabla_{l-1}^\Gamma$ then $f(P) = \vartheta_P(f)+\frac{1}{2}\{f(P_1)+f(P_2)\}$,
    i.e. $\eta_P^l = \gamma_P^{l-1}+\frac{1}{2}\{\eta_{P_1}^{l-1}+\eta_{P_2}^{l-1}\}$
enddo

Clearly, the algorithm in the inner of the do loop requires $O(2^{2l})$ arithmetic operations and the whole algorithm (3.11) no more than $O(2^{2L})$. Due to $\gamma_P^{l-1} = \eta_P^l - \frac{1}{2}\{\eta_{P_1}^{l-1}+\eta_{P_2}^{l-1}\}$ (cf. (3.4)), the inverse $\mathcal{T}_T^{-1}$ is simply a multiplication by a sparse matrix. Hence, the algorithmic complexity of the transforms $\mathcal{T}_T$ and $\mathcal{T}_T^{-1}$ is $O(2^{2L})$. The wavelet transforms $\mathcal{T}_T$ and $\mathcal{T}_T^{-1}$ with the basis functionals ϑ_P replaced by ϑ_P^+ can be treated analogously.

3.4 Wavelet Algorithm

Analogously to the stiffness matrix A_L in Sect. 2.5 we can set up a matrix with respect to the wavelet basis. We introduce A_L^w by

$$A_L^w := \left(a_{P',P}^w\right)_{P',P\in\triangle_L^\Gamma}, \quad a_{P',P}^w := \vartheta_{P'}(A\psi_P). \tag{3.12}$$

Note that $A_L = \mathcal{T}_T A_L^w \mathcal{T}_A$. It will turn out that most of the entries $a^w_{P',P}$ are so small that they can be neglected. Thus in the next subsection we will give an a priori matrix pattern $\mathcal{P} \subset \triangle_L^\Gamma \times \triangle_L^\Gamma$ with no more than $O(2^{2L} L^{1.75})$ elements. We will replace A_L^w by the sparse matrix obtained by the compression

$$A_L^{w,c} := \left(a^{w,c}_{P',P} \right)_{P',P \in \triangle_L^\Gamma}, \quad a^{w,c}_{P',P} := \vartheta_{P'}(a\psi_P) + \begin{cases} \vartheta_{P'}(K\psi_P) & \text{if } (P',P) \in \mathcal{P} \\ 0 & \text{else.} \end{cases} \tag{3.13}$$

In the numerical computation the entries have to be computed by approximating the parametrization and by quadrature. We denote the approximate value for $a^{w,c}_{P',P}$ by $a^{w,c,q}_{P',P}$ and set

$$A_L^{w,c,q} := \left(a^{w,c,q}_{P',P} \right)_{P',P \in \triangle_L^\Gamma}, \quad A_L^c := \mathcal{T}_T A_L^{w,c} \mathcal{T}_A, \quad A_L^{c,q} := \mathcal{T}_T A_L^{w,c,q} \mathcal{T}_A. \tag{3.14}$$

With this notation we can describe two variants of the wavelet algorithm which differ in the iterative solution of the discretized linear systems. The first is designed for integral operators of arbitrary order $\mathbf{r}$ and requires the application of one transform $\mathcal{T}_A^{-1}$ and one transform $\mathcal{T}_T^{-1}$ during the whole algorithm.

First Wavelet Algorithm

i) compute the right-hand side $\gamma := (\vartheta_P(v))_P = \mathcal{T}_T^{-1}(v(P))_P$
ii) compute the sparsity pattern $\mathcal{P}$
iii) assemble $A_L^{w,c,q}$ by a quadrature algorithm
iv) solve $A_L^{w,c,q}\beta = \gamma$ iteratively, e.g. by the diagonally preconditioned GMRes method (3.15)
v) compute $\xi = \mathcal{T}_A^{-1}\beta$
vi) post processing of the values $u(P) \approx \xi_P$, e.g. computation of linear functionals of the solution u

The second is designed for operators of order $\mathbf{r} = 0$. Though an application of the two wavelet transforms $\mathcal{T}_A$ and $\mathcal{T}_T$ is required in each iteration, the corresponding number of all iterations is often much smaller, and the second algorithm is faster.

Second Wavelet Algorithm

i) compute the right-hand side $\eta := (v(P))_P$
ii) compute the sparsity pattern $\mathcal{P}$
iii) assemble $A_L^{w,c,q}$ by a quadrature algorithm
iv) solve $A_L\xi = \eta$ iteratively, e.g. by the GMRes method, whenever a multiplication by matrix A_L is required, then multiply by $\mathcal{T}_A$, by $A_L^{w,c,q}$, and by $\mathcal{T}_T$ (3.16)
v) post processing of the values $u(P) \approx \xi_P$, e.g. computation of linear functionals of the solution u

The GMRes algorithm is described in [42], and the diagonal preconditioner for the algorithm (3.15) will be derived in Sect. 5.4 (cf. (5.37)).

To reduce the complexity of the quadrature algorithm in step iii) of algorithm (3.16), we modify the wavelet algorithm. We split operator A into the sum of a singular near field part A^{sn} and a part $A^{ns,f}$ covering the non-singular near field and the far field part. More precisely, for $P' \in \Gamma$, we introduce the characteristic function $\Xi_{P'}$ of a small neighbourhood of size $O(2^{-L})$ around P' by defining

$$\Xi_{P'}(R) := \begin{cases} 1 & \text{if } R \in \cup_{Q \in \square_L^\Gamma : P' \in \Gamma_Q} \Gamma_Q \\ 0 & \text{else .} \end{cases}$$

Using this cut off function, we set $A^{sn}u(P') := A(\Xi_{P'}u)(P')$ and $A^{ns,f} := A - A^{sn}$. In correspondence to this splitting, we introduce the approximate matrix $[A^{sn}]_L$ for operator A^{sn} as well as the matrices $[A^{ns,f}]_L$ and $[A^{ns,f}]_L^{w,c,q}$ for operator $A^{ns,f}$. By $[A^{sn}]_L^q$ we denote a quadrature approximation of the almost diagonal matrix $[A^{sn}]_L$. Using this notation, we arrive at the following modification of steps iii) and iv) in algorithm (3.16).

$$\begin{array}{ll} \text{iii)} & \text{assemble } [A^{ns,f}]_L^{w,c,q} \text{ and } [A^{sn}]_L^q \text{ by a quadrature algorithm} \\ \text{iv)} & \text{solve } A_L\xi = \eta \text{ iteratively, e.g. by the GMRes method,} \\ & \quad \text{whenever a vector } v_L \text{ is to be multiplied by matrix } A_L, \text{ then:} \\ & \quad \text{compute } \mathcal{T}_A v_L,\ [A^{ns,f}]_L^{w,c,q}\{\mathcal{T}_A v_L\}, \text{ and } \mathcal{T}_T\{[A^{ns,f}]_L^{w,c,q}\mathcal{T}_A v_L\}, \\ & \quad \text{multiply } v_L \text{ by } [A^{sn}]_L^q, \text{ compute the sum} \\ & \quad \{\mathcal{T}_T[A^{ns,f}]_L^{w,c,q}\mathcal{T}_A v_L\} + \{[A^{sn}]_L^q v_L\} \end{array} \tag{3.17}$$

3.5 The Compression Algorithm

In order to introduce the compression pattern $\mathcal{P}$, we need some notation. Let us retain the definition of ∇_l^Γ and $\triangle_L^\Gamma$ from Sect. 2.3. For $P \in \triangle_L^\Gamma$, recall that $l(P)$ is the level of P (cf. the end of Sect. 2.3). By Ψ_P we denote the support of the function ψ_P and by Θ_P the convex hull of the support of the test functional ϑ_P, i.e., $\vartheta_P := \kappa_m(\text{conv}(\kappa_m^{-1}(\text{supp}\,\vartheta_P)))$. Furthermore, we introduce six suitable non-negative parameters a, b, c, $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ and two functions $d = d(L) \geq 1$ and $\tilde{d} = \tilde{d}(L) \geq 1$. Depending on these, the set $\mathcal{P} = \mathcal{P}(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ is the set of all $(P', P) \in \triangle_L^\Gamma \times \triangle_L^\Gamma$ such that Ψ_P is completely contained in the interior of a single parameter patch Γ_m and

$$\text{dist}\,(\Psi_P, \Theta_{P'}) \leq \max\left\{2^{-l(P)}, 2^{-l(P')}, d2^{aL - bl(P) - cl(P')}\right\} \tag{3.18}$$

or such that Ψ_P contains points of at least two parameter patches and

$$\text{dist}\,(\Psi_P, \Theta_{P'}) \leq \max\left\{2^{-l(P)}, 2^{-l(P')}, \tilde{d}2^{\tilde{a}L - \tilde{b}l(P) - \tilde{c}l(P')}\right\}. \tag{3.19}$$

In numerical computations all compression parameters from a to $\tilde{d}$ should be determined by experiments. However, to get an asymptotically optimal compression result, we can choose $a = c = 4/5$, $b = \tilde{b} = 1$, and $\tilde{a} = \tilde{c} = 5/3$. The functions d and $\tilde{d}$ can be defined by $d = CL^{3/8}$ and $\tilde{d} = C\,L^{3/4}$, where C is a sufficiently large constant.

Theorem 3.1 *For the pattern $\mathcal{P} = \mathcal{P}(4/5, 1, 4/5, CL^{3/8}, 5/3, 1, 5/3, CL^{3/4})$, the number of non-zero entries $N_\mathcal{P}$ is less than $CL^{7/4}2^{2L} \sim N[\log N]^{1.75}$, where $N \sim 2^{2L}$ is the number of degrees of freedom. If the piecewise linear collocation is stable, then the collocation method with compression is stable, too. The asymptotic error estimates for the compressed collocation method are the same as for the uncompressed collocation, i.e. (2.8) and (2.9) remain valid.*

Proof. The bound for $N_{\mathcal{P}}$ will follow from Lemma 5.6, and the stability together with the error estimates will be a consequence of Sect. 5.4 and Lemma 5.8. ∎

For the implementation of step ii) in the wavelet algorithms (3.15) and (3.16), the hierarchical structure of the wavelet basis is essential. More precisely, we observe that the pattern $\mathcal{P}$ has the following property. If $(P', P_1) \notin \mathcal{P}$ and $\operatorname{supp} \psi_{P_2} \subseteq \operatorname{supp} \psi_{P_1}$, then $(P', P_2) \notin \mathcal{P}$. To set up a sparsity pattern $\mathcal{P}$ with this property, we can proceed as follows. For each P', we have to determine the set of P with $(P', P) \in \mathcal{P}$. We do this for each level $l = l(P)$ separately. First we check $(P', P) \in \mathcal{P}$ for $l = l(P) = -1$. Then, if the subset of all $P \in \nabla^{\Gamma}_{l-1}$ with $(P', P) \in \mathcal{P}$ is determined, the search for the $P \in \nabla^{\Gamma}_{l}$ can be restricted to all P with

$$\operatorname{supp} \psi_P \ \cap \left[\cup_{R \in \nabla^{\Gamma}_{l-1} :\, (P', R) \in \mathcal{P}} \operatorname{supp} \psi_R \right] \neq \emptyset .$$

Doing this for all $l = 0, \ldots, L-1$ and for all $P' \in \triangle^{\Gamma}_{L}$, only $O(N_{\mathcal{P}})$ of the N^2 pairs (P', P) have to be checked.

Clearly the number of necessary arithmetic operations of all steps in the algorithms (3.15) and (3.16) except the steps iii) and iv) is less than $C\, N_{\mathcal{P}}$. Step iv) requires $C\, N_{\mathcal{P}} \log N$ operations. However, if we solve the systems successively over the grids $\triangle^{\Gamma}_{l}$, $l = 0, \ldots, L$ and if the initial solution for the grid $\triangle^{\Gamma}_{l+1}$ is the final solution from the coarser grid $\triangle^{\Gamma}_{l}$, then the number of necessary iterations is uniformly bounded. This cascadic iteration method requires no more than $C\, N_{\mathcal{P}}$ operations. The key point for a fast algorithm, however, is the implementation of step iii). Usually, this is the most time consuming part of the numerical computation. For its realization and complexity, we refer to the results in Sect. 4 and the proofs in Sect. 6. Further details for the implementation of the wavelet algorithm can be found in [26, 37].

4 Approximation of the Parametrization and Quadrature

4.1 Parametrization and Quadrature for the Far Field

Now we consider the computation of the matrix entries $a^{w,c,q}_{P',P}$ (cf. Sect. 3.4). Obviously, the terms $\vartheta_{P'}(a\psi_P)$ (cf. (3.13)) can be computed without difficulty, and the corresponding number of arithmetic operations is less than $O(N \log N)$. Therefore, we only have to deal with the computation of $\vartheta_{P'}(K\psi_P)$ corresponding to the integral operator K. First we shall indicate the assembling of those entries for which $\operatorname{dist}(\Psi_P, \Theta_{P'})$ is large in a certain sense. We shall fix P' and define a quadrature partition in dependence on P'. Clearly, if a trial function ψ_P has discontinuous first order derivatives over a subdomain, then the standard low order quadrature rules are not very accurate. Therefore, the quadrature partition will be finer than the partition into the patches of linearity, i.e., all trial functions ψ_P with (P', P) in the sparsity pattern $\mathcal{P}$ (cf. Sect. 3.5) will not only be piecewise linear but linear with respect to the parametrization κ_m on each quadrature subdomain. In the class of all partitions, we shall choose the coarsest partition with the just mentioned property. Over the subdomains of this partition we shall approximate the parametrizations κ_m by a low order polynomial interpolation and apply a composite quadrature rule.

Let us define the partition. For $l = 0, \ldots, L$, we introduce the set Qua_l^Γ as the set of all $Q \in \square_l^\Gamma$ such that:

i) There is a $P \in \nabla_{l-1}^\Gamma$ such that $(P', P) \in \mathcal{P}$ and that support Ψ_P intersects the father Γ_{Q^F} of Γ_Q.
ii) If $l < L$, then we suppose that, for any $P \in \nabla_l^\Gamma$ with $(P', P) \in \mathcal{P}$, there holds $\Gamma_Q \cap \Psi_P = \emptyset$.

Lemma 4.1 *The set $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$ is a partition of Γ. For all P with $(P', P) \in \mathcal{P}$ and for all $Q \in \cup_{l=0}^L Qua_l^\Gamma$, the restriction of ψ_P to Γ_Q is linear with respect to the parametrization. Moreover, the partition $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$ is the coarsest partition with this linearity property and with $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\} \subseteq \{\Gamma_Q : Q \in \cup_{l=0}^L \square_l^\Gamma\}$.*

Proof. Clearly, condition i) means that in a partition of Γ the subset Γ_Q cannot be substituted by a larger $\Gamma_{Q'}$ without violating the linearity property. Namely, if Γ_Q would be replaced by $\Gamma_{Q'}$, then $\Gamma_{Q^F} \subseteq \Gamma_{Q'}$ and the function ψ_P with $(P', P) \in \mathcal{P}$ and with $\operatorname{supp} \psi_P \cap \Gamma_{Q^F} \neq \emptyset$ (cf. condition i)) has a discontinuous first derivative over $\Gamma_{Q'}$. On the other hand, condition ii) means that it is not necessary to divide Γ_Q further into smaller subdomains since already all the trial basis function ψ_P with $(P', P) \in \mathcal{P}$ are linear over Γ_Q. Indeed, the wavelet functions of level l with $(P', P) \in \mathcal{P}$ vanish over Γ_Q due to ii), and, due to the definition of $\mathcal{P}$ in (3.18), (3.19), the higher level wavelet functions with $(P', P) \in \mathcal{P}$ vanish over Γ_Q, too. The lower level wavelets, however, are linear on Γ_Q.

To show that $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$ is a partition of Γ, we have to prove that the partition subsets cover Γ and that their interiors are disjoint. Obviously, Γ is covered. Indeed, for any $Q_L \in \square_L^\Gamma$, let us consider the sequence

$$\Gamma_{Q_L},\ \Gamma_{Q_{L-1}} := \text{father of } \Gamma_{Q_L},\ \Gamma_{Q_{L-2}} := \text{father of } \Gamma_{Q_{L-1}},\ \ldots\ , \Gamma_{Q_0} := \text{father of } \Gamma_{Q_1}.$$

In view of the conditions i) and ii), there is exactly one Γ_{Q_m} in this sequence belonging to Qua_m^Γ. Hence, each Γ_{Q_L} is contained in the union of the subdomains $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$. Furthermore, we observe that two sets Γ_Q and $\Gamma_{Q'}$ either have disjoint interiors or one of the two sets is contained in the other. If, for example, $\Gamma_{Q'} \subseteq \Gamma_Q$, then at most one of the sets Γ_Q and $\Gamma_{Q'}$ fulfills i) and ii). Hence, the interiors of the sets in $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$ are disjoint.

Now the first part of this proof implies that the partition $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$ is the coarsest satisfying the desired linearity property. ∎

The partition $\{\Gamma_Q : Q \in \cup_{l=0}^L Qua_l^\Gamma\}$ can be determined analogously to the determination of the sparsity pattern in the step ii) of the algorithms (3.15) and (3.16) described at the end of Sect. 3.5. For each P', we have to determine the sets Qua_l^Γ with $l = 0, \ldots, L$. We do this for each level l separately. First we set up Qua_0^Γ. Then, if the subsets $Qua_{l'}^\Gamma$, $l' = 0, \ldots, l-1$ are determined, the search for the $Q \in \square_l^\Gamma$ satisfying the conditions i) and ii) can be restricted to all $Q \in \square_l^\Gamma$ with

$$\Gamma_Q \subseteq \Gamma \setminus \left[\cup_{l'=0}^{l-1} \cup_{R \in Qua_{l'}^\Gamma} \Gamma_R \right].$$

Doing this for all $l = 1, \ldots, L$ and for all $P' \in \triangle_L^\Gamma$, only $O(N_{\mathcal{P}})$ of the $O(N^2)$ domains Γ_Q have to be checked whether they satisfy the conditions i) and ii) or not.

In view of (3.18) and (3.19), condition i) is equivalent to the existence of a $P \in \nabla^\Gamma_{l-1}$ such that $\Psi_P \cap \Gamma_{Q^F} \neq \emptyset$ and that either

$$\operatorname{dist}(\Psi_P, \Theta_{P'}) \le \max\left\{2^{-(l-1)}, 2^{-l(P')}, d2^{aL-b(l-1)-cl(P')}\right\} \tag{4.1}$$

for Ψ_P contained in the interior of a single parametrization patch Γ_m or

$$\operatorname{dist}(\Psi_P, \Theta_{P'}) \le \max\left\{2^{-(l-1)}, 2^{-l(P')}, \tilde{d}2^{\tilde{a}L-\tilde{b}(l-1)-\tilde{c}l(P')}\right\} \tag{4.2}$$

for Ψ_P not contained in the interior of a single parametrization patch. On the other hand, for an appropriate constant $c_0 > 0$, the diameter of Ψ_P, $P \in \nabla^\Gamma_{l-1}$ is less than $c_0 2^{-(l-1)}$. Hence, the inequalities (4.1) and (4.2) imply either the estimate

$$\operatorname{dist}(\Gamma_Q, \Theta_{P'}) \quad \le \quad (1+c_0)\max\left\{2^{-(l(Q)-1)}, 2^{-l(P')}, d2^{aL-b(l(Q)-1)-cl(P')}\right\} \tag{4.3}$$

or the estimate

$$\operatorname{dist}(\Gamma_Q, \Theta_{P'}) \quad \le \quad (1+c_0)\max\left\{2^{-(l(Q)-1)}, 2^{-l(P')}, \tilde{d}2^{\tilde{a}L-\tilde{b}(l(Q)-1)-\tilde{c}l(P')}\right\}. \tag{4.4}$$

In particular, if Γ_Q is contained in the interior of a single parametrization patch Γ_m and if its distance to the boundary of Γ_m is greater than $c_0 2^{-(l-1)}$, then (4.3) holds.

Condition ii) is satisfied, if and only if, for any $P \in \nabla^\Gamma_l$ with $\Gamma_Q \cap \Psi_P \neq \emptyset$ and with Ψ_P contained in the interior of a single parametrization patch Γ_m, there holds

$$\operatorname{dist}(\Psi_P, \Theta_{P'}) > \max\left\{2^{-l}, 2^{-l(P')}, d2^{aL-bl-cl(P')}\right\}, \tag{4.5}$$

and, for any $P \in \nabla^\Gamma_l$ with $\Gamma_Q \cap \Psi_P \neq \emptyset$ and with Ψ_P not contained in the interior of a single parametrization patch Γ_m, there holds

$$\operatorname{dist}(\Psi_P, \Theta_{P'}) > \max\left\{2^{-l}, 2^{-l(P')}, \tilde{d}2^{\tilde{a}L-\tilde{b}l-\tilde{c}l(P')}\right\}. \tag{4.6}$$

On the other hand, Γ_Q is covered by the Ψ_P with $\Psi_P \cap \Gamma_Q \neq \emptyset$. Hence, the criteria (4.5) and (4.6) ensure either the validity of

$$\operatorname{dist}(\Gamma_Q, \Theta_{P'}) > \max\left\{2^{-l(Q)}, 2^{-l(P')}, d2^{aL-bl(Q)-cl(P')}\right\} \tag{4.7}$$

or the validity of

$$\operatorname{dist}(\Gamma_Q, \Theta_{P'}) > \max\left\{2^{-l(Q)}, 2^{-l(P')}, \tilde{d}2^{\tilde{a}L-\tilde{b}l(Q)-\tilde{c}l(P')}\right\}. \tag{4.8}$$

In particular, if Γ_Q is contained in the interior of a single parametrization patch Γ_m and if its distance to the boundary of Γ_m is greater than $c_0 2^{-l}$, then (4.7) holds. Having in mind the estimates (4.7) and (4.8), we shall call the quadrature subdomains of $\cup_{l=0}^{L-1}\{\Gamma_Q : Q \in Qua^\Gamma_l\}$ the far field subdomains corresponding to the functional $\vartheta_{P'}$. The domains $\{\Gamma_Q : \ Q \in Qua^\Gamma_L\}$ will be referred to as near field subdomains.

In accordance with (3.13) and (2.4), we shall introduce quadrature approximations $a^{w,c,q}_{P',P,Q}$ for

$$\vartheta_{P'}\left(\int_{\Gamma_Q} k(\cdot, R, n_R)\frac{p(\cdot - R)}{|\cdot - R|^\alpha}\psi_P(R)\,\mathrm{d}_R\Gamma\right). \tag{4.9}$$

Here the functional $\vartheta_{P'}$ is applied to the function in brackets depending on the variable indicated by a dot. Using these $a^{w,c,q}_{P',P,Q}$, we define the entries $a^{w,c,q}_{P',P}$ by

$$a^{w,c,q}_{P',P} \;:=\; \vartheta_{P'}(a\psi_P) + \begin{cases} 0 & \text{if } (P',P) \notin \mathcal{P} \\ \sum_{l=0}^{L} \;\sum_{Q\in Qua^\Gamma_l:\,\Gamma_Q\subset \operatorname{supp}\psi_P} a^{w,c,q}_{P',P,Q} & \text{if } (P',P) \in \mathcal{P}. \end{cases} \tag{4.10}$$

We shall defer the definition of the near field terms $a^{w,c,q}_{P',P,Q}$, $Q \in Qua^\Gamma_L$ to Sects. 4.2 and 4.3. In this subsection we introduce the far field terms $a^{w,c,q}_{P',P,Q}$ with $Q \in Qua^\Gamma_l$ and l running from 0 to $L-1$.

Let us fix a far field subdomain Γ_Q with $Q = \kappa_m(\tau) \in Qua^\Gamma_l$. Using the parametrization κ_m over $T_\tau = \kappa_m^{-1}(\Gamma_Q)$, we write the integral of (4.9) in the form

$$\vartheta_{P'}\left(\int_{T_\tau} k(\cdot,\kappa_m(\sigma),n_{\kappa_m(\sigma)})\frac{p\big(\cdot-\kappa_m(\sigma)\big)}{|\cdot-\kappa_m(\sigma)|^\alpha}\tilde{\psi}_P(\sigma)\mathcal{J}_m(\sigma)\,\mathrm{d}\sigma\right), \tag{4.11}$$

where $\mathcal{J}_m(\sigma) := |\partial_{\sigma_1}\kappa_m(\sigma)\times\partial_{\sigma_2}\kappa_m(\sigma)|$ is the Jacobian determinant of the transformation κ_m at $\sigma = (\sigma_1,\sigma_2) \in T_\tau$ and where $\tilde{\psi}_P(\sigma)$ stands for the factor $\psi_P(R) = \psi_P(\kappa_m(\sigma))$ which is independent of the parametrization κ_m (cf. (3.2) and (2.6)). We derive the approximation $a^{w,c,q}_{P',P,Q}$ for (4.11) in three steps.

In the first step, we replace the parametrization κ_m over T_τ by a polynomial interpolation κ'_m of degree $\mathbf{m} := 2-\mathbf{r}$, i.e., we use a cubic interpolation with nine interpolation knots for $\mathbf{r} = -1$ and a quadratic interpolation with six knots for $\mathbf{r} = 0$. For instance, the quadratic interpolation is defined as in [2]. Denoting by τ_i, $i = 1,2,3$ the three corner points and by τ_i, $i = 4,5,6$ the mid-points

$$\tau_4 = \frac{1}{2}(\tau_2+\tau_3), \quad \tau_5 = \frac{1}{2}(\tau_1+\tau_2), \quad \tau_6 = \frac{1}{2}(\tau_1+\tau_3),$$

of the three sides of the triangle $T_\tau = \kappa_m^{-1}(\Gamma_Q)$, we set

$$\begin{aligned} \kappa'_m(\sigma) &= \sum_{i=1}^{6}\kappa_m(\tau_i)\mathcal{L}_i(\sigma), && (4.12)\\ \mathcal{L}_1\Big(\tau_3+s(\tau_1-\tau_3)+t(\tau_2-\tau_3)\Big) &:= s[2s-1], \\ \mathcal{L}_2\Big(\tau_3+s(\tau_1-\tau_3)+t(\tau_2-\tau_3)\Big) &:= t[2t-1], \\ \mathcal{L}_3\Big(\tau_3+s(\tau_1-\tau_3)+t(\tau_2-\tau_3)\Big) &:= (1-s-t)[2(1-s-t)-1], \\ \mathcal{L}_4\Big(\tau_3+s(\tau_1-\tau_3)+t(\tau_2-\tau_3)\Big) &:= 4t(1-s-t), \\ \mathcal{L}_5\Big(\tau_3+s(\tau_1-\tau_3)+t(\tau_2-\tau_3)\Big) &:= 4st, \\ \mathcal{L}_6\Big(\tau_3+s(\tau_1-\tau_3)+t(\tau_2-\tau_3)\Big) &:= 4s(1-s-t). \end{aligned}$$

Hence, we approximate (4.11) by

$$\vartheta_{P'}\left(\int_{T_\tau} k(\cdot,\kappa_m(\sigma),n'_{\kappa'_m(\sigma)})\frac{p\big(\cdot-\kappa'_m(\sigma)\big)}{|\cdot-\kappa'_m(\sigma)|^\alpha}\tilde{\psi}_P(\sigma)\mathcal{J}'_m(\sigma)\,\mathrm{d}\sigma\right), \tag{4.13}$$

where $\mathcal{J}'_m(\sigma) := |\partial_{\sigma_1}\kappa'_m(\sigma) \times \partial_{\sigma_2}\kappa'_m(\sigma)|$ is the Jacobian determinant of the transformation κ'_m at $\sigma = (\sigma_1, \sigma_2) \in T_\tau$. The symbol $n'_{\kappa'_m(\sigma)}$ in the last formula stands for the unit vector at the point $\kappa'_m(\sigma)$ which is normal to the approximating surface $\kappa'_m(T_\tau)$.

