Evolution variational inequalities and multidimensional hysteresis operators.

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Abstract

We give an overview of the theory of multidimensional hysteresis operators defined as solution operators of rate-independent variational inequalities in a Hilbert space X with given convex constraints. Emphasis is put on analytical properties of these operators in the space of functions of bounded variation with values in X, in Sobolev spaces and in the space of continuous functions. We discuss in detail the influence of the geometry of the convex constraint on the input-output behavior. It is shown how multidimensional hysteresis operators arise naturally in constitutive laws of rate-independent plasticity and concrete examples of application of the above theory in material sciences are given.

Introduction

One may wonder why such a particular problem like the variational inequality

$$\langle \dot{u}(t) - \dot{x}(t), x(t) - \tilde{x} \rangle \geq 0 \qquad \forall \tilde{x} \in Z\,, \tag{0.1}$$

where Z is a convex closed subset of a Hilbert space X, u is a given X-valued function of $t \in [0,T]$, x is the unknown function with values in Z and dot denotes the derivative with respect to t, should draw an exceptional attention. As in many analogous cases, it has been extracted as a common feature of different physical models. Its variational character is typically interpreted as a special form of the *maximal dissipation principle* in evolution systems with convex constraints. It turns out that inequalities of the form (0.1) play (explicitly or implicitly) a central role in modeling nonequilibrium processes with rate-independent memory in mechanics of elastoplastic and thermoelastoplastic materials including metals, polymers or for instance bread dough, as well as in ferromagnetism, piezoelectricity or phase transitions (see e.g. [DL, LC, Al, LT, NH, BS, V, Be, KS1, KS2, KS3, KS4, AGM]). They also naturally arise in the analysis of fatigue and damage accumulation, see [BDK, BS].

Another area of application is related to mathematical optimization, where inequality (0.1) is known as a special case of the *Skorokhod problem*, cf. [DI, DN], which consists in approximating a given function $u : [0,T] \to X$ by a function ξ of bounded total variation in a given convex neighborhood of u in such a way that $\dot{\xi}$ (in a generalized sense) points in a prescribed direction. Equation (0.1) corresponds to the case where $\dot{\xi} = \dot{u} - \dot{x}$ belongs to the outward normal cone to Z at the point x. On the other hand, (0.1) is a special case of a 'sweeping process', see [M].

If Z has nonempty interior, the decomposition $u = x + \xi$ defined by inequality (0.1), where x is Z-valued and ξ has bounded variation, can be extended to every continuous function u. Moreover, there are some indications to conjecture that this decomposition is *minimal* in the sense that among all decompositions of u of this form, the total variation of ξ is minimal with respect to a suitable norm in X. In the case $\dim X = 1$, this observation has been made by V. Chernorutskii and a proof can be found in [K]; in higher dimensions, this question seems to be open.

The present text is devoted to a discussion about the influence of the geometry of the convex constraint Z (the *characteristic*) on analytical properties of the mappings $u \mapsto x$

and $u \mapsto \xi$ (the so-called *stop* and *play* operators). They are *hysteresis operators*, that is, according to the classification in [V], operators that are causal and rate-independent (see (1.25), (1.26) below). This terminology is justified by the fact that in the scalar case $\dim X = 1$, hysteresis operators are exactly those that admit a local representation by means of superposition operators in each interval of monotonicity of the input, with a possible branching when the input changes direction.

Most of the material collected here is taken from [KP, K] with some small improvements. More recent contributions ([BK, D, DT]) are referred to in the text, the results of Sections 5 and 7 are new to a large extent.

In Section 1 we present some typical issues of elastoplasticity related to inequality (0.1). Basic elements of convex analysis are recalled in Section 2. In Section 3 we construct the play and stop operators in the space of continuous X-valued functions of bounded variation and prove their continuity in $W^{1,p}(0,T;X)$ for every $1 \leq p < \infty$ and every convex closed set Z. A continuous extension to the space $C([0,T];X)$ of continuous functions is established provided Z has nonempty interior. The uniform continuity in $C([0,T];X)$ is proved in Section 4 under the hypothesis that the set Z is uniformly strictly convex. The local Lipschitz continuity in $W^{1,1}(0,T;X)$ is obtained in Section 5 when the boundary of Z is smooth. If Z is a convex polyhedron, the play and stop are globally Lipschitz in both $C([0,T];X)$ and $W^{1,1}(0,T;X)$; a detailed proof is given in Section 6. In Section 7 we prove a maximal regularity result, namely that the total variation of the derivative of the output can be estimated above by that of the input. Indeed, one cannot expect the output derivative to be continuous across the boundary of Z even if the input is arbitrarily smooth. The last Section 8 gives a brief survey of the theory of Hilbert space-valued functions.

Even in application to elastoplasticity, the investigation of the stop and play operator is not just an academic question. Indeed, the theory of monotone operators provides a traditional tool for solving classical problems ([DL, NH, Al, LT]) without referring to hysteresis operators. The advantage of the hysteresis approach consists however in the fact that additional geometrical considerations allow for solving also nonmonotone problems. Typical examples can be found in [K] and [BK].

We do not give an exhaustive list of related publications and historical references here; an interested reader may consult in particular the pioneering monographs [KP] and [V], or a recent survey paper [Bro].

1 Physical motivation

The equation of motion of a deformable body $\Omega \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}^N$ is the N-dimensional Euclidian space, is in classical continuum mechanics ([LL]) considered in the form

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} + g_i, \quad i = 1, \ldots, N, \tag{1.1}$$

where $x \in \Omega$, $t > 0$ are the space and time variables, respectively, $u = (u_i)$ is the displacement vector, $\rho > 0$ is the density, $\sigma = (\sigma_{ij})$ is the symmetric stress tensor and

$g = (g_i)$ is the volume force density, $i, j = 1, \ldots, N$. The meaningful choice in applications is usually $N = 3$. Equation (1.1) has to be coupled with initial and boundary conditions and with a *constitutive law* between the stress tensor $\sigma = (\sigma_{ij})$ and, for example, the linearized strain tensor $\varepsilon = (\varepsilon_{ij})$ defined as the symmetric derivative of u

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, N. \tag{1.2}$$

While (1.1) is a general physical law, the constitutive relation characterizes specific properties of a given material, subject to time-dependent loading.

In engineering applications, one has always been searching for a mathematically simple phenomenological description of the strain - stress constitutive behavior for a possibly large class of different types of material response including memory effects. Rheological models play a prominent role here and offer one of the main tools in the theory of inelastic constitutive laws (see e.g. [LC], [Al], [LT]). We recall here its main constituents.

1.1 Rheological elements

Let $\mathbb{T}$ be the space of symmetric tensors $\xi = (\xi_{ij})$, $i, j = 1, \ldots, N$, $N \in \mathbb{N}$, $\xi_{ij} = \xi_{ji}$, endowed with the scalar product $\xi : \eta := \sum_{i,j=1}^{N} \xi_{ij}\, \eta_{ij}$, and let δ be the well-known *Kronecker tensor*

$$\delta \in \mathbb{T}, \quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j\,, \\ 1 & \text{if } i = j\,. \end{cases}$$

We split the space $\mathbb{T}$ into the subspace $\mathbb{T}_\delta := \operatorname{span}\{\delta\}$ and its orthogonal complement (the so-called *deviatoric space*) $\mathbb{T}_{\text{dev}} := \mathbb{T}_\delta^{\perp}$. According to this decomposition, we denote by $\xi_I := \xi : \delta$ the first invariant (trace) of a symmetric tensor $\xi \in \mathbb{T}$ and by $\xi_{\text{dev}} := \xi - 1/N\, \xi_I\, \delta \in \mathbb{T}_{\text{dev}}$ the *deviator* of ξ.

