Abstract. This paper studies stochastic particle systems related to the coagulation-fragmentation equation. For a certain class of unbounded coagulation kernels and fragmentation rates, relative compactness of the stochastic systems is established and weak accumulation points are characterized as solutions. These results imply a new existence theorem. Finally a simulation algorithm based on the particle systems is proposed.

Contents

1. Introduction

The continuous coagulation-fragmentation equation

$$\begin{aligned}
\frac{\partial}{\partial t} c(t,x) &= \frac{1}{2}\int_0^x K(x-y,y)\, c(t,x-y)\, c(t,y)\, dy - \int_0^\infty K(x,y)\, c(t,x)\, c(t,y)\, dy \\
&\quad + \int_0^\infty f(x,y)\, c(t,x+y)\, dy - \frac{1}{2}\int_0^x f(x-y,y)\, c(t,x)\, dy\,, \qquad (1.1) \\
c(0,x) &= c_0(x) \geq 0\,,
\end{aligned}$$

describes the time evolution of the average concentration of particles of size $x > 0$. Here $K(x,y)$ denotes the coagulation rate of clusters of size x and y while $f(x,y)$ denotes the

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fragmentation rate of an $(x+y)$-cluster into clusters of size x and y. Both functions are assumed to be non-negative and symmetric. According to (1.1), the concentration $c(x,t)$ can increase

- by coagulation of clusters of size $y < x$ and $x - y$ (first term) or
- by fragmentation of an $(x+y)$-cluster into clusters of size x and y (third term)

and can decrease

- by coagulation of an x-cluster with a cluster of any size y (second term) or
- by fragmentation of an x-cluster into clusters of size $y < x$ and $x - y$ (fourth term).

If the clusters only can take sizes $i = 1, 2, \dots$, then a discrete version of equation (1.1) is obtained, where all integrals are replaced by sums. The discrete coagulation equation was first published by Smoluchowski in [26], and solved for the case of constant coagulation rate. The combined coagulation fragmentation equation appeared in [21]. Both the continuous and the discrete equation have a wide range of applications, e.g., in astrophysics, biology, chemistry and meteorology (see the survey papers by Drake [7] and Aldous [1]).

Stochastic particle systems related to the coagulation equation were introduced by several authors as Marcus [19], Gillespie [12] and Lushnikov [18]. Besides, Filippov [10] used stochastic methods to study the continuous multiple fragmentation equation. Lang and Nguyen [16] gave a rigorous derivation of a spatially inhomogeneous version of Smoluchowski's coagulation equation with constant rate from a system of particles performing Brownian motion.

The stochastic approach to coagulation and fragmentation was reviewed in [1]. Here, in particular, the problem was raised to prove a weak law of large numbers for the relevant stochastic particle systems with general kernels [1, Problem 10(a)]. Recently, several other authors have obtained rigorous results concerning this problem. Guias [14] showed convergence of the particle system to the solution of the discrete coagulation-fragmentation equation for bounded coagulation kernels and bounded total fragmentation rates. Jeon [15] considered the discrete coagulation fragmentation equation and showed (among other results) that weak limit points of the stochastic particle system exist and provide solutions. He assumed that $K(i,j) = o(i)\,o(j)$ and that the total fragmentation rate of an i-cluster is $o(i)$. Norris [22] considered a weak form of the continuous coagulation equation. Among other results, he proved that weak limit points of the corresponding stochastic particle system exist and provide solutions, if K is continuous and satisfies $K(x,y) = o(x)\,o(y)$ $(x, y \to \infty)$.

Beside the derivation of the coagulation-fragmentation equation, stochastic particle systems play a significant role in numerical algorithms for that equation (see [13], [6]). We refer to [23] concerning a survey of Monte Carlo methods and effective stochastic algorithms.

In this paper we consider a weak integral version of (1.1). We use a fragmentation measure instead of the fragmentation kernel $f(x,y)$, since this is convenient for a simultaneous treatment of both the discrete and the continuous equation. Technically, the fragmentation measure has not to be absolutely continuous with respect to Lebesgue measure. We prove tightness of the corresponding stochastic particle systems and characterize the weak limit points as solutions. We require a continuous coagulation kernel satisfying

$$K(x,y) = o(x)\, o(y) \quad \text{for} \quad x,y \to \infty\,,$$

a weakly continuous fragmentation measure for which the total fragmentation rate of a cluster of size x (given as $\frac{1}{2}\int_0^x f(x-y,y)\,dy$ in terms of equation (1.1)) is $o(x)$ as $x \to \infty$, and an initial function with finite zeroth and first moments. In the particular cases of the discrete coagulation-fragmentation equation and of the continuous pure coagulation equation these results basically coincide with the corresponding results in [15, Theorems 1, 2] and [22, Theorem 4.1], respectively. Our approach is related to [27], where stochastic models for the Boltzmann equation were studied.

The above mentioned results on stochastic particle systems imply **existence** of a solution for the coagulation-fragmentation equation. We discuss the relationship with previously known results based on deterministic approaches.

In the **discrete** case there are existence results assuming no restriction to the total fragmentation rates, including the case $K(i,j) = O(i)\,O(j)$ or requiring only finite zeroth moment for the initial data (see [2], [5], [17]). In the **continuous pure coagulation** case [11] gives an existence theorem, where $K(x,y) = o(x)\,o(y)$. In the **continuous pure fragmentation** case [20] contains an existence theorem for the multiple fragmentation equation assuming that the total fragmentation rate is bounded on bounded intervals. Thus, in all these cases we do not get new existence results.

In the **continuous coagulation-fragmentation** case, to our knowledge, the most general existence results are given in [24] and [8]. In [24] the author considers kernels

$$K(x,y) = o(x) + o(y) \quad (x,y \to \infty)\,, \qquad f(x,y) = o(x+y) \quad (x+y \to \infty)$$

and an initial function with finite zeroth and first moments. In [8] the authors consider a coagulation kernel

$$K(x,y) = O(x) + O(y) \quad (x,y \to \infty)$$

and some technical condition which is satisfied if

$$f(x,y) = O((x+y)^\alpha) \quad (x+y \to \infty)\,, \qquad \alpha \geq 0\,,$$

and if the initial function has finite zeroth and r-th moments with $r > 1+\alpha$. Thus, in this case we obtain a new existence result for a certain class of unbounded coagulation kernels and fragmentation rates.

If **uniqueness** of solutions to the weak integral version of the coagulation-fragmentation equation is known, then our results imply **convergence** so that the stochastic particle systems can be used to approximate the solution of the coagulation-fragmentation equation. We do not study the problem of uniqueness in this paper. Uniqueness results can be found for the discrete case in [2], [4], [5] and for the continuous case in [8], [11], [20], [25]. We refer also to the result on convergence to a local solution in [22, Theorem 4.4].

This paper is organized as follows. The main results are formulated in Section 2. In Section 3 we collect some basic properties of the stochastic particle systems. Section 4 contains the proof of the relative compactness. In Section 5 the weak limit points are characterized as solutions. Finally, Section 6 contains some ideas concerning the application of the stochastic system in simulation algorithms.

2. Main results

Let $\mathcal{Z}$ be a closed subset of $[0,\infty)$ such that $x+y\in\mathcal{Z}\,,\ \forall\, x,y\in\mathcal{Z}$ and $x-y\in\mathcal{Z}\,,\ \forall\, y\le x\in\mathcal{Z}$. This assumption allows us to treat both the **continuous** and the **discrete case** simultaneously. Let $\mathcal{B}(\mathcal{Z})$ denote the Borel-σ-algebra, $\mathcal{M}_b(\mathcal{Z})$ the set of all non-negative finite Borel measures on $\mathcal{Z}$ (with weak topology) and $C_b(\mathcal{Z})$ the set of all bounded continuous functions equipped with the supremum norm.