In the second step, we split the integrand of (4.13) into the product $f(\sigma)\tilde{\varrho}(\sigma)$

$$\begin{aligned} f(\sigma) &:= k(\cdot, \kappa_m(\sigma), n_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma), \\ \tilde{\varrho}(\sigma) &:= \varrho\Big(\kappa'_m(\sigma)\Big) = \frac{p\Big(\cdot - \kappa'_m(\sigma)\Big)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\tilde{\psi}_P(\sigma). \end{aligned}$$

Note that f is globally $\mathbf{m}$ times differentiable by assumption whereas ϱ is singular at the points of $\operatorname{supp}\vartheta_{P'}$. We apply a product quadrature with weight $\tilde{\varrho}$ and of order $\mathbf{m}$ to the integral in (4.13). If $\mathbf{r} = -1$, then we choose the six point rule based upon the quadratic interpolation which has been used for (4.12). In case $\mathbf{r} = 0$ we take the three point rule. To simplify the notation, however, we write all the following formulae explicitly for the three point rule. The modifications for the corresponding formulae including the six point rule are straightforward. In the estimates and the convergence results, we always suppose that a quadrature of order $\mathbf{m}$ is in use. The product quadrature rule takes the form

$$\int_{T_\tau} f(\sigma)\tilde{\varrho}(\sigma)\,\mathrm{d}\sigma \approx \sum_{v=1}^{3} f(\tau_v)\int_{T_\tau}\tilde{\phi}_{Q,v}(\sigma)\tilde{\varrho}(\sigma)\,\mathrm{d}\sigma,$$

where $\tilde{\phi}_{Q,v}$ is the linear function on T_τ defined by $\tilde{\phi}_{Q,v}(\tau_{v'}) = \delta_{v,v'}$. In other words, the integral (4.13) is approximated by

$$\vartheta_{P'}\left(\sum_{v=1}^{3} k(\cdot, Q_v, n'_{Q'_v})\mathcal{J}'_m(\tau_v)b^{w,c,q}_{P,Q,v}(\cdot)\right), \tag{4.14}$$

$$b^{w,c,q}_{P,Q,v}(R) := \int_{T_\tau}\tilde{\phi}_{Q,v}(\sigma)\frac{p\Big(R - \kappa'_m(\sigma)\Big)}{|R - \kappa'_m(\sigma)|^\alpha}\tilde{\psi}_P(\sigma)\,\mathrm{d}\sigma, \tag{4.15}$$

where $Q_v := \kappa_m(\tau_v)$ and $Q'_v := \kappa'_m(\tau_v)$ denote the corner points of the triangles $\Gamma_Q = \kappa_m(T_\tau)$ and $\kappa'_m(T_\tau)$, respectively. The symbol $n'_{Q'_v}$ in the last formula stands for the unit vector at the point $Q'_v = \kappa'_m(\tau_v)$ which is normal to the approximating surface $\kappa'_m(T_\tau)$.

In the third and last step we have to compute the quadrature weights $b^{w,c,q}_{P,Q,v}$ of the product rule, i.e. the integrals over T_τ of $g(\sigma) := \tilde{\phi}_{Q,v}(\sigma)\varrho(\kappa'_m(\sigma))$. In some applications these integrals can be computed analytically. For the general case, we have to compute them by quadrature. Note that the weight ϱ is a smooth function on Γ_Q with singularities sufficiently far from Γ_Q. Under these circumstances, the integral of g can be approximated e.g. by panel clustering or multipole techniques (cf. [41, 21]). We, however, describe a third alternative following [20, 23, 32, 44]. To get a quadrature rule over T_τ, we start from the Gauß-Legendre rule over $[0,1]$, i.e., from the interpolatory rule including the zeros σ^k_G, $k = 1, \ldots, n_G$ of the Legendre polynomial as quadrature knots.

$$\int_0^1 F \approx \sum_{k=1}^{n_G} F(\sigma^k_G)\omega^k_G. \tag{4.16}$$

The order n_G will be specified later. Introducing Duffy's coordinates and applying the Gauß type tensor product rule to the resulting double integral, we arrive at

$$\int_{T_\tau} g(\sigma)\,\mathrm{d}\sigma = \int_0^1\int_0^1 g\Big(\tau_3 + \sigma^D_1(\tau_1 - \tau_3) + \sigma^D_1\sigma^D_2(\tau_2 - \tau_3)\Big)\sigma^D_1\,\mathrm{d}\sigma^D_2\,\mathrm{d}\sigma^D_1 \cdot 2\,|T_\tau|$$

$$\approx \sum_{k_1=1}^{n_G}\sum_{k_2=1}^{n_G} g\Big(\tau_3+\sigma_G^{k_1}(\tau_1-\tau_3)+\sigma_G^{k_1}\sigma_G^{k_2}(\tau_2-\tau_3)\Big)\sigma_G^{k_1}\omega_G^{k_1}\omega_G^{k_2}\cdot 2\,|T_\tau|$$

$$=: \sum_{k=1}^{n_G^2} g(\sigma_\tau^k)\omega_\tau^k. \tag{4.17}$$

Note that, for the numerical implementation, one could try to replace the rule (4.17) by triangular rules of high order or e.g. by Stroud's conical product rule (cf. [47]) which is a slight modification of (4.17).

Thus the formulae (4.14), (4.15), and (4.17) together yield

$$a^{w,c,q}_{P',P,Q} \;:=\; \vartheta_{P'}\left(\sum_{v=1}^{3} k(\cdot,Q_v,n'_{Q'_v})\mathcal{J}'_m(\tau_v)\sum_{k=1}^{n_G^2}\tilde\phi_{Q,v}(\sigma_\tau^k)\left[\frac{p\Big(\cdot-\kappa'_m(\sigma_\tau^k)\Big)}{|\cdot-\kappa'_m(\sigma_\tau^k)|^\alpha}\tilde\psi_P(\sigma_\tau^k)\right]\omega_\tau^k\right). \tag{4.18}$$

For $Q\in Qua_l^\Gamma$, we choose the quadrature order n_G in the last formula by

$$n_G \;:=\; n_A+n_B\left\lceil\frac{l}{1+{}^2\log\left(\frac{\operatorname{dist}(\Theta_{P'},\Gamma_Q)}{2^{-l}}\right)}\right\rceil, \tag{4.19}$$

where the integers $n_A>0$ and $n_B>0$ have to be determined by numerical experiments. In Sect. 6.1 we shall prove the existence of positive integers n_A and n_B such that the additional error due to the far field quadrature is, roughly speaking, less than the error of the exact collocation. Analogous error estimates are true also for the approximation of the near field and the singular integrals in the Sects. 4.2 and 4.3. More precisely, to get asymptotically optimal results, we choose the compression parameters $a=c=b=\tilde b=1$, and $\tilde a=\tilde c=5/3$. We define the functions $d=C\,L^{1/8}$ and $\tilde d=C\,L^{1/4}$ with a sufficiently large constant C and get

Theorem 4.1 *For the pattern $\mathcal{P}=\mathcal{P}(1,1,1,CL^{1/8},5/3,1,5/3,CL^{1/4})$, the number of non-zero entries $N_{\mathcal{P}}$ is less than $CL^{9/4}2^{2L}\sim N[\log N]^{2.25}$, where $N\sim 2^{2L}$ is the number of degrees of freedom. If the exact collocation described in Sect. 2.5 is stable, then the compressed collocation with approximation of the boundary and with the quadrature of Sects. 4.1 - 4.3 is stable, too. The error for the collocation solution u_L, including compression, approximation of the parameter mappings, and quadrature, satisfies*

$$\|u-u_L\|_{L^2(\Gamma)} \;\le\; C\,h^2\begin{cases}[\log h^{-1}]^2 & \text{if } \mathbf{r}=0\\ [\log h^{-1}]^{1.625} & \text{if } \mathbf{r}=-1,\end{cases} \tag{4.20}$$

$$\|u-u_L\|_{H^{-1}(\Gamma)} \;\le\; C\,h^3\quad [\log h^{-1}]^{1.625}\quad \text{if } \mathbf{r}=-1. \tag{4.21}$$

The number of quadrature knots and the number of necessary arithmetic operations for the computation of the stiffness matrix $A_L^{w,c,q}$ is less than $C\,N[\log N]^{4.25}$.

Proof. The bound for the number of entries in the compressed stiffness matrix will follow from Lemma 5.6. Stability and error estimates will be a consequence of the Lemmata 5.8, 6.1, 6.3, and 6.5. The complexity bound will be shown in the Lemmata 6.2, 6.4, and 6.6. ∎

Remark 4.1 *A clever code for the computation of the $a^{w,c,q}_{P',P,Q}$ computes first, for fixed $\vartheta_{P'}$ and Q, the quadratures in (4.18) with $\psi_P \circ \kappa_m$ replaced by the three linear basis functions $\phi_{Q,\iota}$, $\iota = 1,2,3$ over T_τ (cf. the basis functions $\phi_{Q,\iota}$ in Sect. 6.1). Then, in a loop over all P with $\Gamma_Q \subset \operatorname{supp}\psi_P$, the values $a^{w,c,q}_{P',P,Q}$ are evaluated as a linear combination of the three quadratures over the basis functions, and $a^{w,c,q}_{P',P,Q}$ is updated to the actual value of the sum (4.10).*

4.2 Parametrization and Quadrature for the Near Field

Let us fix a test functional $\vartheta_{P'}$ and a $Q \in Qua^{\Gamma}_L$, and let us consider the integral (4.9) for which we seek the quadrature $a^{w,c,q}_{P',P,Q}$. Recall from Sect. 3.2 that the test functional $\vartheta_{P'}$ is a linear combination of point evaluation functionals. Thus there are points P_λ and uniformly bounded coefficients μ_λ such that

$$\vartheta_{P'}(f) \;=\; \sum_{\lambda=1}^{\lambda_{P'}} \mu_\lambda f(P_\lambda). \tag{4.22}$$

Obviously, $\lambda_{P'} = 1$ if $P' \in \nabla^{\Gamma}_{-1}$ and $\lambda_{P'} = 3$ else. If the test functional is replaced by $\vartheta^{+}_{P'}$, then we get $\lambda_{P'} = 4,6$ for $P' \in \nabla^{\Gamma}_{P'}$ with $l \geq 2$. In correspondence with (4.22), we can split the unknown quadrature expression $a^{w,c,q}_{P',P,Q}$ into

$$a^{w,c,q}_{P',P,Q} = \sum_{\lambda=1}^{\lambda_{P'}} \mu_\lambda \, a^{w,c,q}_{P',\lambda,P,Q},$$

where $a^{w,c,q}_{P',\lambda,P,Q}$ is defined as a quadrature for the integral

$$\int_{\Gamma_Q} k(P_\lambda, R, n_R) \frac{p(P_\lambda - R)}{|P_\lambda - R|^\alpha} \psi_P(R)\, \mathrm{d}_R\Gamma. \tag{4.23}$$

We distinguish two cases. If P_λ is in Γ_Q, then the integral (4.23) is singular, and we defer the definition of the singular quadrature $a^{w,c,q}_{P',\lambda,P,Q}$ to Sect. 4.3. For $P_\lambda \notin \Gamma_Q$, the integral (4.23) is not singular and the corresponding non-singular near field quadrature $a^{w,c,q}_{P',\lambda,P,Q}$ is treated now. We apply the technique of the previous subsection (cf. the quadrature rule of (4.18)) to (4.23) and get

$$a^{w,c,q}_{P',\lambda,P,Q} \;:=\; \sum_{v=1}^{3} k(P_\lambda, Q_v, n'_{Q'_v}) \mathcal{J}'_m(\tau_v) \sum_{k=1}^{n_G^2} \tilde{\phi}_{Q,v}(\sigma^k_\tau) \left[\frac{p\Big(P_\lambda - \kappa'_m(\sigma^k_\tau)\Big)}{|P_\lambda - \kappa'_m(\sigma^k_\tau)|^\alpha} \tilde{\psi}_P(\sigma^k_\tau) \right] \omega^k_\tau, \tag{4.24}$$

where this time the order n_G is chosen by $n_G := n_C + L n_D$. In practical computations the integers $n_C > 0$ and $n_D > 0$ have to be determined by experiments. However, in Sect. 6.2 we shall prove the existence of positive integers n_C and n_D such that the additional error due to the non-singular near field quadrature is, roughly speaking, less than the error of the exact collocation.

4.3 Parametrization and Quadrature for Entries with Singular Integrals

4.3.1. First we consider the case of weakly singular integrals. This occurs if $\mathbf{r} = -1$ or if $\mathbf{r} = 0$ and the kernel function depending on the variables P and R contains a factor

$n_P \cdot (P-R)$ or $n_R \cdot (P-R)$. For definiteness, we restrict our consideration to the case of an additional factor $n_R \cdot (P-R)$. More precisely, we suppose that the kernel takes the form

$$k(P,R,n_R)\,\frac{p(P-R)}{|P-R|^\alpha} \;=\; \tilde{k}(P,R,n_R)\,\frac{\tilde{p}(P-R)[n_R \cdot (P-R)]^{1+\mathbf{r}}}{|P-R|^\alpha}\,. \tag{4.25}$$

Here $\tilde{k}=k$ and $\tilde{p}=p$ if $\mathbf{r}=-1$. For $\mathbf{r}=0$, we assume that $\tilde{k}$ fulfills all the assumptions made for k in Sect. 2.2 and that $\tilde{p}$ is a homogeneous polynomial of degree $\deg(\tilde{p}) = \deg(p)-1$, i.e., $\deg(\tilde{p})-\alpha=-3$. Hence, for a suitable constant $C>0$, we get

$$\begin{aligned} |n_R \cdot (P-R)| &\le C\,|P-R|^2, \\ \left|\tilde{k}(P,R,n_R)\frac{\tilde{p}(P-R)[n_R\cdot(P-R)]^{1+\mathbf{r}}}{|P-R|^\alpha}\right| &\le C\,|P-R|^{-1}, \end{aligned}$$

and our kernel (4.25) is weakly singular, indeed. Notice that the kernel of the double layer integral operator K_d (cf. Sect. 2.2) can be represented as in (4.25) if $\mathbf{r}$ is set to zero.

Now we fix the test functional $\vartheta_{P'}$, a point $P_\lambda \in \operatorname{supp}\vartheta_{P'}$, and a triangle $\Gamma_Q = \kappa_m(T_\tau)$ with $Q=\kappa_m(\tau)\in \square_L^\Gamma$ and $P_\lambda \in \Gamma_Q$. Clearly, the grid point P_λ is one of the corner points of Γ_Q. We denote the three corners of T_τ by τ_ι, $\iota=1,2,3$ and suppose $\kappa_m(\tau_3)=P_\lambda$. In the triangles T_τ and Γ_Q we introduce Duffy's coordinates.

$$\begin{aligned} \delta\left(\sigma^D\right) &:= \delta\left(\sigma_1^D,\sigma_2^D\right) := \tau_3 + \sigma_1^D(\tau_1-\tau_3) + \sigma_1^D\sigma_2^D(\tau_2-\tau_3), \\ \tilde{\kappa}_m\left(\sigma^D\right) &:= \kappa_m\Big(\delta\left(\sigma^D\right)\Big). \end{aligned} \tag{4.26}$$

The Jacobian determinant corresponding to Duffy's coordinate in T_τ is given by $\mathcal{J}_\delta(\sigma^D) = |(\tau_1-\tau_3)\times(\tau_2-\tau_3)|\sigma_1^D = 2\,|T_\tau|\sigma_1^D$ and the Jacobian $\tilde{\mathcal{J}}_m(\sigma^D)$ of $\tilde{\kappa}_m$ is equal to the product $\mathcal{J}_m(\delta(\sigma^D))\mathcal{J}_\delta(\sigma^D)$. We seek an approximation $a^{w,c,q}_{P',\lambda,P,Q}$ for the integral

$$\begin{aligned} &\int_{\Gamma_Q} \tilde{k}(P_\lambda,R,n_R)\frac{\tilde{p}(P_\lambda-R)[n_R\cdot(P_\lambda-R)]^{1+\mathbf{r}}}{|P_\lambda-R|^\alpha}\psi_P(R)\,\mathrm{d}_R\Gamma \;= \\ &\int_0^1\int_0^1 \Bigg\{\tilde{k}(P_\lambda,\tilde{\kappa}_m(\sigma^D),n_{\tilde{\kappa}_m(\sigma^D)})\frac{\tilde{p}\Big(P_\lambda-\tilde{\kappa}_m(\sigma^D)\Big)\Big[n_{\tilde{\kappa}_m(\sigma^D)}\cdot\Big(P_\lambda-\tilde{\kappa}_m(\sigma^D)\Big)\Big]^{1+\mathbf{r}}}{|P_\lambda-\tilde{\kappa}_m(\sigma^D)|^\alpha}\,\cdot \\ &\qquad \mathcal{J}_m\Big(\delta(\sigma^D)\Big)\mathcal{J}_\delta(\sigma^D)\tilde{\psi}_P^D(\sigma^D)\Bigg\}\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D, \end{aligned} \tag{4.27}$$

where $\tilde{\psi}_P^D(\sigma^D) := \psi_P(\tilde{\kappa}_m(\sigma^D))$. Due to the additional factor σ_1^D in $\mathcal{J}_\delta(\sigma^D)$, the weak singularity of the kernel function is cancelled.

Similarly as before, we proceed in three steps. First, we replace the parametrization $\tilde{\kappa}_m$ by the approximate parametrization in Duffy coordinates $\tilde{\kappa}'_m := \kappa'_m \circ \delta$, where κ'_m is the polynomial interpolation to κ_m of polynomial degree $\mathbf{m} = 2-\mathbf{r}$. We suppose that P_λ is one of the interpolation knots. Second, we apply a product rule of order $\mathbf{m}$. To this end the integrand in (4.27) with $\tilde{\kappa}_m$ replaced by $\tilde{\kappa}'_m$ is split into the product $f\cdot\varrho$ with

$$\begin{aligned} f(\sigma^D) &:= \tilde{k}(P_\lambda,\tilde{\kappa}_m(\sigma^D),n'_{\tilde{\kappa}'_m(\sigma^D)})\mathcal{J}'_m\Big(\delta(\sigma^D)\Big), \\ \varrho(\sigma^D) &:= \frac{\tilde{p}\Big(P_\lambda-\tilde{\kappa}'_m(\sigma^D)\Big)\Big[n'_{\tilde{\kappa}'_m(\sigma^D)}\cdot\Big(P_\lambda-\tilde{\kappa}_m(\sigma^D)\Big)\Big]^{1+\mathbf{r}}}{|P_\lambda-\tilde{\kappa}_m(\sigma^D)|^\alpha}\mathcal{J}_\delta(\sigma^D)\tilde{\psi}_P^D(\sigma^D). \end{aligned}$$

For $\mathbf{r} = -1$, the quadrature rule could be the tensor product variant of a quadratic interpolatory rule and, for $\mathbf{r} = 0$, we simply take the tensor product linear interpolatory rule.

$$\int_0^1 \int_0^1 f(\sigma^D)\varrho(\sigma^D)\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D \;\approx\; \sum_{\upsilon=1}^{4} f(\tau_\upsilon^D) \int_0^1 \int_0^1 \tilde{\phi}_\upsilon^D(\sigma^D)\varrho(\sigma^D)\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D,$$

where τ_υ^D, $\upsilon = 1,\ldots,4$ denote the four corners of $[0,1]\times[0,1]$ and $\tilde{\phi}_\upsilon^D$ is the bilinear function defined by $\tilde{\phi}_\upsilon^D(\tau_{\upsilon'}^D) = \delta_{\upsilon,\upsilon'}$. Again, to simplify the notation we shall write the subsequent formulae with the linear interpolatory rule. The modifications for the tensor product of the quadratic interpolatory rule are straightforward. In the third and last step we apply the tensor product variant of the Gauß-Legendre rule (4.16) of order n_G

$$\int_0^1 \int_0^1 g(\sigma^D)\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D \;\approx\; \sum_{k_1=1}^{n_G} \sum_{k_2=1}^{n_G} g\left(\sigma_G^{k_1}, \sigma_G^{k_2}\right)\omega_G^{k_1}\omega_G^{k_2} \;=:\; \sum_{k=1}^{n_G^2} g(\tilde{\sigma}^k)\tilde{\omega}^k,$$

with order $n_G = n_E + L\,n_F$ to compute the integral of the function $g(\sigma^D) = \tilde{\phi}_\upsilon^D(\sigma^D)\varrho(\sigma^D)$. Finally, we arrive at

$$a^{w,c,q}_{P',\lambda,P,Q} \;:=\; \sum_{\upsilon=1}^{4} \tilde{k}(P_\lambda, Q_\upsilon^D, n'_{R_\upsilon^D})\mathcal{J}'_m\left(\delta\left(\tau_\upsilon^D\right)\right)\;\cdot$$
$$\sum_{k=1}^{n_G^2} \tilde{\phi}_\upsilon^D(\tilde{\sigma}^k)\frac{\tilde{p}\big(P_\lambda - \tilde{\kappa}'_m(\tilde{\sigma}^k)\big)\left[n'_{\tilde{\kappa}'_m(\tilde{\sigma}^k)}\cdot\big(P_\lambda - \tilde{\kappa}'_m(\tilde{\sigma}^k)\big)\right]^{1+\mathbf{r}}}{|P_\lambda - \tilde{\kappa}'_m(\tilde{\sigma}^k)|^\alpha}\mathcal{J}_\delta(\tilde{\sigma}^k)\tilde{\psi}_P^D(\tilde{\sigma}^k)\tilde{\omega}^k.$$

Here we have set $Q_\upsilon^D := \tilde{\kappa}_m(\tau_\upsilon^D)$ and $R_\upsilon^D := \tilde{\kappa}'_m(\tau_\upsilon^D)$, and $n'_{Q''}$ denotes the unit normal to the approximate surface at Q''. Note that the Jacobian of $\tilde{\kappa}'_m$ takes the form $\mathcal{J}'_m(\delta(\sigma^D))\mathcal{J}_\delta(\sigma^D)$. The numbers n_E and n_F in the definition of n_G are to be determined by numerical experiments. However, in Sect. 6.3 we shall prove the existence of values of n_E and n_F ensuring asymptotically optimal error estimates.

4.3.2. Now let us consider $\mathbf{r} = 0$ and suppose the integral operator is strongly singular. If the value $\psi_P(P_\lambda)$ vanishes or if, according to Remark 4.1, ψ_P is replaced by a linear basis function $\phi_{Q,\iota}$ and $\phi_{Q,\iota}(P_\lambda) = 0$, then this additional zero turns the strongly singular integral into a weakly singular, and we may apply the same procedure as for the weakly singular case treated before. For $\psi_P(P_\lambda) \neq 0$ or $\phi_{Q,\iota}(P_\lambda) \neq 0$, we substitute $\psi_P = \psi_P(P_\lambda) + (\psi_P - \psi_P(P_\lambda))$ resp. $\phi_{Q,\iota} = \phi_{Q,\iota}(P_\lambda) + (\phi_{Q,\iota} - \phi_{Q,\iota}(P_\lambda))$ into the singular integral. This way the integral splits into two parts, where the integral containing the functions $(\psi_P - \psi_P(P_\lambda))$ resp. $(\phi_{Q,\iota} - \phi_{Q,\iota}(P_\lambda))$ can be approximated like in the case $\psi_P(P_\lambda) = 0$. The only strongly singular case occurs if $\psi_P(P_\lambda) \neq 0$ resp. $\phi_{Q,\iota}(P_\lambda) \neq 0$ and if the function ψ_P resp. $\phi_{Q,\iota}$ are replaced the constants $\psi_P(P_\lambda)$ resp. $\phi_{Q,\iota}(P_\lambda)$. Without loss of generality we set these constants to one.

4.3.3. For the computation of the corresponding singular integrals, there exist several techniques (cf. e.g. [25, 43]). Here we shall present a quadrature algorithm similar to that in [8, 45] since this seems to require less assumptions on the smoothness. We consider a fixed singularity point P_λ. Since the singular integral is to be understood in the sense of Cauchy's principal value, we have to treat the quadrature for all Γ_Q with $P_\lambda \in \Gamma_Q$ simultaneously. Let m_0 stand for the smallest positive integer such that $P_\lambda \in \Gamma_{m_0}$. Beside m_0 we consider an arbitrary m and an arbitrary Γ_Q such that $P_\lambda \in \Gamma_Q \subseteq \Gamma_m$, i.e. $P_\lambda = \kappa_m(\tau_3)$ for a corner τ_3 of $T_\tau = \kappa_m^{-1}(\Gamma_Q)$. Note that the parameter value τ_3 in $P_\lambda = \kappa_m(\tau_3)$ depends, of course, on the parametrization κ_m and on the triangle Γ_Q. However, to simplify the notation, we do not indicate this dependence. By the assumption of Sect. 2.1 the parametrization κ_{m_0} mapping T onto Γ_{m_0} extends to a neighbourhood of T. Hence, we can define

$$\begin{aligned} T(P_\lambda, m, \varepsilon) &:= \left\{\sigma : \left|\nabla\left(\kappa_{m_0}^{-1} \circ \kappa_m\right)(\tau_3) \cdot (\sigma - \tau_3)\right| \le \varepsilon\right\}, \\ \Gamma(P_\lambda, \varepsilon) &:= \bigcup_{m=1,\dots,m_\Gamma:\, P_\lambda \in \Gamma_m} \kappa_m\Big(T(P_\lambda, m, \varepsilon)\Big) \approx \{\kappa_{m_0}(\sigma) : |\sigma - \tau_3| \le \varepsilon\}. \end{aligned}$$

By assumption the polynomial part p of the kernel function is odd. For such kernels, it is not hard to see that (cf. [29], Chapter XI, Sect. 1)

$$\left| \int_{\Gamma(P_\lambda, \varepsilon)} k(P_\lambda, R, n_R) \frac{p(P_\lambda - R)}{|P_\lambda - R|^\alpha} \, \mathrm{d}_R\Gamma \right| \le C\, \varepsilon. \tag{4.28}$$

We seek a quadrature with error less than $C\, 2^{-2L}$. Therefore, the integral over Γ can be replaced by that over $\Gamma \setminus \Gamma(P_\lambda, 2^{-2L})$, and it remains to approximate the integral

$$\begin{aligned} &\int_{\Gamma_Q \setminus \Gamma(P_\lambda, 2^{-2L})} k(P_\lambda, R, n_R) \frac{p(P_\lambda - R)}{|P_\lambda - R|^\alpha} \, \mathrm{d}_R\Gamma = \qquad (4.29) \\ &\int_{T_\tau \setminus T(P_\lambda, m, 2^{-2L})} k\left(\kappa_m(\tau_3), \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right) \frac{p\Big(\kappa_m(\tau_3) - \kappa_m(\sigma)\Big)}{|\kappa_m(\tau_3) - \kappa_m(\sigma)|^\alpha} \mathcal{J}_m(\sigma) \, \mathrm{d}\sigma, \end{aligned}$$

for each Γ_Q with $P \in \Gamma_Q$. We replace the parametrization κ_m over $T_\tau \setminus T(P_\lambda, m, 2^{-2L})$ by the quadratic interpolation κ'_m defined over T_τ in (4.12), and it remains to compute

$$\int_{T_\tau \setminus T'(P_\lambda, m, 2^{-2L})} k\left(\kappa_m(\tau_3), \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right) \frac{p\Big(\kappa'_m(\tau_3) - \kappa'_m(\sigma)\Big)}{|\kappa'_m(\tau_3) - \kappa'_m(\sigma)|^\alpha} \mathcal{J}'_m(\sigma) \, \mathrm{d}\sigma, \tag{4.30}$$

$$T'(P_\lambda, m, \varepsilon) := \left\{\sigma : \left|\nabla\left([\kappa'_{m_0}]^{-1} \circ \kappa'_m\right)(\tau_3) \cdot (\sigma - \tau_3)\right| \le \varepsilon\right\}. \tag{4.31}$$

Similarly to the product rule in Sect. 4.1, we approximate the last integral over the domain $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$ by

$$\begin{aligned} a^{w,c,q}_{P',\lambda,P,Q} &:= \sum_{v=1}^{3} k\left(\kappa_m(\tau_3), \kappa_m(\tau_v), n'_{\kappa'_m(\tau_v)}\right) \mathcal{J}'_m(\tau_v)\, b^{w,c,q}_{P',\lambda,Q,v}, \qquad (4.32) \\ b^{w,c,q}_{P',\lambda,Q,v} &\approx \int_{T_\tau \setminus T'(P_\lambda, m, 2^{-2L})} \tilde{\phi}_{Q,v}(\sigma) \frac{p\Big(\kappa'_m(\tau_3) - \kappa'_m(\sigma)\Big)}{|\kappa'_m(\tau_3) - \kappa'_m(\sigma)|^\alpha} \, \mathrm{d}\sigma. \end{aligned}$$

In contrast to the third step for the far field integrals, the quadrature approximation $b^{w,c,q}_{P',\lambda,Q,v}$ will be computed by introducing a geometric mesh and by applying high order quadrature rules over each subdomain. Fixing a grading parameter $0 < q < 1$, we denote the largest ι such that (for δ cf. (4.26))

$$T'(P_\lambda, m, 2^{-2L}) \subseteq \left\{\delta(\sigma^D) \in T_\tau : 0 \le \sigma^D_1 \le q^{\iota - 1},\ 0 \le \sigma^D_2 \le 1\right\}$$

by ι_0. Clearly, $\iota_0 \sim L$. We divide the domain of integration $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$ into the subdomains

$$\begin{aligned} T_\tau \setminus T'(P_\lambda, m, 2^{-2L}) &= \cup_{\iota=1}^{\iota_0} T_{\tau,\iota}, \qquad (4.33) \\ T_{\tau,\iota} &:= \left\{\delta(\sigma^D) \in T_\tau : q^\iota < \sigma^D_1 \le q^{\iota-1},\ 0 \le \sigma^D_2 \le 1\right\}, \quad \iota = 1, \dots, \iota_0 - 1, \\ T_{\tau,\iota_0} &:= \left\{\delta(\sigma^D) \in T_\tau : 0 \le \sigma^D_1 \le q^{\iota_0 - 1},\ 0 \le \sigma^D_2 \le 1\right\} \setminus T'(P_\lambda, m, 2^{-2L}). \end{aligned}$$

The optimal grading parameter q should be determined by numerical experiments. Note that for a different kind of integrals the choice $q = 0.15$ is optimal (cf. e.g. [45]). For fixed ι with $1 \le \iota \le \iota_0$, we observe that $T_{\tau,\iota} = \{\delta(\sigma^D) : \ 0 \le \sigma_2^D \le 1, \ S_a(\sigma_2^D) \le \sigma_1^D \le S_b\}$, where S_b is equal to $q^{\iota-1}$ and $S_a(\sigma_2^D) := q^\iota$ for $\iota < \iota_0$. The bound $S_a(\sigma_2^D)$ for $\iota = \iota_0$ is the solution σ_1^D of the equation $|\nabla([\kappa'_{m_0}]^{-1} \circ \kappa'_m)(\tau_3) \cdot (\delta(\sigma^D) - \tau_3)| = 2^{-2L}$, i.e., the boundary curve $\sigma_2^D \mapsto \delta(S_a(\sigma_2^D), \sigma_2^D)$ of the domain $T'(P_\lambda, m, 2^{-2L})$ is an ellipse. We may write the integral restricted to $T_{\tau,\iota}$ in the form

$$
\begin{aligned}
&\int_{T_{\tau,\iota}} \tilde{\phi}_{Q,v}(\sigma) \frac{p\Big(\kappa'_m(\tau_3) - \kappa'_m(\sigma)\Big)}{|\kappa'_m(\tau_3) - \kappa'_m(\sigma)|^\alpha} \,\mathrm{d}\sigma \ = \\
&\int_0^1 \int_{S_a(\sigma_2^D)}^{S_b} \tilde{\phi}_{Q,v}\left(\delta(\sigma^D)\right) \frac{p\left(\kappa'_m(\tau_3) - \tilde{\kappa}'_m(\sigma^D)\right)}{|\kappa'_m(\tau_3) - \tilde{\kappa}'_m(\sigma^D)|^\alpha} \mathcal{J}_\delta(\sigma^D) \,\mathrm{d}\sigma_1^D \,\mathrm{d}\sigma_2^D .
\end{aligned}
$$

Applying the tensor product variant of the Gauß-Legendre rule (4.16) to the last integral, we complete the formula (4.32) by the quadrature

$$
\begin{aligned}
b^{w,c,q}_{P',\lambda,Q,v} \ &:= \ \sum_{\iota=1}^{\iota_0} \sum_{k_2=1}^{n_G} \sum_{k_1=1}^{n_G} \tilde{\phi}_{Q,v}\left(\delta(\sigma^D_{k_1,k_2})\right) \frac{p\left(\kappa'_m(\tau_3) - \tilde{\kappa}'_m(\sigma^D_{k_1,k_2})\right)}{\left|\kappa'_m(\tau_3) - \tilde{\kappa}'_m(\sigma^D_{k_1,k_2})\right|^\alpha} \ . \\
&\qquad\qquad \mathcal{J}_\delta(\sigma^D_{k_1,k_2}) \left| S_b - S_a(\sigma_G^{k_2}) \right| \omega^G_{k_1} \omega^G_{k_2}, \\
\sigma^D_{k_1,k_2} \ &:= \ \left(S_a(\sigma_G^{k_2}) + \sigma_G^{k_1} \left[S_b - S_a(\sigma_G^{k_2}) \right], \sigma_G^{k_2} \right) .
\end{aligned}
\tag{4.34}
$$

The order n_G in (4.34) is chosen to be $n_G = n_E + L n_F$ again.