The strain and stress tensors ε and σ, respectively, are in general functions of the space variable $x \in \Omega \subset \mathbb{R}^N$ and time variable $t \geq 0$ with values in $\mathbb{T}$. We consider here only *homogeneous media*, where the constitutive law is independent of the spatial variable x which thus plays the role of a parameter.

Definition 1.1 *A system consisting of*

$$\begin{array}{ll} \text{(i)} & a \ \text{ constitutive relation } \ \textit{between } \varepsilon \textit{ and } \sigma, \\ \text{(ii)} & a \ \text{ potential energy } \ U \geq 0 \end{array} \tag{1.3}$$

is called a rheological element.

A rheological element is said to be thermodynamically consistent, *if the quantity*

$$\dot{q} := \dot{\varepsilon} : \sigma - \dot{U} \tag{1.4}$$

called dissipation rate, *where the dot denotes the time derivative, is nonnegative in the sense of distributions for all ε, σ, U satisfying (1.3).*

Example 1.2 The elastic element $\mathcal{E}$.

Elastic materials are characterized by a *linear stress-strain relation* and by the *complete reversibility* of dynamical processes. In mathematical terminology, it is assumed that there exists a matrix $A = (A_{ijk\ell})$ over $\mathbb{T}$ such that

$$\sigma = A\,\varepsilon \quad \text{or equivalently} \quad \sigma_{ij} = \sum_{k,\ell=1}^{N} A_{ijk\ell}\,\varepsilon_{k\ell},\ i,j = 1,\ldots,N. \tag{1.5}$$

Reversibility means that the potential energy U involves no memory and can be chosen in such a way that the dissipation rate $\dot{q}$ vanishes, i.e. the value of $U(t)$ for each $t > 0$ depends only on the instantaneous value of $\varepsilon(t)$ and $\dot{U} = \dot{\varepsilon} : A\varepsilon$ almost everywhere for every absolutely continuous ε. This necessarily implies that the matrix A is symmetric with respect to the scalar product '$:$' and U has the form

$$U = \frac{1}{2} A\varepsilon : \varepsilon \tag{1.6}$$

up to an additive constant. Indeed, for an arbitrary $\varepsilon \in W^{1,1}(0,T;\mathbb{T})$ and $t \in]0,T[$ put $\tilde{\varepsilon}(\tau) := \varepsilon(0) + \tau/t\,(\varepsilon(t) - \varepsilon(0))$ for $\tau \in [0,t]$. We can choose the initial value for U arbitrarily, for instance $U(0) := 1/2\,A\varepsilon(0) : \varepsilon(0)$. We have by hypothesis

$$U(t) = U(0) + \int_0^t \dot{\tilde{\varepsilon}}(\tau) : A\tilde{\varepsilon}(\tau)\,d\tau = \frac{1}{2}\varepsilon(t) : A\varepsilon(t) + \frac{1}{2}\varepsilon(t) : (A - A^T)\varepsilon(0), \tag{1.7}$$

where $(A^T)_{ijk\ell} = A_{k\ell ij}$, hence

$$\dot{U}(t) = \dot{\varepsilon}(t) : A\varepsilon(t) + \frac{1}{2}\dot{\varepsilon}(t) : (A - A^T)(\varepsilon(0) - \varepsilon(t))$$

and we easily conclude that the matrix A is symmetric and (1.6) holds.

To guarantee that the stress-strain relation is one-to-one and the material law is deterministic, we assume that the matrix A is *positive definite*.

The elastic element is said to be *isotropic*, if the matrix A has the form

$$A = 2\mu\, I + N\,\lambda\, P_\delta, \tag{1.8}$$

where μ, λ are positive numbers called *Lamé's constants* (see [Ra]), I is the identity matrix $I\xi = \xi$ and P_δ is the orthogonal projection onto $\mathbb{T}_\delta$, that is $P_\delta \xi = 1/N\,\xi_I\,\delta$.

Example 1.3 The viscous element $\mathcal{E}$.

Modeling of *rate-dependent* relaxation effects makes often use of the concept of *viscosity* based on the hypothesis that there exist two coefficients $\eta > 0, \zeta > 0$ of proportionality between the deviators and first invariants of the strain rate and stress, that is

$$\sigma_{\text{dev}} = \eta\dot{\varepsilon}_{\text{dev}},\ \sigma_I = \zeta\dot{\varepsilon}_I. \tag{1.9}$$

The assumption that no reversible energy can be stored by the viscous element ($U = 0$) ensures its thermodynamical consistency.

Example 1.4 The rigid-plastic element $\mathcal{R}$.

The basic concept in plasticity is the *yield surface* in the stress space which can be described as the boundary ∂Z of a convex closed set $Z \subset \mathbb{T}$.

The rigid-plastic behavior consists of two different phases characterized by the instantaneous value σ of the stress tensor. The material remains rigid as long as $\sigma \in \mathrm{Int}\ Z$ (the *interior* of Z). In this case no deformation occurs and $\dot{\varepsilon} = 0$. The material becomes plastic if σ reaches the boundary ∂Z of Z. Plasticity is governed by three physical principles: the stress values remain confined to the set Z, no reversible energy is stored, and the dissipation rate is maximal with respect to all admissible stress values. Mathematically, this means

(1.10) $$\sigma \in Z\,,$$

(1.11) $$U = 0\,,$$

(1.12) $$\dot{\varepsilon} : (\sigma - \tilde{\sigma}) \geq 0 \qquad \forall \tilde{\sigma} \in Z\,,$$

Geometrically, $\dot{\varepsilon}$ points in the direction of the outward normal cone, and condition (1.12) is also called *the normality rule*. We see that the variational inequality (1.12) includes the rigid behavior (for $\sigma \in \mathrm{Int}\ Z$ it entails $\dot{\varepsilon} = 0$). In order to ensure the thermodynamical consistency, we assume $0 \in Z$. In fact, it is natural to assume that no deformation occurs for $\sigma = 0$. This is equivalent to the hypothesis $0 \in \mathrm{Int}\ Z$ which, as we show in the next sections, has a considerable impact on the regularity in the mathematical setting.