The **coagulation kernel** $K:\mathcal{Z}\times\mathcal{Z}\to\mathbb{R}_+$ is assumed to satisfy the conditions

$$K(x,y)\quad\text{is measurable in}\quad(x,y)\,,\tag{2.1}$$

$$K(x,y)=K(y,x)\,,\qquad x,y\in\mathcal{Z}\,,\tag{2.2}$$

$$K(x,y)\quad\text{is bounded on compact sets}\,.\tag{2.3}$$

The **fragmentation measure** $F:\mathcal{Z}\times\mathcal{B}(\mathcal{Z})\to\mathbb{R}_+$ is assumed to satisfy the conditions

$$F(x,\cdot)\in\mathcal{M}_b(\mathcal{Z})\,,\quad\forall x\in\mathcal{Z}\,,\tag{2.4}$$

$$F(\cdot,B)\quad\text{is measurable for all}\quad B\in\mathcal{B}(\mathcal{Z})\,,\tag{2.5}$$

$$F(x,\cdot)\quad\text{has support on}\quad[0,x]\cap\mathcal{Z}\,,\tag{2.6}$$

$$\int_{\mathcal{Z}}\varphi(x-y)\,F(x,dy)=\int_{\mathcal{Z}}\varphi(y)\,F(x,dy)\,,\qquad\forall x\in\mathcal{Z}\,,\quad\varphi\in C_b(\mathcal{Z})\,,\tag{2.7}$$

$$F(\cdot,\mathcal{Z})\quad\text{is bounded on compact sets}\,.\tag{2.8}$$

We consider the **weak integral version of the coagulation-fragmentation equation**

$$\begin{aligned}\int_{\mathcal{Z}}\varphi(x)\,P(t,dx)=&\int_{\mathcal{Z}}\varphi(x)\,P_0(dx)+\\&\int_0^t\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\frac{1}{2}\,\varphi(x+y)-\varphi(x)\Big]K(x,y)\,P(s,dy)\,P(s,dx)\,ds+\\&\int_0^t\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(y)-\frac{1}{2}\,\varphi(x)\Big]F(x,dy)\,P(s,dx)\,ds\,,\qquad\forall t\ge0\,,\quad\varphi\in C_b(\mathcal{Z})\,.\end{aligned}\tag{2.9}$$

We call $P\in\mathbb{C}([0,\infty),\mathcal{M}_b(\mathcal{Z}))$ a **measure-valued solution** for initial measure $P_0\in\mathcal{M}_b(\mathcal{Z})$ if P satisfies equation (2.9) and

$$\int_0^t\Big[\int_{\mathcal{Z}}\int_{\mathcal{Z}}K(x,y)\,P(s,dx)\,P(s,dy)\;+\;\int_{\mathcal{Z}}F(x,\mathcal{Z})\,P(s,dx)\Big]\,ds<\infty\,,\qquad\forall t\ge0\,.\tag{2.10}$$

To show the **connection with the standard form** (1.1) of the coagulation-fragmentation equation we consider the case $\mathcal{Z}=[0,\infty)$ and assume

$$F(x,dy)=1_{\{y\le x\}}(y)f(x-y,y)\,dy\,,\qquad f(x,y)=f(y,x)\ge0\,,$$

$$P_0(dx) = c_0(x)\,dx \quad \text{and} \quad P(t,dx) = c(t,x)\,dx\,.$$

Using the identity

$$\int_0^\infty \int_0^\infty \psi(x,y)\,dy\,dx = \int_0^\infty \int_0^x \psi(x-y,y)\,dy\,dx\,,$$

equation (2.9) takes the form

$$\begin{aligned}\int_0^\infty \varphi(x)\; c(t,x)\; dx \;&=\; \int_0^\infty \varphi(x)\; c_0(x)\; dx + \int_0^\infty \varphi(x) \int_0^t \\ &\Bigg[\frac{1}{2}\int_0^x K(x-y,y)\,c(s,x-y)\,c(s,y)\,dy\,ds - \int_0^\infty K(x,y)\; c(s,x)\; c(s,y)\; dy \\ &+ \int_0^\infty f(x,y)\; c(s,x+y)\; dy - \frac{1}{2}\int_0^x f(x-y,y)\; c(s,x)\; dy\Bigg]\; ds\; dx\,,\end{aligned}$$

and (1.1) follows under appropriate regularity assumptions.

We want to approximate the measure-valued solution of equation (2.9) by means of a **particle system with variable particle number**. For this reason we define a sequence of jump processes on a suitable space of discrete measures.

For every $N \in \mathbb{N}$ define

$$\mathcal{S}^N = \left\{ p = \frac{1}{N}\sum_{i=1}^n \delta_{x_i}\,,\; x_i \in \mathcal{Z}\,,\; n = 1,2,\ldots \,:\, p(\mathcal{Z}) \le c_N\,,\; \int_{\mathcal{Z}} x\,p(dx) \le M \right\}, \tag{2.11}$$

where $M > 0$ and $c_N \in (0,\infty)$ such that

$$\lim_{N\to\infty} c_N = \infty\,. \tag{2.12}$$

On $C_b(\mathcal{S}^N)$ define the coagulation operator $\mathcal{K}^N$ by

$$\mathcal{K}^N\Phi(p) = \frac{1}{2N}\sum_{1\le i\ne j\le n}\Big[\Phi(J_K(p,i,j)) - \Phi(p)\Big]K(x_i,x_j)\,, \tag{2.13}$$

where

$$J_K(p,i,j) = p + \frac{1}{N}(\delta_{x_i+x_j} - \delta_{x_i} - \delta_{x_j})\,, \qquad i \ne j\,, \tag{2.14}$$

and the fragmentation operator $\mathcal{F}^N$ by

$$\mathcal{F}^N\Phi(p) = \frac{1}{2}\sum_{i=1}^n \int_{\mathcal{Z}} \Big[\Phi(J_F(p,i,y)) - \Phi(p)\Big]F(x_i,dy)\,, \tag{2.15}$$

where

$$J_F(p,i,y) = \begin{cases} p + \frac{1}{N}(\delta_y + \delta_{x_i-y} - \delta_{x_i})\,, & \text{if} \;\; p(\mathcal{Z}) \le c_N - \frac{1}{N}\,, \\ p\,, & \text{otherwise.}\end{cases} \tag{2.16}$$

Note that $y \in [0, x_i] \cap \mathcal{Z}$ in (2.16), according to (2.6). Moreover $J_K(p,i,j)$, $J_F(p,i,y) \in \mathcal{S}^N$ since

$$\int_{\mathcal{Z}} x \; J_K(p,i,j)(dx) = \int_{\mathcal{Z}} x \; J_F(p,i,y)(dx) = \int_{\mathcal{Z}} x \; p(dx) \,, \qquad \forall p \in \mathcal{S}^N \,, \tag{2.17}$$

and

$$J_K(p,i,j)(\mathcal{Z}) = p(\mathcal{Z}) - \frac{1}{N} \,, \quad J_F(p,i,y)(\mathcal{Z}) = \begin{cases} p(\mathcal{Z}) + \frac{1}{N} \,, & \text{if} \quad p(\mathcal{Z}) \le c_N - \frac{1}{N} \,, \\ p(\mathcal{Z}) \,, & \text{otherwise.} \end{cases} \tag{2.18}$$

According to (2.11) we obtain

$$|\mathcal{K}^N \Phi(p) + \mathcal{F}^N \Phi(p)| \;\; \le \;\; \|\Phi\| \, N \, \varrho(p) \,, \qquad \forall p \in \mathcal{S}^N \,, \tag{2.19}$$

where

$$\varrho(p) \;\; = \;\; \int_{\mathcal{Z}} \int_{\mathcal{Z}} K(x,y) \, p(dx) \, p(dy) \; + \; \int_{\mathcal{Z}} F(x, \mathcal{Z}) \, p(dx) \,, \qquad p \in \mathcal{M}_b(\mathcal{Z}) \,, \tag{2.20}$$

and

$$x_i \le N \int_{\mathcal{Z}} x \, p(dx) \le N \, M \,, \qquad \forall p \in \mathcal{S}^N \,, \tag{2.21}$$

which implies

$$\sup_{p \in \mathcal{S}^N} \varrho(p) \;\; \le \;\; c_N^2 \; \sup\{K(x,y) : x, y \in \mathcal{Z} \cap [0, NM]\} + c_N \; \sup\{F(x, \mathcal{Z}) : x \in \mathcal{Z} \cap [0, NM]\} \,.$$

Thus, according to (2.19), (2.3) and (2.8), the operator

$$\mathcal{G}^N = \mathcal{K}^N + \mathcal{F}^N \tag{2.22}$$

is bounded for fixed N and an $\mathcal{S}^N$-valued jump process $U^N(t)$ with generator $\mathcal{G}^N$ exists for any random initial state $U^N(0)$ in $\mathcal{S}^N$ (cf. [9, p.162]). This process is mass preserving, i.e.

$$\int_{\mathcal{Z}} x \; U^N(t, dx) = \int_{\mathcal{Z}} x \; U^N(0, dx) \,, \qquad t \ge 0 \,, \tag{2.23}$$

according to (2.17). It has trajectories in the Skorokhod space $\mathbb{D}([0,\infty), \mathcal{D}_M(\mathcal{Z}))$, where

$$\mathcal{D}_M(\mathcal{Z}) = \left\{ p \in \mathcal{M}_b(\mathcal{Z}) \; : \; \int_{\mathcal{Z}} x \; p(dx) \le M \right\} . \tag{2.24}$$

Theorem 2.1 *Assume*

$$\sup_N \mathbb{E} \left[U^N(0, \mathcal{Z}) \right]^2 < \infty \,, \tag{2.25}$$

$$K(x,y) \;\; \le \;\; C_K \left[x \, y + x + y + 1 \right] \,, \qquad \textit{for some} \quad C_K \ge 0 \,, \tag{2.26}$$

and

$$F(x, \mathcal{Z}) \;\; \le \;\; C_F \left[x + 1 \right] , \qquad \textit{for some} \quad C_F \ge 0 \,. \tag{2.27}$$

Then U^N is relatively compact in $\mathbb{D}([0,\infty), \mathcal{D}_M(\mathcal{Z}))$.