5 The Analysis of the Wavelet Compression

5.1 The Properties of the Three-Point Hierarchical Basis

The three-point hierarchical basis is well analyzed in the case of a hierarchy of uniform triangulations over the plane (cf. [24, 46, 27]). The triangles of level l in this hierarchy are obtained by splitting the level $l-1$ triangles into four subtriangles. This splitting is realized by connecting the three midpoints of the three sides. Unfortunately, we are not able to prove Riesz stability for the corresponding three-point hierarchical wavelets over triangles and manifolds. The reason is that the grids, where three straight lines meet in each grid point, are not suitable for the symmetric extensions which we present after Lemma 5.1. Therefore, we define our basis over the triangulations $\{T_\tau : \ \tau \in \square_l^{\mathbb{R}^2}\}$ (cf. Figure 1). For these partitions, the triangles of level l are obtained from those of level $(l-1)$ by cutting each triangle along the lines connecting one midpoint of a side with the opposite corner and with the two other midpoints. Fortunately, the techniques of proof from e.g. [46] apply also to our situation. To describe the results we need some notation. To avoid ambiguities we write $\psi_\tau^{\mathbb{R}^2}$ for ψ_τ in this section. We define the level $l(\tau)$ of τ by $l(\tau) := l$ if τ in $\nabla_l^{\mathbb{R}^2}$. From now on C stands for a generic constant the value of which varies from instance to instance. For two expressions E_1 and E_2, we write $E_1 \sim E_2$ if there is a constant independent of the parameters involved in E_1 and E_2 such that $E_1/C \le E_2 \le C\, E_1$. We get

Lemma 5.1 *For $-\alpha_H < s < 1.5$, the basis $\{\psi_\tau^{\mathbb{R}^2} : \tau \in \cup_{L=0}^\infty \triangle_L^{\mathbb{R}^2}\}$ is a Riesz basis, i.e., for any vector of real numbers $(\xi_\tau)_\tau$ we get*

$$\left\| \sum_{\tau \in \cup_{L=0}^\infty \triangle_L^{\mathbb{R}^2}} \xi_\tau \psi_\tau^{\mathbb{R}^2} \right\|_{H^s(\mathbb{R}^2)} \sim \sqrt{\sum_{\tau \in \cup_{L=0}^\infty \triangle_L^{\mathbb{R}^2}} 2^{2l(\tau)(s-1)} |\xi_\tau|^2}. \tag{5.1}$$

The positive real constant α_H is greater or equal to $0.559\ldots$.

Proof. i) In this proof we shall use the technique of Stevenson [46]. The reader is supposed to be familiar with that paper. Following [46] we introduce the quadrature approximation of the L^2-scalar product and the norm

$$\begin{aligned} \langle u, v\rangle_{\triangle_l^{\mathbb{R}^2}} &:= 2^{-2l} \left\{ \frac{2}{3} \sum_{\tau \in {}^1\triangle_l^{\mathbb{R}^2}} u(\tau)\overline{v(\tau)} + \frac{1}{3} \sum_{\tau \in {}^2\triangle_l^{\mathbb{R}^2}} u(\tau)\overline{v(\tau)} \right\}, \\ \|u\|_{\triangle_l^{\mathbb{R}^2}} &:= \sqrt{\langle u, u\rangle_{\triangle_l^{\mathbb{R}^2}}}. \end{aligned}$$

With respect to this scalar product the basis $\{\varphi_\tau^l : \tau \in \triangle_l^{\mathbb{R}^2}\}$ is orthogonal, it is $\langle\cdot,\cdot\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}$-biorthogonal to the basis $\{\varphi_\tau^{l+1} : \tau \in \triangle_l^{\mathbb{R}^2}\}$, and the wavelet functions can be represented as

$$\psi_\tau^{\mathbb{R}^2} = \varphi_\tau^{l+1} - \sum_{\tau' \in \triangle_l^{\mathbb{R}^2}} \frac{\langle \varphi_\tau^{l+1}, \varphi_{\tau'}^l\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}}{\langle \varphi_{\tau'}^{l+1}, \varphi_{\tau'}^l\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}} \varphi_{\tau'}^{l+1}, \quad \tau \in \nabla_l^{\mathbb{R}^2}.$$

In other words, the wavelets $\psi_\tau^{\mathbb{R}^2}$, $\tau \in \nabla_l^{\mathbb{R}^2}$ are orthogonal to the space $Lin_l^{\mathbb{R}^2} = \operatorname{span}\{\varphi_\tau^l : \tau \in \triangle_l^{\mathbb{R}^2}\}$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}$, i.e., they are prewavelets (semi-orthogonal wavelets) with respect to a non-standard scalar product.
We introduce the mappings $\tilde{m}_l : Lin_l^{\mathbb{R}^2} \longrightarrow Lin_l^{\mathbb{R}^2}$ and $\tilde{Y}_l : Lin_{l+1}^{\mathbb{R}^2} \longrightarrow Lin_l^{\mathbb{R}^2}$ by

$$\begin{aligned} \langle \tilde{m}_l u_l, v_l\rangle_{\triangle_l^{\mathbb{R}^2}} &= \langle u_l, v_l\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}, \qquad u_l, v_l \in Lin_l^{\mathbb{R}^2}, \\ \left\langle \tilde{Y}_l u_{l+1}, v_l\right\rangle_{\triangle_l^{\mathbb{R}^2}} &= \langle u_{l+1}, v_l\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}, \quad u_{l+1} \in Lin_{l+1}^{\mathbb{R}^2},\ v_l \in Lin_l^{\mathbb{R}^2}. \end{aligned}$$

For a function $v_l \in Lin_l^{\mathbb{R}^2}$, we observe that $\langle v_l, v_l\rangle_{\triangle_l^{\mathbb{R}^2}} - \langle v_l, v_l\rangle_{\triangle_{l+1}^{\mathbb{R}^2}}$ is equivalent to $2^{-2l}\langle \nabla v_l, \nabla v_l\rangle_{L^2}$, i.e., $\|\nabla v_l\|^2 \sim \langle 2^{2l}(I - \tilde{m}_l)v_l, v_l\rangle_{\triangle_l^{\mathbb{R}^2}}$. Thus we introduce the norms

$$\|v_l\|_{\triangle_l^{\mathbb{R}^2}, s} := \left\| \left[I + 2^{2l}(I - \tilde{m}_l)\right]^{s/2} v_l \right\|_{\triangle_l^{\mathbb{R}^2}} \tag{5.2}$$

which are equivalent to $\|v_l\|_{H^s(\mathbb{R}^2)}$. Then it is proved in [46] (cf. the paragraph before [46], Theorem 4.7) that the norm equivalences (5.1) hold for

$$-1 + {}^2\log\left\{ \frac{1}{2} \sup_{l=0,1,\ldots} \left\| \tilde{Y}_l \tilde{m}_{l+1}^{-1} \right\|_{\triangle_l^{\mathbb{R}^2}, -2 \leftarrow \triangle_{l+1}^{\mathbb{R}^2}, -2} \right\} < s < \frac{3}{2}. \tag{5.3}$$

Moreover, a simple modification of the derivation of (5.3) yields even the s-range

$$-1 + {}^2\log\left\{ \frac{1}{2} \sup_{l=l_0,l_0+1,\ldots} \left\| \prod_{j=l+1}^{l+k} \tilde{Y}_j \tilde{m}_{j+1}^{-1} \right\|_{\triangle_{l+1}^{\mathbb{R}^2}, -2 \leftarrow \triangle_{l+k+1}^{\mathbb{R}^2}, -2}^{1/k} \right\} < s < \frac{3}{2}, \tag{5.4}$$

where l_0 and k are arbitrarily fixed positive integers. All what left is to compute the lower bound of the s-range, i.e., to estimate $\| \prod \tilde{Y}_j \tilde{m}_{j+1}^{-1} \|^{1/k}$.

ii) Now we derive the standard representation of $\tilde{Y}_j$ and $\tilde{m}_j$ from the theory of wavelets (cf. [19]). We consider the $\langle \cdot, \cdot \rangle_{\triangle^{\mathbb{R}^2}_{l+1}}$-orthonormalized bases

$$\left\{ {}^1\Phi^l_k := \sqrt{\frac{3}{2}} 2^l \varphi^l_{2^{-l}k} : \ k \in \mathbb{Z}^2 \right\} \cup \left\{ {}^2\Phi^l_k := \sqrt{3} 2^l \varphi^l_{(2^{-l-1}, -2^{-l-1}) + 2^{-l}k} : \ k \in \mathbb{Z}^2 \right\}$$

of the spaces $Lin^{\mathbb{R}^2}_l$ and represent the mappings $\tilde{m}_l$ and $\tilde{Y}_l$ as matrices with respect to these bases. As mentioned already by Stevenson, we get $\tilde{m}_l = p_l^* p_l$ and $\tilde{Y}_l = p_l^*$, where p_l is the matrix of the embedding operator $Lin^{\mathbb{R}^2}_l \longrightarrow Lin^{\mathbb{R}^2}_{l+1}$. Due to the refinement equations

$$\begin{aligned} {}^1\Phi^0_{(0,0)} &= \frac{1}{2} {}^1\Phi^1_{(0,0)} + \frac{1}{4} \left\{ {}^1\Phi^1_{(0,1)} + {}^1\Phi^1_{(0,-1)} + {}^1\Phi^1_{(1,0)} + {}^1\Phi^1_{(-1,0)} \right\} + \\ &\qquad \frac{\sqrt{2}}{8} \left\{ {}^2\Phi^1_{(0,0)} + {}^2\Phi^1_{(-1,0)} + {}^2\Phi^1_{(0,1)} + {}^2\Phi^1_{(-1,1)} \right\}, \\ {}^2\Phi^0_{(0,0)} &= \frac{\sqrt{2}}{2} {}^1\Phi^1_{(1,-1)} + \frac{1}{4} \left\{ {}^2\Phi^1_{(0,0)} + {}^2\Phi^1_{(1,0)} + {}^2\Phi^1_{(0,-1)} + {}^2\Phi^1_{(1,-1)} \right\}, \end{aligned}$$

we get (cf. e.g. [19]), for the function $u_0 = \sum_{k \in \mathbb{Z}^2} [\xi^1_k \, {}^1\Phi^0_k + \xi^2_k \, {}^2\Phi^0_k]$ embedded as $u_0 = \sum_{k \in \mathbb{Z}^2} [\eta^1_k \, {}^1\Phi^1_k + \eta^2_k \, {}^2\Phi^1_k]$,

$$\begin{aligned} \eta_k &= \sum_{k' \in \mathbb{Z}^2} h^T_{k-2k'} \xi_{k'}, \quad \eta_k := \begin{pmatrix} \eta^1_k \\ \eta^2_k \end{pmatrix}, \quad \xi_k = \begin{pmatrix} \xi^1_k \\ \xi^2_k \end{pmatrix}, \quad h^T_k := \begin{pmatrix} h^{1,1}_k & h^{2,1}_k \\ h^{1,2}_k & h^{2,1}_k \end{pmatrix}, \\ h^{i,j}_k &:= \begin{cases} \frac{1}{2} & \text{if } i=j=1, \ k=(0,0) \\ \frac{\sqrt{2}}{2} & \text{if } i=2, \ j=1, \ k=(1,-1) \\ \frac{1}{4} & \text{if } i=j=1, \ k \in \{(0,1),(0,-1),(1,0),(-1,0)\} \\ & \text{ or } i=j=2, \ k \in \{(0,0),(1,0),(0,-1),(1,-1)\} \\ \frac{\sqrt{2}}{8} & \text{if } i=1, \ j=2, \ k \in \{(0,0),(-1,0),(0,1),(-1,1)\}. \end{cases} \end{aligned} \tag{5.5}$$

As usually in the theory of wavelets, we identify the coefficient vectors $(\xi_k)_{k \in \mathbb{Z}^2}$ and $(\eta_k)_{k \in \mathbb{Z}^2}$ with the generator functions $\xi(x,y) := \sum \xi_{(k_1,k_2)} e^{i2\pi k_1 x} e^{i2\pi k_2 y}$ and $\eta(x,y) := \sum \eta_{(k_1,k_2)} e^{i2\pi k_1 x} e^{i2\pi k_2 y}$, respectively. Then the l^2 spaces of coefficient vectors are isometric to the space of L^2 functions over $\mathbb{R}^2/\mathbb{Z}^2$, and (5.5) is equivalent to the equation $\eta(x,y) = h^T(x,y)\xi(2x,2y)$ with the matrix function

$$\begin{aligned} h^T(x,y) &:= \sum_{(k_1,k_2) \in \mathbb{Z}^2} h^T_{(k_1,k_2)} e^{i2\pi k_1 x} e^{i2\pi k_2 y} \\ &= \begin{pmatrix} \frac{1}{2}\{1 + \cos(2\pi x) + \cos(2\pi y)\} & \frac{\sqrt{2}}{2} e^{i2\pi(x-y)} \\ \frac{\sqrt{2}}{2} e^{i\pi(y-x)} \cos(\pi x)\cos(\pi y) & e^{i\pi(x-y)} \cos(\pi x)\cos(\pi y) \end{pmatrix}. \end{aligned}$$

In other words, the embedding operator $(\xi_k)_k \mapsto (\eta_k)_k = p_l(\xi_k)_k$ corresponds to the multiplication operator $\xi(x,y) \mapsto \eta(x,y) := h^T(x,y)\xi(2x,2y)$. We denote the adjoint matrix function of $(x,y) \mapsto h^T(x,y)$ by $(x,y) \mapsto \bar{h}(x,y)$. The formula $\tilde{m}_l = p_l^* p_l$ and an easy computation reveal that the operator $\tilde{m}_l$ acting in the space of generator functions is simply the operator of multiplication by the invertible matrix function

$$\begin{aligned}\tilde{m}(x,y) &:= \frac{1}{4}\sum_{i,j=0}^{1} \bar{h}\Big(\frac{x}{2}+\frac{i}{2},\frac{y}{2}+\frac{j}{2}\Big)h^T\Big(\frac{x}{2}+\frac{i}{2},\frac{y}{2}+\frac{j}{2}\Big)\\ &= \begin{pmatrix} \frac{5}{8}+\frac{1}{8}\{\cos(2\pi x)+\cos(2\pi y)\} & \frac{\sqrt{2}}{8}e^{\mathrm{i}\pi(x-y)}\cos(\pi x)\cos(\pi y)\\ \frac{\sqrt{2}}{8}\,e^{\mathrm{i}\pi(y-x)}\cos(\pi x)\cos(\pi y) & \frac{3}{4}\end{pmatrix}.\end{aligned}$$

We denote the self adjoint and non-negative matrix $I-\tilde{m}(x,y)$ by $a(x,y)$ and conclude that $\tilde{m}_{l+1}^{-1}p_l$ corresponds to

$$\xi(x,y)\mapsto \tilde{m}^{-1}(x,y)h^T(x,y)\xi(2x,2y).$$

The H^{-2} operator norm $\|\tilde{Y}_l\tilde{m}_{l+1}^{-1}\|$ is equal to the H^2 operator norm $\|\tilde{m}_{l+1}^{-1}p_l\|$, and, due to the norm definition in (5.2), the last is equal to the operator norm of the multiplication operator

$$\xi(x,y)\mapsto \Big[I+2^{2(l+1)}a(x,y)\Big]\,\tilde{m}^{-1}(x,y)h^T(x,y)\,\Big[I+2^{2l}a(2x,2y)\Big]^{-1}\xi(2x,2y) \qquad (5.6)$$

acting in the L^2 space over $\mathbb{R}^2/\mathbb{Z}^2$. Thus, to compute the lower bound in (5.4), we have to estimate the norm of the operators (5.6) depending on l and the norm of their products, respectively.

iii) To estimate the norm of (5.6), we introduce the auxiliary operator Te^ϵ depending on a non-negative parameter ϵ by

$$Te^\epsilon\xi(x,y) \;:=\; [\epsilon I+4a(x,y)]\,\tilde{m}^{-1}(x,y)h^T(x,y)\,[\epsilon I+a(2x,2y)]^{-1}\,\xi(2x,2y) \qquad (5.7)$$

and observe that the operator in (5.6) is Te^ϵ for $\epsilon=2^{-2l}$. This 2^{-2l} can be made small by choosing l_0 large in (5.4). In what follows we shall derive an estimate for Te^0. We shall split Te^ϵ for $\epsilon=2^{-2l}$ into the sum of three terms, and, using the bound for Te^0, we shall estimate each term separately.

Following the announced program, we observe

$$\begin{aligned} Te^0\xi(x,y) &= Ma(x,y)\xi(2x,2y),\\ Ma(x,y) &:= 4a(x,y)\tilde{m}^{-1}(x,y)h^T(x,y)a(2x,2y)^{-1}. \end{aligned} \qquad (5.8)$$

The determinant $\det(a(x,y))$ of $a(x,y)$ has a zero only at $(x,y)=(0,0)$ and $\det(\tilde{m}(x,y))$ does not vanish at all. Moreover, we get $\det(a(x,y))\sim x^2+y^2$ for $(x,y)\longrightarrow(0,0)$. Since $a(x,y)^{-1}=a(x,y)^A/\det(a(x,y))$ with

$$a(x,y)^A \;:=\; \begin{pmatrix} a_{2,2}(x,y) & -a_{1,2}(x,y)\\ -a_{2,1}(x,y) & a_{1,1}(x,y)\end{pmatrix},$$

we arrive at $Ma(x,y)=4a(x,y)\tilde{m}(x,y)^Ah^T(x,y)a(2x,2y)^A/[\det(a(2x,2y))\det(\tilde{m}(x,y))]$. A lengthy but trivial calculation shows that all entries of $a(x,y)\tilde{m}(x,y)^Ah^T(x,y)a(2x,2y)^A$ vanish together with their first derivatives at the points $(0,0)$, $(1/2,0)$, $(0,1/2)$, and $(1/2,1/2)$, where $\det(a(2x,2y))$ has its zeros. Hence, $Ma(x,y)$ is bounded over $\mathbb{R}^2/\mathbb{Z}^2$. Using the periodicity of the function $\xi\in L^2(\mathbb{R}^2/\mathbb{Z}^2)$, the norm of operator Te^0 can be estimated as follows.

$$
\begin{aligned}
\|Ma(x,y)\xi(2x,2y)\|^2_{L^2} &= \int_0^1\int_0^1 \Big\langle [Ma^*Ma](x,y)\xi(2x,2y),\xi(2x,2y)\Big\rangle\,\mathrm{d}x\mathrm{d}y\\
&= \int_0^1\int_0^1 \left\langle \frac14\sum_{i=0}^1\sum_{j=0}^1[Ma^*Ma]\left(\frac{x}{2}+\frac{i}{2},\frac{y}{2}+\frac{i}{2}\right)\xi(x,y),\xi(x,y)\right\rangle \mathrm{d}x\mathrm{d}y\\
\|Te^0\| &\le \sup_{(x,y)}\left\|\frac14\sum_{i,j=0}^1[Ma^*Ma]\left(\frac{x}{2}+\frac{i}{2},\frac{y}{2}+\frac{j}{2}\right)\right\|^{1/2}. \qquad (5.9)
\end{aligned}
$$

The matrix norm on the right-hand side of the last equation is the l^2 matrix norm, i.e., the operator norm in the two-dimensional Euclidean space. A numerical evaluation of (5.9) yields $\|Te^0\|\le 10.37\ldots$.

Next we fix a small positive number δ and introduce the cut off function $\chi(x,y)$ on $\mathbb{R}^2/\mathbb{Z}^2$ which is equal to one for $|x|,|y|\le\delta$ and zero else. Using this function, we split

$$
\begin{aligned}
Te^\epsilon &= \sum_{i=1}^3 Te^\epsilon_i, \qquad (5.10)\\
Te^\epsilon_1\xi(x,y) &:= \Big(1-\chi(2x,2y)\Big)\,Te^\epsilon\xi(x,y),\\
Te^\epsilon_2\xi(x,y) &:= \chi(2x,2y)\left[4a(x,y)\tilde{m}^{-1}(x,y)h^T(x,y)a(2x,2y)^{-1}\right]\\
&\qquad\qquad a(2x,2y)\left[\epsilon I+a(2x,2y)\right]^{-1}\xi(2x,2y),\\
Te^\epsilon_3\xi(x,y) &:= \chi(2x,2y)\left[\tilde{m}^{-1}(x,y)h^T(x,y)\right]\,\epsilon I\left[\epsilon I+a(2x,2y)\right]^{-1}\xi(2x,2y).
\end{aligned}
$$

Since $\chi^2=\chi$ and since

$$
\begin{aligned}
Te^\epsilon_i[\chi\xi](x,y) &= \chi(2x,2y)Te^\epsilon_i\xi(x,y),\\
\|\xi(x,y)\|^2 &= \|\chi(2x,2y)\xi(x,y)\|^2+\|(1-\chi(2x,2y))\xi(x,y)\|^2,\\
\|\xi(x,y)\|^2 &= \|\chi(x,y)\xi(x,y)\|^2+\|(1-\chi(x,y))\xi(x,y)\|^2,
\end{aligned}
$$

we conclude

$$
\begin{aligned}
\|Te^\epsilon\xi\|^2 &= \|\chi(2\cdot,2\cdot)Te^\epsilon\xi\|^2+\|[1-\chi(2\cdot,2\cdot)]Te^\epsilon\xi\|^2\\
&= \|Te^\epsilon_1[\chi\xi]\|^2+\|[Te^\epsilon_2+Te^\epsilon_3][(1-\chi)\xi]\|^2\\
&\le \max\Big\{\|Te^\epsilon_1\|,\|Te^\epsilon_2+Te^\epsilon_3\|\Big\}^2\Big\{\|\chi\xi\|^2+\|(1-\chi)\xi\|^2\Big\},\\
\|Te^\epsilon\| &\le \max\Big\{\|Te^\epsilon_1\|,\|Te^\epsilon_2\|+\|Te^\epsilon_3\|\Big\}. \qquad (5.11)
\end{aligned}
$$

The matrices $a(2x,2y)$ are invertible on the support of $(1-\chi(2x,2y))$ and the inverses are uniformly bounded. Hence,

$$
\begin{aligned}
\|Te^\epsilon_1-Te^0_1\| &\le C\,\epsilon,\\
\|Te^\epsilon_1\| &\le \|Te^0_1\|+C\,\epsilon\;\le\;10.37\ldots+C\,\epsilon. \qquad (5.12)
\end{aligned}
$$

Clearly, the last constant C depends on the δ from the definition of the cut off function χ. On the other hand, the adjoint operator $[Te^\epsilon_2]^*$ is given by

$$
[Te^\epsilon_2]^*\xi(x,y) = \chi(x,y)a(x,y)\left[\epsilon I+a(x,y)\right]^{-1}[Te^0]^*\xi(x,y).
$$

From this and from the matrix inequality $a(x,y)\,[\epsilon I + a(x,y)]^{-1} \le I$ we obtain $\|[Te_2^\epsilon]^*\| \le \|[Te^0]^*\|$ and

$$\|Te_2^\epsilon\| \;\le\; \|Te^0\| \;\le\; 10.37\dots \;. \tag{5.13}$$

Now we turn to $\|Te_3^\epsilon\|$. The non-negative self adjoint matrix $a(x,y)$ can be represented as $a(x,y) = \lambda(x,y)q(x,y) + \mu(x,y)o(x,y)$, where $\lambda(x,y)$ and $\mu(x,y)$ are the eigenvalues of $a(x,y)$. The matrices $q(x,y)$ and $o(x,y) = I - q(x,y)$ are the orthogonal projections onto the spaces of eigenvectors. In particular, we get

$$a(0,0) = \begin{pmatrix} \frac{1}{8} & -\frac{\sqrt{2}}{8} \\ -\frac{\sqrt{2}}{8} & \frac{1}{4} \end{pmatrix}, \quad q(0,0) = \begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} \end{pmatrix}, \quad o(0,0) = \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix},$$
$$\lambda(0,0) = \frac{3}{8}, \quad \mu(0,0) = 0.$$

Since $\lambda(x,y)$ is separated from 0 by a positive constant, we get

$$\begin{aligned} \epsilon I\,[\epsilon + a(2x,2y)]^{-1} \;&=\; \epsilon\,[\epsilon + \lambda(2x,2y)]^{-1}\,q(2x,2y) + \epsilon\,[\epsilon + \mu(2x,2y)]^{-1}\,o(2x,2y), \\ \left\|\epsilon\,[\epsilon + \lambda(2x,2y)]^{-1}\,q(2x,2y)\right\| \;&\le\; C\epsilon. \end{aligned}$$

Consequently, we arrive at

$$\begin{aligned} \|Te_3^\epsilon - Te_4^\epsilon\| \;&\le\; C\epsilon, \\ Te_4^\epsilon\xi(x,y) \;&:=\; \chi(2x,2y)\left[\tilde{m}^{-1}(x,y)h^T(x,y)\right]\,\epsilon\,[\epsilon + \mu(2x,2y)]^{-1}\,o(2x,2y)\xi(2x,2y). \end{aligned}$$