It has been observed that volume changes are negligible during plastic deformation ([Ra]). Combining constitutive relation (1.10) - (1.12) with the *volume invariance condition*

(1.13) $$\dot{\varepsilon}_I = 0\,,$$

we conclude from Proposition 2.9 below that Z has the form of a cylinder

(1.14) $$Z = Z_0 + \mathbb{T}_\delta\,,$$

where $Z_0 \subset \mathbb{T}_{\mathrm{dev}}$ is a convex closed set. In applications, it is often assumed that Z_0 is bounded. The classical models of Tresca and von Mises are special cases of (1.10)–(1.14) with (von Mises) $Z_0 = B_r(0) \cap \mathbb{T}_{\mathrm{dev}}$ (ball centered at 0 with radius r) or (Tresca) $Z_0 := \{\xi \in \mathbb{T}_{\mathrm{dev}}\,;\, \sum_{k=1}^N |\xi_k| \leq r\}$ for some $r > 0$, where $\{\xi_k\}$ are the eigenvalues of the symmetric matrix $\xi = (\xi_{ij})$. Note that we have $\sum_{k=1}^N \xi_k = 0$ for $\xi \in \mathbb{T}_{\mathrm{dev}}$. The Tresca set Z_0 is usually represented for $N = 3$ by a hexagon in the plane $\xi_1 + \xi_2 + \xi_3 = 0$.

Example 1.5 The rigid-plastic element with isotropic hardening $\mathcal{J}$.

In many materials, the yield surface does not remain fixed in time, but changes according to the loading history. This phenomenon is called hardening (softening). We first recall the concept of *isotropic hardening*, where the yield surface evolution is a simple dilation governed by a scalar hardening parameter α. Following [NH] we assume analogously as in Example 1.4 that a bounded convex closed set $Z_0 \subset \mathbb{T}_{\mathrm{dev}}$ is given such that $0 \in \mathrm{Int}\ Z_0$, and we denote by $M_0 : \mathbb{T}_{\mathrm{dev}} \to [0, \infty[$ the Minkowski functional associated to Z_0 by Definition 2.18 below. Let further a concave nondecreasing function $\varphi : [1, \infty[\to [1, \infty[$ be given, $\varphi(1) = 1$.

We denote by $\mathbb{T}_1$ the space $\mathbb{T}\times\mathbb{R}^1$ endowed with the natural scalar product $\langle(\xi,\beta),(\eta,\gamma)\rangle := \xi:\eta+\beta\gamma$ for $\xi,\eta\in\mathbb{T}$, $\beta,\gamma\in\mathbb{R}^1$, and by Z_1 the convex closed subset of $\mathbb{T}_1$

$$(1.15) \qquad Z_1 := \{(\xi,\alpha)\in\mathbb{T}_1\,;\,\alpha\geq 0,\ M_0(\xi_{\mathrm{dev}})\leq\varphi(1+\alpha)\}\,.$$

The constitutive relations are analogous to (1.10)-(1.12), namely

$$(1.16) \qquad (\sigma,\alpha)\in Z_1\,,$$

$$(1.17) \qquad U=0\,,\ \alpha(0)=0\,,$$

$$(1.18) \qquad \langle(\dot\varepsilon,-(1/c)\,\dot\alpha),(\sigma,\alpha)-(\tilde\sigma,\tilde\alpha)\rangle\geq 0 \quad \forall(\tilde\sigma,\tilde\alpha)\in Z_1\,,$$

where $c>0$ is a given physical constant.

We immediately observe that choosing $\tilde\sigma=\sigma$ in (1.18), we obtain $\dot\alpha(\alpha-\tilde\alpha)\leq 0$ for every $\tilde\alpha\geq\alpha$, hence $\dot\alpha\geq 0$.

Let $Z^\alpha := \{\xi\in\mathbb{T}\,;\,(\xi,\alpha)\in Z_1\}$ be the domain of admissible stresses for an instantaneous value α of the hardening parameter. We see that Z^α increases without changing its shape with increasing α.

1.2 Composition of rheological elements

A large variety of models for the material behavior can be obtained by composing rheological elements from Examples 1.2 - 1.5 in series or in parallel.

Let G_1, G_2 be two rheological elements and let $\varepsilon_i,\sigma_i,U_i$ be the strain, stress and potential energy, respectively, corresponding to the elements G_i, $i=1,2$.

The total strain ε, stress σ and potential energy U for the combination in parallel $G_1|G_2$ and in series G_1-G_2 are defined by the following relations

$G_1\vert G_2$	G_1-G_2
$\varepsilon=\varepsilon_1=\varepsilon_2$	$\varepsilon=\varepsilon_1+\varepsilon_2$
$\sigma=\sigma_1+\sigma_2$	$\sigma=\sigma_1=\sigma_2$
$U=U_1+U_2$	$U=U_1+U_2$

in analogy with the theory of electrical circuits. It is easy to see that every combination of thermodynamically consistent elements is thermodynamically consistent.

Example 1.6 Elastoplastic models $\mathcal{E}-\mathcal{R},\mathcal{E}/\mathcal{R}$.

There are good reasons for rewriting constitutive variational inequalities in plasticity in operator form. This enables us to distinguish clearly between input and output quantities: while the input can be controlled, the output is determined by solving the constitutive equation.

Let us compare the constitutive relations for two elastoplastic models $\mathcal{E}|\mathcal{R}$, $\mathcal{E}-\mathcal{R}$. We denote by ε^e,σ^e and ε^p,σ^p the strain and stress on the elastic and rigid-plastic element, respectively.

$$
\begin{array}{lcl}
\underline{\mathcal{E}|\mathcal{R}} & & \underline{\mathcal{E}-\mathcal{R}} \\
\varepsilon = \varepsilon^e = \varepsilon^p & & \varepsilon = \varepsilon^e + \varepsilon^p \\
\sigma = \sigma^e + \sigma^p & & \sigma = \sigma^e = \sigma^p \\
\sigma^e = A\varepsilon & & \sigma = A\varepsilon^e \\
\sigma^p \in Z & & \sigma \in Z \\
\dot\varepsilon : (\sigma^p - \tilde\sigma) \geq 0 & \forall \tilde\sigma \in Z & \dot\varepsilon^p : (\sigma - \tilde\sigma) \geq 0 \\
U = \frac{1}{2}\varepsilon : \sigma^e & & U = \frac{1}{2}\varepsilon^e : \sigma
\end{array}
$$

Recall that $Z \subset \mathbb{T}$ is a given convex closed set, $0 \in \operatorname{Int} Z$. We see that both models are governed by a variational inequality of the same type, namely

$$
\begin{array}{rl}
\mathcal{E}|\mathcal{R}: & (A^{-1}(\dot\sigma - \dot\sigma^p)) : (\sigma^p - \tilde\sigma) \geq 0 \\
\mathcal{E}-\mathcal{R}: & (A^{-1}(A\dot\varepsilon - \dot\sigma)) : (\sigma - \tilde\sigma) \geq 0
\end{array}
\qquad \forall \tilde\sigma \in Z. \tag{1.19}
$$

The solvability of such equations is ensured by the following theorem whose detailed proof (in a more general setting) will be given in Section 3. Definition and general information about the space $W^{1,1}(0,T;X)$ of absolutely continuous Hilbert space valued functions is given in Section 8.