Theorem 2.2 *Assume (2.25),*

$$U^N(0) \Rightarrow P_0 \qquad \textit{(convergence in distribution)}\,, \tag{2.28}$$

$$K(\cdot,\cdot) \quad \textit{is continuous on} \quad \mathcal{Z}\times\mathcal{Z}\,, \tag{2.29}$$

and

$$\int_{\mathcal{Z}} \varphi(y)\,F(\cdot,dy) \quad \textit{is continuous on } \mathcal{Z} \textit{ for any} \quad \varphi\in C_b(\mathcal{Z})\,. \tag{2.30}$$

Suppose there is a continuous function $h:\mathcal{Z}\to[0,\infty)$*, with* $h(x)=o(x)$ *for* $x\to\infty$*, such that*

$$K(x,y) \;\le\; h(x)\,h(y)\,, \qquad x,y\in\mathcal{Z}\,, \tag{2.31}$$

and

$$F(x,\mathcal{Z}) \;\le\; h(x)\,, \qquad x\in\mathcal{Z}\,. \tag{2.32}$$

Then each weak accumulation point X *of* U^N *solves the coagulation-fragmentation equation (2.9) a.e. for initial measure* P_0.

Corollary 2.3 *If* $\int_{\mathcal{Z}} x\,P_0(dx)<\infty$ *and (2.29)-(2.32) are fulfilled, then Theorems 2.1 and 2.2 imply* **existence** *of at least one solution to equation (2.9). Note that* $h(x)\le C\,(1+x)$ *for some* $C>0$ *and therefore (2.31) and (2.32) imply (2.26) and (2.27), respectively. Assumptions (2.25) and (2.28) are satisfied if, e.g.,* $U^N(0)$ *is deterministic and converges weakly to* P_0.

Corollary 2.4 *If equation (2.9) has at most one solution, then Theorems 2.1 and 2.2 imply* **convergence** *of the sequence* U^N *to the unique solution.*

3. Basic properties of the particle systems

Lemma 3.1 (Martingale representation) *Assume (2.1)-(2.8) and let (cf. (2.11))*

$$\Phi(p)=\langle\varphi,p\rangle\,, \qquad p\in\mathcal{S}^N\,, \quad \varphi\in C_b(\mathcal{Z})\,. \tag{3.1}$$

where the notation

$$\langle f,\nu\rangle=\int_{\mathcal{Z}} f(z)\,\nu(dz)\,, \qquad f\in C_b(\mathcal{Z})\,, \quad \nu\in\mathcal{M}_b(\mathcal{Z})\,, \tag{3.2}$$

is used. Then (cf. (2.22))

$$M_\varphi^N(t) \;=\; \langle\varphi,U^N(t)\rangle-\langle\varphi,U^N(0)\rangle-\int_0^t(\mathcal{G}^N\Phi)(U^N(s))\,ds \tag{3.3}$$

is a martingale and we have the representations (cf. (2.13), (2.15))

$$\mathcal{K}^N\Phi(p) = \tag{3.4}$$
$$\int_{\mathcal{Z}}\int_{\mathcal{Z}} K(x,y)\Big[\frac{1}{2}\,\varphi(x+y)-\varphi(x)\Big]p(dx)\,p(dy)-\frac{1}{2N}\int_{\mathcal{Z}} K(x,x)\Big[\varphi(2x)-2\,\varphi(x)\Big]p(dx)$$

and

$$\mathcal{F}^N\Phi(p) = \int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(y)-\frac{1}{2}\,\varphi(x)\Big]F(x,dy)\,p(dx)\;\cdot\;\chi_{\{p(\mathcal{Z})\le c_N-\frac{1}{N}\}}(p)\,, \tag{3.5}$$

where χ_A denotes the indicator function of a set A. Moreover, the following inequalities hold (cf. (2.20))

$$|\mathcal{G}^N\Phi(p)| \le \frac{3}{2}\,\|\varphi\|\,\varrho(p)\,, \tag{3.6}$$

$$\mathbb{E}\left[M_\varphi^N(T)\right]^2 \;\le\; \frac{9\,\|\,\varphi\,\|^2}{2N}\,\mathbb{E}\left[\int_0^T \varrho(U^N(r))\,dr\right]. \tag{3.7}$$

Proof: The definition of weak convergence and the boundedness of measures in $\mathcal{S}^N$ imply $\Phi\in C_b(\mathcal{S}^N)$ for $\varphi\in C_b(\mathcal{Z})$ and accordingly M_φ^N is a martingale. Note that (2.13), (2.14), (3.1) imply

$$\mathcal{K}^N\Phi(p) = \frac{1}{2N^2}\sum_{1\le i\ne j\le n} K(x_i,x_j)\Big[\varphi(x_i+x_j)-\varphi(x_i)-\varphi(x_j)\Big] \tag{3.8}$$

and (3.4) follows from (2.2). Analogously, (2.15), (2.16) and (3.1) imply

$$\mathcal{F}^N\Phi(p) = \begin{cases} \frac{1}{2N}\sum_{i=1}^n\int_{\mathcal{Z}}\Big[\varphi(y)+\varphi(x_i-y)-\varphi(x_i)\Big]F(x_i,dy)\,, & \text{if}\quad p(\mathcal{Z})\le c_N-\frac{1}{N}\,,\\ 0\,, & \text{otherwise}\,,\end{cases}$$

and (3.5) follows from (2.7). Estimate (3.6) is a consequence of (2.22), (3.8), (3.5) and (2.20). Since $\Phi^2\in C_b(\mathcal{S}^N)$, we obtain

$$\mathbb{E}\left[M_\varphi^N(t)-M_\varphi^N(s)\right]^2 \;=\; \mathbb{E}\int_s^t\left[\mathcal{G}^N\Phi^2-2\,\Phi\,\mathcal{G}^N\Phi\right](U^N(r))\,dr\,. \tag{3.9}$$

Let $\widehat{\mathcal{Z}}$ be a locally compact separable metric space. For every operator J on $C_b(\widehat{\mathcal{Z}})$ of the form

$$(J\psi)(z) \;=\; \int_{\widehat{\mathcal{Z}}}[\psi(y)-\psi(z)]\,\nu(z,dy)\,,\qquad z\in\widehat{\mathcal{Z}}\,, \tag{3.10}$$

where $\nu(z,\widehat{\mathcal{Z}})\le\widehat{c}$, $z\in\widehat{\mathcal{Z}}$, and $\nu(\cdot,B)$, $B\in\mathcal{B}(\widehat{\mathcal{Z}})$, is measurable, one easily computes

$$[J\psi^2-2\,\psi\,J\psi](z) \;=\; \int_{\widehat{\mathcal{Z}}}[\psi(y)-\psi(z)]^2\,\nu(z,dy)\,. \tag{3.11}$$

Both $\mathcal{K}^N$ and $\mathcal{F}^N$ can be represented in the form (3.10) so that (2.13), (2.15) and formula (3.11) imply

$$\Big[\mathcal{K}^N\Phi^2 - 2\Phi\,\mathcal{K}^N\Phi\Big](p) = \frac{1}{2N^3}\sum_{1\le i\neq j\le n} K(x_i,x_j)\Big[\varphi(x_i+x_j)-\varphi(x_j)-\varphi(x_i)\Big]^2 \qquad (3.12)$$

and

$$\Big[\mathcal{F}^N\Phi^2 - 2\Phi\,\mathcal{F}^N\Phi\Big](p)$$
$$= \frac{1}{2\,N^2}\sum_{i=1}^{n}\int_{\mathcal{Z}}\Big[\varphi(y)+\varphi(x_i-y)-\varphi(x_i)\Big]^2 F(x_i,dy)\,\cdot\,\chi_{\{p(\mathcal{Z})\le c_N-\frac{1}{N}\}}(p)\,. \qquad (3.13)$$