In other words, the norm $\|Te_3^\epsilon\|$ is less than $C\epsilon$ plus the norm $\|Te_5\|$ of the operator

$$Te_5:\; \xi(x,y) \mapsto \chi(2x,2y)\left[\tilde{m}^{-1}h^T\right](x,y)\,o(2x,2y)\xi(2x,2y),$$

and we even get $\|Te_3^\epsilon\| \le C\epsilon + C\delta + \|Te_6\|$ with Te_6 defined by

$$\xi(x,y) \mapsto \begin{cases} \left[\tilde{m}^{-1}h^T\right](0,0)\;\; o(0,0)\xi(2x,2y) & \text{if } |2x| \le \delta \text{ and } |2y| \le \delta \\ \left[\tilde{m}^{-1}h^T\right]\left(\frac{1}{2},0\right) o(0,0)\xi(2x,2y) & \text{if } |2x-1| \le \delta \text{ and } |2y| \le \delta \\ \left[\tilde{m}^{-1}h^T\right]\left(0,\frac{1}{2}\right) o(0,0)\xi(2x,2y) & \text{if } |2x| \le \delta \text{ and } |2y-1| \le \delta \\ \left[\tilde{m}^{-1}h^T\right]\left(\frac{1}{2},\frac{1}{2}\right) o(0,0)\xi(2x,2y) & \text{if } |2x-1| \le \delta \text{ and } |2y-1| \le \delta \\ 0 & \text{else .} \end{cases} \tag{5.14}$$

Since we have

$$\begin{aligned} h^T(0,0)\,o(0,0) \;&=\; 2\,o(0,0), & h^T\left(\tfrac{1}{2},0\right)o(0,0) \;&=\; 0, \\ h^T\left(0,\tfrac{1}{2}\right)o(0,0) \;&=\; 0, & h^T\left(\tfrac{1}{2},\tfrac{1}{2}\right)o(0,0) \;&=\; 0, \\ \tilde{m}^{-1}(0,0)\,o(0,0) \;&=\; o(0,0), \end{aligned} \tag{5.15}$$

we conclude

$$Te_6:\; \xi(x,y) \mapsto \begin{cases} 2\,o(0,0)\xi(2x,2y) & \text{if } |2x| \le \delta \text{ and } |2y| \le \delta \\ 0 & \text{else} \end{cases}$$

and $\|Te_3^\epsilon\| \le 2 + C\epsilon + C\delta$. This and the estimates (5.11), (5.12), and (5.13), lead us to $\|Te^\epsilon\| \le 12.37\ldots + C\epsilon + C\delta$. Choosing δ small and choosing ϵ small in comparison to δ, we get $\|Te^\epsilon\| \le 12.37\ldots$. Using (5.4) with $k = 1$ and sufficiently large l_0, the Riesz property (5.1) follows for "$1.62\ldots < s < 1.5$".

iv) To improve the lower bound of the Sobolev range, we apply (5.4) with larger k. Analogously to (5.7) and (5.8), we define

$$\begin{aligned} Te^\epsilon\xi(x,y) &:= \Big[\epsilon I + 4^k a(x,y)\Big] \prod_{i=0}^{k-1} \left\{ \left[\tilde{m}^{-1}h^T\right] \left(2^i x, 2^i y\right) \right\} \Big[\epsilon I + a(2^k x, 2^k y)\Big]^{-1} \xi(2^k x, 2^k y) \\ Ma(x,y) &:= 4^k a(x,y) \prod_{i=0}^{k-1} \left\{ \left[\tilde{m}^{-1}h^T\right] \left(2^i x, 2^i y\right) \right\} a(2^k x, 2^k y)^{-1}. \end{aligned}$$

For $k = 10$, numerical computations lead us to the estimate (compare (5.9))

$$\sup_{(x,y)} \left\| \frac{1}{4^k} \sum_{j,j'=0}^{2^k-1} [Ma^* Ma] \left(\frac{x}{2^k} + \frac{j}{2^k}, \frac{y}{2^k} + \frac{j'}{2^k} \right) \right\|^{1/2} \le 20661.3\ldots .$$

Analogously to (5.14), we define Te_6 by

$$\xi(x,y) \mapsto \begin{cases} \prod_{i=0}^{k-1} \left\{ \left[\tilde{m}^{-1}h^T\right] \left(2^i \frac{j}{2^k}, 2^i \frac{j'}{2^k}\right) \right\} o(0,0)\xi\big(2^k x, 2^k y\big) & \text{if } |2^k x - j| \le \delta \\ & \text{and } |2^k y - j'| \le \delta \\ 0 & \text{else} . \end{cases} \quad (5.16)$$

In view of (5.15) we conclude

$$Te_6 : \ \xi(x,y) \mapsto \begin{cases} 2^k o(0,0)\xi\big(2^k x, 2^k y\big) & \text{if } |2^k x| \le \delta \\ & \text{and } |2^k y| \le \delta \\ 0 & \text{else} , \end{cases}$$

and the arguments from part iii) of the present proof lead us to the estimate $\|Te^\epsilon\| \le 20661.3\ldots + 2^{10} + C\epsilon + C\delta$. Choosing small values ϵ and δ, we get $\|Te^\epsilon\| \le 21685.3\ldots$, and (5.4) implies the Riesz property (5.1) for $-0.559\ldots < s < 1.5$. ∎

Next we deal with functions over the triangle T. In the construction of Sect. 3.1 we need basis functions which admit symmetric (even) or antisymmetric (odd) extensions with respect to the boundary of T. To construct such functions, we shall extend the piecewise linear functions on T by symmetry mappings to periodic functions over the plane $\mathbb{R}^2$. More precisely, we shall suppose that a subset of the three sides of T is given through which the functions should possess an even extension. Through the rest of the sides there should exist odd extensions. In accordance to these symmetry properties, we shall define an extension procedure from functions over T to periodic functions over $\mathbb{R}^2$. For the periodic extensions, however, there exists a natural basis. Restricting this basis to the triangle, we shall arrive at our basis over T.

In view of the assumptions in Sect. 2.1, the two shorter sides $\{(s,s) :\ 0 \le s \le 0.5\}$ and $\{(s,1-s) :\ 0.5 \le s \le 1\}$ simultaneously belong to the fixed subset of sides or not. For the sake of definiteness, we suppose the only side with odd extension is the lower side

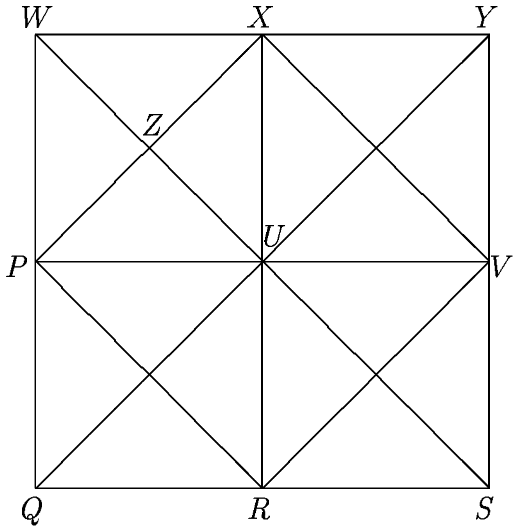

Figure 9: Torus $\mathbb{T}$.

$\{(s,0):\ 0 \le s \le 1\}$. To prepare the definition of the extension, we introduce the points (cf. Figure 9)

$$\begin{array}{rcrrcrrcr} P &:=& (0,0), & U &:=& (1,0), & Z &:=& (0.5,0.5), \\ W &:=& (0,1), & X &:=& (1,1), & Q &:=& (0,-1), \\ R &:=& (1,-1), & S &:=& (2,-1), & Y &:=& (2,1), \\ & & & V &:=& (2,0). & & & \end{array}$$

Clearly, a piecewise linear function u_L on T admits a continuous extension through the boundary if and only if u_L vanishes on the side of odd extension. If a function u_L vanishing on $\{(s,0):\ 0 \le s \le 1\}$ is given, then we can extend u_L to triangle PZW by symmetry with respect to the line through P and Z, i.e. $v_L(s,t) := u_L(t,s)$. The extended function on triangle PUW will be denoted by v_L. We can extend v_L to triangle WUX as a function symmetric with respect to the line through W and U by $v_L(s,t) := v_L(1-t, 1-s)$. Similarly, we extend v_L to the square $QRUP$ as a function antisymmetric with respect to the line through T and U by $v_L(s,t) := -v_L(s,-t)$. Again we extend v_L to the rectangle $RSYX$ as a function antisymmetric with respect to the line through R and X by $v_L(s,t) := -v_L(2-s,t)$. In other words, the function u_L is extended to a continuous piecewise linear function v_L on the square $QSYW$. This function extends to a function which is 2-periodic with respect to both variables, and we denote the periodic extension v_L by u_L^{ext}.

Let us consider the periodic functions more carefully. Periodicity of a piecewise linear function w_L means that w_L satisfies

$$w_L(s,t) \;=\; w_L(s+2k, t+2k'), \quad (k,k') \in \mathbb{Z}^2.$$

The periodic functions are functions defined on the torus, i.e., on the quotient space

$$\mathbb{T} \;:=\; \mathbb{R}^2 / \left\{(2k,2k'):\ (k,k') \in \mathbb{Z}^2\right\}.$$

We denote the space of periodic linear functions by $Lin_L^{\mathbb{T}}$. To get periodic basis functions, we take periodizations ψ_τ^{per} of $\psi_\tau^{\mathbb{R}^2}$ defined by

$$\psi_\tau^{per}(s,t) \;:=\; \sum_{(k,k')\in\mathbb{Z}^2} \psi_\tau^{\mathbb{R}^2}(s+2k, t+2k') \;=\; \sum_{(k,k')\in\mathbb{Z}^2} \psi_{\tau+(2k,2k')}(s,t).$$

If we define the grid $\triangle_L^{\mathbb{T}}$ by

$$\triangle_L^{\mathbb{T}} := \left\{(s,t) \in \triangle_L^{\mathbb{R}^2} :\ 0 \le s,t < 2\right\},$$

then $\{\psi_\tau^{per} :\ \tau \in \triangle_L^{\mathbb{T}}\}$ is a finite system of basis functions of $Lin_L^{\mathbb{T}}$. It is well known that Lemma 5.1 remains true for periodic functions and for the Sobolev spaces over the torus, i.e., for $-0.559\ldots < s < 1.5$ and for all vectors of coefficients ξ_τ,

$$\left\| \sum_{\tau \in \cup_{L=0}^\infty \triangle_L^{\mathbb{T}}} \xi_\tau \psi_\tau^{per} \right\|_{H^s(\mathbb{T})} \sim \sqrt{\sum_{\tau \in \cup_{L=0}^\infty \triangle_L^{\mathbb{T}}} 2^{2l(\tau)(s-1)} |\xi_\tau|^2}. \tag{5.17}$$

On the other hand, the extension $v_L = u_L^{ext}$ of a linear function u_L on triangle T belongs to the subspace

$$Lin_L^{Sym} := \left\{w_L \in Lin_L^{\mathbb{T}} :\ [w_L|_T]^{ext} = w_L\right\}$$

which is determined by the properties of symmetry included in the extension procedure $[w_L|_T] \mapsto [w_L|_T]^{ext}$. For a point $\tau \in \triangle_L^T \setminus \{(s,0) :\ 0 \le s \le 1\}$, we denote by $\tau_1^\tau, \ldots, \tau_k^\tau$ those points of $\triangle_L^{\mathbb{T}}$ for which the function value $[u_L]^{ext}(\tau_i^\tau)$ is set to $\pm u_L(\tau)$ in the extension procedure $u_L \mapsto [u_L]^{ext}$. We define $\lambda_i \in \{1,-1\}$ by $[u_L]^{ext}(\tau_i^\tau) = \lambda_i u_L(\tau)$. Clearly, the points τ_i^τ are obtained by the symmetric reflections mapping the triangle T to the subtriangles of the quadrangle $QSYW$. The number of these points is $k = 16$ if τ is an interior point of T, $k = 8$ if τ is on a side of T, and $k = 4$ if τ is the corner Z. Now a function w_L belongs to Lin_L^{Sym}, if and only if, $w_L(\tau) = \lambda_i w_L(\tau_i^\tau)$, $i = 1,\ldots,k$. Obviously, the set of functions $\sum_{i=1}^k \lambda_i [\varphi_{\tau_i^\tau}^L]^{per}$ with $\tau \in \triangle_L^T \setminus \{(s,0) :\ 0 \le s \le 1\}$ forms a basis of Lin_L^{Sym} and the cardinality of $\triangle_L^T \setminus \{(s,0) :\ 0 \le s \le 1\}$ is the dimension of Lin_L^{Sym}. Another basis is formed by $\sum_{i=1}^k \lambda_i \psi_{\tau_i^\tau}^{per}$ with $\tau \in \triangle_L^T \setminus \{(s,0) :\ 0 \le s \le 1\}$. Indeed this system of functions has the right cardinality, all its elements belong to the space Lin_L^{Sym}, and they are linearly independent since the functions ψ_τ^{per}, $\tau \in \triangle_T^\Gamma \setminus \{(s,0) :\ 0 \le s \le 1\}$ are linearly independent. We introduce the functions

$$\psi_\tau^{ext} \;:=\; \sum_{i=1}^k \lambda_i \psi_{\tau_i^\tau}^{per}, \quad \psi_\tau^T \;:=\; \psi_\tau^{ext}|_T, \quad \tau \in \triangle_L^T \setminus \{(s,0) :\ 0 \le s \le 1\}$$

and obtain $w_L = \sum_{\tau \in \triangle_L^T \setminus \{(s,0):\, 0 \le s \le 1\}} \xi_\tau \psi_\tau^{ext}$. Applying this to the extension $w_L = [u_L]^{ext}$ of a function u_L on T, we arrive at $u_L = \sum_{\tau \in \triangle_L^T \setminus \{(s,0):\, 0 \le s \le 1\}} \xi_\tau \psi_\tau^T$. It turns out that $\{\psi_\tau^T :\ \tau \in \triangle_L^T\}$ is a basis of the space of piecewise linear functions over T vanishing over the side $\{(s,0) :\ 0 \le s \le 1\}$. Using $\|u_L\|_{H^s(T)} \sim \|[u_L]^{ext}\|_{H^s(\mathbb{T})}$, the Riesz property (5.17) implies

$$\left\| \sum_{\tau \in \cup_{L=0}^\infty \triangle_L^T \setminus \{(s,0):\, 0 \le s \le 1\}} \xi_\tau \psi_\tau^T \right\|_{H^s(T)} \sim \sqrt{\sum_{\tau \in \cup_{L=0}^\infty \triangle_L^T \setminus \{(s,0):\, 0 \le s \le 1\}} 2^{2l(\tau)(s-1)} |\xi_\tau|^2} \tag{5.18}$$

for $-0.559\ldots < s < 1.5$. We note that, for $\tau \in \triangle_L^T \setminus \{(s,0) :\ 0 \le s \le 1\}$,

$$\psi_\tau^T \quad := \quad \begin{cases} \varphi_\tau^0|_T & \text{if } \tau \in \triangle_L^T \cap \nabla_{-1}^T \\ \varphi_\tau^{l+1}|_T - \frac{1}{2}\left\{\varepsilon^{\tau,\tau_1}\varphi_{\tau_1}^{l+1}|_T + \varepsilon^{\tau,\tau_2}\varphi_{\tau_2}^{l+1}|_T\right\} & \text{if } \tau \in \triangle_L^T \cap {}^1\nabla_l^T, \\ & \qquad l = 0,\ldots,L-1 \\ \varphi_\tau^{l+1}|_T - \frac{1}{4}\left\{\varepsilon^{\tau,\tau_1}\varphi_{\tau_1}^{l+1}|_T + \varepsilon^{\tau,\tau_2}\varphi_{\tau_2}^{l+1}|_T\right\} & \text{if } \tau \in \triangle_L^T \cap {}^2\nabla_l^T, \\ & \qquad l = 0,\ldots,L-1, \end{cases} \tag{5.19}$$

$$\varepsilon^{\tau,\tau'} := \begin{cases} 1 & \text{if} \quad \tau \text{ and } \tau' \text{ belong to the interior of } T \\ & \quad\ \text{or there exists a side of } T \text{ such that } \tau \text{ and } \tau' \\ & \quad\ \text{belong to the interior of this side} \\ 2 & \text{if} \quad \tau \text{ is an interior point of } T \text{ and } \tau' \text{ belongs to} \\ & \quad\ \text{a side of } T \\ & \quad\ \text{or } \tau' = Z \text{ and } \tau \text{ is on a side of } T \\ 4 & \text{if} \quad \tau' = Z \text{ and } \tau \text{ is an interior point of } T \\ 0 & \text{else.} \end{cases}$$

With ψ_τ^T we have constructed a three-point wavelet basis for the space of linear functions on T vanishing on $\{(s,0):\ 0 \le s \le 1\}$. Completely analogously, we can construct a basis for the linear functions on T vanishing on three, two or no sides. These functions are the basis ingredients for the wavelet basis on the manifold. Indeed, as indicated in Sect. 3.1, the three-point hierarchical basis of (3.2) is constructed as follows.

We start with functions ψ_P such that $P \in \triangle_L^\Gamma \cap \Gamma_1$. We just take the basis $\{\psi_\tau^T\}$ on T with no zero condition for boundary sides. For $P = \kappa_1(\tau)$, we take the composition $\psi_P = \psi_\tau^T \circ \kappa_1^{-1}$ to get functions over the parametrization patch Γ_1. To get continuous trial functions, we extend these functions ψ_P with $P \in \nabla_l^\Gamma \cap \Gamma_1 \subset \triangle_{l+1}^\Gamma \cap \Gamma_1$ from Γ_1 to Γ such that the extension is piecewise linear on the partition $\{\Gamma_Q :\ Q \in \square_{l+1}^\Gamma\}$ corresponding to the grid $\triangle_{l+1}^\Gamma$ and vanishes at all grid points from $\triangle_{l+1}^\Gamma \setminus \Gamma_1$. This simply means that, if $\psi_\tau^T = \varphi_\tau^{l+1} - \frac{1}{2}\{\varepsilon^{\tau,\tau_1}\varphi_{\tau_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{\tau_2}^{l+1}\}$ resp. $\psi_\tau^T = \varphi_\tau^{l+1} - \frac{1}{4}\{\varepsilon^{\tau,\tau_1}\varphi_{\tau_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{\tau_2}^{l+1}\}$, then $\psi_P = \varphi_P^{l+1} - \frac{1}{2}\{\varepsilon^{\tau,\tau_1}\varphi_{P_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{P_2}^{l+1}\}$ resp. $\psi_P = \varphi_P^{l+1} - \frac{1}{4}\{\varepsilon^{\tau,\tau_1}\varphi_{P_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{P_2}^{l+1}\}$, where φ_P^{l+1} and $\varphi_{P_i}^{l+1}$ are the continuous hat functions introduced in Sect. 2.4.

Next we define the functions ψ_P for $P \in \triangle_L^\Gamma \cap \Gamma_2 \setminus \Gamma_1$. The patch Γ_2 has one or no common side with Γ_1. We take the basis $\{\psi_\tau^T\}$ on T which vanishes on those sides (one ore maybe no side) which are mapped by κ_2 into a side common with Γ_1. Again we take the composition with κ_2^{-1} to get functions over the parametrization patch Γ_2 which vanish over $\Gamma_2 \cap \Gamma_1$. To get continuous trial functions, we extend these functions ψ_P with $P \in \nabla_l^\Gamma \cap \Gamma_2 \setminus \Gamma_1 \subset \triangle_{l+1}^\Gamma \cap \Gamma_2$ from Γ_2 to Γ such that the extension is piecewise linear on the grid $\triangle_{l+1}^\Gamma$ and vanishes at all grid points from $\triangle_{l+1}^\Gamma \setminus \Gamma_2$. In other words, if $\psi_\tau^T = \varphi_\tau^{l+1} - \frac{1}{2}\{\varepsilon^{\tau,\tau_1}\varphi_{\tau_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{\tau_2}^{l+1}\}$ resp. $\psi_\tau^T = \varphi_\tau^{l+1} - \frac{1}{4}\{\varepsilon^{\tau,\tau_1}\varphi_{\tau_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{\tau_2}^{l+1}\}$, then $\psi_P = \varphi_P^{l+1} - \frac{1}{2}\{\varepsilon^{\tau,\tau_1}\varphi_{P_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{P_2}^{l+1}\}$ resp. $\psi_P = \varphi_P^{l+1} - \frac{1}{4}\{\varepsilon^{\tau,\tau_1}\varphi_{P_1}^{l+1} + \varepsilon^{\tau,\tau_2}\varphi_{P_2}^{l+1}\}$, where φ_P^{l+1} and $\varphi_{P_i}^{l+1}$ are the continuous hat functions introduced in Sect. 2.4.

Analogously to the previous step, we define the functions ψ_P for $P \in \triangle_L^\Gamma \cap \Gamma_3 \setminus \{\Gamma_1 \cup \Gamma_2\}$ which vanish over $[\cup_{m=1}^2 \Gamma_m] \cap \Gamma_3$. Then we construct the functions ψ_P with $P \in \triangle_L^\Gamma \cap \Gamma_4 \setminus \{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3\}$ vanishing over $[\cup_{m=1}^3 \Gamma_m] \cap \Gamma_4$ and so on. Finally, we define ψ_P with $P \in \triangle_L^\Gamma \cap \Gamma_{m_\Gamma} \setminus \cup_{m=1}^{m_\Gamma - 1} \Gamma_m$ vanishing over the boundary of Γ_{m_Γ}. We arrive at the basis of (3.2). If the level $l(P)$ of P is defined by $l(P) := l$ for P in ∇_l^Γ, then we get

Lemma 5.2 *i) For $-0.5 < s < 1.5$, the basis $\{\psi_P :\ P \in \cup_{L=0}^\infty \triangle_L^\Gamma\}$ is a Riesz basis, i.e., for any vector of real numbers $(\xi_P)_P$, we get*

$$\left\| \sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P \right\|_{H^s(\Gamma)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)} |\xi_P|^2}. \tag{5.20}$$

ii) For the Sobolev space orders $s \le t \le 2,\ s < 1.5$, the functions from Lin_L^Γ fulfil the approximation property (Jackson type theorem)

$$\inf_{u_L \in Lin_L^\Gamma} \|u - u_L\|_{H^s(\Gamma)} \ \le\ C 2^{-L(t-s)} \|u\|_{H^t(\Gamma)}. \tag{5.21}$$

iii) For the interpolation projection R_L defined in Sect. 2.5, for $u \in H^t(\Gamma)$, and for the Sobolev space orders $0 \le s \le t \le 2,\ s < 1.5,\ t > 1$, we get

$$\|u - R_L u\|_{H^s(\Gamma)} \ \le\ C 2^{-L(t-s)} \|u\|_{\oplus_{m=1}^{m_\Gamma} H^t(\Gamma_m)}. \tag{5.22}$$

iv) For the $L^2(\Gamma)$ orthogonal projection P_L and for the Sobolev space orders $-2 \le s \le t \le 2,\ s < 1.5,\ t > -1.5$, we get

$$\|u - P_L u\|_{H^s(\Gamma)} \ \le\ C 2^{-L(t-s)} \|u\|_{H^t(\Gamma)}. \tag{5.23}$$

v) For the Sobolev space orders $s \le t < 1.5$, the functions u_L from Lin_L^Γ fulfil the inverse property (Bernstein inequality)

$$\|u_L\|_{H^t(\Gamma)} \ \le\ C 2^{L(t-s)} \|u_L\|_{H^s(\Gamma)}. \tag{5.24}$$

Proof. The assertions ii) - v) are well known. It remains to proof the Riesz property. Let $-0.5 < s < 1.5$ and $f = \sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P$. Since $\psi_P = \psi_\tau \circ \kappa_1^{-1}$ for any $P = \kappa_1(\tau) \in \Gamma_1 \cap \triangle_L^\Gamma$ and since all the ψ_P with $P \notin \Gamma_1$ vanish over Γ_1, the corresponding estimate over Γ_1 analogous to (5.18) implies

$$\left\| \sum_{P \in \triangle_L^\Gamma \cap \Gamma_1} \xi_P \psi_P \right\|_{H^s(\Gamma_1)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma \cap \Gamma_1} 2^{2l(P)(s-1)} |\xi_P|^2}. \tag{5.25}$$

Now we set $f_2^+ := \sum_{P \in \triangle_L^\Gamma \cap \Gamma_2 \setminus \Gamma_1} \xi_P \psi_P$ and $f_2^- := (f - f_2^+)|_{\Gamma_2}$. Clearly, the second function f_2^- is $\sum_{P \in \triangle_L^\Gamma \cap \Gamma_1} \xi_P \psi_P|_{\Gamma_2}$, and we observe that, for each restriction $\psi_P|_{\Gamma_2},\ P \in \triangle_L^\Gamma \cap \Gamma_1$, the function $\psi_P \circ \kappa_2$ is equal to a restriction to T of a wavelet $\psi_\tau^{\mathbb{R}^2}$ or at least to the linear combinations of three restrictions to T of wavelets $\psi_\tau^{\mathbb{R}^2}$. First suppose all $\psi_P \circ \kappa_2|_T$ with $P \in \triangle_L^\Gamma \cap \Gamma_1$ are restrictions of wavelets $\psi_\tau^{\mathbb{R}^2}$. Then the upper estimate of the Riesz properties (5.1) applied to the $\psi_P \circ \kappa_2|_T$ and the lower estimate (5.25) yield (cf. also (2.3))

$$\begin{aligned}
\|f\|_{H^s(\Gamma_2)} \ &\le\ \|f_2^+\|_{H^s(\Gamma_2)} + \|f_2^-\|_{H^s(\Gamma_2)} \\
&\le\ \|f_2^+\|_{H^s(\Gamma_2)} + C \sqrt{\sum_{P \in \triangle_L^\Gamma \cap \Gamma_1} 2^{2l(P)(s-1)} |\xi_P|^2} \\
&\le\ \|f_2^+\|_{H^s(\Gamma_2)} + C\, \|f\|_{H^s(\Gamma_1)}, && (5.26) \\
\|f\|_{H^s(\Gamma_2)} \ &\ge\ \|f_2^+\|_{H^s(\Gamma_2)} - C\, \|f\|_{H^s(\Gamma_1)}. && (5.27)
\end{aligned}$$

The case that not all $\psi_P \circ \kappa_2|_T$ are restrictions of wavelets $\psi_\tau^{\mathbb{R}^2}$ occurs only if $P = \kappa_2(\tau)$ is at the boundary of $\kappa_2(T)$, if $\psi_P = \varphi_P^{l+1} - \frac{1}{4}\{\varphi_{P_1}^{l+1} + \varphi_{P_2}^{l+1}\}$ resp. $\psi_P = \varphi_P^{l+1} - \frac{1}{2}\{\varphi_{P_1}^{l+1} + \varphi_{P_2}^{l+1}\}$, and if the corresponding wavelet on $\mathbb{R}^2$ is $\psi_\tau^{\mathbb{R}^2} = \varphi_\tau^{l+1} - \frac{1}{2}\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\}$ resp. $\psi_\tau^{\mathbb{R}^2} = \varphi_\tau^{l+1} - \frac{1}{4}\{\varphi_{\tau_1}^{l+1} + \varphi_{\tau_2}^{l+1}\}$. The functions $\varphi_{\tau_i}^{l+1}|_T$, however, are restrictions to T of wavelets $\psi_{\tau_i'}^{\mathbb{R}^2}$ with $\tau_i' \in \mathbb{R}^2 \setminus T$ and $\tau_i' \in \nabla_l^{\mathbb{R}^2}$. Moreover these $\varphi_{\tau_i}^{l+1}|_T$ coincide with the restrictions of $\psi_{P_i} \circ \kappa_2|_T$ for certain $P_i \in \triangle_L^\Gamma \cap \Gamma_1$. Hence, for an upper bound of $\|f_2^-\|^2$, we get the sum of terms $2^{2l(P)(s-1)}|\xi_P|^2$ and $2^{2l(P_i)(s-1)}|\xi_{P_i} \pm \frac{1}{4}\xi_P|^2$, and the estimates (5.26) and (5.27) remain valid. From these and (5.25) we get that

$$\|f\|_{H^s(\Gamma_1)} + \|f\|_{H^s(\Gamma_2)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma \cap \Gamma_1} 2^{2l(P)(s-1)}|\xi_P|^2 + \|f_2^+\|_{H^s(\Gamma_2)}},$$

and the estimate over Γ_2 analogous to (5.18) leads to

$$\|f\|_{H^s(\Gamma_1)} + \|f\|_{H^s(\Gamma_2)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma \cap [\Gamma_1 \cup \Gamma_2]} 2^{2l(P)(s-1)}|\xi_P|^2}.$$

Repeating the last arguments with Γ_1 replaced by $\Gamma_1 \cup \Gamma_2$ and Γ_2 replaced by Γ_3, we arrive at

$$\|f\|_{H^s(\Gamma_1)} + \|f\|_{H^s(\Gamma_2)} + \|f\|_{H^s(\Gamma_3)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma \cap [\Gamma_1 \cup \Gamma_2 \cup \Gamma_3]} 2^{2l(P)(s-1)}|\xi_P|^2}.$$

Further applications of the arguments lead finally to

$$\sum_{m=1}^{m_\Gamma} \|f\|_{H^s(\Gamma_m)} \sim \sqrt{\sum_{P \in \triangle_L^\Gamma \cap \cup_{m=1}^{m_\Gamma} \Gamma_m} 2^{2l(P)(s-1)}|\xi_P|^2} = \sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)}|\xi_P|^2}$$

∎

The Riesz property implies the existence of a projection Q_L, which is defined by

$$u = \sum_{P \in \cup_{l=0}^\infty \triangle_l^\Gamma} \xi_P \psi_P \mapsto Q_L u := \sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P$$

and which is bounded in H^s, $-0.5 < s < 1.5$. For the wavelet coefficients of smooth functions, we obtain the following decay estimate.