Theorem 1.7 *Let X be a real separable Hilbert space endowed with a scalar product $\langle .,. \rangle$. Let $Z \subset X$ be a convex closed set, $0 \in Z$, and let $x^0 \in Z$ be a given element. Then for every function $u \in W^{1,1}(0,T;X)$ there exists a unique $x \in W^{1,1}(0,T;Z)$ satisfying the variational inequality*

$$
\langle \dot u(t) - \dot x(t), x(t) - \tilde x \rangle \geq 0 \quad a.e. \quad \forall \tilde x \in Z \tag{1.20}
$$

and the initial condition

$$
x(0) = x^0. \tag{1.21}
$$

We define the solution operators $\mathcal{S}, \mathcal{P}: Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ of the problem (1.20), (1.21) by the formula

$$
\mathcal{S}(x^0,u) := x, \quad \mathcal{P}(x^0,u) := u - \mathcal{S}(x^0,u). \tag{1.22}
$$

According to [KP], the operators $\mathcal{S}, \mathcal{P}$ are called *stop* and *play*, respectively. The set Z is called the *characteristic* of $\mathcal{S}$ and $\mathcal{P}$.

The constitutive relations for the elastoplastic models above can be written in the form

$$
\left\{
\begin{array}{lll}
\mathcal{E}/\mathcal{R}: & \varepsilon = A^{-1}\mathcal{P}(\sigma_0^p,\sigma), & U = \frac{1}{2}\left(A^{-1}\mathcal{P}(\sigma_0^p,\sigma)\right) : \mathcal{P}(\sigma_0^p,\sigma), \\
\mathcal{E}-\mathcal{R}: & \sigma = \mathcal{S}(\sigma^0, A\varepsilon), & U = \frac{1}{2}\left(A^{-1}\mathcal{S}(\sigma^0,A\varepsilon)\right) : \mathcal{S}(\sigma^0,A\varepsilon),
\end{array}
\right. \tag{1.23}
$$

where $\mathcal{S}, \mathcal{P}$ are the stop and play in $X = \mathbb{T}$ endowed with the scalar product $\langle \xi, \eta \rangle := (A^{-1}\xi) : \eta$, and σ_0^p, σ^0 are given initial output values.

It is clear that the roles of input and output in the models $\mathcal{E}|\mathcal{R}$ and $\mathcal{E}-\mathcal{R}$ cannot be reversed.

The definition immediately suggests that the stop has the

Semigroup property : For $u \in W^{1,1}(0,T;X)$, $s \in]0,T[$ and $t \in [0,T-s]$ put $u_s(t) := u(s+t)$. Then for every $x^0 \in Z$ we have

$$\mathcal{S}(x^0,u)(t+s) = \mathcal{S}(\mathcal{S}(x^0,u)(s),u_s)(t). \tag{1.24}$$

An operator F acting in some space $R(0,T\,;X)$ of functions $[0,T] \to X$ is called

Rate-independent , if for every $u \in R(0,T\,;X)$ and every nondecreasing mapping α of $[0,T]$ onto $[0,T]$ such that $u_\alpha(t) := u(\alpha(t))$ belongs to $R(0,T\,;X)$ we have

$$F(u_\alpha)(t) = F(u)(\alpha(t)) \quad \text{for all } t \in [0,T]\,, \tag{1.25}$$

Causal , if the implication

$$u(t) = v(t) \quad \forall\, t \in [0,t_0] \;\Rightarrow\; F(u)(t_0) = F(v)(t_0)\,. \tag{1.26}$$

Rate-independence and causality characterize *hysteresis operators* according to the classification of [V]. By definition, the stop and play are hysteresis operators in $W^{1,1}(0,T\,;X)$ (we will see in the next section that they can be extended to the space of continuous functions $C([0,T];X)$). Indeed, the concept of 'hysteresis branching' or 'hysteresis loops' is meaningful only in the scalar case $\dim X = 1$. However, the play operator turns out to be the main building block for a very large family of scalar hysteresis models used in elastoplasticity (Prandtl-Ishlinskii model), ferromagnetism (Preisach and Della Torre models), fatigue analysis (the 'rainflow' method) and many others. A more complete information can be found in [BS] and [K]. Recent applications to thermoplasticity ([KS1, KS2]) and phase transitions ([KS3, KS4]) also confirm its universal character.

We remain here within the multidimensional framework and give some examples of application of hysteresis operators for modeling the kinematic and isotropic hardening in elastoplastic materials.

1.3 Linear kinematic hardening

Let us consider the so-called Prager model $\mathcal{E}-(\mathcal{E}|\mathcal{R})$. The general rheological rules yield

$$\begin{aligned}
\sigma &= \sigma^e + \sigma^p\,,\\
\varepsilon &= \varepsilon^e + \varepsilon^p\,,\\
\sigma^e &= A\varepsilon^p\,,\\
\sigma &= B\varepsilon^e\,,\\
\sigma^p &\in Z\,,\\
\dot\varepsilon^p : (\sigma^p - \tilde\sigma) &\geq 0 \quad \forall \tilde\sigma \in Z\,,\\
U &= \frac{1}{2}\left(\varepsilon^e : \sigma + \varepsilon^p : \sigma^e\right)
\end{aligned}$$

where A, B are given constant symmetric positive definite matrices and $Z \subset \mathbb{T}$ is a convex closed set, $0 \in \text{Int}\, Z$. For $t \in [0,T]$ put

$$Z(t) := Z + \sigma^e(t). \tag{1.27}$$

Then $\sigma(t) \in Z(t)$ for all $t \in [0,T]$. Relation (1.27) can be interpreted as a translation of Z in the stress space $\mathbb{T}$ driven by the elastic component σ^e of the stress without changing shape and size. This phenomenon is called *kinematic hardening* and is typical for metals, see [LC]. The word 'linear' is related to the linear dependence between σ^e and ε^p.

The evolution of σ^e is governed by variational inequality of the form (1.20), namely

$$\left(A^{-1}\dot{\sigma}^e\right) : (\sigma^p - \tilde{\sigma}) \geq 0, \quad \forall \tilde{\sigma} \in Z. \tag{1.28}$$

Inequality (1.28) can be interpreted as a normality condition for the hardening rate $\dot{\sigma}^e$ with respect to the scalar product $\left\langle \cdot\,, \cdot \right\rangle_A := (A^{-1} \cdot : \cdot)$; both the hardening rate $\dot{\sigma}^e$ and the plastic strain rate $\dot{\varepsilon}^p$ have the outward normal direction to ∂Z at the point σ, but *with respect to different scalar products*.

With the intention to deal with several scalar products in $\mathbb{T}$ we introduce the subscript A for the play $\mathcal{P}_A$ and stop $\mathcal{S}_A$ corresponding to the scalar product $\left\langle \cdot\,, \cdot \right\rangle_A$.

Using (1.23) we can express the constitutive law for the model $\mathcal{E} - (\mathcal{E} \,|\, \mathcal{R})$ in the form

$$\varepsilon = B^{-1}\sigma + A^{-1}\mathcal{P}_A(\sigma_0^p, \sigma) \tag{1.29}$$

with input σ and output ε. We now prove that the constitutive operator $B^{-1} + A^{-1}\mathcal{P}_A$ is invertible. Identity (1.31) below gives an equivalent expression for (1.29) with input ε and output σ.