From (3.12), (3.13) we get

$$\begin{aligned}\Big|\Big[\mathcal{G}^N\Phi^2 - 2\Phi\,\mathcal{G}^N\Phi\Big](p)\Big| &\le \frac{1}{2N}\int_{\mathcal{Z}}\int_{\mathcal{Z}} K(x,y)\Big[\varphi(x+y)-\varphi(x)-\varphi(y)\Big]^2 p(dx)\,p(dy)\\ &\quad+\frac{1}{2N}\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(y)+\varphi(x-y)-\varphi(x)\Big]^2 F(x,dy)\,p(dx)\\ &\le \frac{9\,\|\varphi\|^2}{2N}\,\varrho(p)\,. \end{aligned} \qquad (3.14)$$

From (3.9) and (3.14) we obtain inequality (3.7). □

Corollary 3.2 *Applying Doob's inequality and (3.7), we obtain*

$$\Big(\mathbb{E}\sup_{t\le T}\big|M_\varphi^N(t)\big|\Big)^2 \le 4\,\mathbb{E}\left[M_\varphi^N(T)\right]^2 \le \frac{18\,\|\varphi\|^2}{N}\,\mathbb{E}\left[\int_0^T \varrho(U^N(t))\,dt\right]. \qquad (3.15)$$

Lemma 3.3 *Assume (2.1)-(2.8) and (2.27). Then*

$$\mathbb{E}\left[U^N(t,\mathcal{Z})\right]^2 \le \Big(\mathbb{E}\left[U^N(0,\mathcal{Z})\right]^2+1\Big)\exp(2\,C_F\,(M+1)\,t)\,, \qquad t\ge 0\,. \qquad (3.16)$$

Proof: Define $\Phi(p)=p(\mathcal{Z})^2\,,\ p\in\mathcal{S}^N$ (cf. (2.11)). According to (2.22), (2.13)-(2.16) one obtains

$$\begin{aligned}(\mathcal{G}^N\Phi)(p) &= \frac{1}{2N}\sum_{1\le i\neq j\le n}\left[J_K^2(p,i,j)(\mathcal{Z})-p^2(\mathcal{Z})\right]K(x_i,x_j)\\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\int_{\mathcal{Z}}\left[J_F^2(p,i,y)(\mathcal{Z})-p^2(\mathcal{Z})\right]F(x_i,dy)\\ &\le \frac{1}{2N}\sum_{1\le i\neq j\le n}\left[-\frac{2}{N}\,p(\mathcal{Z})+\frac{1}{N^2}\right]K(x_i,x_j)+\frac{1}{2}\sum_{i=1}^{n}\int_{\mathcal{Z}}\left[\frac{2}{N}\,p(\mathcal{Z})+\frac{1}{N^2}\right]F(x_i,dy)\\ &\le \left[p(\mathcal{Z})+\frac{1}{2N}\right]\int_{\mathcal{Z}}F(x,\mathcal{Z})\,p(dx)\le C_F\left[p(\mathcal{Z})+1\right]\left[M+p(\mathcal{Z})\right]\\ &\le 2\,C_F\,(M+1)\left[p(\mathcal{Z})^2+1\right]. \end{aligned} \qquad (3.17)$$

Now (3.3) and (3.17) imply

$$\mathbb{E}\left[U^N(t,\mathcal{Z})\right]^2 \le \mathbb{E}\left[U^N(0,\mathcal{Z})\right]^2 + 2\,C_F\,(M+1)\int_0^t\Big(\mathbb{E}\left[U^N(s,\mathcal{Z})\right]^2+1\Big)\,ds\,.$$

Thus (3.16) follows from Gronwall's inequality (see, e.g., [9, p.498]). □

4. Relative compactness

Let $C(\mathcal{Z})$ the set of all continuous functions on $\mathcal{Z}$, $C_0(\mathcal{Z})$ the set of all $f \in C(\mathcal{Z})$ vanishing at infinity, and $C_c(\mathcal{Z})$ the set of all $f \in C(\mathcal{Z})$ having compact support. A sequence $\nu_n \in \mathcal{M}_b(\mathcal{Z})$ is called weakly convergent to $\nu \in \mathcal{M}_b(\mathcal{Z})$ (denoted by $\nu_n \stackrel{w}{\to} \nu$) if

$$\langle f, \nu_n \rangle \to \langle f, \nu \rangle \,, \qquad \forall f \in C_b(\mathcal{Z}) \,,$$

where the notation (3.2) is used. The space $C_c(\mathcal{Z})$ is separable that is there is a sequence $\{\varphi_k\}_{k=1}^{\infty} \subset C_c(\mathcal{Z})$ that is dense with respect to supremum norm. The distance

$$d_{weak}(\nu, \mu) \;=\; \sum_{k=0}^{\infty} \frac{1}{2^k} \min\{1, |\langle \varphi_k, \nu \rangle - \langle \varphi_k, \mu \rangle|\} \,, \qquad \nu, \mu \in \mathcal{M}_b(\mathcal{Z}) \,, \tag{4.1}$$

with $\varphi_0 \equiv 1$, metricizes the weak topology (cf. [3, Theorems 30.8, 31.5]). Since $\mathcal{Z}$ is Polish, the space $(\mathcal{M}_b(\mathcal{Z}), d_{weak})$ is also Polish.

Lemma 4.1 *Suppose $\nu_n \stackrel{w}{\to} \nu$ in $\mathcal{M}_b(\mathcal{Z})$ and let $G \in C(\mathcal{Z})$ be a non-negative function such that $\langle G, \nu_n \rangle \le m$, for some $m > 0$. Then $\langle G, \nu \rangle \le m$ and for any $g \in C(\mathcal{Z})$ with $\frac{g}{G} \in C_0(\mathcal{Z})$ we have*

$$\langle g, \nu_n \rangle \;\to\; \langle g, \nu \rangle \,. \tag{4.2}$$

Proof: Define non-negative Borel measures $\widetilde{\nu}_n$, $\widetilde{\nu}$ by

$$\widetilde{\nu}_n(B) = \int_B G \, d\nu_n \,, \qquad \widetilde{\nu}(B) = \int_B G \, d\nu \,, \qquad B \in \mathcal{B}(\mathcal{Z}) \,,$$

and note that $\widetilde{\nu}_n(\mathcal{Z}) \le m$. For any $\varphi \in C_c(\mathcal{Z})$ the product $\varphi \cdot G$ belongs to $C_c(\mathcal{Z})$ and thus

$$\langle \varphi, \widetilde{\nu_n} \rangle \;=\; \langle \varphi \cdot G, \nu_n \rangle \;\to\; \langle \varphi \cdot G, \nu \rangle \;=\; \langle \varphi, \widetilde{\nu} \rangle \,.$$

By [3, Lemma 30.3] we obtain

$$\langle G, \nu \rangle = \widetilde{\nu}(\mathcal{Z}) \;\le\; \liminf_n \widetilde{\nu}_n(\mathcal{Z}) \le m$$

and, by [3, Theorem 30.6],

$$\langle g, \nu_n \rangle \;=\; \langle \frac{g}{G}, \widetilde{\nu}_n \rangle \;\to\; \langle \frac{g}{G}, \widetilde{\nu} \rangle \;=\; \langle g, \nu \rangle$$

so that (4.2) is established. □

Lemma 4.2 *The space $(\mathcal{D}_M(\mathcal{Z}), d_{weak})$ is Polish (cf. (2.24)), and*

$$\mathcal{S}_{c,M} = \{p \in \mathcal{D}_M(\mathcal{Z}) \,:\, p(\mathcal{Z}) \le c\} \,, \qquad c > 0 \,, \tag{4.3}$$

are compact subsets.