Lemma 5.3 *Suppose the continuous function u belongs to $\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)$ for an s with $-0.5 < s \leq 2$ and suppose $\sum_{P \in \triangle_L^\Gamma} \xi_P \psi_P$ is the representation of either the interpolation $R_L u$ or the orthogonal projection $P_L u$ or the projection $Q_L u$. Then*

$$\sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2l(P)(s-1)}|\xi_P|^2} \leq C \, \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)} \cdot \begin{cases} 1 & \text{if } -0.5 < s < 1.5 \\ \sqrt{L} & \text{if } 1.5 \leq s \leq 2. \end{cases} \tag{5.28}$$

Proof. The case $-0.5 < s < 1.5$ follows immediately from the Riesz property (5.20), and it remains to consider $1.5 \leq s \leq 2$. First we suppose that $\sum \xi_P \psi_P$ is the projection $Q_L u$. The Riesz property and the approximation property of Lemma 5.2, iii), which remains valid for R_L replaced by the uniformly bounded Q_L (cf. (5.20)), imply

$$\begin{aligned} \sqrt{\sum_{P \in \nabla_{l-1}^\Gamma} 2^{-2(l-1)}|\xi_P|^2} &\sim \|Q_l u - Q_{l-1} u\|_{L^2} \leq \|Q_l u - u\|_{L^2} + \|u - Q_{l-1} u\|_{L^2} \\ &\leq C \, 2^{-ls} \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)}, \\ \sqrt{\sum_{P \in \nabla_{l-1}^\Gamma} 2^{2(l-1)(s-1)}|\xi_P|^2} &\leq C \, \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)}. \end{aligned}$$

Passing to the squares and summing up over $l = -1, \ldots, L-1$, we get the upper bound $C\, L\, \|u\|^2$. Taking square roots we obtain the assertion for $1.5 \le s \le 2$.

Now we denote the coefficients of $R_L u$ by $\tilde{\xi}_P$ in order to distinguish them from those of $Q_L u$. From the assertion with $Q_L u$ and from Lemma 5.2 ii) and iii) we get

$$\begin{aligned}
\sqrt{\sum_{P\in\triangle_L^\Gamma} 2^{-2l(P)}|\xi_P - \tilde{\xi}_P|^2} \quad &\sim \quad \|Q_L u - R_L u\|_{L^2} \;\le\; \|Q_L u - u\|_{L^2} + \|u - R_L u\|_{L^2} \\
&\le \quad C\, 2^{-Ls} \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)}, \\
\sqrt{\sum_{P\in\triangle_L^\Gamma} 2^{2l(P)(s-1)}|\xi_P - \tilde{\xi}_P|^2} \quad &\le \quad C\, \|u\|_{\oplus_{m=1}^{m_\Gamma} H^s(\Gamma_m)}.
\end{aligned}$$

This together with the estimate (5.28) for the coefficients ξ_P of $Q_L u$ implies (5.28) for the coefficients $\tilde{\xi}_P$ of $R_L u$. Similarly we can prove the assertion for the orthogonal projection.

∎

5.2 The Properties of the Wavelet Basis in the Test Space

The properties of the basis of test wavelets introduced in Sect. 3.2 can be described using the predual basis. We simply define the classical hierarchical basis by $\chi_P := \varphi_P^{l+1}$ for $P \in \nabla_l^\Gamma$ and observe

$$\langle \vartheta_P, \chi_{P'} \rangle \;:=\; \vartheta_P(\chi_{P'}) \;=\; \delta_{P,P'} \tag{5.29}$$

as well as $\operatorname{span}\{\chi_P :\; P \in \triangle_L^\Gamma\} = Lin_L^\Gamma$. The interpolation projection can be represented as

$$R_L u \;=\; \sum_{P\in\triangle_L^\Gamma} u(P)\varphi_P^L \;=\; \sum_{P\in\triangle_L^\Gamma} \langle \vartheta_P, u\rangle\, \chi_P .$$

The following properties are well known.

Lemma 5.4 *i) For $1 < s < 1.5$, the basis $\{\chi_P :\; P \in \cup_{L=0}^\infty \triangle_L^\Gamma\}$ is a Riesz basis, i.e., for any vector of real numbers $(\xi_P)_P$, we get*

$$\left\| \sum_{P\in\triangle_L^\Gamma} \xi_P \chi_P \right\|_{H^s(\Gamma)} \sim \sqrt{\sum_{P\in\triangle_L^\Gamma} 2^{2l(P)(s-1)}|\xi_P|^2}. \tag{5.30}$$

ii) The approximation and inverse properties for the space predual to the test functionals are formulated in Lemma 5.2 ii)-iv).

The second basis $\{\vartheta_P^+\}$ is a slight modification of $\{\vartheta_P\}$. In fact the basis transform from $\{\vartheta_P\}$ to $\{\vartheta_P^+\}$ is the identity matrix plus an upper triangular matrix with only one entry 0.25 in each row and no more than six entries 0.25 in each column. Hence, the basis transform is invertible. A dual system $\{\chi_P^+\}$ for a fixed L can easily be constructed from $\{\chi_P\}$ by applying the inverse adjoint basis transform. Moreover, if we change the basis $\{\vartheta_P\}$ to the H^s scaled basis $\{2^{l(P)(s-1)}\vartheta_P\}$ and $\{\vartheta_P^+\}$ to $\{2^{l(P)(s-1)}\vartheta_P^+\}$, then the basis transform is the identity matrix plus an upper triangular matrix with only one entry $0.25 \cdot 2^{s-1}$ in each row and no more than six entries $0.25 \cdot 2^{s-1}$ in each column. Due to Schur's lemma the norm of the triangular matrix is less than

$$\sqrt{[0.25 \cdot 2^{s-1}] \cdot 1}\sqrt{[0.25 \cdot 2^{s-1}] \cdot 6} \;\; \leq \;\; \sqrt{0.75} \; < \; 1.$$

Thus the basis transform is stable even for $1 < s < 1.5$, and assertion i) of Lemma 5.4 remains true if we replace $\{\vartheta_P\}$ by $\{\vartheta_P^+\}$.

We finish this subsection with a result on the boundedness of the wavelet transform $\mathcal{T}_T$.

Lemma 5.5 *Suppose that* $u_L = \sum_{P\in\triangle_P^\Gamma} \eta_P \varphi_P^L$ *and that* $\eta = (\eta_P)_{P\in\triangle_L^\Gamma} = \mathcal{T}_T\gamma$ *with* $\gamma = (\gamma_P)_{P\in\triangle_L^\Gamma}$. *Here the wavelet transform* $\mathcal{T}_T$ *from Sect. 3.3 could be defined also with* ϑ_P *replaced by* ϑ_P^+. *Then, we get*

$$\|u_L\|_{H^s(\Gamma)} \leq C\sqrt{\sum_{P\in\triangle_L^\Gamma} 2^{2l(P)(s-1)}|\gamma_P|^2}\begin{cases} \sqrt{L} & \text{if } 0 \leq s \leq 1 \\ 1 & \text{if } 1 < s < 1.5\,. \end{cases} \tag{5.31}$$

Proof. Since the basis transform from $\{\chi_P : \; P \in \triangle_L^\Gamma\}$ to $\{\chi_P^+ : \; P \in \triangle_L^\Gamma\}$ and its inverse is stable in H^s, we may suppose, without loss of generality, that the wavelet transform is defined with ϑ_P. The case $1 < s < 1.5$ follows from Lemma 5.4, i). For $0 \leq s \leq 1$, we conclude

$$\begin{aligned} \|u_L\|_{H^s(\Gamma)} &= \|\sum_{P\in\triangle_L^\Gamma} \gamma_P\chi_P\|_{H^s(\Gamma)} \leq \sum_{l=-1}^{L-1} \|\sum_{P\in\nabla_l^\Gamma} \gamma_P\chi_P\|_{H^s(\Gamma)} \\ &\leq C\sqrt{L}\sqrt{\sum_{l=-1}^{L-1} \|\sum_{P\in\nabla_l^\Gamma} \gamma_P\chi_P\|_{H^s(\Gamma)}^2}. \end{aligned}$$

Now it remains to apply the inverse property and a discrete norm estimate for shifts of hat functions on one level.

$$\|u_L\|_{H^s(\Gamma)} \;\leq\; C\sqrt{L}\sqrt{\sum_{l=-1}^{L-1} 2^{2sl}\|\sum_{P\in\nabla_l^\Gamma} \gamma_P\chi_P\|_{L^2(\Gamma)}^2} \;\leq\; C\sqrt{L}\sqrt{\sum_{l=-1}^{L-1} 2^{2l(s-1)}\sum_{P\in\nabla_l^\Gamma} |\gamma_P|^2}.$$

■

5.3 The Complexity of the Compression Algorithm

Lemma 5.6 *The number* $N_{\mathcal{P}}$ *of non-zero entries in the compressed matrix* $A_L^{w,c}$ *corresponding to the compression pattern* $\mathcal{P}(a,b,c,d,\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ *(cf. Sects. 3.4 and 3.5) satisfies*

$$\begin{aligned} N_{\mathcal{P}} \;\leq\;& C\,L\,2^{2L} + C\,d^2\,2^{2L[a+(1-c)_+ + (1-b)_+]}\begin{cases} 1 & \text{if } c \neq 1,\; b \neq 1 \\ L^2 & \text{if } c = b = 1 \\ L & \text{else} \end{cases} \\ &+ C\,\tilde{d}\,2^{2L[\tilde{a}/2+(1-\tilde{c}/2)_+ + (1/2-\tilde{b}/2)_+]}\begin{cases} 1 & \text{if } \tilde{c} \neq 2,\; \tilde{b} \neq 1 \\ L^2 & \text{if } \tilde{c} = 2,\; \tilde{b} = 1 \\ L & \text{else}\,. \end{cases} \end{aligned} \tag{5.32}$$

In the last formula $(\ldots)_+$ *stands for the positive part of* $(\ldots)$, *i.e.,* $(\ldots)_+$ *is equal to* $(\ldots)$ *if* $(\ldots) \geq 0$ *and* $(\ldots)_+$ *is zero else.*

Proof. First we count the entries from (3.18) and denote their number by $N^1_{\mathcal{P}}$. For a fixed test functional $\vartheta_{P'}$, the number of entries with column indices P such that $l(P)=l$ and (3.18) hold is less than

$$C\left[\frac{\max\left\{2^{-l},2^{-l(P')},d2^{aL-bl-cl(P')}\right\}}{2^{-l}}\right]^2 .$$

We estimate the maximum of the three numbers by the square root of the sum of the squares. Then we sum up over all levels l, over the $O(2^{2l'})$ test functionals with level $l(P')=l'$, and over all levels l'. We arrive at

$$\begin{aligned} N^1_{\mathcal{P}} &\leq \sum_{l'=-1}^{L-1} 2^{2l'} \sum_{l=-1}^{L-1} C\left\{1+2^{2(l-l')}+d^2 2^{2aL+2(1-b)l-2cl'}\right\} \\ &\leq C\,L\,2^{2L}+C\,d^2\,2^{2L[a+(1-c)_+ +(1-b)_+]}\begin{cases} 1 & \text{if } c\neq 1,\ b\neq 1 \\ L^2 & \text{if } c=b=1 \\ L & \text{else .}\end{cases} \end{aligned} \tag{5.33}$$

Next we count the entries from (3.19) and denote their number by $N^2_{\mathcal{P}}$. For a fixed test functional $\vartheta_{P'}$, the number of entries with column indices P such that $l(P)=l$ and (3.19) hold is less than

$$C\frac{\max\left\{2^{-l},2^{-l(P')},\tilde d 2^{\tilde aL-\tilde bl-\tilde cl(P')}\right\}}{2^{-l}}$$

since all the ψ_P intersecting the common boundary of two parametrization patches are located along a one dimensional submanifold. Estimating the maximum of the three numbers by their sum and summing up over all levels l, over the $O(2^{2l'})$ test functionals with level $l(P')=l'$, and over all levels l', leads to

$$\begin{aligned} N^2_{\mathcal{P}} &\leq \sum_{l'=-1}^{L-1} 2^{2l'} \sum_{l=-1}^{L-1} C\left\{1+2^{(l-l')}+\tilde d 2^{\tilde aL+(1-\tilde b)l-\tilde cl'}\right\} \\ &\leq C\,L\,2^{2L}+C\,\tilde d\,2^{L[\tilde a+(2-\tilde c)_+ +(1-\tilde b)_+]}\begin{cases} 1 & \text{if } \tilde c\neq 2,\ \tilde b\neq 1 \\ L^2 & \text{if } \tilde c=2,\ \tilde b=1 \\ L & \text{else .}\end{cases} \end{aligned} \tag{5.34}$$

The estimates (5.33) and (5.34) together with $N_{\mathcal{P}}=N^1_{\mathcal{P}}+N^2_{\mathcal{P}}$ imply (5.32). ∎

5.4 General Error Estimates for the Numerical Solution and Preconditioning

In this subsection we recall well-known error estimates for stable numerical methods. We give the precise assumptions on the stability and derive necessary conditions which ensure that the numerical methods, perturbed by compression and by boundary and quadrature approximation, admit the same asymptotic orders of convergence as the unperturbed methods. Moreover, we give necessary conditions which ensure the existence of diagonal preconditioners for the matrix $A^{w,c,q}$ of the compressed and approximated collocation method.

The collocation method for the equation $Au = v$ defines an approximate solution $u_L \in Lin_L^\Gamma$ by $R_L A u_L = R_L v$ (cf. Sect. 2.5). This method is called stable in the space $H^s(\Gamma)$ if the approximate operators $R_L A: \ Lin_L^\Gamma \longrightarrow Lin_L^\Gamma$ are invertible for sufficiently large L and if their inverses are bounded, i.e.,

$$\left\| \left(R_L A|_{Lin_L^\Gamma} \right)^{-1} w_L \right\|_{H^{s+\mathbf{r}}(\Gamma)} \ \leq \ C \, \|w_L\|_{H^s(\Gamma)}, \quad w_L \in Lin_L^\Gamma.$$

We suppose that the collocation method is stable for $s = 0$. Additionally, if $\mathbf{r} = -1$ or if the algorithm (3.15) is applied to an operator A of order $\mathbf{r} = 0$, then we suppose stability also for $s = 1.1$ (or for an arbitrary s with $1 < s < 1.5$ instead of 1.1). Note that stability is well known for second kind integral operators including compact integral operators. In particular this is true for double layer operators over smooth boundaries (cf. e.g. [2]). For first kind operators and operators involving strongly singular integral operators, the question of stability is not solved yet. A first step toward the solution is done in [34, 35, 10, 13]. Note that, since our trial space Lin_L^Γ is generated by two scaling functions, the stability is needed for a multiwavelet space (cf. the univariate multiwavelet paper [36]). Though a rigorous proof of stability is missing engineers frequently use collocation methods without observing instabilities.

To simplify the notation, let us denote the operator $R_L A|_{Lin_L^\Gamma}$ by A_L, i.e., by the same symbol as for its matrix with respect to the basis $\{\varphi_P^L : \ P \in \triangle_L^\Gamma\}$ (cf. Sect. 2.5). Similarly, we denote by A_L^c and $A_L^{c,q}$ the operators in Lin_L^Γ the matrix of which with respect to $\{\varphi_P^L : \ P \in \triangle_L^\Gamma\}$ is A_L^c and $A_L^{c,q}$, respectively (cf. (3.14)). Using the L^2 orthogonal projection P_L, we represent the error $u - u_L$ of the fully discretized and compressed method $A_L^{c,q} u_L = R_L v$ as

$$\begin{aligned} u - u_L &= u - P_L u - (A_L^{c,q})^{-1} \left\{ R_L A u - A_L^{c,q} P_L u \right\} \\ &= u - P_L u - (A_L^{c,q})^{-1} \left\{ [A_L - A_L^{c,q}] P_L u + A(I - P_L) u - (I - R_L) A (I - P_L) u \right\}. \end{aligned}$$

We apply the boundedness assumption on A (cf. Sect. 2.2), assume the stability of $A_L^{c,q}$ for Sobolev index $s = 0$, and use Lemma 5.2 to get

$$\begin{aligned} \|u - u_L\|_{H^{\mathbf{r}}(\Gamma)} &\leq \|u - P_L u\|_{H^{\mathbf{r}}(\Gamma)} + C \left\{ \|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} + \right. \\ &\qquad \left. \|(I - P_L) u\|_{H^{\mathbf{r}}(\Gamma)} + 2^{-1.1L} \, \|A(I - P_L) u\|_{H^{1.1}(\Gamma)} \right\} \\ &\leq C 2^{-(2-\mathbf{r})L} \, \|u\|_{H^2(\Gamma)} + C \, \|[A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)}. \end{aligned}$$

In other words, to ensure the optimal convergence order $2 - \mathbf{r}$, we need the estimate

$$\| \, [A_L - A_L^{c,q}] P_L u\|_{H^0(\Gamma)} \ \leq \ C 2^{-(s-\mathbf{r})L} \|u\|_{H^s(\Gamma)} \tag{5.35}$$

for $s = 2$ and the stability of $A_L^{c,q}$. Since A_L is stable by assumption, for the stability of $A_L^{c,q}$, it will be sufficient to require

$$\|A_L - A_L^{c,q}\|_{H^0(\Gamma) \leftarrow H^{\mathbf{r}}(\Gamma)} \ \leq \ \frac{1}{2} \sup_{L' = L_0, L_0+1, \ldots} \|A_{L'}^{-1}\|^{-1}_{H^{\mathbf{r}}(\Gamma) \leftarrow H^0(\Gamma)}.$$

In view of the inverse property v) of Lemma 5.2 the last condition is a consequence of (5.35) with the choice $s = 1.1$ if we show that the constant C in (5.35) can be made smaller than any prescribed positive number. It will be the task of the next sections to prove estimate (5.35) for $s = 2$ and $s = 1.1$.

The issue of wavelet preconditioners has been addressed by many authors (cf. e.g. [12, 14, 27, 48]) and we will follow the same ideas. In the case $\mathbf{r} = 0$ the stability of $A_L^{c,q}$ implies that the matrix $A_L^{c,q}$ has a condition number which is already uniformly bounded with respect to L. Thus, for the algorithm (3.16), no preconditioning is needed, and we can restrict our consideration to algorithm (3.15). Unfortunately, the wavelet transform $\mathcal{T}_T^{-1}$ (cf. Sect. 3.3) has not a uniformly bounded condition number with respect to Euclidean matrix norm. Therefore, preconditioning is needed even for $\mathbf{r} = 0$, and the preconditioner is to be derived from the stability for a different Sobolev index. We choose e.g. $s = 1.1$.

Let us consider an operator A of order $\mathbf{r} = 0, -1$ and suppose the stability of A_L in the Sobolev space $H^{1.1}(\Gamma)$. If we could prove

$$\|A_L - A_L^{c,q}\|_{H^{1.1}(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} \;\leq\; \frac{1}{2} \sup_{L'=L_0,L_0+1,\ldots} \|A_{L'}^{-1}\|^{-1}_{H^{1.1+\mathbf{r}}(\Gamma)\leftarrow H^{1.1}(\Gamma)}, \tag{5.36}$$

then $A_L^{c,q}$ is stable in $H^{1.1}(\Gamma)$, too. From Sects. 3.1 and 5.2, we recall that $A_L^{w,c,q}$ is the matrix of the operator $A_L^{c,q}$ with respect to the bases $\{\psi_P : \; P \in \triangle_L^\Gamma\}$ and $\{\chi_P : \; P \in \triangle_L^\Gamma\}$. Under assumption (5.36), the assertions i) of the Lemmata 5.4 and 5.2 imply that the matrices

$$\left(\delta_{P,P'} 2^{l(P')(1.1-1)}\right)_{P,P'\in\triangle_L^\Gamma} A_L^{w,c,q} \left(\delta_{P,P'} 2^{-l(P)(\mathbf{r}+1.1-1)}\right)_{P,P'\in\triangle_L^\Gamma} \tag{5.37}$$

have condition numbers which are uniformly bounded with respect to L, i.e. the matrix $A_L^{w,c,q}$ admits a diagonal preconditioning. The boundedness of the condition number ensures the fast convergence of the iterative solver in the wavelet algorithm (3.15). In other words, for the fast iterative solution of the linear systems $A_L^{w,c,q}\beta = \gamma$ (cf. part iv) of (3.15)) using preconditioning, we only have to prove (5.36). This will be done in the next two sections.

5.5 The Estimate of the Compression Error

The fundamental relation for the compression is the following decay property of the entries $a^w_{P',P}$ of the stiffness matrix (3.12) with respect to the wavelet bases. The decay estimates rely on the assumptions for the kernel function (cf. Sect. 2.2) and on the vanishing moment properties for the wavelets (cf. Sects. 3.1 and 3.2). Let $\mathbf{m}$ stand for the number of vanishing moments of the test functionals. For $\mathbf{r} = 0$, we use the test functionals $\vartheta_{P'}$ and get $\mathbf{m} = 2$. If $\mathbf{r} = -1$, then we use the test functionals $\vartheta_{P'}$ and set $\mathbf{m} = 3$. In any case $\mathbf{r} + \mathbf{m} = 2$ and $\mathbf{m} = 2 - \mathbf{r}$. The support $\Theta_{P'}$ of $\vartheta_{P'}$ resp. $\vartheta^+_{P'}$ is supposed to be defined like in the beginning of Sect. 3.5.

Lemma 5.7 *If the support Ψ_P of the trial function ψ_P is contained in the interior of a single patch Γ_m of the boundary and if the distance of Ψ_P to the support $\Theta_{P'}$ of the test functional $\vartheta_{P'}$ resp. $\vartheta^+_{P'}$ is positive, then we get*

$$\left|a^w_{P',P}\right| \;\leq\; C\, 2^{-\mathbf{m}l(P')} 2^{-4l(P)} \mathrm{dist}\left(\Theta_{P'}, \Psi_P\right)^{-\mathbf{r}-4-\mathbf{m}}. \tag{5.38}$$

If Ψ_P is not contained in the interior of a single patch Γ_m and if the distance of Ψ_P to $\Theta_{P'}$ is positive, then we get

$$\left|a^{w}_{P',P}\right| \;\leq\; C\,2^{-\mathbf{m}l(P')}2^{-2l(P)}\mathrm{dist}\,(\Theta_{P'},\Psi_P)^{-\mathbf{r}-2-\mathbf{m}}. \tag{5.39}$$

Proof. For a rigorous proof of such estimates we refer e.g. to [14, 44]. We give only a short explanation for the estimates (5.38) and (5.39). Since the kernel k in (2.4) is bounded and since $p(P-Q)/|P-Q|^\alpha$ behaves like $|P-Q|^{-2-\mathbf{r}}$, the estimate $C2^{-2l}\mathrm{dist}(P',\mathrm{supp}\,\varphi^l_P)^{-2-\mathbf{r}}$ for the entry $(K\varphi^l_P)(P')$ is standard. If we change φ^l_P into the wavelet ψ_P with two vanishing moments, then the integration against ψ_P is like applying a second order derivative to the kernel, multiplying by the factor $2^{-2l(P)}$, and integrating over the support Ψ_P. Using the bound $C\,|P-Q|^{-4-\mathbf{r}}$ for the second order derivative of the kernel function in (2.4), we arrive at the estimate $C2^{-4l(P)}\mathrm{dist}(P',\Psi_P)^{-4-\mathbf{r}}$ for the entry $(K\psi_P)(P')$. Replacing the Dirac delta functional at P' by the wavelet functional $\vartheta_{P'}$ with $\mathbf{m}$ vanishing moments is like applying an $\mathbf{m}$-th order derivative to the kernel and multiplying by the factor $2^{-\mathbf{m}l(P')}$. Thus the entry $\vartheta_{P'}(K\psi_P)$ is bounded by the right-hand side of (5.38). Similarly, we get (5.39) for a wavelet ψ_P without vanishing moments. ∎

Now we suppose that the entries $a^{w,c}_{P',P}$ of $A^{w,c}_L$ are computed exactly. In this case the missing estimate (5.36) and the inequality (5.35) with the Sobolev indices $s=2$ and $s=1.1$ follow from

Lemma 5.8 *Suppose $A_L\in\mathcal{L}(Lin^\Gamma_L)$ is the approximate operator of the collocation method (cf. Sect. 2.5) and A^c_L the operator of the compressed collocation method (cf. Sect. 5.4) including the sparsity pattern $\mathcal{P}=\mathcal{P}(a,b,c,d,\tilde a,\tilde b,\tilde c,\tilde d)$ (cf. Sect. 3.5) with the parameters $b=\tilde b=1$, $a=c>0.75$, and $\tilde a=\tilde c>1.5$. Then we get*

$$\begin{aligned}
\|A_L-A^c_L\|_{H^0(\Gamma)\leftarrow H^2(\Gamma)} &\leq C\,\{d^{-4}+\tilde d^{-2}\}L^{1.5}2^{-(2-\mathbf{r})L}, &(5.40)\\
\|A_L-A^c_L\|_{H^0(\Gamma)\leftarrow H^{1.1}(\Gamma)} &\leq C\,\{d^{-4}+\tilde d^{-2}\}\sqrt{L}\,2^{-(1.1-\mathbf{r})L}, &(5.41)\\
\|A_L-A^c_L\|_{H^{1.1}(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} &\leq C\,\{d^{-4}+\tilde d^{-2}\}. &(5.42)
\end{aligned}$$

Proof. First we consider (5.40) and (5.41). We set

$$b_{P',P} \;:=\; \sqrt{L}2^{(0-1)l(P')}\left|a^{w}_{P',P}-a^{w,c}_{P',P}\right|2^{-(s-1)l(P)}\begin{cases}\sqrt{L} & \text{if } s=2\\ 1 & \text{if } s=1.1\,.\end{cases} \tag{5.43}$$

In view of Lemmata 5.3 and 5.5 the norm of $\|A_L-A^c_L\|$ can be majorized by the Euclidean norm of $(b_{P',P})_{P',P}$. Schur's lemma gives

$$\begin{aligned}
\left\|(b_{P',P})_{P',P}\right\| &\leq \sqrt{\Sigma_1\cdot\Sigma_2},\\
\Sigma_1 &:= \max_{P'\in\triangle^\Gamma_L}2^{l(P')}\sum_{P\in\triangle^\Gamma_L}b_{P',P}2^{-l(P)}, &(5.44)\\
\Sigma_2 &:= \max_{P\in\triangle^\Gamma_L}2^{l(P)}\sum_{P'\in\triangle^\Gamma_L}b_{P',P}2^{-l(P')},
\end{aligned}$$

and it remains to estimate Σ_1 and Σ_2. Let us set dist $:=\mathrm{dist}(\Theta_{P'},\Psi_P)$ and $\max_1:=\max\{2^{-l},2^{-l(P')},d2^{aL-bl-cl(P')}\}$ as well as $\max_2:=\max\{2^{-l},2^{-l(P')},\tilde d2^{\tilde aL-\tilde bl-\tilde cl(P')}\}$. By $\odot^\Gamma_l$

we denote the set of $P \in \nabla_l^\Gamma$ such that Ψ_P is not contained in the interior of a single patch Γ_m. Furthermore, we set $\oslash_l^\Gamma := \nabla_l^\Gamma \setminus \odot_l^\Gamma$. Finally we write $\tilde{L} = L$ for $s = 2$ and $\tilde{L} = \sqrt{L}$ if $s = 1.1$. Now the compression criteria (3.18) and (3.19) as well as Lemma 5.7 imply

$$\begin{aligned}
\Sigma_1 &\leq C\,\tilde{L} \max_{P' \in \triangle_L^\Gamma} \sum_{l=-1}^{L-1} 2^{-sl} \sum_{P \in \nabla_l^\Gamma} \left| a_{P',P}^{w} - a_{P',P}^{w,c} \right| \\
&\leq C\,\tilde{L} \max_{P' \in \triangle_L^\Gamma} \left\{ \sum_{l=-1}^{L-1} 2^{-(s-2)l} 2^{-2l} \sum_{P \in \oslash_l^\Gamma:\,\mathrm{dist} > \max_1} 2^{-\mathbf{m}l(P')} 2^{-4l} \mathrm{dist}^{-\mathbf{r}-4-\mathbf{m}} + \right. \\
&\qquad \left. \sum_{l=-1}^{L-1} 2^{-(s-1)l} 2^{-l} \sum_{P \in \odot_l^\Gamma:\,\mathrm{dist} > \max_2} 2^{-\mathbf{m}l(P')} 2^{-2l} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}} \right\} \\
&\leq C\,\tilde{L} \max_{P' \in \triangle_L^\Gamma} \left\{ \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l(P')} 2^{-(s+2)l} \max\nolimits_1^{-\mathbf{r}-2-\mathbf{m}} + \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l(P')} 2^{-(s+1)l} \max\nolimits_2^{-\mathbf{r}-1-\mathbf{m}} \right\} \\
&\leq C\,\tilde{L} \max_{P' \in \triangle_L^\Gamma} \left\{ \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l(P')} 2^{-(s+2)l} \left[d 2^{aL - bl - cl(P')} \right]^{-\mathbf{r}-2-\mathbf{m}} + \right. \\
&\qquad \left. \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l(P')} 2^{-(s+1)l} \left[\tilde{d} 2^{\tilde{a}L - \tilde{b}l - \tilde{c}l(P')} \right]^{-\mathbf{r}-1-\mathbf{m}} \right\} \\
&\leq C\,\tilde{L} \left\{ d^{-4} 2^{-a(\mathbf{r}+\mathbf{m}+2)L} \max_{l'=-1,\ldots,L-1} 2^{[c(\mathbf{r}+\mathbf{m}+2)-\mathbf{m}]l'} \sum_{l=-1}^{L-1} 2^{[b(\mathbf{r}+\mathbf{m}+2)-(2+s)]l} + \right. \\
&\qquad \left. \tilde{d}^{-3} 2^{-\tilde{a}(\mathbf{r}+\mathbf{m}+1)L} \max_{l'=-1,\ldots,L-1} 2^{[\tilde{c}(\mathbf{r}+\mathbf{m}+1)-\mathbf{m}]l'} \sum_{l=-1}^{L-1} 2^{[\tilde{b}(\mathbf{r}+\mathbf{m}+1)-(1+s)]l} \right\}.
\end{aligned}$$