Lemma 1.8 *Let $\sigma_0^p \in Z$ be given and let A, C be given constant matrices such that A, CA are symmetric and positive definite. Put $\hat{A} := A + CA$. Then for all $\sigma \in W^{1,1}(0,T\,; \mathbb{T})$ we have*

$$\mathcal{S}_{\hat{A}}(\sigma_0^p, \sigma + C\mathcal{P}_A(\sigma_0^p, \sigma)) = \mathcal{S}_A(\sigma_0^p, \sigma)\,. \tag{1.30}$$

Proof. Put $x := \mathcal{S}_A(\sigma_0^p, \sigma)$, $y := \mathcal{S}_{\hat{A}}(\sigma_0^p, \sigma + C\mathcal{P}_A(\sigma_0^p, \sigma))$. Then $y = \mathcal{S}_{\hat{A}}(\sigma_0^p,\, (I+C)\sigma - Cx)$, where I is the identity matrix. Putting $\tilde{\sigma} := (x+y)/2$ in the variational inequalities

$$\begin{aligned} A^{-1}(\dot{\sigma} - \dot{x}) : (x - \tilde{\sigma}) &\geq 0\,, \\ \hat{A}^{-1}((I+C)\dot{\sigma} - C\dot{x} - \dot{y}) : (y - \tilde{\sigma}) &\geq 0\,, \end{aligned}$$

and using the identity $\hat{A}^{-1} + \hat{A}^{-1}C = A^{-1}$, we conclude $\left\langle \dot{x} - \dot{y}, x - y \right\rangle_{\hat{A}} \leq 0$, hence $x = y$.

❒

We now apply Lemma 1.8 with $C = BA^{-1}$ to the constitutive equation (1.29). We obtain

$$\mathcal{S}_{\hat{A}}(\sigma_0^p, B\varepsilon) = \mathcal{S}_A(\sigma_0^p, \sigma) \quad \text{for} \quad \hat{A} = A + B,$$

hence $(I + BA^{-1})\sigma = B\varepsilon + BA^{-1}\mathcal{S}_{\hat{A}}(\sigma_0^p, B\varepsilon)$, or equivalently

$$\sigma = (A^{-1} + B^{-1})^{-1}\varepsilon + B\hat{A}^{-1}\mathcal{S}_{\hat{A}}(\sigma_0^p, B\varepsilon) = B\varepsilon - B\hat{A}^{-1}\mathcal{P}_{\hat{A}}(\sigma_0^p, B\varepsilon)\,, \tag{1.31}$$

where ε is the input and σ is the output.

In the particular case $B = I, A = \frac{1}{\gamma} I$ for some $\gamma > 0$, we obtain $\mathcal{P}_A = \mathcal{P}_{\hat{A}} = \mathcal{P}_I$ and the inversion formula

$$(1.32) \qquad (\mathcal{I} + \gamma \mathcal{P}_I(x^0, .))^{-1} = \mathcal{I} - \frac{\gamma}{1+\gamma} \mathcal{P}_I(x^0, .)$$

holds for all $x^0 \in Z$, where $\mathcal{I}$ is the identity mapping in $W^{1,1}(0,T\,; X)$.

Exercise 1.9 Assume that the matrices A, B commute, i.e. $AB = BA$. Prove that (1.31) is the constitutive equation of the model $\mathcal{E}\,|(\mathcal{E} - \mathcal{R})$ with

$$\begin{aligned} \sigma^e &= \tilde{A}\varepsilon\,, & \tilde{A} &= (A^{-1} + B^{-1})^{-1}\,, \\ \sigma^p &= \tilde{B}\varepsilon^e\,, & \tilde{B} &= B^2(A+B)^{-1}\,, \\ \dot{\varepsilon}^p : (\sigma^p - \tilde{\sigma}) &\geq 0 & \forall \tilde{\sigma} &\in \tilde{Z}\,,\ \tilde{Z} = B(A+B)^{-1}(Z)\,. \end{aligned}$$

Hint. Use the identity $C\mathcal{S}_A(x^0, u) = \tilde{\mathcal{S}}_{CAC}(Cx^0, Cu)$ for each positive definite symmetric matrix C, where $\tilde{\mathcal{S}}$ is the stop with characteristic $\tilde{Z} = C(Z)$.

The commutativity hypothesis $AB = BA$ is satisfied for instance if both elastic elements are isotropic. In this case the models $\mathcal{E} - (\mathcal{E}\,|\,\mathcal{R})$ and $\mathcal{E}\,|(\mathcal{E} - \mathcal{R})$ are equivalent.

1.4 Isotropic and kinematic hardening

Let us consider now the model $\mathcal{E} - (\mathcal{E}\,|\,\mathcal{J})$. With the notation taken from Example 1.5, the constitutive relations are analogous to the model $\mathcal{E} - (\mathcal{E}\,|\,\mathcal{R})$, namely

$$\begin{aligned} &(1.33) \qquad & (\sigma, 0) &= (\sigma^e, -\alpha) + (\sigma^p, \alpha)\,, & (\sigma^p, \alpha) &\in Z_1\,, \\ &(1.34) \qquad & (\varepsilon, -(1/c)\alpha) &= (\varepsilon^e, 0) + (\varepsilon^p, -(1/c)\alpha)\,, & \sigma &= B\varepsilon^e, \sigma^e = A\varepsilon^p\,, \\ &(1.35) \qquad & \langle(\dot{\varepsilon}^p, -(1/c)\dot{\alpha}), (\sigma^p, \alpha) - (\tilde{\sigma}, \tilde{\alpha})\rangle &\geq 0 & \forall (\tilde{\sigma}, \tilde{\alpha}) &\in Z_1\,, \end{aligned}$$

where A, B are symmetric positive definite matrices.

Let $A_1, B_1 : \mathbb{T}_1 \to \mathbb{T}_1$ be the linear mappings defined by the identities $A_1(\xi, \alpha) := (A\xi, c\,\alpha)$, $B_1(\xi, \alpha) := (B\xi, c\,\alpha)$. We have

$$\left\langle A_1^{-1}\left((\dot{\sigma}, 0) - (\dot{\sigma}^p, \dot{\alpha})\right), (\sigma^p, \alpha) - (\tilde{\sigma}, \tilde{\alpha})\right\rangle \geq 0 \quad \forall (\tilde{\sigma}, \tilde{\alpha}) \in Z_1\,,$$

hence $(\sigma^p, \alpha) = \mathcal{S}_1\left((\sigma^p_0, 0), (\sigma, 0)\right)$, $(\sigma^e, -\alpha) = \mathcal{P}_1\left((\sigma^p_0, 0), (\sigma, 0)\right)$, where $\mathcal{S}_1, \mathcal{P}_1$ are the stop and play in $\mathbb{T}_1$ endowed with the scalar product $\left\langle A_1^{-1}\cdot, \cdot\right\rangle$, with characteristic Z_1 and with a given initial condition $(\sigma^p_0, 0) \in Z_1$. The constitutive equation has the form

$$(1.36) \qquad (\varepsilon, -(1/c)\,\alpha) = B_1^{-1}(\sigma, 0) + A_1^{-1}\mathcal{P}_1\left((\sigma^p_0, 0), (\sigma, 0)\right)\,.$$

We derive now some consequences of the constitutive equation. For a function $f : [0,T] \to \mathbb{R}^1$ and $t \in [0,T]$ we denote $\|f\|_{[0,t]} := \sup\{|f(\tau)|\,;\, \tau \in [0,t]\}$.