Proof: According to **Lemma 4.1**, $\mathcal{D}_M(\mathcal{Z})$ and $\mathcal{S}_{c,M}$ are closed subsets of the Polish space $\mathcal{M}_b(\mathcal{Z})$. Thus it remains to show, that any sequence (p_n) from $\mathcal{S}_{c,M}$ is relatively compact. Assume without restriction $p_n(\mathcal{Z}) > 0$ and choose a subsequence (p_{n_k}) such that $\lim_{k\to\infty} p_{n_k}(\mathcal{Z}) > 0$. If there are no such subsequences, then (p_n) converges weakly to the zero measure. The sequence of probability measures given by $q_k(B) = \frac{1}{p_{n_k}(\mathcal{Z})}\, p_{n_k}(B)$, $B \in \mathcal{B}(\mathcal{Z})$, is tight, since

$$q_k((z,\infty)\cap\mathcal{Z}) \;\leq\; \frac{1}{z\, p_{n_k}(\mathcal{Z})}\int_{\mathcal{Z}} x\, p_{n_k}(dx)\,, \qquad z > 0\,.$$

By Prohorov's theorem there is a subsequence weakly converging to some probability measure q. The corresponding subsequence of (p_n) is also weakly converging. □

Lemma 4.3 *If (2.26) and (2.27) hold, then (cf. (2.20), (4.3))*

$$\sup_{p\in\mathcal{S}_{c,M}} \varrho(p) < \infty\,, \qquad \forall c \in (0,\infty)\,. \tag{4.4}$$

If, in addition, (2.25) holds, then

$$\sup_N \mathbb{E}\left[\int_0^T \varrho(U^N(t))\,dt\right] \;<\; \infty\,, \qquad T \geq 0\,. \tag{4.5}$$

Proof: For $p \in \mathcal{D}_M(\mathcal{Z})$ one obtains from (2.26), (2.27)

$$\varrho(p) \leq C_K\left[M^2 + 2\,Mp(\mathcal{Z}) + [p(\mathcal{Z})]^2\right] + C_F\,[M + p(\mathcal{Z})] \tag{4.6}$$

so that (4.4) follows. Moreover, (4.6) implies

$$\mathbb{E}\left[\int_0^T \varrho(U^N(t))\;dt\right] \leq$$
$$(C_K + C_F)\left[M\,(M+1)\,T + (2M+1)\,\mathbb{E}\int_0^T U^N(t,\mathcal{Z})\;dt + \mathbb{E}\int_0^T [U^N(t,\mathcal{Z})]^2\,dt\right].$$

So by **Lemma 3.3** and assumption (2.25) condition (4.5) is satisfied. □

Lemma 4.4 (Compact containment) *Assume (2.25), (2.26) and (2.27). Then, for any $T > 0$ and $\eta > 0$, there is $c > 0$ such that*

$$\inf_N P(U^N(t) \in \mathcal{S}_{c,M},\; 0 \leq t \leq T) \;\geq\; 1 - \eta\,. \tag{4.7}$$

Proof: Let $\varphi \equiv 1$ and $\Phi(p) = \langle\varphi, p\rangle = p(\mathcal{Z})$. From (3.3) we obtain

$$\begin{aligned} P\left(\sup_{t\leq T} U^N(t,\mathcal{Z}) \geq c\right) &= P\left(\sup_{t\leq T}\left[M_\varphi^N(t) + U^N(0,\mathcal{Z}) + \int_0^t (\mathcal{G}^N\Phi)(U^N(s))\;ds\right] \geq\; c\right) \\ &\leq\; P\left(\sup_{t\leq T}\left|M_\varphi^N(t)\right| + \int_0^T \left|(\mathcal{G}^N\Phi)(U^N(s))\right|ds + U^N(0,\mathcal{Z}) \geq c\right). \end{aligned}$$

Applying Tschebyscheff's inequality, (3.15), (3.6) and $2\sqrt{x} \leq 1 + x$, one obtains

$$
\begin{aligned}
P\left(\sup_{t\leq T} U^N(t,\mathcal{Z}) \geq c\right) &\leq \frac{1}{c}\left[\mathbb{E}\sup_{t\leq T}\left|M_\varphi^N(t)\right| + \mathbb{E}\int_0^T \left|(\mathcal{G}^N\Phi)(U^N(s))\right| ds + \mathbb{E}U^N(0,\mathcal{Z})\right] \\
&\leq \frac{1}{c}\left[\frac{1}{2} + \left(\frac{9}{N}+\frac{3}{2}\right)\mathbb{E}\int_0^T \varrho(U^N(s))\, ds + \mathbb{E}U^N(0,\mathcal{Z})\right].
\end{aligned}
$$

Thus, by (4.5) and (2.25), we can choose $c > 0$ such that the right-hand side of the last inequality becomes smaller than η for all N and (4.7) results. □

Proof of Theorem 2.1: For $y \in \mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z}))$, $\delta > 0$ and $T > 0$, define the modulus of continuity $w(y,\delta,T)$ by (cf. (4.1))

$$
w(y,\delta,T) = \inf_{\{t_i\}}\max_i \sup_{s,t\in[t_{i-1},t_i)} d_{weak}(y(s),y(t)),
$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \cdots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \delta$ and $n \geq 1$. By **Lemma 4.2** the space $\mathcal{D}_M(\mathcal{Z})$ is Polish and $\mathcal{S}_{c,M}$ is a compact subset. **Lemma 4.4** gives the tightness of $U^N(t)$ for any $t \geq 0$. Thus, by [9, Corollary 3.7.4] it is enough to show that

$$
\forall\, T,\eta > 0 \quad \exists \delta > 0 : \limsup_{N\to\infty} P(w(U^N,\delta,T) \geq \eta) \leq \eta .
$$

Let $T,\eta > 0$. For $y \in \mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z}))$, $\Delta t > \delta$ and the concrete partition $t_i = i\Delta t$, $i = 0,1,\ldots$, we obtain

$$
w(y,\delta,T) \leq 2 \max_{t_i<T} \sup_{s\in[t_i,t_{i+1})} d_{weak}(y(s),y(t_i)).
$$

So it remains to show

$$
\limsup_{N\to\infty} P\left(\max_{t_i<T} \sup_{s\in[t_i,t_{i+1})} d_{weak}(U^N(s),U^N(t_i)) \geq \eta\right) \leq \eta \tag{4.8}
$$

for sufficiently small Δt. Let $\varphi_0 \equiv 1$ and $\{\varphi_k\}_{k=1}^\infty \subset C_c(\mathcal{Z})$ as in (4.1). Define

$$
\Phi_k(p) = \langle \varphi_k, p\rangle, \qquad p \in \mathcal{S}^N, \qquad k = 0,1,\ldots .
$$

Using (3.3) and (4.1) we obtain

$$
\begin{aligned}
d_{weak}(U^N(s),U^N(t)) &= \sum_{k=0}^\infty \frac{1}{2^k}\min\left\{1,\left|\langle\varphi_k,U^N(s)\rangle - \langle\varphi_k,U^N(t)\rangle\right|\right\} \\
&\leq \sum_{k=0}^\infty \frac{1}{2^k}\min\left\{1,\left|M_{\varphi_k}^N(s) - M_{\varphi_k}^N(t)\right| + \int_t^s \left|\left(\mathcal{G}^N\Phi_k\right)(U^N(r))\right| dr\right\}.
\end{aligned} \tag{4.9}
$$

According to **Lemma 4.4** there is $c > 0$ such that

$$
\inf_N P(U^N(t) \in \mathcal{S}_{c,M},\ 0 \leq t \leq T+\Delta t) \geq 1 - \frac{\eta}{2}. \tag{4.10}
$$

By (4.10), (4.9), (3.6) and Tschebyscheff's inequality we obtain

$$
\begin{aligned}
& P\left(\max_{t_i<T}\sup_{s\in[t_i,t_{i+1})} d_{weak}(U^N(s),U^N(t_i))\geq\eta\right)\\
\leq\ & P\left(\left\{\max_{t_i<T}\sup_{s\in[t_i,t_{i+1})} d_{weak}(U^N(s),U^N(t_i))\geq\eta\right\}\cap\left\{U^N(t)\in\mathcal{S}_{c,M},\ 0\leq t\leq T+\Delta t\right\}\right)+\frac{\eta}{2}\\
\leq\ & P\left(\sum_{k=0}^{\infty}\frac{1}{2^k}\min\left\{1,\max_{t_i<T}\sup_{s\in[t_i,t_{i+1})}\left|M^N_{\varphi_k}(s)-M^N_{\varphi_k}(t_i)\right|+\Delta t\,\frac{3}{2}\,\|\varphi_k\|\sup_{p\in\mathcal{S}_{c,M}}\varrho(p)\right\}\geq\eta\right)+\frac{\eta}{2}\\
\leq\ & \frac{1}{\eta}\sum_{k=0}^{\infty}\frac{1}{2^k}\min\left\{1,2\,\mathbb{E}\Big[\sup_{s\leq T+\Delta t}\left|M^N_{\varphi_k}(s)\right|\Big]+\Delta t\,\frac{3}{2}\,\|\varphi_k\|\sup_{p\in\mathcal{S}_{c,M}}\varrho(p)\right\}+\frac{\eta}{2}\,.
\end{aligned}
$$

Thus, (3.15) and **Lemma 4.3** imply (4.8) for sufficiently small Δt. □

5. Characterization of weak limit points

Lemma 5.1 (path property) *If a subsequence U^{N_k} weakly converges to X then*

$$P(X\in\mathbb{C}([0,\infty),\mathcal{D}_M(\mathcal{Z})))\ =\ 1\,.$$

Proof: Consider $p\in\mathcal{S}^N$ and let p' be any possible consecutive state. According to (2.14) and (2.16) we obtain (cf. (4.1))

$$d_{weak}(p,p')\ \leq\ \sum_{k=0}^{\infty}\frac{1}{2^k}\min\left\{1,\frac{3\,\|\varphi_k\|}{N}\right\}.$$