Note that, in the step from line two to three of the preceding estimation, we have used

$$\begin{aligned}
2^{-2l} \sum_{P \in \oslash_l^\Gamma:\,\mathrm{dist} > \max_1} \mathrm{dist}^{-\mathbf{r}-4-\mathbf{m}} &\leq C \int_{\{P \in \Gamma:\, |P'-P| > \max_1\}} \frac{\mathrm{d}_P \Gamma}{|P'-P|^{\mathbf{r}+4+\mathbf{m}}} \\
&\leq C \max\nolimits_1^{-\mathbf{r}-2-\mathbf{m}}, \\
2^{-l} \sum_{P \in \odot_l^\Gamma:\,\mathrm{dist} > \max_2} \mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}} &\leq C \sum_{m,m'=1}^{m_\Gamma} \int_{\{P \in \Gamma_m \cap \Gamma_{m'}:\, |P'-P| > \max_2\}} \frac{\mathrm{d}_P \{\Gamma_m \cap \Gamma_{m'}\}}{|P'-P|^{\mathbf{r}+2+\mathbf{m}}} \\
&\leq C \max\nolimits_2^{-\mathbf{r}-1-\mathbf{m}}.
\end{aligned}$$

For $s = 2$ and $s = 1.1$ and for our special choice of the parameters a, b, c, $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$, we get $c(\mathbf{r}+\mathbf{m}+2) - \mathbf{m} \geq 0$ and $b(\mathbf{r}+\mathbf{m}+2) - (2+s) \geq 0$ as well as $\tilde{c}(\mathbf{r}+\mathbf{m}+1) - \mathbf{m} \geq 0$ and $\tilde{b}(\mathbf{r}+\mathbf{m}+1) - (1+s) \geq 0$. Hence, we may continue

$$\Sigma_1 \leq C \{ d^{-4} + \tilde{d}^{-3} \} 2^{-(s-\mathbf{r})L} \begin{cases} L^2 & \text{if } s = 2 \\ \sqrt{L} & \text{if } s = 1.1\,. \end{cases}$$

Let us turn to Σ_2. We set $\oslash := \cup_{l=-1}^{L-1} \oslash_l^\Gamma$ as well as $\odot := \cup_{l=-1}^{L-1} \odot_l^\Gamma$, and, similarly to the estimation for Σ_1, we get

$$\Sigma_2 \leq C\,\tilde{L} \max_{P \in \triangle_L^\Gamma} \sum_{l=-1}^{L-1} 2^{-(s-2)l(P)} 2^{-2l} \sum_{P' \in \nabla_l^\Gamma} \left| a_{P',P}^{w} - a_{P',P}^{w,c} \right|$$

$$
\begin{aligned}
&\le C\,\tilde L \max_{P\in\oslash}\left\{ 2^{-(s-2)l(P)}\sum_{l=-1}^{L-1} 2^{-2l}\sum_{P'\in\nabla_l^\Gamma:\,\mathrm{dist}>\max_1} 2^{-\mathbf{m}l}2^{-4l(P)}\mathrm{dist}^{-\mathbf{r}-4-\mathbf{m}}\right\}\\
&\quad +C\,\tilde L \max_{P\in\odot}\left\{ 2^{-(s-2)l(P)}\sum_{l=-1}^{L-1} 2^{-2l}\sum_{P'\in\nabla_l^\Gamma:\,\mathrm{dist}>\max_2} 2^{-\mathbf{m}l}2^{-2l(P)}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}}\right\}\\
&\le C\,\tilde L \max_{P\in\oslash}\left\{ \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l}2^{-(s+2)l(P)}\max{}_1^{-\mathbf{r}-2-\mathbf{m}}\right\}\\
&\quad +C\,\tilde L \max_{P\in\odot}\left\{ \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l}2^{-sl(P)}\max{}_2^{-\mathbf{r}-\mathbf{m}}\right\}\\
&\le C\,\tilde L \max_{P\in\oslash}\left\{ \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l}2^{-(2+s)l(P)}\left[d2^{aL-bl(P)-cl}\right]^{-\mathbf{r}-2-\mathbf{m}}\right\}\\
&\quad +C\,\tilde L \max_{P\in\odot}\left\{ \sum_{l=-1}^{L-1} 2^{-\mathbf{m}l}2^{-sl(P)}\left[\tilde d2^{\tilde aL-\tilde bl(P)-\tilde cl}\right]^{-\mathbf{r}-\mathbf{m}}\right\}\\
&\le C\,\tilde L\, d^{-4}2^{-a(\mathbf{r}+\mathbf{m}+2)L}\max_{l'=-1,\dots,L-1} 2^{[b(\mathbf{r}+\mathbf{m}+2)-(2+s)]l'}\sum_{l=-1}^{L-1}2^{[c(\mathbf{r}+\mathbf{m}+2)-\mathbf{m}]l}\\
&\quad +C\,\tilde L\, \tilde d^{-2}2^{-\tilde a(\mathbf{r}+\mathbf{m})L}\max_{l'=-1,\dots,L-1} 2^{[\tilde b(\mathbf{r}+\mathbf{m})-s]l'}\sum_{l=-1}^{L-1}2^{[\tilde c(\mathbf{r}+\mathbf{m})-\mathbf{m}]l}
\end{aligned}
$$

For $s=2$ and $s=1.1$ and for our special choice of the parameters a, b, c, $\tilde a$, $\tilde b$, and $\tilde c$, we get $c(\mathbf{r}+\mathbf{m}+2)-\mathbf{m}\ge 0$ and $b(\mathbf{r}+\mathbf{m}+2)-(2+s)\ge 0$ as well as $\tilde c(\mathbf{r}+\mathbf{m})-\mathbf{m}\ge 0$ and $\tilde b(\mathbf{r}+\mathbf{m})-s\ge 0$. Hence, we may continue

$$
\Sigma_2 \;\le\; \left\{C\,d^{-4}2^{-\mathbf{m}L}+C\,\tilde d^{-2}2^{-\mathbf{m}L}\right\}\left\{\begin{array}{ll} L & \text{if } s=2\\ \sqrt{L} & \text{if } s=1.1\ ,\end{array}\right.
$$

and the assertions (5.40) and (5.41) follow.

Now we turn to (5.42). The estimation is analogous to that of (5.40). Instead of (5.43) we set

$$
b_{P',P} \;:=\; 2^{(1.1-1)l(P')}\left|a^w_{P',P}-a^{w,c}_{P',P}\right|2^{-(1.1+\mathbf{r}-1)l(P)},
$$

and, proceeding analogously to the preceding estimation of Σ_1 and Σ_2, we arrive at

$$
\Sigma_1 \;\le\; C\,d^{-4}+C\tilde d^{-3},\qquad \Sigma_2\;\le\; C\,d^{-4}+C\tilde d^{-2}.
$$

This implies (5.42). ∎

6 The Estimation of the Errors due to the Approximate Parametrization and due to the Quadrature

6.1 The Far Field Estimate

In this subsection we suppose that the near field and the singular integrations are performed exactly and derive the convergence estimates for the far field case. The error estimate for the near field and for the singular integrals will be considered in Sects. 6.2 and 6.3, respectively. In view of Sect. 5.4 and Lemma 5.8, it remains to prove

Lemma 6.1 *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P} = \mathcal{P}(a,b,c,d,\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ (cf. Sect. 3.5) with $a = b = c = \tilde{b} = 1$ and $\tilde{a} = \tilde{c} > 1.5$. If $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of Sect. 4.1, then we get*

$$\begin{aligned}
\|A_L^c - A_L^{c,w}\|_{H^0(\Gamma)\leftarrow H^2(\Gamma)} &\leq C\{d^{-(2-\mathbf{r})} + \tilde{d}^{-(2-\mathbf{r})}\}L^2 2^{-(2-\mathbf{r})L}, && (6.1)\\
\|A_L^c - A_L^{c,w}\|_{H^0(\Gamma)\leftarrow H^{1.1}(\Gamma)} &\leq C\{d^{-(2-\mathbf{r})} + \tilde{d}^{-(2-\mathbf{r})}\}L^2\, 2^{-(2-\mathbf{r})L}, && (6.2)\\
\|A_L^c - A_L^{c,w}\|_{H^{1.1}(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} &\leq C\{d^{-(2-\mathbf{r})} + \tilde{d}^{-(2-\mathbf{r})}\}L^2\, 2^{-(0.9-\mathbf{r})L}. && (6.3)
\end{aligned}$$

Proof. i) The three estimates (6.1)-(6.3) follow from the inverse property v) of Lemma 5.2, from the property $\|f\|_{H^{s'}(\Gamma)} < C\|f\|_{H^s(\Gamma)}$ corresponding to the continuous embedding $H^s(\Gamma) \subset H^{s'}(\Gamma)$ for $s > s'$, and from the estimate

$$\|A_L^c - A_L^{c,w}\|_{H^0(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} \leq C\{d^{-(2-\mathbf{r})} + \tilde{d}^{-(2-\mathbf{r})}\}L^2 2^{-(2-\mathbf{r})L}. \qquad (6.4)$$

Hence, the only thing left to be proved is (6.4).

To estimate (6.4), we need new functions spanning the trial space. We shall represent the operator of quadrature errors $A_L^c - A_L^{c,w}$ as a matrix $\tilde{A}_L$ with respect to this system of functions, and $\tilde{A}_L$ will be estimated just like the compression error $A_L - A_L^c$ in the proof to Lemma 5.8. The new functions $\phi_{Q,\iota}$ are defined as follows. The space of linear functions over a triangle T_τ with $\tau \in \square_l^T$ is spanned by the three basis functions which vanish at two corners and take the value 1 at the third. We denote these three functions by $\phi_{\tau,\iota}$, $\iota = 1,2,3$ and extend them by zero over the rest of T. The point where $\phi_{\tau,\iota}$ is one will be denoted by τ_ι. For $Q = \kappa_m(\tau)$, we set $\phi_{Q,\iota}(\kappa_m(\sigma)) := \tilde{\phi}_{Q,\iota}(\sigma) := \phi_{\tau,\iota}(\sigma)$ over T_τ. Notice that the function $\tilde{\phi}_{Q,\iota}$ has been defined already in Sect. 4.1. We extend the function $\phi_{Q,\iota}$ from Γ_Q to Γ by setting it to zero over $\Gamma \setminus \Gamma_Q$. The point $\kappa_m(\tau_\iota)$ depending on $Q = \kappa_m(\tau)$ and on ι will be denoted by Q_ι. Clearly, $\phi_{Q,\iota}(Q_{\iota'}) = \delta_{\iota,\iota'}$ and the system $\{\phi_{Q,\iota} : \; Q \in \square_l^\Gamma, \; \iota = 1,2,3\}$ is a basis of the space of all discontinuous piecewise linear functions subordinate to the partition $\{\Gamma_Q : \; Q \in \square_l^\Gamma\}$ of Γ. The system $\{\phi_{Q,\iota} : \; \iota = 1,2,3, \; Q \in \square_l^\Gamma, \; l = 0,\dots,L\}$ is a generating system for the piecewise continuous and piecewise linear functions over the triangulation $\{\Gamma_Q : \; Q \in \square_L^\Gamma\}$.

To prepare the derivation of a representation for $A_L^c - A_L^{c,w}$ with respect to this new generating system $\{\phi_{Q,\iota}\}$, we first represent the trial functions with respect to this system. If a function $u_L = \sum \xi_P \psi_P$ is given, then, in the quadrature algorithm $A_L^{c,w} u_L$ for the computation of $A_L u_L$, the function u_L is compressed, and then it is split into the sum of the restrictions to smaller integration domains Γ_Q on which a quadrature rule is applied. More precisely, for a fixed test functional $\vartheta_{P'}$, we get

$$u_L \;\approx\; u_L^c := \sum_{P\in\triangle_L^\Gamma:\,(P',P)\in\mathcal{P}} \xi_P \psi_P \;=\; \sum_{l=0}^{L} \sum_{Q\in Qua_l^\Gamma} \sum_{\iota=1}^{3} u_L^c(Q_\iota)\phi_{Q,\iota}, \qquad (6.5)$$

where the splitting depends on $\vartheta_{P'}$. Due to the definition of Qua_l^Γ in Sect. 4.1 we get $l = l(Q) > l(P)$ for all $P \in \triangle_L^\Gamma$ with $(P',P) \in \mathcal{P}$ and for $Q \in Qua_l^\Gamma$ with $\Gamma_Q \cap \operatorname{supp} \psi_P \neq \emptyset$

(cf. conditions i) and ii) before Lemma 4.1). Thus, to estimate the quadrature error for a fixed $u_L = \sum \xi_P \psi_P$, we define the majorant function $u_L^m := \sum \eta_{Q,\iota} \phi_{Q,\iota}$ of u_L^c by

$$\eta_{Q,\iota} \;:=\; \sum_{P \in \triangle_L^\Gamma :\, Q \in \Psi_P \text{ and } l(Q) > l(P)} |\xi_P| \; |\psi_P(Q_\iota)| \tag{6.6}$$

with $\Psi_P := \operatorname{supp} \psi_P$. This majorant u_L^m is independent of $\vartheta_{P'}$, and its "norm" is almost less than the norm of u_L (cf. the subsequent estimate (6.8)). In part ii) of the present proof we shall estimate the operator norm $\|A_L^c - A_L^{c,w}\|$ of the quadrature error by the Euclidean matrix norm of a matrix acting on the coefficients $\eta_{Q,\iota}$ of u_L^m. This matrix will be treated by the wavelet compression technique, i.e. analogously to the proof of Lemma 5.8.

To show that the "norm" of u_L^m is almost less than the norm of u_L, we formally introduce the norms

$$\begin{aligned}
\left\| (\eta_{Q,\iota})_{Q,\iota} \right\|_{H^0} &:= \sqrt{\sum_{l=0}^{L} \sum_{Q \in \square_l^\Gamma} \sum_{\iota=1}^{3} 2^{-2l} \left|\eta_{Q,\iota}\right|^2}, \\
\left\| (\xi_P)_{P \in \triangle_L^\Gamma} \right\|_{H^s} &:= \sqrt{\sum_{P \in \triangle_L^\Gamma} 2^{2(s-1)l(P)} \left|\xi_P\right|^2}, \\
\left\| (\zeta_P)_{P \in \triangle_L^\Gamma} \right\|_{\tilde{H}^0} &:= \left\| \sum_{P \in \triangle_L^\Gamma} \zeta_P \chi_P \right\|_{H^0(\Gamma)} .
\end{aligned} \tag{6.7}$$

Recall that the H^s norm of $(\xi_P)_P$ is equivalent to the H^s norm of $\sum \xi_P \psi_P$ by assertion i) of Lemma 5.2. We get

$$\left\| (\eta_{Q,\iota})_{Q,\iota} \right\|_{H^0} \;\leq\; C \left\| (\xi_P)_{P \in \triangle_L^\Gamma} \right\|_{H^s} \begin{cases} L & \text{if } s = 0 \\ \sqrt{L} & \text{if } 0 < s < \frac{3}{2} . \end{cases} \tag{6.8}$$

Indeed, from (6.6) and the boundedness of the functions ψ_P, we conclude

$$\left|\eta_{Q,\iota}\right|^2 \;\leq\; C\,\tilde{L} \sum_{P \in \triangle_L^\Gamma :\, Q \in \Psi_P \text{ and } l(Q) > l(P)} 2^{2sl(P)} \left|\xi_P\right|^2, \quad \tilde{L} \;:=\; \begin{cases} L & \text{if } s = 0 \\ 1 & \text{if } 0 < s < \frac{3}{2} , \end{cases}$$

$$\begin{aligned}
\sum_{(Q,\iota)} 2^{-2l(Q)} \left|\eta_{Q,\iota}\right|^2 &\leq C\,\tilde{L} \sum_{P \in \triangle_L^\Gamma} 2^{2sl(P)} \left|\xi_P\right|^2 \sum_{(Q,\iota) :\, Q \in \Psi_P \text{ and } l(Q) > l(P)} 2^{-2l(Q)} \\
&\leq C\,\tilde{L} \sum_{P \in \triangle_L^\Gamma} 2^{2sl(P)} \left|\xi_P\right|^2 \sum_{l=l(P)+1}^{L} 2^{-2l} \sum_{Q \in \square_l^\Gamma :\, Q \in \Psi_P} \sum_{\iota=1}^{3} 1 \\
&\leq C\,\tilde{L} \sum_{P \in \triangle_L^\Gamma} 2^{2sl(P)} \left|\xi_P\right|^2 \sum_{l=l(P)+1}^{L} 2^{-2l} \left[\frac{2^{-l(P)}}{2^{-l}} \right]^2 \\
&\leq C\,\tilde{L} L \sum_{P \in \triangle_L^\Gamma} 2^{2(s-1)l(P)} \left|\xi_P\right|^2
\end{aligned}$$

which proves the estimate (6.8) for $\eta_{Q,\iota}$ defined by (6.6).

ii) Let us introduce the matrix $\tilde{A}_L$ the norm of which majorizes the norm of operator $A_L^c - A_L^{c,w}$ and let us estimate this norm $\|\tilde{A}_L\|$. By $\tilde{a}_{P',(Q,\iota)}$ we denote the absolute value of the quadrature error in the far field integral

$$\vartheta_{P'}\left(\int_\Gamma k(\cdot, R, n_R)\frac{p(\cdot - R)}{|\cdot - R|^\alpha}\phi_{Q,\iota}(R)\,\mathrm{d}_R\Gamma\right) \tag{6.9}$$

where $Q \in Qua_l^\Gamma$ and $l < L$ (cf. Sect. 4.1), and we set $\tilde{a}_{P',(Q,\iota)} = 0$ for $Q \in Qua_L^\Gamma$. We denote the matrix $(\tilde{a}_{P',(Q,\iota)})_{P',(Q,\iota)}$ by $\tilde{A}_L$. Due to (6.5) and (6.6), each component of the vector of quadrature errors $[A_L^{w,c} - A_L^{w,c,q}](\xi_P)_P$ is less or equal to the corresponding entry of the vector $\tilde{A}_L(\eta_{Q,\iota})_{Q,\iota}$. In other words, we obtain

$$\begin{aligned}\left\|[A_L^{w,c} - A_L^{w,c,q}]\,(\xi_P)_{P\in\triangle_L^\Gamma}\right\|_{\tilde{H}^0} &\le \left\|\tilde{A}_L\,(\eta_{Q,\iota})_{Q,\iota}\right\|_{\tilde{H}^0} \le \left\|\tilde{A}_L\right\|_{\tilde{H}^0\leftarrow H^0}\left\|(\eta_{Q,\iota})_{Q,\iota}\right\|_{H^0}\\ &\le C\sqrt{L}\left\|\tilde{A}_L\right\|_{\tilde{H}^0\leftarrow H^0}\left\|(\xi_P)_{P\in\triangle_L^\Gamma}\right\|_{H^{1.1+\mathbf{r}}}.\end{aligned} \tag{6.10}$$

It remains to estimate the norm $\|\tilde{A}_L\|$. In view of Lemma 5.5, the definition of the norm $\|\cdot\|_{\tilde{H}^0}$, and the estimate (6.10), we set

$$b_{P',(Q,\iota)} \;:=\; \sqrt{L}\,\sqrt{L}\,2^{(0-1)l(P')}\;\tilde{a}_{P',(Q,\iota)}\;2^{l(Q)} \tag{6.11}$$

and get that the upper bound $\sqrt{L}\,\|\tilde{A}_L\|$ on the right-hand side of (6.10) is less than the Euclidean matrix norm of the matrix $(b_{P',(Q,\iota)})_{P',(Q,\iota)}$. Now, to get the estimate (6.4), we can proceed analogously as in the proof to Lemma 5.8. We shall prove

$$\tilde{a}_{P',(Q,\iota)} \;\le\; C2^{-\mathbf{m}l(P')}2^{-(4-\mathbf{r})l(Q)}\mathrm{dist}\,(\Theta_{P'},\Gamma_Q)^{-2-\mathbf{m}}. \tag{6.12}$$

This estimate (compare (5.38) and (5.39)), the relations (4.7) and (4.8) (compare (3.18) and (3.19)), and the proof of Lemma 5.8 imply (6.4).

iii) Let us prove (6.12). This, however, is a consequence of $\mathrm{dist}\,(\Theta_{P'},\Gamma_Q) < C$ and of the stronger resp. equivalent estimate

$$\tilde{a}_{P',(Q,\iota)} \;\le\; C2^{-\mathbf{m}l(P')}2^{-(4-\mathbf{r})l(Q)}\mathrm{dist}\,(\Theta_{P'},\Gamma_Q)^{-\mathbf{r}-2-\mathbf{m}}. \tag{6.13}$$

It remains to derive (6.13). The approximation to (6.9) (cf. (4.18)) is obtained by interpolating the parametrization κ_m, by applying a $2-\mathbf{r}$ order product rule to the integral over T_τ of the integrand $\sigma \mapsto k(\cdot, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma)$, and by applying an n_G order quadrature to the integrals of the weight functions $\sigma \mapsto \tilde{\phi}_{Q,v}(\sigma)p(\cdot - \kappa'_m(\sigma))\,|\cdot - \kappa'_m(\sigma)|^{-\alpha}\,\phi_{Q,\iota}(\kappa_m(\sigma))$ (cf. Remark 4.1). Let us make this more precise. It is not hard to see that the test functional $\vartheta_{P'}$ is a scaled version of a difference formula and that it satisfies a certain Leibniz rule of the form

$$\vartheta_{P'}(fg) \;=\; \sum_{i=1}^{i_{P'}}\vartheta_{P',1,i}(f)\vartheta_{P',2,i}(g), \tag{6.14}$$

where the $\vartheta_{P',j,i}$ are, just like the $\vartheta_{P'}$, finite linear combination of Dirac delta functionals with bounded coefficients and with $\mathrm{supp}\,\vartheta_{P',j,i} \subseteq \mathrm{supp}\,\vartheta_{P'}$. Moreover, the sum $\mathbf{m}_{P',1,i} + \mathbf{m}_{P',2,i}$ of the vanishing moments $\mathbf{m}_{P',j,i}$ for $\vartheta_{P',j,i}$ is equal to the number $\mathbf{m} := 2-\mathbf{r}$ of vanishing moments for $\vartheta_{P'}$. Applying (6.14) to (6.9), we get the integrand

$$\sum_{i=1}^{i_{P'}}\int_{\Gamma_Q}k(\vartheta_{P',1,i}\,, R, n_R)\vartheta_{P',2,i}\left(\frac{p(\cdot - R)}{|\cdot - R|^\alpha}\right)\phi_{Q,\iota}(R)\,\mathrm{d}_R\Gamma.$$

Consequently, the term $\tilde{a}_{P',(Q,\iota)}$ is the sum over i of errors due to replacing the parameter mapping κ_m by its interpolation κ'_m, due to applying a $2-\mathbf{r}$ order product rule to the integral over T_τ of the integrand $\sigma \mapsto k(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)})\mathcal{J}'_m(\sigma)$, and due to applying a tensor product variant of Gauß quadrature of order n_G to the integrals of the corresponding weight functions $\sigma \mapsto \tilde{\phi}_{Q,v}(\sigma)\vartheta_{P',2,i}(p(\cdot - \kappa'_m(\sigma))\,|\cdot - \kappa'_m(\sigma)|^{-\alpha})\,\tilde{\phi}_{Q,\iota}(\sigma)$ for $v = 1,2,3$. Indeed, this splitting (6.14) into a sum over $i = 1, \ldots, i_{P'}$ has to be included into the derivation of formula (4.18). We have not mentioned this since the splitting is not seen explicitly in the final formula and since we did not want to overload the presentation in Sect. 4.1 by these technical details.

Clearly, concerning the replacement of κ_m, we get $|\kappa_m(\sigma) - \kappa'_m(\sigma)| \le C\,2^{-(\mathbf{m}+1)l(Q)}$ for $\sigma \in T_\tau = \kappa_m^{-1}(\Gamma_Q)$ and $|\nabla_\sigma \kappa_m(\sigma) - \nabla_\sigma \kappa'_m(\sigma)| \le C\,2^{-\mathbf{m}l(Q)}$ if ∇_σ is the gradient with respect to σ. From the smoothness assumptions on κ_m in Sect. 2.1 and on the integral kernel in Sect. 2.2, we conclude (cf. the proof of Lemma 5.7)

$$|\mathcal{J}_m(\sigma) - \mathcal{J}'_m(\sigma)| \le C2^{-\mathbf{m}l(Q)}, \quad |\mathcal{J}_m(\sigma)| \le C, \quad |\mathcal{J}'_m(\sigma)| \le C,$$

$$\begin{aligned}
\left|k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right) - k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right)\right| &\le C2^{-\mathbf{m}l(Q)}2^{-\mathbf{m}_{P',1,i}\,l(P')}, \\
\left|k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right)\right| &\le C2^{-\mathbf{m}_{P',1,i}\,l(P')}, \\
\left|k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right)\right| &\le C2^{-\mathbf{m}_{P',1,i}\,l(P')},
\end{aligned} \tag{6.15}$$

$$\begin{aligned}
\left|\vartheta_{P',2,i}\left(\frac{p\left(\cdot - \kappa_m(\sigma)\right)}{|\cdot - \kappa_m(\sigma)|^\alpha}\right) - \vartheta_{P',2,i}\left(\frac{p\left(\cdot - \kappa'_m(\sigma)\right)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\right)\right| &\le C\frac{2^{-(\mathbf{m}+1)l(Q)}\,2^{-\mathbf{m}_{P',2,i}\,l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}+1}} \\
&\le C\frac{2^{-\mathbf{m}l(Q)}\,2^{-\mathbf{m}_{P',2,i}l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}}}\,, \\
\left|\vartheta_{P',2,i}\left(\frac{p\left(\cdot - \kappa_m(\sigma)\right)}{|\cdot - \kappa_m(\sigma)|^\alpha}\right)\right| &\le C\frac{2^{-\mathbf{m}_{P',2,i}\,l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}}}, \\
\left|\vartheta_{P',2,i}\left(\frac{p\left(\cdot - \kappa'_m(\sigma)\right)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\right)\right| &\le C\frac{2^{-\mathbf{m}_{P',2,i}\,l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}_{P',2,i}}},
\end{aligned}$$

where we have used the notation $\mathrm{dist} := \mathrm{dist}(\Theta_{P'}, \Gamma_Q)$ and the estimate $\mathrm{dist} > 2^{-l(Q)}$ (cf. (4.7) and (4.8)). Hence, we arrive at

$$\begin{aligned}
&\Bigg|k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n_{\kappa_m(\sigma)}\right)\vartheta_{P',2,i}\left(\frac{p\left(\cdot - \kappa_m(\sigma)\right)}{|\cdot - \kappa_m(\sigma)|^\alpha}\right)\mathcal{J}_m(\sigma)\phi_{\tau,\iota}(\sigma) - \\
&\quad k\left(\vartheta_{P',1,i}, \kappa_m(\sigma), n'_{\kappa'_m(\sigma)}\right)\vartheta_{P',2,i}\left(\frac{p\left(\cdot - \kappa'_m(\sigma)\right)}{|\cdot - \kappa'_m(\sigma)|^\alpha}\right)\mathcal{J}'_m(\sigma)\phi_{\tau,\iota}(\sigma)\Bigg| \\
&\qquad \le C\frac{2^{-\mathbf{m}l(Q)}\,2^{-\mathbf{m}l(P')}}{\mathrm{dist}^{2+\mathbf{r}+\mathbf{m}}}\,,
\end{aligned}$$

and the integral over T_τ of this difference is less than the right-hand side of (6.13).