Lemma 1.10 *Let $\sigma \in W^{1,1}(0,T;\mathbb{T})$ be given and assume $\sigma(0) = \sigma_0^p = 0$. Let ε, α be given by the equation (1.36). Then we have*

$$\varphi(1+\alpha(t)) = \max\left\{1, \|M_0(\sigma^p_{\text{dev}})\|_{[0,t]}\right\}, \tag{1.37}$$

where φ, M_0 are as in (1.15) and σ^p_{dev} is the deviator of the plastic stress σ^p.

Proof. We have $(\sigma^p(t), \alpha(t)) \in Z_1$ for all $t \in [0,T]$, hence $M_0(\sigma^p_{\text{dev}}(t)) \le \varphi(1+\alpha(t))$ by definition. The fact that α is nondecreasing (cf. (1.18)) entails $\|M_0(\sigma^p_{\text{dev}})\|_{[0,t]} \le \varphi(1+\alpha(t))$. In the case $\|M_0(\sigma^p_{\text{dev}})\|_{[0,t]} < 1$ we obviously have $\alpha(t) = 0$ and (1.37) holds. Let us assume now $1 \le \|M_0(\sigma^p_{\text{dev}})\|_{[0,t]} < \varphi(1+\alpha(t))$ for some $t \in]0,T[$. Then there exists $\tau \in]0,t[$ such that $\dot\alpha(\tau) > 0$ and $\|M_0(\sigma^p_{\text{dev}})\|_{[0,\tau]} < \varphi(1+\alpha(\tau))$, hence $(\sigma^p(\tau), \alpha(\tau)) \in \text{Int } Z_1$. From (1.35) we conclude $\dot\alpha(\tau) = 0$, which is a contradiction. ❒

According to general rheological principles, we associate to the model $\mathcal{E} - (\mathcal{E} \mid \mathcal{J})$ the potential energy $U = \left(\langle \varepsilon^e, \sigma\rangle + \langle \varepsilon^p, \sigma^e\rangle\right)/2$. The dissipated energy $q(t)$ is then equal to the plastic work $\int_0^t \dot\varepsilon^p(\tau) : \sigma^p(\tau)\, d\tau$ and is related to $\alpha(t)$ by the following identity.

Proposition 1.11 *Let the assumptions of Lemma 1.10 hold. Put $r := \inf\{\beta > 0; \varphi'(1+\beta) = 0\} \in [0,\infty]$. For $p \in [0,r]$ put*

$$\Phi(p) := \int_0^p \frac{\varphi(1+\beta)}{c\varphi'(1+\beta)}\, d\beta\,.$$

Then we have $\alpha(t) \in [0,r]$ for all $t \in [0,T]$ and

$$q(t) = \Phi(\alpha(t)) \quad \textit{provided} \quad \alpha(t) \in [0,r[. \tag{1.38}$$

Proof. Assume $\alpha(t) > r$ for some $t \in]0,T[$. Then there exists $\tau < t$ such that $\dot\alpha(\tau) > 0$ and $\alpha(\tau) > r$. Putting $\tilde\sigma := \sigma^p(\tau), \tilde\alpha = r$ we have $\varphi(1+\tilde\alpha) = \varphi(1+\alpha(\tau))$, hence $(\tilde\sigma, \tilde\alpha) \in Z_1$ and (1.35) yields $\dot\alpha(\tau) \le 0$, which is a contradiction.

Identity (1.38) can be equivalently written in the form

$$\dot q(t) = \dot\alpha(t) \frac{\varphi(1+\alpha(t))}{c\varphi'(1+\alpha(t))} \quad \text{a.e. provided } \alpha(t) < r. \tag{1.39}$$

To prove (1.39) we distinguish two cases.

a) $\dot\alpha(t) = 0$. We choose $a > 0$ sufficiently small and $b > 0$ sufficiently large such that $\tilde\sigma := (1+a)\sigma^p(t)$ and $\tilde\alpha := \alpha(t) + b$ satisfy $M_0(\tilde\sigma_{\text{dev}}) - \varphi(1+\tilde\alpha) = (1+a)\big(M_0(\sigma^p_{\text{dev}}(t)) - \varphi(1+\alpha(t))\big) + (1+a)\varphi(1+\alpha(t)) - \varphi(1+\alpha(t)+b) \le 0$, that is, $(\tilde\sigma, \tilde\alpha) \in Z_1$. From inequality (1.35) we infer $a\,\dot\varepsilon^p : \sigma^p \le 0$, hence $\dot q(t) = 0$.

b) $\dot\alpha(t) > 0$. We will see in Section 3 that the play depends continuously on the characteristic with respect to the Hausdorff distance. It therefore suffices to assume that φ and M_0 are smooth functions. We have $(\sigma^p(t), \alpha(t)) \in \partial Z_1$ and according to

(1.35), the vector $(\dot\varepsilon^p(t), -(1/c)\dot\alpha(t))$ points in the direction of the outward normal vector to Z_1. In other words, we have

$$(\dot\varepsilon^p(t), -(1/c)\dot\alpha(t)) = \frac{\dot\alpha(t)}{c\varphi'(1+\alpha(t))}\left(\partial M_0(\sigma^p_{\mathrm{dev}}(t)), -\varphi'(1+\alpha(t))\right),$$

where ∂M_0 is the gradient of M_0. This yields

$$\dot q(t) = \dot\varepsilon^p(t) : \sigma^p(t) = \dot\varepsilon^p(t) : \sigma^p_{\mathrm{dev}}(t) = \frac{\dot\alpha(t)}{c\varphi'(1+\alpha(t))}\,\partial M_0(\sigma^p_{\mathrm{dev}}(t)) : \sigma^p_{\mathrm{dev}}(t).$$

We have $\partial M_0(\sigma^p_{\mathrm{dev}}) : \sigma^p_{\mathrm{dev}} = M_0(\sigma^p_{\mathrm{dev}})$ by Lemma 2.21 and $M_0(\sigma^p_{\mathrm{dev}}(t)) = \varphi(1+\alpha(t))$ by hypothesis, hence identity (1.39) holds.

❐

As a consequence of Proposition 1.11, we see that the isotropic hardening can be equivalently characterized by the plastic work (or dissipation) q. For this reason it is sometimes referred to as *work hardening*, see [NH], [LC].