Thus the distance between arbitrary successive states uniformly vanishes and the claim follows from [9, Theorem 3.10.2(a)]. □

For $p\in\mathcal{M}_b(\mathcal{Z})$ with $\varrho(p)<\infty$ (cf. (2.20)) and $\varphi\in C_b(\mathcal{Z})$ define

$$\mathcal{K}(\varphi,p)=\int_{\mathcal{Z}}\int_{\mathcal{Z}}\left[\frac{1}{2}\,\varphi(x+y)-\varphi(x)\right]K(x,y)\,p(dy)\,p(dx)\tag{5.1}$$

and

$$\mathcal{F}(\varphi,p)=\int_{\mathcal{Z}}\int_{\mathcal{Z}}\left[\varphi(y)-\frac{1}{2}\,\varphi(x)\right]F(x,dy)\,p(dx)\,.\tag{5.2}$$

Note that

$$|\mathcal{K}(\varphi,p)+\mathcal{F}(\varphi,p)|\leq\frac{3}{2}\,\|\varphi\|\,\varrho(p)\,.\tag{5.3}$$

Remark 5.2 *Since*

$$\sup_{t\leq T}\,y(t,\mathcal{Z})<\infty\,,\qquad\forall T\geq 0\,,\quad y\in\mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z}))\,,$$

assumptions (2.26) and (2.27) imply

$$\sup_{t\leq T}\,\varrho(y(t))<\infty\,,\qquad\forall T\geq 0\,,\quad y\in\mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z}))\,,\tag{5.4}$$

according to (4.4).

With the notations (5.1), (5.2), equation (2.9) takes the form

$$\langle \varphi, P(t) \rangle = \langle \varphi, P_0 \rangle + \int_0^t \Big[\mathcal{K}(\varphi, P(s)) + \mathcal{F}(\varphi, P(s)) \Big] ds \,, \qquad t \ge 0 \,. \tag{5.5}$$

According to Remark 5.2, condition (2.10) is fulfilled for all $P \in \mathbb{D}([0,\infty), \mathcal{D}_M(\mathcal{Z}))$ under the assumptions (2.26) and (2.27).

For each $\varphi \in C_b(\mathcal{Z})$ and $y \in \mathbb{D}([0,\infty), \mathcal{D}_M(\mathcal{Z}))$, we define

$$M_\varphi(y,T) = \langle \varphi, y(T) \rangle - \langle \varphi, y(0) \rangle - \int_0^T \Big[\mathcal{K}(\varphi, y(t)) + \mathcal{F}(\varphi, y(t)) \Big] dt \,, \qquad T \ge 0 \,. \tag{5.6}$$

Lemma 5.3 *Assume (2.26), (2.27) and let*

$$\varphi_m \stackrel{bp}{\to} \varphi \,, \qquad \varphi_m \,,\, \varphi \in C_b(\mathcal{Z}) \,, \tag{5.7}$$

that is $\sup_m \|\varphi_m\| < \infty$ *and* $\varphi_m(x) \to \varphi(x)$ *for every* $x \in \mathcal{Z}$ *(cf. [9, p.495]). Then*

$$M_{\varphi_m}(y,T) \;\; \to \;\; M_\varphi(y,T) \,, \qquad \forall T \ge 0 \,, \quad y \in \mathbb{D}([0,\infty), \mathcal{D}_M(\mathcal{Z})) \,. \tag{5.8}$$

Proof: The dominated convergence theorem (cf. [9, p.492]) and (5.7) imply

$$\langle \varphi_m, p \rangle \;\; \to \;\; \langle \varphi, p \rangle \,, \qquad \forall p \in \mathcal{D}_M(\mathcal{Z}) \,. \tag{5.9}$$

Consider $p \in \mathcal{D}_M(\mathcal{Z})$ and define non-negative Borel measures by

$$\mu_1(dx,dy) = K(x,y)\, p(dx)\, p(dy) \,, \qquad \mu_2(dx,dy) = F(x,dy)\, p(dx) \,.$$

Note that μ_1, $\mu_2 \in \mathcal{M}_b(\mathcal{Z} \times \mathcal{Z})$ because of (2.26), (2.27). Using (5.7) one obtains

$$g_m(x,y) = \frac{1}{2}\, \varphi_m(x+y) - \varphi_m(x) \quad \stackrel{bp}{\to} \quad g(x,y) = \frac{1}{2}\, \varphi(x+y) - \varphi(x)$$

and

$$h_m(x,y) = \varphi_m(y) - \frac{1}{2}\, \varphi_m(x) \quad \stackrel{bp}{\to} \quad h(x,y) = \varphi(y) - \frac{1}{2}\, \varphi(x) \,.$$

By the dominated convergence theorem we obtain (cf. (5.1), (5.2))

$$\mathcal{K}(\varphi_m, p) = \langle g_m, \mu_1 \rangle \;\; \to \;\; \langle g, \mu_1 \rangle = \mathcal{K}(\varphi, p) \tag{5.10}$$

and

$$\mathcal{F}(\varphi_m, p) = \langle h_m, \mu_2 \rangle \;\; \to \;\; \langle h, \mu_2 \rangle = \mathcal{F}(\varphi, p) \,. \tag{5.11}$$

Using (5.3) and (5.4), one obtains

$$\sup_{t \le T} \Big| \mathcal{K}(\varphi_m, y(t)) + \mathcal{F}(\varphi_m, y(t)) \Big| \;\le\; \frac{3}{2}\, \|\varphi_m\| \, \sup_{t \le T} \varrho(y(t)) \;<\; \infty \,,$$

and, because of (5.10), (5.11) and (5.9), one further application of the dominated convergence theorem gives (5.8). □

Lemma 5.4 *Assume (2.29), (2.30), (2.31) and (2.32). Then*

$$M_\varphi(U^{N_k},T) \Rightarrow M_\varphi(X,T)\,, \qquad \forall\, T \geq 0\,, \quad \varphi \in C_b(\mathcal{Z})\,, \tag{5.12}$$

whenever

$$U^{N_k} \Rightarrow X \quad in \quad \mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z}))\,.$$

Proof: We will check that the mapping (cf. (5.6))

$$M_\varphi(\cdot,T)\,:\;\mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z})) \to \mathbb{R}$$

is a.s. continuous with respect to the limiting distribution for all $T \geq 0$ and $\varphi \in C_b(\mathcal{Z})$. By **Lemma 5.1** it is sufficient to show that

$$M_\varphi(y^N,T) \;\to\; M_\varphi(y,T) \tag{5.13}$$

whenever

$$y^N \in \mathbb{D}([0,\infty),\mathcal{D}_M(\mathcal{Z})) \;\to\; y \in \mathbb{C}([0,\infty),\mathcal{D}_M(\mathcal{Z}))\,. \tag{5.14}$$

Let $p^N \xrightarrow{w} p$ in $\mathcal{D}_M(\mathcal{Z})$ and define $G(x,y) = (1+x)(1+y)$. Since $\mathcal{Z}$ is separable, the product measures $p^N \times p^N$ weakly converge to $p \times p \in \mathcal{M}_b(\mathcal{Z}\times\mathcal{Z})$. Furthermore,

$$\sup_N \int_{\mathcal{Z}}\int_{\mathcal{Z}} G(x,y)\,p^N(dx)\,p^N(dy) \;\leq\; \sup_N \left(p^N(\mathcal{Z})+M\right)^2 \;<\; \infty\,.$$

According to (2.29), the function

$$g(x,y) = K(x,y)\Big[\frac{1}{2}\,\varphi(x+y) - \varphi(x)\Big]\,, \qquad x,y \in \mathcal{Z}\,, \quad \varphi \in C_b(\mathcal{Z})\,,$$

is continuous and $\frac{g}{G} \in C_0(\mathcal{Z}\times\mathcal{Z})$ because of (2.31). By **Lemma 4.1** (cf. (5.1))

$$\mathcal{K}(\varphi,p^N) = \langle g, p^N \times p^N\rangle \;\to\; \langle g, p\times p\rangle = \mathcal{K}(\varphi,p)\,. \tag{5.15}$$

Now consider the functions $G'(x) = 1+x$ and

$$g'(x) = \int_{\mathcal{Z}} \Big[\varphi(y) - \frac{1}{2}\,\varphi(x)\Big]\,F(x,dy)\,, \qquad x \in \mathcal{Z}\,, \quad \varphi \in C_b(\mathcal{Z})\,.$$