On the other hand, the error of the product rule can be estimated by the supremum norm interpolation error of the integrand multiplied by the weighted measure of the integration domain. Using the smoothness assumptions on κ_m from Sect. 2.1 and on the kernel function k from Sect. 2.2 as well as the definition of κ'_m as an $\mathbf{m}+1=3-\mathbf{r}$ order interpolation to κ_m, we observe that the interpolation error due to the product integration is less than $2^{-(2-\mathbf{r})l(Q)}$. Note that, again, from the rate of convergence $O(2^{-(3-\mathbf{r})l(Q)})$ for the approximation of the geometry a factor $2^{-l(Q)}$ is lost since the integrand contains first order derivatives. Estimating the integrals over the weight functions of the product rule with the help of (6.15), we get an upper estimate $C2^{-\mathbf{m}_{P',2,i}l(P')}2^{-2l(Q)}\mathrm{dist}^{-\mathbf{r}-2-\mathbf{m}_{P',2,i}}$ for them, and the error of the product rule is less or equal to the right-hand side of (6.13).

iv) Let us turn to the quadrature error of the n_G-th order quadrature applied to the integral over the weight function and show that this is less than the right-hand side of (6.13), too. The tensor product Gauß rule (4.17) with n_G from (4.19) is a very strong tool for producing a small quadrature error. Since we believe that the values n_A and n_B should be determined by numerical tests, we shall not try here to derive the theoretically optimal values for them. This allows us to simplify the estimation. To deduce an error estimate for (4.17), we start from a univariate estimate for the Gauß rule. If I is the identity operator and I_G the operator of polynomial interpolation at the Gauß-Legendre knots σ_G^k, then

$$\sum_{k=1}^{n_G} F(\sigma_G^k)\omega_G^k = \int_0^1 (I_G F).$$

For any bivariate function $(\sigma_1^D,\sigma_2^D)\mapsto \tilde f(\sigma_1^D,\sigma_2^D)$, we conclude

$$\begin{aligned}\left|\int_0^1\int_0^1 \tilde f - \sum_{k_1,k_2=1}^{n_G} \tilde f\left(\sigma_G^{k_1},\sigma_G^{k_2}\right)\omega_G^{k_1}\omega_G^{k_1}\right| &\le \sup_{[0,1]\times[0,1]}\left|\tilde f - [I_G\otimes I_G]\,\tilde f\right| \\ &\le C\left\{\sup_{[0,1]\times[0,1]}\left|[(I-I_G)\otimes I]\,\tilde f\right| + \right. \\ &\qquad\left.\sup_{[0,1]\times[0,1]}\left|[I_G\otimes(I-I_G)]\,\tilde f\right|\right\}.\end{aligned}$$

In view of the well-known fact that the norm of I_G in L^∞ is less than $C\log n_G$ and using the simple estimate

$$\sup_{[0,1]}|(I-I_G)F| \le \frac{C}{n_G!}\sup_{[0,1]}\left|F^{(n_G)}\right|,$$

we continue

$$\left|\int_0^1\int_0^1 \tilde f - \sum_{k_1,k_2=1}^{n_G} \tilde f\left(\sigma_G^{k_1},\sigma_G^{k_2}\right)\omega_G^{k_1}\omega_G^{k_1}\right| \le \frac{C\log n_G}{n_G!}\left\{\sup_{[0,1]\times[0,1]}\left|\partial_{\sigma_1^D}^{n_G}\tilde f\right| + \sup_{[0,1]\times[0,1]}\left|\partial_{\sigma_2^D}^{n_G}\tilde f\right|\right\}. \tag{6.16}$$

In particular, setting $\tilde f(\sigma_1^D,\sigma_2^D) := 2|T_\tau| f(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3))\sigma_1^D$, the rule (4.17) applied to function f is the tensor product Gauß rule applied to $\tilde f$, and we get

$$\begin{aligned}\left|\int_{T_\tau} f - \sum_{k=1}^{n_G^2} f(\sigma_\tau^k)\omega_\tau^k\right| &\le 2\,|T_\tau|\,\frac{C\log n_G}{n_G!}\left\{\sup\left|\partial_{\sigma_1^D}^{n_G}\tilde f\right| + \sup\left|\partial_{\sigma_2^D}^{n_G}\tilde f\right|\right\}, \\ \partial_{\sigma_2^D}^{n_G}\tilde f(\sigma^D) &= 2|T_\tau|\,\partial_{\sigma^\ddagger}^{n_G} f\left(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\right)\sigma_1^D\left|\sigma_1^D(\tau_2-\tau_3)\right|^{n_G}, \\ \partial_{\sigma_1^D}^{n_G}\tilde f(\sigma^D) &= 2|T_\tau|\,\partial_{\sigma^\dagger}^{n_G} f\left(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\right)\sigma_1^D\;\cdot \\ &\qquad \left|(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)\right|^{n_G} + \\ &\quad n_G\cdot 2|T_\tau|\,\partial_{\sigma^\dagger}^{n_G-1} f\left(\tau_3+\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\right)\;\cdot \\ &\qquad \left|(\tau_1-\tau_3)+\sigma_2^D(\tau_2-\tau_3)\right|^{n_G-1},\end{aligned}$$

where ∂_{σ^+} and $\partial_{\sigma^\dagger}$ stand for the derivatives in the directions of $(\tau_2 - \tau_3)/|\tau_2 - \tau_3|$ and

$$\frac{(\tau_1 - \tau_3) + \sigma_2^D(\tau_2 - \tau_3)}{|(\tau_1 - \tau_3) + \sigma_2^D(\tau_2 - \tau_3)|},$$

respectively. Hence, using the relations $|\tau_2 - \tau_3| \sim 2^{-l(Q)}$ and $|(\tau_1 - \tau_3) + \sigma_2^D(\tau_2 - \tau_3)| \sim 2^{-l(Q)}$, we conclude

$$\left| \int_{T_\tau} f - \sum_{k=1}^{n_G^2} f(\sigma_\tau^k)\omega_\tau^k \right| \;\leq\; 2\,|T_\tau|\,\frac{C\log n_G}{n_G!} \sup_{\substack{n=n_G-1,n_G \\ \tilde\sigma=\sigma^+,\sigma^\dagger}} n_G\, 2^{-nl(Q)} \sup_{T_\tau} |\partial_{\tilde\sigma}^n f|\,. \quad (6.17)$$

Now consider the weight function to which we apply the tensor product Gauß rule, i.e., we consider

$$f(\sigma) \;:=\; \tilde\phi_{Q,\upsilon}(\sigma)\,\vartheta_{P',2,i}\left(\frac{p\Big(\,\cdot - \kappa_m'(\sigma)\Big)}{|\,\cdot - \kappa_m'(\sigma)|^\alpha}\right)\tilde\phi_{Q,\iota}(\sigma). \quad (6.18)$$

We shall show next that the directional derivative of order n to f is less than the expression $C2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\mathrm{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-n}$ including a small fixed constant $\varepsilon > 0$. Using $2^{-l(Q)} \leq C\,\mathrm{dist}$ (cf. (4.1) and (4.2)), we arrive at a quadrature error of at most

$$C\;\; 2^{-2l(Q)}\;\frac{\log n_G\, 2^{-(n_G-1)l(Q)}}{(n_G-1)!}\;\; 2^{-\mathbf{m}l(P')}2^{2l(Q)}[\varepsilon\mathrm{dist}]^{-\mathbf{r}-\mathbf{m}-(n_G-1)}.$$

The last expression is less than the right-hand side of (6.13) if

$$(n_G-1)!\,\frac{1}{\log n_G}\,\left[\frac{\varepsilon\mathrm{dist}}{2^{-l(Q)}}\right]^{n_G-3} \;\geq\; C2^{(2-\mathbf{r})l(Q)}.$$

Passing to the logarithms and using Stirling's formula for the logarithm of $(n_G-1)!$, we get the sufficient condition

$$\begin{aligned}\left(n_G - \frac{1}{2}\right)\log(n_G-1) - (n_G-1) - \log\log n_G + (n_G-3)\log\varepsilon + (n_G-3)\log\left[\frac{\mathrm{dist}}{2^{-l(Q)}}\right] \\ \geq \log 2\,\Big\{C + (2-\mathbf{r})l(Q)\Big\} \qquad (6.19)\end{aligned}$$

Choosing n_A sufficiently large in (4.19), the Gauß order n_G is large and we can replace the first part

$$\left(n_G - \frac{1}{2}\right)\log(n_G-1) - (n_G-1) - \log\log n_G + (n_G-3)\log\varepsilon$$

on the left-hand side of (6.19) by the smaller term $(n_G-3)\log 2$. This leads to the sufficient condition

$$(n_G-3)\left\{1 + {}^2\!\log\left[\frac{\mathrm{dist}}{2^{-l(Q)}}\right]\right\} \;\geq\; C + (2-\mathbf{r})l(Q). \quad (6.20)$$

In other words, choosing n_A sufficiently large and setting $n_B = 2 - \mathbf{r}$ in (4.19), the number n_G fulfills (6.20), and the estimate (6.13) is proved if only the upper estimate for the derivative to the function in (6.18) holds

v) Let us show the estimate $C2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\mathrm{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-n}$ for the n-th order derivative of the function in (6.18). To simplify the notation we prove the estimate for the directional derivatives only for the partial derivative with respect to the coordinate t_1 of $\sigma=(t_1,t_2)\in T_\tau$. Clearly, due to the linearity, the absolute value of a j-th order derivative of $\tilde{\phi}_{Q,\iota}$ with $Q\in Qua_l^\Gamma$ is bounded by $C2^{lj}$ for $j=0,1$, and is zero for $j>1$. To show the uniform boundedness of the derivatives to $\sigma\mapsto\vartheta_{P',2,i}(p(\cdot-\kappa_m'(\sigma))\mid\cdot-\kappa_m'(\sigma)|^{-\alpha})$, we fix a t_2 and consider the function

$$I\ni t_1\mapsto\frac{p\left(P_\lambda-\kappa_m'(t_1,t_2)\right)}{|P_\lambda-\kappa_m'(t_1,t_2)|^\alpha}\quad=:\;\frac{p\big(p_2(t_1)\big)}{|p_2(t_1)|^\alpha},\quad I\;:=\;\{t_1:\,(t_1,t_2)\in T_\tau\}\tag{6.21}$$

and its extension to the complex plane. We fix a point $t_I\in I$. For the polynomial p_2 of degree $\deg(p_2)$ less or equal to the degree $2-\mathbf{r}$ of the interpolation, the standard estimates for interpolation imply

$$\begin{aligned}\left(\frac{\partial}{\partial t_1}\right)^k\big(P_\lambda-\kappa_m'(t_I,t_2)\big)\;&\sim\;\left(\frac{\partial}{\partial t_1}\right)^k\big(P_\lambda-\kappa_m(t_I,t_2)\big),\quad k=0,1,\ldots,\deg(p_2),\\ \left|\left(\frac{\partial}{\partial t_1}\right)^k p_2(t_I)\right|\;&\sim\;\begin{cases}|P_\lambda-\kappa_m(t_I,t_2)| & \text{if } k=0\\ \left|\left(\frac{\partial}{\partial t_1}\right)^k\kappa_m(t_I,t_2)\right| & \text{if } k=1,\ldots,\deg(p_2)\end{cases}\\ &\sim\;\begin{cases}\mathrm{dist} & \text{if } k=0\\ C & \text{if } k=1,\ldots,\deg(p_2).\end{cases}\end{aligned}$$

Consequently, for any complex t_1 with $\mathrm{dist}(t_1,I)\le\varepsilon\mathrm{dist}$ and with a constant $\varepsilon>0$ sufficiently small, we get

$$\begin{aligned}p_2(t_1)\;&=\;\sum_{k=0}^{\deg(p_2)}\frac{\partial_{t_1}^k p_2(t_I)}{k!}(t_1-t_I)^k,\\ |p_2(t_1)|\;&\ge\;|p_2(t_I)|-\sum_{k=1}^{\deg(p_2)}\frac{\left|\partial_{t_1}^k p_2(t_I)\right|}{k!}|t_1-t_I|^k\;\ge\;\frac{1}{C}\mathrm{dist}-O(\varepsilon\mathrm{dist})\;\ge\;\frac{1}{2C}\mathrm{dist},\\ |p_2(t_1)|\;&\le\;C\mathrm{dist}.\end{aligned}$$

In other words, the function $p(p_2(t_1))|p_2(t_1)|^{-\alpha}$ is analytic for t_1 with $\mathrm{dist}(t_1,I)<\varepsilon\mathrm{dist}$, and, using the estimate $p(p_2(t_1))\le\mathrm{dist}^{\deg(p)}$, we conclude

$$\left|\frac{p\big(p_2(t_1)\big)}{|p_2(t_1)|^\alpha}\right|\;\le\;C\,\mathrm{dist}^{-2-r}.\tag{6.22}$$

If we apply the functional $\vartheta_{P',2,i}$ to $p(\cdot-\kappa_m'(\sigma))\mid\cdot-\kappa_m'(\sigma)|^{-\alpha}$, then we apply a difference formula with a scaling factor of order $\sim 2^{-l(P')\mathbf{m}_{P',2,i}}$. Since the difference scheme can be represented as a derivative taken at an intermediate point, we can write the function $\vartheta_{P',2,i}(p(\cdot-\kappa_m'(\sigma))\mid\cdot-\kappa_m'(\sigma)|^{-\alpha})$ as a sum of functions similar to that in (6.21). Analogously to (6.22), we arrive at the estimate

$$\left|\vartheta_{P',2,i}\left(\frac{p\big(\cdot-\kappa_m'(t_1,t_2)\big)}{|\cdot-\kappa_m'(t_1,t_2)|^\alpha}\right)\right|\;\le\;C\,2^{-l(P')\mathbf{m}_{P',2,i}}\,\mathrm{dist}^{-2-r-\mathbf{m}_{P',2,i}}\tag{6.23}$$

valid for the complex extension to all t_1 with $\operatorname{dist}(t_1, I) < \varepsilon\operatorname{dist}$. Now we represent the analytic function by Cauchy's integral over a closed countour C around I with distance $\varepsilon\operatorname{dist}$ to I, i.e., by

$$\vartheta_{P',2,i}\left(\frac{p\left(\cdot-\kappa_m'(t_1,t_2)\right)}{|\cdot-\kappa_m'(t_1,t_2)|^{\alpha}}\right) = \frac{1}{2\pi\mathbf{i}}\int_C \left\{\vartheta_{P',2,i}\left(\frac{p\left(\cdot-\kappa_m'(t,t_2)\right)}{|\cdot-\kappa_m'(t,t_2)|}\right)\right\}\frac{1}{t-t_1}\,\mathrm{d}t\,.$$

Differentiating this equation with respect to t_1, restricting t_1 to I, and using (6.23), we get

$$\left|\frac{\partial^k}{\partial t_1^k}\vartheta_{P',2,i}\left(\frac{p\left(\cdot-\kappa_m'(t_1,t_2)\right)}{|\cdot-\kappa_m'(t_1,t_2)|^{\alpha}}\right)\right| \leq C\,2^{-l(P')\mathbf{m}_{P',2,i}}\,[\varepsilon\operatorname{dist}]^{-2-r-\mathbf{m}_{P',2,i}-k},\quad (t_1,t_2)\in T_\tau.$$

This together with the estimate $C\,2^{l(Q)j}$ for the j-th derivatives of the functions $\phi_{Q,\iota}$ and $\phi_{Q,v}$, and with $\operatorname{dist}^{-1} \leq 2^{l(Q)}$ (cf. (4.7) and (4.8)) proves that the n-th order derivatives of the function f in (6.18) are indeed less than $C 2^{-\mathbf{m}_{P',2,i}l(P')}2^{2l(Q)}[\varepsilon\operatorname{dist}]^{-\mathbf{r}-\mathbf{m}_{P',2,i}-n}$. ∎

Lemma 6.2 *The number of necessary arithmetic operations for setting up the far field part of the stiffness matrix $A_L^{w,c,q}$, including the sparsity pattern $\mathcal{P} = \mathcal{P}(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ with $a = b = c = \tilde{b} = 1$ and $1.5 < \tilde{a} = \tilde{c} < 2$ is less than $C\{d^2L^4 + \tilde{d}L^3\}2^{2L}$.*

Proof. Clearly, if the test functional $\vartheta_{P'}$ and the domain of integration Γ_Q is fixed, then the number of operations is less than a constant multiple of the number of quadrature knots plus the number of trial functions ψ_P with $\Gamma_Q \subseteq \Psi_P$. Thus, for fixed $\vartheta_{P'}$ and Γ_Q, no more than $C\,L^2$ operations are needed. The number of all arithmetic operations is less than $C\,L^2$ times $\sum_{P'}\sum_l \#Qua_l^\Gamma$ where $\#Qua_l^\Gamma$ is the number of domains Γ_Q in Qua_l^Γ. We only have to count the number of domains Γ_Q in Qua_l^Γ. The estimates (4.3) (compare (3.18)) and (4.4) (compare (3.19)) together with the proof to Lemma 5.6 imply our assertion. ∎

6.2 The Near Field Estimate

In this subsection we suppose that the far field integration and the integration of the singular integrals are performed exactly and derive the convergence estimates for the non-singular near field case. The non-singular near field, however, can be treated by the same method as the far field. In view of Sect. 5.4 and Lemma 5.8, it remains to prove

Lemma 6.3 *Suppose $A_L^{\mathrm{c}} \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P} = \mathcal{P}(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ (cf. Sect. 3.5) with $a = b = c = \tilde{b} = 1$ and $\tilde{a} = \tilde{c} > 1.5$. If $A_L^{\mathrm{c},q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of Sect. 4.2, then we get the estimates*

$$\|A_L^{\mathrm{c}} - A_L^{\mathrm{c},w}\|_{H^0(\Gamma)\leftarrow H^2(\Gamma)} \leq C\,2^{-(2-\mathbf{r})L}\,\tilde{L}, \tag{6.24}$$

$$\|A_L^{\mathrm{c}} - A_L^{\mathrm{c},w}\|_{H^0(\Gamma)\leftarrow H^{1.1}(\Gamma)} \leq C\,2^{-(2-\mathbf{r})L}\,\tilde{L}, \tag{6.25}$$

$$\|A_L^{\mathrm{c}} - A_L^{\mathrm{c},w}\|_{H^{1.1}(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} \leq C\,2^{-(0.9-\mathbf{r})L}\,\tilde{L}, \tag{6.26}$$

$$\tilde{L} := \begin{cases} L^2 & \text{if } \mathbf{r} = 0 \\ L^{3/2} & \text{if } \mathbf{r} = -1. \end{cases}$$

Proof. We proceed analogously to Lemma 6.1. Clearly, it is sufficient to show the analogue of (6.4) which takes the form

$$\|A_L^{c} - A_L^{c,w}\|_{H^0(\Gamma)\leftarrow H^{1.1+\mathbf{r}}(\Gamma)} \;\leq\; C\,\tilde{L}\,2^{-(2-\mathbf{r})L}\,. \tag{6.27}$$

For the near field estimate, however, we change the definition (6.6) to

$$\eta_{Q,\iota} \;:=\; \begin{cases} \sum_{P\in\triangle_L^\Gamma:\,Q\in\Psi_P} |\xi_P|\;|\psi_P(Q_\iota)| & \text{if } Q\in\square_L^\Gamma \\ 0 & \text{if } Q\in\square_l^\Gamma,\ l<L, \end{cases} \tag{6.28}$$

and we define the entries $\tilde{a}_{P',(Q,\iota)}$ of the matrix $\tilde{A}_L$ to be zero if $Q\in Qua_l^\Gamma$ for some $l<L$ and to be the absolute value of the non-singular near field quadrature error if $Q\in Qua_L^\Gamma$. If we take into account the decomposition (4.22) and if we repeat the arguments leading to (6.13), then we obtain

$$\tilde{a}_{P',(Q,\iota)} \;\leq\; C 2^{-(4-\mathbf{r})L} \sum_{\lambda=1,\dots,\lambda_{P'}:\,P_\lambda\notin\Gamma_Q} \operatorname{dist}\left(P_\lambda,\Gamma_Q\right)^{-\mathbf{r}-2} \tag{6.29}$$

for the errors of the non-singular quadrature including approximate parametrizations, product rule, and tensor product Gauß rule of order n_G defined with sufficiently large n_C and n_D. The estimate (6.8) can be replaced by one of the following estimates.

$$\left\|\left(\eta_{Q,\iota}\right)_{Q,\iota}\right\|_{H^0} \;\leq\; C\,\left\|(\xi_P)_{P\in\triangle_L^\Gamma}\right\|_{H^s},\quad 0<s<1.5\,, \tag{6.30}$$

$$\sup_{Q,\iota}|\eta_{Q,\iota}| \;\leq\; C\,\left\|(\xi_P)_{P\in\triangle_L^\Gamma}\right\|_{H^s},\quad 1<s<1.5\,. \tag{6.31}$$

Note that (6.30) follows analogously to (6.8), and (6.31) is easy to prove. Moreover, we get the inequality

$$\begin{aligned} \left\|\sum_{P'\in\triangle_L^\Gamma}\zeta_{P'}\chi_{P'}\right\|_{L^2} \;&\leq\; C\left\|\sum_{P'\in\triangle_L^\Gamma}\zeta_{P'}\chi_{P'}\right\|_{L^\infty} \;\leq\; C\sum_{l=-1}^{L-1}\left\|\sum_{P'\in\nabla_l^\Gamma}\zeta_{P'}\chi_{P'}\right\|_{L^\infty} \\ &\leq\; C\sum_{l=-1}^{L-1}\sup_{P'\in\nabla_l^\Gamma}|\zeta_{P'}| \;\leq\; C\,L\sup_{P'\in\triangle_L^\Gamma}|\zeta_{P'}|\,. \end{aligned} \tag{6.32}$$

to estimate the L^2 norm of a function $\sum_{P'}\zeta_{P'}\chi_{P'}$.

Now suppose $\mathbf{r}=0$. Instead of (6.10), we derive from (6.32) that

$$\begin{aligned} \left\|\left[A_L^{w,c}-A_L^{w,c,q}\right](\xi_P)_{P\in\triangle_L^\Gamma}\right\|_{\tilde{H}^0} \;&\leq\; C\,L\sup_{P'}\left|\left[\tilde{A}_L\left(\eta_{Q,\iota}\right)_{Q,\iota}\right]_{P'}\right| \;\leq\; C\,L\,\left\|\tilde{A}_L\right\|_{l^\infty\leftarrow l^\infty}\sup_{Q,\iota}|\eta_{Q,\iota}| \\ &\leq\; C\,L\left\|\tilde{A}_L\right\|_{l^\infty\leftarrow l^\infty}\left\|(\xi_P)_{P\in\triangle_L^\Gamma}\right\|_{H^{1.1}}, \end{aligned} \tag{6.33}$$

and the inequality (6.27) for $\mathbf{r}=0$ follows if we prove that the right-hand side of (6.27) is an upper bound for $L\|\tilde{A}_L\|$. Analogously to (5.44), we set

$$\Sigma_1^0 \;:=\; L\,\left\|\tilde{A}_L\right\|_{l^\infty\leftarrow l^\infty} \;=\; \max_{P'\in\triangle_L^\Gamma}\sum_{Q\in\square_L^\Gamma}\sum_{\iota=1}^{3} L\,\tilde{a}_{P',(Q,\iota)}. \tag{6.34}$$

Similarly to the estimation of Σ_1 in the proof to Lemma 5.8, we conclude that

$$\begin{aligned}
\Sigma_1^0 &\leq C\,L2^{-2L} \sup_{P'} \sum_{\lambda=1}^{\lambda_{P'}} \left\{ 2^{-2L} \sum_{Q\in\square_L^\Gamma:\, P_\lambda \notin \Gamma_Q} \operatorname{dist}(P_\lambda, \Gamma_Q)^{-2} \right\} \\
&\leq C\,L2^{-2L} \int_{\{R\in\Gamma:\, 2^{-L}\leq|R-P_\lambda|\leq C\}} |R-P_\lambda|^{-2}\,\mathrm{d}_R\Gamma \;\leq\; C\,L2^{-2L}L. \qquad (6.35)
\end{aligned}$$

In view of (6.33) - (6.35), the estimate (6.27) for $\mathbf{r}=0$ follows.