1.5 Nonlinear kinematic hardening

In order to account for the phenomenon of *ratchetting* which is manifested by the accumulation of the plastic strain under cyclic stress loading, Armstrong and Frederick proposed in [AF] a modification of the Prager model from Example 1.3, replacing the linear relation between σ^e and ε^p by a nonlinear one, namely

$$\dot\sigma^e = \gamma\left(R\,\dot\varepsilon^p - \sigma^e\,|\dot\varepsilon^p|\right) \tag{1.40}$$

with given positive constants γ, R. It is assumed that the convex closed set Z is the von Mises cylinder of radius $r > 0$,

$$Z = (B_r(0) \cap \mathbb{T}_{\mathrm{dev}}) + \mathbb{T}_\delta\,. \tag{1.41}$$

The normality rule here implies $\dot\varepsilon^p = \sigma^p_{\mathrm{dev}}\,|\dot\varepsilon^p|/r$, hence (1.40) is equivalent to

$$\dot\sigma^e = \gamma\left((R+r)\,\dot\varepsilon^p - \sigma_{\mathrm{dev}}\,|\dot\varepsilon^p|\right)\,. \tag{1.42}$$

Introducing an auxiliary function u by the formula

$$u := \gamma\,(R+r)\,\varepsilon^p + \sigma^p_{\mathrm{dev}}\,, \tag{1.43}$$

we see that the variational inequality

$$\dot\varepsilon^p : (\sigma^p - \tilde\sigma) \ge 0 \qquad \forall \tilde\sigma \in Z \tag{1.44}$$

can be rewritten as

$$(\dot u - \dot\sigma^p_{\mathrm{dev}}) : (\sigma^p_{\mathrm{dev}} - \tilde\sigma) \ge 0 \qquad \forall \tilde\sigma \in B_r(0) \cap \mathbb{T}_{\mathrm{dev}}\,, \tag{1.45}$$

hence $\sigma^p_{\text{dev}} = \mathcal{S}(\sigma^p_{0\,\text{dev}}, u)$, $\varepsilon^p = (1/(\gamma(R+r)))\,\mathcal{P}(\sigma^p_{0\,\text{dev}}, u)$ according to the above notation for a given initial plastic stress σ^p_0. The function u is to be determined as the solution of the operator - differential equation

$$\dot{u} = \dot{\sigma}_{\text{dev}} + \frac{\sigma_{\text{dev}}}{R+r}\left|\frac{d}{dt}\,\mathcal{P}(\sigma^p_{0\,\text{dev}}, u)\right| \tag{1.46}$$

for each given stress input σ and with an appropriate initial condition. The constitutive operator of the stress-controlled Armstrong-Frederick model thus contains a superposition of the play operator to the solution operator $\sigma \mapsto u$ of the equation (1.46). It is shown in [BK] that the model is well posed. The strain-controlled case leads to similar considerations. We do not give the details here; let us just point out that the solvability of equation (1.46) is closely related to the local Lipschitz estimate (5.6) for the play operator in $W^{1,1}(0,T\,;\mathbb{T}_{\text{dev}})$.

2 Convex sets

The aim of this section is to recall some basic elements of convex analysis in Hilbert spaces. Most of the results are well-known. We present them in order to fix the notation and to keep the presentation consistent (for more information we refer the reader to the monographs [Ro] and [AE]). The complementary function of a convex set (Definition 2.4 below) has been introduced in [K].

Throughout the section, X denotes a real separable Hilbert space endowed with a scalar product $\langle\cdot,\cdot\rangle$ and norm $|x| := \langle x, x\rangle^{1/2}$. By Z we denote a convex closed subset of X such that $0 \in Z$. We fix the number

$$m := \operatorname{dist}(0, \partial Z) := \inf\{|z|\,;\, z \in \partial Z\} \geq 0. \tag{2.1}$$

It is clear that $m > 0$ if and only if $0 \in \operatorname{Int} Z$.

We start with a simple lemma.

Lemma 2.1 *For each $x \in X$ there exists a unique $z \in Z$ such that* $|x-z| = \operatorname{dist}(x, Z) = \min\{|x-y|\,;\, y \in Z\}$.

Proof. Let $x \in X$ be given. Put $p := \inf\{|x-y|\,;\, y \in Z\}$ and let $\{y_n\}$ be a sequence in Z such that $|x - y_n| \to p$. From the identity

$$|u-v|^2 + |u+v|^2 = 2(|u|^2 + |v|^2) \tag{2.2}$$

for $u = x - y_n$, $v = x - y_k$, it follows

$$\frac{1}{2}|y_n - y_k|^2 = |x-y_n|^2 + |x-y_k|^2 - 2\left|x - \frac{y_n + y_k}{2}\right|^2 \leq |x-y_n|^2 + |x-y_k|^2 - 2p^2,$$

hence $\{y_n\}$ is a convergent sequence and it suffices to put $z := \lim_{n\to\infty} y_n$. Uniqueness is obtained analogously. ❐

Using Lemma 2.1 we can define the projection $Q : X \to Z$ onto Z and its complement $P := I - Q$ (I is the identity) by the formulae

$$Qx \in Z\,, \ |Px| = \mathrm{dist}\,(x, Z) \quad \text{for } x \in X\,. \tag{2.3}$$

In the sequel, we call (P, Q) the *projection pair* associated to Z. We make extensive use of the following lemma.

Lemma 2.2 *For every $x, y \in X$ we have*

(i) $\langle Px, Qx - z\rangle \geq 0 \quad \forall z \in Z$,

(ii) $\langle Px - Py, Qx - Qy\rangle \geq 0$,

(iii) $\langle Px, x\rangle \geq m|Px| + |Px|^2$ *with m given by (2.1),*

(iv) $Q(x + \alpha Px) = Qx \quad \forall \alpha \geq -1$.

Proof. (i) For $z \in Z$, $z \neq Qx$ and $\gamma \in]0, 1[$ we have $|x - \gamma z - (1-\gamma)Qx|^2 > |Px|^2$, hence $2\langle Px, Qx - z\rangle + \gamma|Qx - z|^2 > 0$ and the assertion follows easily. Statement (ii) is an obvious consequence of (i). We obtain (iii) from (i) by putting $z := m\,Px/|Px|$ if $x \notin Z$, the case $x \in Z$ is trivial. To prove (iv) we notice that for all $z \in Z$ we have $|x + \alpha Px - z|^2 = |Qx - z|^2 + (1+\alpha)^2|Px|^2 + 2(1+\alpha)\langle Px, Qx - z\rangle$, hence the minimum of $|x + \alpha Px - z|$ is attained for $z = Qx$. ❐

2.1 Recession cone

Definition 2.3 *A nonempty closed convex set $C \subset X$ is called* a cone, *if the implication $x \in C \Rightarrow \alpha x \in C$ holds for all $x \in X$ and $\alpha \geq 0$.*

Definition 2.4 *Let $Z \subset X$ be a convex closed set, $0 \in Z$. The set*

$$C_Z := \{x \in Z\,;\, \alpha x \in Z \quad \forall \alpha \geq 0\} \tag{2.4}$$

is called the recession cone *of Z and the function $K_Z : [0, \infty[\to [0, \infty[$ defined by the formula*

$$K_Z(r) := \sup\,\{\mathrm{dist}\,(x, C_Z)\,;\, x \in Z \cap B_r(0)\} \quad \textit{for } r \geq 0 \tag{2.5}$$

is called the complementary function of Z, *where*

$$B_r(x_0) := \{x \in X\,;\, |x - x_0| \leq r\} \tag{2.6}$$

denotes the ball centered at x_0 with radius r.

The following properties of the complementary function are proved in [K].