According to (2.30) the function g' is continuous. Since $\sup_N \int_{\mathcal{Z}} G'(x)\,p^N(dx) < \infty$ and since $\frac{g'}{G'} \in C_0(\mathcal{Z})$ because of (2.32), we can apply **Lemma 4.1** once again to get (cf. (5.2))

$$\mathcal{F}(\varphi,p^N) = \langle g', p^N\rangle \;\to\; \langle g', p\rangle = \mathcal{F}(\varphi,p)\,. \tag{5.16}$$

From (5.14) one obtains $y^N(t) \xrightarrow{w} y(t)$ and

$$\langle \varphi, y^N(t)\rangle \;\to\; \langle \varphi, y(t)\rangle\,, \qquad \forall t \geq 0\,, \quad \varphi \in C_b(\mathcal{Z})\,. \tag{5.17}$$

On the other hand

$$\mathcal{K}(\varphi, y^N(t)) + \mathcal{F}(\varphi, y^N(t)) \quad \to \quad \mathcal{K}(\varphi, y(t)) + \mathcal{F}(\varphi, y(t)) \,, \qquad \forall t \geq 0 \,, \quad \varphi \in C_b(\mathcal{Z}) \,, \tag{5.18}$$

by (5.15) and (5.16). Due to uniform convergence on finite time intervals we obtain

$$\sup_N \sup_{t \leq T} y^N(t, \mathcal{Z}) < \infty \,. \tag{5.19}$$

Using (5.3) and (5.19) one obtains

$$\sup_N \sup_{t \leq T} \left| \mathcal{K}(\varphi, y^N(t)) + \mathcal{F}(\varphi, y^N(t)) \right| \leq \frac{3}{2} \, \|\varphi\| \, \sup_N \sup_{t \leq T} \varrho(y^N(t)) < \infty \,,$$

according to (4.4) (cf. Corollary 2.3). Applying (5.17), (5.18) and the dominated convergence theorem we obtain (5.13) (cf. (5.6)). □

Lemma 5.5 *Assume (2.25) and (2.27). Then*

$$\mathbb{E}\left[\int_0^T \left(\chi_{\{U^N(t,\mathcal{Z}) > c_N - \frac{1}{N}\}} \int_{\mathcal{Z}} F(x, \mathcal{Z}) \, U^N(t, dx) \right) dt \right] \quad \to \quad 0 \,, \qquad \forall \, T \geq 0 \,. \tag{5.20}$$

Proof: By (2.11) and Tschebyscheff's inequality one obtains

$$\begin{aligned}
& \mathbb{E} \int_0^T \left(\chi_{\{U^N(t,\mathcal{Z}) > c_N - \frac{1}{N}\}} \int_{\mathcal{Z}} F(x, \mathcal{Z}) \, U^N(t, dx) \right) dt \leq \\
& \qquad C_F \, \mathbb{E} \int_0^T \left(\chi_{\{U^N(t,\mathcal{Z}) > c_N - \frac{1}{N}\}} U^N(t, \mathcal{Z}) \right) dt + C_F \, M \, \mathbb{E} \int_0^T \left(\chi_{\{U^N(t,\mathcal{Z}) > c_N - \frac{1}{N}\}} \right) dt \\
& \leq \quad C_F \, (c_N + M) \int_0^T P\left(U^N(t, \mathcal{Z}) > c_N - \frac{1}{N} \right) dt \\
& \leq \quad C_F \, (c_N + M) \, \frac{\int_0^T \mathbb{E} \left[U^N(t, \mathcal{Z}) \right]^2 \, dt}{(c_N - \frac{1}{N})^2} \,.
\end{aligned}$$

Thus **Lemma 3.3**, (2.25) and (2.12) imply (5.20). □

Lemma 5.6 *Assume (2.25), (2.27) and suppose there is a continuous function $h : \mathcal{Z} \to [0, \infty)$ such that $h(x) = o(x)$ for $x \to \infty$ and*

$$K(x, y) \quad \leq \quad C \left[h(x) \, h(y) + x + y \right] \tag{5.21}$$

Then

$$\frac{1}{N} \, \mathbb{E} \left[\int_0^T \int_{\mathcal{Z}} K(x, x) \, U^N(t, dx) \, dt \right] \quad \to \quad 0 \,, \qquad \forall \, T \geq 0 \,. \tag{5.22}$$

Proof: Assumption (5.21) implies

$$\frac{1}{N} \, \mathbb{E} \left[\int_0^T \int_{\mathcal{Z}} K(x, x) \, U^N(t, dx) \, dt \right] \leq \frac{C}{N} \, \mathbb{E} \left[\int_0^T \int_{\mathcal{Z}} h(x)^2 \, U^N(t, dx) \, dt \right] + \frac{2 \, C \, M \, T}{N} \,.$$

Define for $t \geq 0$ the intensity measure $\nu^N(t) \in \mathcal{M}_b(\mathcal{Z})$ by

$$\nu^N(t,B) \;=\; \mathbb{E}U^N(t,B), \qquad B \in \mathcal{B}(\mathcal{Z}).$$

It remains to show that

$$\frac{1}{N}\int_0^T \int_{\mathcal{Z}} h(x)^2\, \nu^N(t,dx)\, dt \;\to\; 0. \tag{5.23}$$

Define $\mu^N(t,dx) = \frac{1}{N}\nu^N(t,dx)$ and $G(x) = (x+1)^2$. Note that $\nu^N(t) \in \mathcal{D}_M(\mathcal{Z})$ and $\nu^N(t)$ has support on $[0,NM] \cap \mathcal{Z}$ according to (2.21). Using these properties of $\nu^N(t)$, **Lemma 3.3** and (2.25), we obtain

$$\sup_N \sup_{t\leq T} \int_{\mathcal{Z}} G(x)\,\mu^N(t,dx) \;\leq\; \sup_N \sup_{t\leq T} \frac{1}{N}\left[NM^2 + 2M + \nu^N(t,\mathcal{Z})\right] \;<\; \infty. \tag{5.24}$$

Since $\mu^N(t)$ weakly converges to the zero measure and $\frac{h^2}{G} \in C_0(\mathcal{Z})$, **Lemma 4.1** gives

$$\int_{\mathcal{Z}} h^2(x)\,\mu^N(t,dx) \;\to\; 0, \qquad t \geq 0.$$

By (5.24) and the dominated convergence theorem we obtain (5.23). □

Proof of Theorem 2.2: Let a subsequence U^{N_k} converge in distribution to X. In the following we omit the index k. We will prove that (cf. (5.5), (5.6))

$$M_\varphi(X,T) = 0, \qquad \forall T \geq 0, \quad \varphi \in C_b(\mathcal{Z}) \quad \text{a.e.} \tag{5.25}$$

Considering the function $\Phi(p) = \langle \varphi, p\rangle$ on $\mathcal{S}^N$, one obtains from (5.1), (5.2) and (3.3), (3.4), (3.5) that

$$\begin{aligned}
&\mathbb{E}\,|M_\varphi(U^N,T)| = \\
&\qquad \mathbb{E}\left|\langle\varphi,U^N(T)\rangle - \langle\varphi,U^N(0)\rangle - \int_0^T \mathcal{K}(\varphi,U^N(s))\,ds - \int_0^T \mathcal{F}(\varphi,U^N(s))\,ds\right| \\
&\leq\; \mathbb{E}\left|\langle\varphi,U^N(T)\rangle - \langle\varphi,U^N(0)\rangle - \int_0^T (\mathcal{K}^N\Phi)(U^N(s))\,ds - \int_0^T (\mathcal{F}^N\Phi)(U^N(s))\,ds\right| \\
&\qquad +\mathbb{E}\left|\int_0^T (\mathcal{K}^N\Phi)(U^N(s))\,ds - \int_0^T \mathcal{K}(\varphi,U^N(s))\,ds\right| \\
&\qquad +\mathbb{E}\left|\int_0^T (\mathcal{F}^N\Phi)(U^N(s))\,ds - \int_0^T \mathcal{F}(\varphi,U^N(s))\,ds\right| \\
&=\; \mathbb{E}\,|M_\varphi^N(T)| + \frac{1}{2N}\,\mathbb{E}\left|\int_0^T\int_{\mathcal{Z}} K(x,x)\,[\varphi(2x)-2\varphi(x)]\,U^N(s,dx)\,ds\right| \\
&\qquad +\mathbb{E}\left|\int_0^T \mathcal{F}(\varphi,U^N(s))\,\chi_{\{U^N(s,\mathcal{Z})>c_N-\frac{1}{N}\}}\,ds\right|.
\end{aligned} \tag{5.26}$$