For $\mathbf{r}=-1$, we conclude from (6.30) that

$$\begin{aligned}
\left\| [A_L^{w,c} - A_L^{w,c,q}]\,(\xi_P)_{P\in\triangle_L^\Gamma} \right\|_{\tilde{H}^0} &\leq C\,L \sup_{P'} \left| \left[\tilde{A}_L\,(\eta_{Q,\iota})_{Q,\iota} \right]_{P'} \right| \;\leq\; \Sigma_1^{-1} \left\| (\eta_{Q,\iota})_{Q,\iota} \right\|_{H^0} \\
&\leq C\Sigma_1^{-1} \left\| (\xi_P)_{P\in\triangle_L^\Gamma} \right\|_{H^{0.1}}, \qquad (6.36) \\
\Sigma_1^{-1} &:= C\,L \max_{P'\in\triangle_L^\Gamma} \sqrt{ \sum_{Q\in\square_L^\Gamma} \sum_{\iota=1}^{3} \left| \tilde{a}_{P',(Q,\iota)} \right|^2 2^{2l(Q)} }. \qquad (6.37)
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\Sigma_1^{-1} &\leq C\,L2^{-3L} \sup_{P'} \sqrt{ \sum_{\lambda=1}^{\lambda_{P'}} \left\{ 2^{-2L} \sum_{Q\in\square_L^\Gamma:\, P_\lambda\notin\Gamma_Q} \operatorname{dist}(P_\lambda, \Gamma_Q)^{-2} \right\} } \\
&\leq C\,L2^{-3L} \sqrt{ \int_{\{R\in\Gamma:\, 2^{-L}\leq|R-P_\lambda|\leq C\}} |R-P_\lambda|^{-2}\,\mathrm{d}_R\Gamma } \;\leq\; C\,L2^{-3L}\sqrt{L}. \qquad (6.38)
\end{aligned}$$

The estimate (6.27) for $\mathbf{r}=-1$ follows from (6.36) - (6.38) and the proof is completed. ■

Lemma 6.4 *The number of necessary arithmetic operations for setting up the non-singular near field part of the stiffness matrix $A_L^{w,c,q}$ including $\mathcal{P}=\mathcal{P}(a,b,c,d,\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ with $a=b=c=\tilde{b}=1$ and $1.5<\tilde{a}=\tilde{c}<2$ is less than $C\{d^2L^4+\tilde{d}L^3\}2^{2L}$.*

Proof. Similarly to Sect. 6.1, the number of operations is less than $C\,L^2$ times the number of domains Γ_Q in Qua_L^Γ. Thus we only have to count the number of domains Γ_Q in Qua_L^Γ. In view of (4.3) and (4.4), the proof of Lemma 5.6 implies our assertion. ■

6.3 The Singular Case

In this subsection we suppose that the far field integration and the integration of the non-singular integrals are performed exactly and derive the convergence estimates for the singular near field case. The singular near field, however, can be treated by the same method as the far field. In view of Sect. 5.4 and Lemma 5.8, it remains to prove

Lemma 6.5 *Suppose $A_L^c \in \mathcal{L}(Lin_L^\Gamma)$ is the approximate operator of the compressed collocation method including the sparsity pattern $\mathcal{P}=\mathcal{P}(a,b,c,d,\tilde{a},\tilde{b},\tilde{c},\tilde{d})$ (cf. Sect. 3.5) with $a=b=c=\tilde{b}=1$ and $\tilde{a}=\tilde{c}>1.5$. If $A_L^{c,q}$ is the operator of the compressed collocation method including the approximation of the parameter mappings and the quadrature of Sect. 4.3, then, for $\mathbf{r}=-1$ and for the case of $\mathbf{r}=0$ with weakly singular kernels of* *the form (4.25), the estimates (6.24)-(6.26) remain valid. For the strongly singular case, (6.24)-(6.26) hold with $\tilde{L}$ replaced by L^3. Now let us turn to the operator $A_L^{c,q}$ of the modified second algorithm (3.17), i.e., $A_L^{c,q}$ is the operator whose matrix with respect to the basis $\{\varphi_P^L\}$ is $[A^{sn}]_L^q + \mathcal{T}_T[A^{ns,f}]_L^{w,c,q}\mathcal{T}_A$. If $\mathbf{r}=0$ and the kernel is of the form (4.25), then (6.24)-(6.26) hold even with $\tilde{L}$ replaced by one. If $\mathbf{r}=0$ and the kernel is strongly singular, then (6.24)-(6.26) hold with $\tilde{L}$ replaced by L.*

Proof. i) Without loss of generality we suppose $\tau_3 = 0$ and $P_\lambda = \kappa_m(0)$ in the formulae of Sect. 4.3. First we consider the case of weakly singular integrals and consider the error for fixed $\vartheta_{P'}$, fixed $P_\lambda \in \operatorname{supp}\vartheta_{P'}$, and fixed (Q,ι) with $P_\lambda \in \Gamma_Q$ and $Q \in \Box_L^\Gamma$, i.e., we consider the error for the integral in (4.27) with $\tilde{\psi}_P^D$ replaced by $\tilde{\Phi}_{Q,\iota} := \phi_{Q,\iota}\circ\tilde{\kappa}_m$ (cf. Remark 4.1). We shall show that the error of approximation is less than $O(2^{-\mathbf{m}L})$. To this end we consider the errors due to the approximation of κ_m, due to the product integration, and due to the approximation of the quadrature weights separately.

ii) To estimate the error due to the replacement of κ_m by κ_m' in this integral, we need a few technical inequalities (cf. the subsequent formulae (6.39)-(6.54)). We observe

$$\begin{aligned}\tilde{\kappa}_m(\sigma^D)-\tilde{\kappa}_m(0) &= \int_0^1 \nabla\tilde{\kappa}_m(\lambda\sigma^D)\,\mathrm{d}\lambda\cdot\sigma^D \qquad (6.39)\\ &= \int_0^1 \Big\{\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\cdot\\ &\qquad \Big(\,(\tau_1-\tau_3)+\lambda\sigma_2^D(\tau_2-\tau_3)\,,\ \lambda\sigma_1^D(\tau_2-\tau_3)\,\Big)\Big\}\,\mathrm{d}\lambda\cdot\begin{pmatrix}\sigma_1^D\\ \sigma_2^D\end{pmatrix}\\ &= \int_0^1 \Big\{\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\cdot\\ &\qquad \Big(\,(\tau_1-\tau_3)+2\lambda\sigma_2^D(\tau_2-\tau_3)\,\Big)\Big\}\,\mathrm{d}\lambda\,\sigma_1^D.\end{aligned}$$

This and the corresponding relation for $\tilde{\kappa}_m$ replaced by $\tilde{\kappa}_m'$ imply

$$\Big|\tilde{\kappa}_m(\sigma^D)-\tilde{\kappa}_m(0)\Big| \sim 2^{-L}\sigma_1^D, \qquad (6.40)$$

$$\Big|\tilde{\kappa}_m'(\sigma^D)-\tilde{\kappa}_m'(0)\Big| \sim 2^{-L}\sigma_1^D, \qquad (6.41)$$

$$\Big|\tilde{p}\left(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)\right)\Big| \sim \left[2^{-L}\sigma_1^D\right]^{\deg(\tilde{p})}, \qquad (6.42)$$

$$\Big|\tilde{p}\left(\tilde{\kappa}_m'(0)-\tilde{\kappa}_m'(\sigma^D)\right)\Big| \sim \left[2^{-L}\sigma_1^D\right]^{\deg(\tilde{p})}. \qquad (6.43)$$

By assumption, we get that $\mathcal{J}_m\circ\delta$ and k are bounded. Since κ_m' approximates κ_m over T_τ with order $\mathbf{m}+1$ and since the gradient $\nabla\kappa_m'$ approximates $\nabla\kappa_m$ over T_τ with order $\mathbf{m} = 2-\mathbf{r}$, formula (6.39) leads us to

$$\Big|\tilde{\kappa}_m(\sigma^D)-\tilde{\kappa}_m'(\sigma^D)\Big| \le C\,2^{-(3-\mathbf{r})L}\sigma_1^D, \qquad (6.44)$$

$$\Big|\mathcal{J}_m\Big(\delta(\sigma^D)\Big)-\mathcal{J}_m'\Big(\delta(\sigma^D)\Big)\Big| \le C\,2^{-(2-\mathbf{r})L}, \qquad (6.45)$$

$$\Big|k(P_\lambda,\tilde{\kappa}_m(\sigma^D),n_{\tilde{\kappa}_m(\sigma^D)})-k(P_\lambda,\tilde{\kappa}_m(\sigma^D),n'_{\tilde{\kappa}_m'(\sigma^D)})\Big| \le C\,2^{-(2-\mathbf{r})L}. \qquad (6.46)$$

Moreover, from (6.40), (6.41), and (6.44) it is not hard to conclude

$$\Big|\tilde{p}\left(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)\right)-\tilde{p}\left(\tilde{\kappa}_m'(0)-\tilde{\kappa}_m'(\sigma^D)\right)\Big| \le C\,2^{-(3-\mathbf{r})L}\sigma_1^D\left[2^{-L}\sigma_1^D\right]^{\deg(\tilde{p})-1} \qquad (6.47)$$

$$\leq \; C\,2^{-(2-\mathbf{r})L}\left[2^{-L}\sigma_1^D\right]^{\deg(\tilde{p})}, \qquad (6.48)$$

$$\left|\left|\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)\right|^{-\alpha}-\left|\tilde{\kappa}_m'(0)-\tilde{\kappa}_m'(\sigma^D)\right|^{-\alpha}\right| \;\leq\; C\,2^{-(3-\mathbf{r})L}\sigma_1^D\left[2^{-L}\sigma_1^D\right]^{-\alpha-1} \qquad (6.49)$$

$$\leq \; C\,2^{-(2-\mathbf{r})L}\left[2^{-L}\sigma_1^D\right]^{-\alpha}. \qquad (6.50)$$

To estimate $n_{\tilde{\kappa}_m(\sigma^D)}\cdot(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D))$, we observe $n_{\tilde{\kappa}_m(\sigma^D)}\cdot\nabla\kappa_m(\delta(\sigma^D))=0$, and the equation (6.39) leads us to

$$\begin{aligned} n_{\tilde{\kappa}_m(\sigma^D)}\cdot\Big(\tilde{\kappa}_m(\sigma^D)-\tilde{\kappa}_m(0)\Big) = n_{\tilde{\kappa}_m(\sigma^D)}\cdot\int_0^1\Bigg\{\Big[&\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\\ &-\nabla\kappa_m\Big(\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\Big]\cdot\\ &\Big((\tau_1-\tau_3)+2\lambda\sigma_2^D(\tau_2-\tau_3)\Big)\Bigg\}\,\mathrm{d}\lambda\,\sigma_1^D. \end{aligned} \qquad (6.51)$$

Analogously to Equation (6.39), we write

$$\begin{aligned} &\nabla\kappa_m\Big(\lambda\sigma_1^D(\tau_1-\tau_3)+\lambda^2\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)-\nabla\kappa_m\Big(\sigma_1^D(\tau_1-\tau_3)+\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\\ &=\int_0^1\nabla^2\kappa_m\Big([1+\mu(\lambda-1)]\sigma_1^D(\tau_1-\tau_3)+[1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D(\tau_2-\tau_3)\Big)\,\mathrm{d}\mu\;\cdot\\ &\quad\Big[(\lambda-1)(\tau_1-\tau_3)+(\lambda^2-1)\sigma_2^D(\tau_2-\tau_3)\Big]\cdot\sigma_1^D, \end{aligned} \qquad (6.52)$$

and, from inserting this into the representation of $n_{\tilde{\kappa}_m(\sigma^D)}\cdot(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D))$ as well as from the analogous formula for the expression $n'_{\tilde{\kappa}_m'(\sigma^D)}\cdot(\tilde{\kappa}_m'(0)-\tilde{\kappa}_m'(\sigma^D))$, we obtain

$$\left|n_{\tilde{\kappa}_m(\sigma^D)}\cdot\Big(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)\Big)\right| \;\leq\; C\,\left[2^{-L}\sigma_1^D\right]^2, \qquad (6.53)$$

$$\left|n_{\tilde{\kappa}_m(\sigma^D)}\cdot\Big(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)\Big)-n'_{\tilde{\kappa}_m'(\sigma^D)}\cdot\Big(\tilde{\kappa}_m'(0)-\tilde{\kappa}_m'(\sigma^D)\Big)\right| \;\leq\; C\,2^{1-\mathbf{r}}\left[2^{-L}\sigma_1^D\right]^2. \qquad (6.54)$$

Now, using (6.39)-(6.54), the error due to the replacement of κ_m by κ_m' can be represented as the sum of the errors corresponding to the replacements in the several factors of the integrand in (4.27). These factors are $\tilde{k}(P_\lambda,\tilde{\kappa}_m(\sigma^D),n_{\tilde{\kappa}_m(\sigma^D)})$, $\tilde{p}(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)$, $|\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D)|^{-\alpha}$, $[n_{\tilde{\kappa}_m(\sigma^D)}\cdot(\tilde{\kappa}_m(0)-\tilde{\kappa}_m(\sigma^D))]^{1+\mathbf{r}}$, and $\mathcal{J}_m(\delta(\sigma^D))$, respectively. The last factor $\mathcal{J}_\delta(\sigma^D)\tilde{\Phi}_{Q,\iota}(\sigma^D)$ needs no replacement of κ_m. We arrive at the estimate

$$\begin{aligned} C\int_0^1\int_0^1\Bigg\{&\Big[2^{-(2-\mathbf{r})L}\;[2^{-L}\sigma_1^D]^{\deg(\tilde{p})}\;[2^{-L}\sigma_1^D]^{-\alpha}\;[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\;C\Big]\\ &+\Big[C\;2^{-(2-\mathbf{r})L}[2^{-L}\sigma_1^D]^{\deg(\tilde{p})}\;[2^{-L}\sigma_1^D]^{-\alpha}\;[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\;C\Big]\\ &+\Big[C\;[2^{-L}\sigma_1^D]^{\deg(\tilde{p})}\;2^{-(2-\mathbf{r})L}[2^{-L}\sigma_1^D]^{-\alpha}\;[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\;C\Big]\\ &+\Big[C\;[2^{-L}\sigma_1^D]^{\deg(\tilde{p})}\;[2^{-L}\sigma_1^D]^{-\alpha}\;[2^{-(1-\mathbf{r})L}]^{1+\mathbf{r}}[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\;C\Big]\cdot\delta_{\mathbf{r},0}\\ &+\Big[C\;[2^{-L}\sigma_1^D]^{\deg(\tilde{p})}\;[2^{-L}\sigma_1^D]^{-\alpha}\;[2^{-L}\sigma_1^D]^{2(1+\mathbf{r})}\;2^{-(2-\mathbf{r})L}\Big]\Bigg\}2^{-2L}\;\sigma_1^D\,\mathrm{d}\sigma_2^D\,\mathrm{d}\sigma_1^D\\ \leq\;C&\begin{cases}2^{-4L} & \text{if } \mathbf{r}=-1\\ 2^{-2L} & \text{if } \mathbf{r}=0.\end{cases} \end{aligned} \qquad (6.55)$$

This completes the estimate for the first step in approximating the integral.

iii) The second step is the product integration of order $\mathbf{m} = 2 - \mathbf{r}$. Analogously to the derivation of (6.55) from (6.39)-(6.54), we conclude that the integral over the weight function $\tilde{\phi}_\tau^D \tilde{p} \,|\ldots|^{-\alpha}[\ldots]^{1+\mathbf{r}} J_\delta \tilde{\phi}_{Q,\iota}$ is less than 2^{-L}. Hence, it remains to estimate the interpolation error for the $\mathbf{m}$-th order interpolation which defines the product rule. Clearly, the interpolation error is less than a constant times the supremum of the derivatives to the integrand function $\tilde{k}(P_\lambda, \tilde{\kappa}_m(\sigma^D), n'_{\tilde{\kappa}'_m(\sigma^D)})\ \mathcal{J}'_m(\sigma^D)$ if the derivatives are taken with respect to σ_1^D or σ_2^D up to the $\mathbf{m}$-th order. Since our product rule relies up on tensor product interpolation, mixed derivatives need not to be considered. The integrand is a composite function of the outer functions $\tilde{k}$, κ'_m, and $\mathcal{J}'_m$ and of the inner function δ. By assumption (cf. Sects. 2.1 and 2.2) the corresponding derivatives of κ'_m, $\mathcal{J}'_m$, and $\tilde{k}$ do exist and they are uniformly bounded. For the inner function δ, each order of derivative with respect to σ_1^D and σ_2^D brings a factor $(\tau_1 - \tau_3) + \sigma_2^D(\tau_2 - \tau_3)$ and $\sigma_1^D(\tau_2 - \tau_3)$, respectively. Thus the derivatives of order $\mathbf{m}$ are less than $2^{-\mathbf{m}L}$, and the estimate on the right-hand side of (6.55) is an upper bound also for the error of product integration in the second step of approximation. We even get the better bound 2^{-3L} for $\mathbf{r} = 0$.

iv) To analyze the third step, we introduce the notation

$$\begin{aligned} H(\lambda,\mu) &:= \lambda(\tau_1-\tau_3)+\mu(\tau_2-\tau_3),\\ \tilde{H}(\lambda,\mu) &:= \lambda\frac{\tau_1-\tau_3}{|\tau_1-\tau_3|}+\mu\frac{\tau_2-\tau_3}{|\tau_1-\tau_3|}. \end{aligned}$$

In this last step an n_G-th order rule is applied to the integral of the weight function from the previous step, i.e., to

$$\begin{aligned}
&\int_0^1\int_0^1 \Bigg\{ \tilde{\phi}^D_{Q,\upsilon}(\sigma^D) \frac{\tilde{p}\big(\tilde{\kappa}'_m(0)-\tilde{\kappa}'_m(\sigma^D)\big)}{|\tilde{\kappa}'_m(0)-\tilde{\kappa}'_m(\sigma^D)|^\alpha} \Big[n_{\tilde{\kappa}'_m(\sigma^D)} \cdot \big(\tilde{\kappa}'_m(0)-\tilde{\kappa}'_m(\sigma^D)\big)\Big]^{1+\mathbf{r}} \cdot \\
&\qquad\qquad \mathcal{J}_\delta(\sigma^D)\tilde{\Phi}_{Q,\iota}(\sigma^D) \Bigg\} \,\mathrm{d}\sigma_1^D\,\mathrm{d}\sigma_2^D \\
&= \int_0^1\int_0^1 \Bigg\{ \tilde{\phi}^D_{Q,\upsilon}(\sigma^D) \frac{\tilde{p}\Big(\int_0^1 \nabla\kappa'_m\big(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\big)\cdot H(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big)}{\Big|\int_0^1 \nabla\kappa'_m\big(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\big)\cdot H(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big|^\alpha} \cdot \\
&\qquad \Big[n_{\tilde{\kappa}'_m(\sigma^D)} \cdot \int_0^1\Big\{ \int_0^1 \nabla^2\kappa'_m\Big(H\big([1+\mu(\lambda-1)]\sigma_1^D, [1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D\big)\Big)\,\mathrm{d}\mu \\
&\qquad H\big(\lambda-1,(\lambda^2-1)\sigma_2^D\big)\, H\big(1,2\lambda\sigma_2^D\big)\Big\}\,\mathrm{d}\lambda\Big]^{1+\mathbf{r}} 2|T_\tau|\tilde{\Phi}_{Q,\iota}(\sigma^D)\Bigg\}\,\mathrm{d}\sigma_1^D\,\mathrm{d}\sigma_2^D \\
&= \frac{2|T_\tau|}{|\tau_1-\tau_3|}\int_0^1\int_0^1 \Bigg\{ \tilde{\phi}^D_{Q,\upsilon}(\sigma^D) \frac{\tilde{p}\Big(\int_0^1 \nabla\kappa'_m\big(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\big)\cdot \tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big)}{\Big|\int_0^1 \nabla\kappa'_m\big(H(\lambda\sigma_1^D,\lambda^2\sigma_1^D\sigma_2^D)\big)\cdot \tilde{H}(1,2\lambda\sigma_2^D)\,\mathrm{d}\lambda\Big|^\alpha} \cdot \\
&\qquad \Big[n_{\tilde{\kappa}'_m(\sigma^D)} \cdot \int_0^1\Big\{ \int_0^1 \nabla^2\kappa'_m\Big(H\big([1+\mu(\lambda-1)]\sigma_1^D, [1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D\big)\Big)\,\mathrm{d}\mu \\
&\qquad \tilde{H}\big(\lambda-1,(\lambda^2-1)\sigma_2^D\big)\, \tilde{H}\big(1,2\lambda\sigma_2^D\big)\Big\}\,\mathrm{d}\lambda\Big]^{1+\mathbf{r}} \tilde{\Phi}_{Q,\iota}(\sigma^D)\Bigg\}\,\mathrm{d}\sigma_1^D\,\mathrm{d}\sigma_2^D, \qquad (6.56)
\end{aligned}$$

where the equalities $\mathcal{J}_\delta(\sigma^D) = 2|T_\tau|\sigma_1^D$, (6.39), (6.51), and (6.52) have been substituted into the first integral. The last integrand is a function which can be treated as the integrand in part v) of the proof to Lemma 6.1. Indeed, to apply (6.16), we need an estimate for the derivatives. Without loss of generality we consider the derivative with respect to σ_1^D. For the k-th order derivatives of $\tilde{\phi}_{Q,v}^D$ and $\tilde{\Phi}_{Q,\iota}$, we get the bound $C2^{kL}$ if $k=0,1$ and the bound zero if $k \geq 2$. Similarly to (6.21), we fix σ_2^D and set

$$\begin{aligned}
p_2(\sigma_1^D) &:= \int_0^1 \nabla\kappa_m' \left(H(\lambda\sigma_1^D, \lambda^2\sigma_1^D\sigma_2^D)\right) \cdot \tilde{H}(1, 2\lambda\sigma_2^D)\,\mathrm{d}\lambda, \\
p_3(\sigma_1^D) &:= \Big[n_{\tilde{\kappa}_m'(\sigma^D)} \cdot \textstyle\int_0^1 \Big\{ \int_0^1 \nabla^2\kappa_m' \Big(H\left([1+\mu(\lambda-1)]\sigma_1^D, [1+\mu(\lambda^2-1)]\sigma_1^D\sigma_2^D \right) \Big)\,\mathrm{d}\mu \\
&\qquad \tilde{H}\left(\lambda-1, (\lambda^2-1)\sigma_2^D\right) \tilde{H}\left(1, 2\lambda\sigma_2^D\right) \Big\}\,\mathrm{d}\lambda \Big]^{1+\mathrm{r}}
\end{aligned}$$

and consider

$$[0,1] \ni \sigma_1^D \;\mapsto\; \frac{\tilde{p}(p_2(\sigma_1^D))}{|p_2(\sigma_1^D)|^\alpha} p_3(\sigma_1^D) \tag{6.57}$$

together with its extension to the complex plane. Since the parametrizations κ_m are injective mappings, we get $\|\kappa_m(\sigma)\xi\| \geq \|\xi\|$, $\forall \xi \in \mathbb{R}^2$ and

$$\begin{aligned}
p_2(\tilde{\sigma}_1^D) &\sim \int_0^1 \nabla\kappa_m \left(H(\lambda\tilde{\sigma}_1^D, \lambda^2\tilde{\sigma}_1^D\sigma_2^D)\right) \cdot \tilde{H}(1, 2\lambda\sigma_2^D)\,\mathrm{d}\lambda \\
&\sim \nabla\kappa_m\Big(H(0,0)\Big) \int_0^1 \tilde{H}(1, 2\lambda\sigma_2^D)\,\mathrm{d}\lambda, \\
\left|p_2(\tilde{\sigma}_1^D)\right| &\geq 1/C
\end{aligned}$$

for a $\tilde{\sigma}_1^D$ such that $0 \leq \tilde{\sigma}_1^D \leq 1$. On the other hand, the k-th order derivative of the interpolation κ_m' to κ_m is bounded by $C2^{kL}$ if k is less or equal to the total degree of the polynomial κ_m', and the k-th order derivative of $H(\cdot,\cdot)$ is less than $C2^{-kL}$. Consequently, the k-th order derivative of p_2 at σ_1^D with $k \leq \deg(p_2)$ and $0 \leq \sigma_1^D \leq 1$ is less then a constant. We obtain

$$\begin{aligned}
p_2(\sigma_1^D) &= \sum_{k=0}^{\deg(p_2)} \frac{\partial_{\sigma_1^D}^k p_2(\tilde{\sigma}_1^D)}{k!} (\sigma_1^D - \tilde{\sigma}_1^D)^k, \\
\left|p_2(\sigma_1^D)\right| &\geq \left|p_2(\tilde{\sigma}_1^D)\right| - \sum_{k=1}^{\deg(p_2)} \frac{\left|\partial_{\sigma_1^D}^k p_2(\tilde{\sigma}_1^D)\right|}{k!} \left|\sigma_1^D - \tilde{\sigma}_1^D\right|^k \\
&\geq 1/C - \sum_{k=1}^{\deg(p_2)} C \left|\sigma_1^D - \tilde{\sigma}_1^D\right|^k,
\end{aligned}$$

where $\tilde{\sigma}_1^D$ with $0 \leq \tilde{\sigma}_1^D \leq 1$ can be chosen such that $|\sigma_1^D - \tilde{\sigma}_1^D| \leq \operatorname{dist}(\sigma_1^D, [0,1])$. Hence, we can take a sufficiently small $\varepsilon > 0$ and observe $|p_2(\sigma_1^D)| \geq 1/(2C)$ for any complex σ_1^D with $\operatorname{dist}(\sigma_1^D, [0,1]) \leq \varepsilon$. Similarly, we obtain $|p_2(\sigma_1^D)| \leq C$ and $|p_3(\sigma_1^D)| \leq C$. Analogously to part v) of the proof to Lemma 6.1, we arrive at the estimate $C\varepsilon^{-(k+1)}$ for the k-th order derivative of (6.57) and at the bound $C2^{2L}\varepsilon^{-(n_G-1)}$ for the n_G-th order derivative of the integrand in (6.56). The estimate $C2^{-L}$ for the factor $2|T_\tau|\;|\tau_1 - \tau_3|^{-1}$ and the error estimate (6.16) applied to the quadrature approximation of (6.56) yield the bound

$$C\frac{\log n_G}{n_G!} 2^{-L} 2^{2L} \varepsilon^{-(n_G-1)} \;\leq\; C 2^{L - {}^2\log \varepsilon\,[n_G-1] + {}^2\log e\,[\log\log n_G - (n_G + \frac{1}{2})\log n_G + n_G]}.$$

The last bound is less than $2^{-(3-\mathbf{r})L}$ if we set $n_F := 4 - \mathbf{r}$ and choose n_E sufficiently large in $n_G = n_E + Ln_F$. Hence, we get the estimate on the right-hand side of (6.55) for the quadrature error of the Gauß rules. We even get the better bound 2^{-3L} for $\mathbf{r} = 0$.

v) Now let us estimate the entries in the case of strongly singular integral operators. We assume $\mathbf{r} = 0$ and distinguish the two cases $\phi_{Q,\iota}(P_\lambda) = 0$ and $\phi_{Q,\iota}(P_\lambda) \neq 0$. If $\phi_{Q,\iota}(P_\lambda) = \tilde{\Phi}_{Q,\iota}(0,0) = 0$, then we can repeat the estimate from above. Indeed, the obvious estimate $|\phi_{Q,\iota}(R)| \leq C\,2^L\,|R - P_\lambda|$ provides us with a factor $|R - P_\lambda|$ which cancels one factor $|R - P_\lambda|$ from the denominator $|R - P_\lambda|^\alpha$. Though we have $\mathbf{r} = 0$, there is no factor $n_R \cdot (R - P_\lambda)$ this time. Hence, we get the estimate $C2^{-3L}$ in (6.55) which is to be multiplied by the factor 2^L from the estimate $|\phi_{Q,\iota}(R)| \leq C\,2^L\,|R - P_\lambda|$. In other words the final estimate for the matrix entries is again $C\,2^{-2L}$.

Finally, we turn to the case $\phi_{Q,\iota}(P_\lambda) \neq 0$ and consider the error of the approximation (4.32) and (4.34). The first part of the error is due to restricting the domain of integration from T_τ to $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$. This is less than $C2^{-2L}$ by (4.28). The second part of the error is caused by the replacement of the parametrization in the kernel function. Writing the difference of (4.29) and (4.30) in Duffy's coordinates and using the equations (6.44)-(6.49) with the polynomial $\tilde{p}$ replaced by p, we obtain the bound

$$C\,2^{-2L}\int_{\delta^{-1}[T\setminus T(P_\lambda,m,2^{-2L})]} |\sigma_1^D|^{-1}\,\mathrm{d}\sigma^D \;\leq\; C\,2^{-2L}\int_{2^{-2L}}^{1} |\sigma_1^D|^{-1}\,\mathrm{d}\sigma_1^D \;\leq\; C\,L\,2^{-2L}. \quad (6.58)$$

By simple estimates analogous to those in [29], Chapter XI, Sect.1, the third part of the error due to the change of the parametrization in the integration domain $T_\tau \setminus T'(P_\lambda, m, 2^{-2L})$ is less than $C2^{-2L}$. The error bound (6.58) for the fourth part due to product integration follows as in the case $\mathbf{r} = -1$. Finally, it remains to estimate the error of the tensor product Gauß rule in (4.34). This however can be treated as in the parts iv) and v) of the proof to Lemma 6.1 and as in part iv) of the present proof since the ratio of the diameter of $T_{\tau,\iota}$ to the distance of $T_{\tau,\iota}$ to the singularity point τ_3 is bounded from below and since the variable integration bound $S_a(\sigma_2^D)$ for the inner integration is analytic. Indeed, the function $S_a(\sigma_2^D)$ for $\iota = \iota_0$ depending on the parameter $\varepsilon = 2^{-2L}$ (cf. (4.31)) is of the form $S_a(\sigma_2^D) = 2^{-2L}S(2^{2L}\sigma_2^D)$ with an S such that $\sigma \mapsto \delta(S(\sigma), \sigma)$ describes the boundary curve of an ellipse. The summation over all ι from one to $\iota_0 = O(L)$ leads to an additional factor $C\,L$.

vi) In other words, for the algorithms (3.15) and (3.16) without modification, we have the same estimate like for the almost singular entries in (6.29). Only for the strong singular case we have an additional factor L. Hence, the proof to Lemma 6.3 completes the proof of the corresponding assertions of Lemma 6.5. For $\mathbf{r} = 0$ and the modified second algorithm, we estimate the Euclidean norm of the error matrix $A_L^{\mathrm{c}} - A_L^{\mathrm{c},q}$ with respect to the basis $\{\varphi_P^L\}$. The singular near field part $A_L^{\mathrm{c}} - A_L^{\mathrm{c},q}$, however, is a matrix whose columns and rows contain only a small number of entries depending on the geometry of Γ. Hence, the matrix norm is less than constant times the supremum norm of the entries. Moreover, the entries are just the errors for the computation of the integral in (4.27) with ψ_P replaced by φ_P^L. The parts ii)-v) of the present proof imply that these entries are less than $C2^{-2L}$ if the kernel is of the form (4.25) and less than $CL2^{-2L}$ if the kernel is strongly singular. The corresponding assertions of Lemma 6.5 follow analogously to the derivation of Lemma 6.1. ∎

Lemma 6.6 *If* $\mathbf{r} = -1$ *or if* $\mathbf{r} = 0$ *and the operator has a kernel function of the form (4.25), then the number of necessary arithmetic operations for setting up the singular near field part of the stiffness matrix* $A_L^{w,c,q}$ *including* $\mathcal{P} = \mathcal{P}(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ *with* $a = b = c = \tilde{b} = 1$ *and* $1.5 < \tilde{a} = \tilde{c} < 2$ *is less than* $CL^2 2^{2L}$ *. If* $\mathbf{r} = 0$ *and if the kernel function is strongly singular, then no more than* $CL^3 2^{2L}$ *arithmetic operations are required.*

Proof. First we consider the case that the kernel function is weakly singular and that it is of the form (4.25). Then the number of all P_λ is less than $C\,2^{2L}$, and for each point there is only a bounded number of Q with $P_\lambda \in \Gamma_Q$ and $l(Q) = L$. For each Γ_Q, there are no more than $C\,L^2$ quadrature knots in Γ_Q and no more than $C\,L$ functions ψ_P and φ_P^L such that $\Gamma_Q \subseteq \operatorname{supp} \psi_P$ resp. $\Gamma_Q \subseteq \operatorname{supp} \varphi_P^L$. Thus the number of operations is less than $C\,L^2 2^{2L}$. In case that the operator has a strongly singular kernel, Γ_Q is divided in $\iota_0 \sim L$ subdomains, and the number of quadrature knots is bounded by $C\,L^2$ for each subdomain. Thus the whole number of knots is bounded by $C\,L^3 2^{2L}$. ∎

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