By (3.15) and (4.5) the first summand in (5.26) vanishes for $N \to \infty$. Note that (2.31) implies (5.21). Thus, according to (5.22), the second term in (5.26) vanishes. By (5.20) the third term vanishes. Because of (5.12) we can apply Fatou's lemma (cf. [9, p.492]) to obtain from (5.26)

$$\mathbb{E}\,|M_\varphi(X,T)| \;\le\; \liminf_N \mathbb{E}\,|M_\varphi(U^N,T)| \;=\; 0$$

so that

$$M_\varphi(X,T) = 0 \quad \text{a.e.}\,, \tag{5.27}$$

where the exception set of measure zero depends on T and φ . Let $\{t_n\}_{n=1}^\infty$ be dense in the time interval $[0,\infty)$ and $\{\varphi_k\}_{k=1}^\infty$ dense in $C_c(\mathcal{Z})$ (cf. (4.1)). It follows from (5.27) that

$$M_{\varphi_k}(X,T) = 0\,, \qquad \forall\, k \in \mathbb{N}\,, \quad T \in \{t_n\}_{n=1}^\infty \qquad \text{a.e.} \tag{5.28}$$

Every $\varphi \in C_b(\mathcal{Z})$ can be approximated in the sense of (5.7) by functions from $\{\varphi_k\}_{k=1}^\infty$. Therefore, (5.28) and **Lemma 5.3** imply

$$M_\varphi(X,T) = 0\,, \qquad \forall\, \varphi \in C_b(\mathcal{Z})\,, \quad T \in \{t_n\}_{n=1}^\infty \qquad \text{a.e.} \tag{5.29}$$

Note that $M_\varphi(y,\cdot)$ is continuous for $y \in \mathbb{C}([0,\infty),\mathcal{D}_M(\mathcal{Z}))$. Thus, according to **Lemma 5.1**, $M_\varphi(X,\cdot)$ has almost surely continuous paths and (5.29) implies (5.25). It follows from (2.28) that $X(0) = P_0$ a.e. Note that X satisfies (2.10) a.e. according to Remark 5.2. Thus X is almost surely a solution of the coagulation-fragmentation equation for initial measure P_0 . □

6. Simulation algorithms

We will describe a class of simulation algorithms related to the stochastic particle systems considered in this paper. The infinitesimal generator (2.22) (cf. (2.13), (2.15)) does not change if one adds terms of the form

$$\frac{1}{2N} \sum_{1\le i\ne j\le n} \Big[\Phi(p) - \Phi(p)\Big]\Big[\hat{K}(x_i,x_j) - K(x_i,x_j)\Big]$$

and

$$\frac{1}{2}\sum_{i=1}^{n} \Big[\Phi(p) - \Phi(p)\Big]\Big[\hat{F}(x_i) - F(x_i,\mathcal{Z})\Big],$$

where $p \in \mathcal{S}^N$ and $\hat{K}$, $\hat{F}$ are appropriate functions such that

$$K(x_i,x_j) \;\le\; \hat{K}(x_i,x_j)\,, \qquad 1 \le i \ne j \le n\,,$$

and

$$F(x_i,\mathcal{Z}) \;\le\; \hat{F}(x_i)\,, \qquad 1 \le i \le n\,.$$

However this introduction of artificial "fictitious" jumps (cf., e.g., [9, p. 163]) provides a variety of ways to generate trajectories of the process. The efficiency of the simulation procedure depends on the choice of the functions $\hat{K}$ and $\hat{F}$. We first describe the general procedure before turning to some special cases.

0. Generate the initial state $U^N(0,dx) = p(dx) \in \mathcal{S}^N$ (cf. (2.11)).

1. Wait an exponentially distributed time with parameter

$$\hat{\varrho}(p) = \hat{\varrho}_K(p) + \hat{\varrho}_F(p)\,,$$

where

$$\hat{\varrho}_K(p) = \frac{1}{2N} \sum_{1 \le i \ne j \le n} \hat{K}(x_i, x_j) \tag{6.1}$$

and

$$\hat{\varrho}_F(p) = \frac{1}{2} \sum_{i=1}^{n} \hat{F}(x_i)\,. \tag{6.2}$$

2. With probability

$$\frac{\hat{\varrho}_K(p)}{\hat{\varrho}(p)}$$

go to step 3, else go to step 4.

3. (a) Choose a pair of indices according to the distribution

$$\frac{\hat{K}(x_i, x_j)}{2N\,\hat{\varrho}_K(p)}\,, \qquad 1 \le i \ne j \le n\,. \tag{6.3}$$

(b) Reject the coagulation with probability

$$1 - \frac{K(x_i, x_j)}{\widehat{K}(x_i, x_j)}\,,$$

else replace p by the new state $J_K(p,i,j)$ (cf. (2.14)), i.e. remove the clusters x_i and x_j and add the cluster $x_i + x_j$.

(c) Go to step 1.

4. (a) If $p(\mathcal{Z}) > c_N - \frac{1}{N}$, then go to step 1 (cf. (2.16), (2.18)).

(b) Choose an index according to the distribution

$$\frac{\hat{F}(x_i)}{2\,\hat{\varrho}_F(p)}\,, \qquad 1 \le i \le n\,. \tag{6.4}$$

(c) Reject the fragmentation with probability

$$1 - \frac{F(x_i, \mathcal{Z})}{\hat{F}(x_i)}\,,$$

else choose a fragmentation part y according to the distribution

$$\frac{F(x_i, dy)}{F(x_i, \mathcal{Z})} \quad \text{on} \quad [0, x_i] \cap \mathcal{Z}\,,$$

and replace p by the new state $J_F(p,i,y)$ (cf. (2.16)), i.e. remove the cluster x_i and add the clusters y and $x_i - y$.

(d) Go to step 1.

The **special case**

$$\hat{K}(x,y) = K(x,y)\,, \qquad \hat{F}(x) = F(x,\mathcal{Z})\,, \qquad x,y \in \mathcal{Z}\,,$$

corresponds to the direct "physical" simulation of the process (see [13], [6]). Here no fictitious jumps occur, and the time steps are as large as possible. However, the calculation of (6.1) and the generation of the distribution (6.3) are very time consuming if n is large and K has a complicated structure.

In the **special case**

$$\hat{K}(x,y) = K_{max} = \max_{i,j} K(x_i,x_j)\,, \qquad \hat{F}(x) = F_{max} = \max_{i} F(x_i,\mathcal{Z})\,, \qquad x,y \in \mathcal{Z}\,,$$

one obtains a simulation procedure (see , e.g., [13], [23]) which is in some sense opposite to the direct simulation. Namely, the calculation of (6.1) and the generation of the distribution (6.3) are extremely simple. On the other hand, the time steps are very small, and many fictitious jumps occur if K is unbounded and clusters of significantly different sizes are contained in the system.

Consider the **special case**

$$\hat{K}(x,y) = C_K\,(x+y+1)\,, \qquad \hat{F}(x) = C_F\,(x+1)\,, \qquad x,y \in \mathcal{Z}\,.$$

One obtains (cf. (6.1))

$$\hat{\varrho}_K(p) \quad = \quad \frac{C_K}{2N}\Big[2\,N\,m\,(n-1) + n\,(n-1)\Big]$$

and (cf. (6.2))

$$\hat{\varrho}_F(p) \quad = \quad \frac{C_F}{2}\Big[N\,m + n\Big],$$

where

$$m \quad = \quad \int_{\mathcal{Z}} x\,U^N(0,dx)\,.$$

Here the mass conservation (2.23) has been used. The distribution (6.3) takes the form

$$\frac{x_i + x_j + 1}{2\,N\,m\,(n-1) + n\,(n-1)} = \frac{x_i + x_j}{2\,N\,m\,(n-1)}\,\frac{2\,N\,m}{2\,N\,m + n} + \frac{1}{n\,(n-1)}\,\frac{n}{2\,N\,m + n}\,.$$

Accordingly, with probability

$$\frac{n}{2\,N\,m + n}\,,$$

the indices i,j are generated uniformly. Otherwise, one index is generated according to the distribution

$$\frac{x_i}{N\,m}\,, \qquad i = 1,\ldots,n\,, \tag{6.5}$$

the other uniformly.

This simulation procedure is extremely simple compared with the direct simulation. For an efficient generation of the distribution (6.5), some refined acceptance-rejection techniques can be applied. On the other hand, the number of fictitious jumps is reduced significantly for systems with unbounded kernels and strongly varying cluster sizes. Note that the case

$$\hat{K}(x,y) = C_K\,(x\,y+1)\,, \qquad \hat{F}(x) = C_F\,(x+1)\,, \qquad x,y \in \mathcal{Z}\,,$$

can be handled in an analogous way.

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