Proposition 2.5 *Let $Z \subset X$ be a convex closed set with $0 \in \operatorname{Int} Z$ and with the recession cone C_Z and complementary function K_Z. Then*

(i) $x + y \in Z \quad \forall x \in C_Z,\ \forall y \in B_m(0)$, *where m is given by (2.1),*

(ii) *K_Z is nondecreasing in $[0,\infty[$, $1 \geq K_Z(s)/s \geq K_Z(r)/r$ for $0 < s < r$,*

(iii) *if $\dim X < \infty$, then*

$$\lim_{r\to\infty} \frac{K_Z(r)}{r} = 0. \tag{2.7}$$

Let us note that by Proposition 2.5 (ii) we have $K_Z(r) - K_Z(s) \leq (r-s)\, K_Z(r)/r$ for $r > s > 0$, hence K_Z is Lipschitz.

Property (2.7) is crucial for the extension of hysteresis operators to the space of continuous functions. We therefore introduce the following terminology.

Definition 2.6 *A convex closed set $Z \subset X$ is called a* recession set *if $0 \in \operatorname{Int} Z$ and the complementary function K_Z satisfies (2.7).*

Indeed, every convex closed set $Z \subset X$ with $0 \in \operatorname{Int} Z$ is a recession set if $\dim X < \infty$. This is not true for infinitely dimensional spaces, where there exist unbounded sets Z with $C_Z = \{0\}$, but condition (2.7) holds for instance for all sets of the form $Z = C + Z_B$, where C is a cone and Z_B is bounded, $0 \in \operatorname{Int} Z_B$.

2.2 Tangent and normal cones

A natural generalization of normal vectors and tangent hyperplanes which in general are not uniquely determined, is the concept of *normal cone* $N_Z(x)$ and *tangent cone* $T_Z(x)$ to a convex closed set $Z \subset X$ at a point $x \in Z$. They are defined by the formula

$$\begin{cases} N_Z(x) := \{y \in X;\ \langle y, x - z\rangle \geq 0 \quad \forall z \in Z\}, \\ T_Z(x) := \{w \in X;\ \langle w, y\rangle \leq 0 \quad \forall y \in N_Z(x)\}. \end{cases} \tag{2.8}$$

Every element $u \in X$ admits a unique orthogonal decomposition into the sum $u = v + w$ of the normal component $v \in N_Z(x)$ and the tangential component $w \in T_Z(x)$, namely $v = Q_N(u)$, $w = P_N(u)$, where (P_N, Q_N) is the projection pair associated to $N_Z(x)$. Indeed, by Lemma 2.2 (i) we have $\langle w, (1-\alpha)v\rangle \geq 0$ for all $\alpha \geq 0$, hence $\langle w, v\rangle = 0$ and $\langle w, y\rangle \leq 0$ for every $y \in N_Z(x)$. Uniqueness is easy: assume $v_1 + w_1 = v_2 + w_2$ for some $v_i \in N_Z(x)$, $w_i \in T_Z(x)$, $\langle w_i, v_i\rangle = 0$, $i = 1,2$. Then $0 \leq \langle w_1 - w_2, v_1 - v_2\rangle \leq -|w_1 - w_2|^2$, hence $w_1 = w_2$, $v_1 = v_2$.

For $x \in \operatorname{Int} Z$ we obviously have $N_Z(x) = \{0\}$, $T_Z(x) = X$. One might expect that for $x \in \partial Z$ the normal cone should contain nonzero elements. The example $Z := \{x \in X; |\langle x, e_k\rangle| \leq 1/k\ \forall k \in \mathbb{N}\}$, where $\{e_k\}$ is an orthonormal basis, shows that this conjecture is false, since $0 \in \partial Z$ and $N_Z(0) = \{0\}$. In regular cases this cannot happen.

Proposition 2.7 *Assume* Int $Z \neq \emptyset$. *Then for every* $x \in \partial Z$ *we have* $N_Z(x) \setminus \{0\} \neq \emptyset$.

Proof. Let $\{z_n; n \in \mathbb{N}\}$ be a sequence in $X \setminus Z$ such that $\lim_{n\to\infty} |z_n - x| = 0$. Put $\varepsilon_n := |Pz_n| > 0$, $y_n := z_n + 1/\varepsilon_n \, Pz_n$. We have $\varepsilon_n \leq |z_n - x|$ and Lemma 2.2 (iv) yields $Qy_n = Qz_n$, $Py_n = (1 + 1/\varepsilon_n)\, Pz_n$. By Lemma 2.2 (i) we further have $|Qy_n - x|^2 = |Qz_n - x|^2 = |z_n - x|^2 - |Pz_n|^2 - 2\langle Pz_n, Qz_n - x\rangle \leq |z_n - x|^2$ and

$$\langle Py_n, Qy_n - z\rangle \geq 0 \quad \forall z \in Z, \quad \forall n \in \mathbb{N}. \tag{2.9}$$

Passing to subsequences we can assume that $\{Py_n\}$ converges weakly to an element ξ which belongs to $N_Z(x)$ by (2.9). It remains to verify that $\xi \neq 0$. We fix an arbitrary ball $B_\delta(x_0) \subset \text{Int}\, Z$. Putting $z := x_0 + \delta/(1+\varepsilon_n)\, Py_n$ in (2.9) we obtain $\delta \leq \langle \xi, x - x_0\rangle$, hence $\xi \neq 0$. ❐

Let us mention the important particular case of *cylinders* in X.

Definition 2.8 *Let* $Y \subset X$ *be a closed subspace of* X, *let* $Y^\perp$ *be its orthogonal complement and let* $\tilde{Z} \subset Y$ *be a convex closed set. Then the set* $Z := \tilde{Z} + Y^\perp$ *is called a* convex cylinder *in* X.

Proposition 2.9 *A convex closed set* $Z \subset X$ *is a convex cylinder of the form* $Z = \tilde{Z} + Y^\perp$ *with* $\tilde{Z} \subset Y$ *if and only if* $N_Z(x) \subset Y$ *for all* $x \in Z$.

Proof. The 'only if' part is trivial. To prove the converse we put $\tilde{Z} := Z \cap Y$ and choose arbitrarily $u \in \tilde{Z}$ and $w \in Y^\perp$. From Lemma 2.2 (i) we infer $\langle P(u+w), Q(u+w) - u\rangle \geq 0$, hence $|P(u+w)|^2 \leq \langle P(u+w), w\rangle$. On the other hand, we have $P(u+w) \in N_Z(Q(u+w)) \subset Y$, and we conclude $\langle P(u+w), w\rangle = |P(u+w)|^2 = 0$. Consequently, $\tilde{Z} + Y^\perp \subset Z$ and equality follows from the convexity of Z. ❐

Remark 2.10 Cylinders of the form $Z = \tilde{Z} + Y^\perp$ with $\tilde{Z} \subset Y$ are characterized by the condition $Px \in Y$ for all $x \in X$. Denoting by $(\tilde{P}, \tilde{Q})$ the projection pair associated to $\tilde{Z}$ in Y, we obtain for every $x \in X$ of the form $x = u + w$, $u \in Y$, $w \in Y^\perp$ the identities $Px = \tilde{P}u$, $Qx = \tilde{Q}u + w$.

2.3 Strict convexity

In general, the boundary ∂Z of a convex closed set $Z \subset X$ can contain straight segments. We recall two criteria for their existence. It is easy to verify that ∂Z contains a segment of length $r > 0$ if one of the following conditions is satisfied.

Internal criterion There exist $x, y \in \partial Z$ such that $|x - y| = r$, $(x+y)/2 \in \partial Z$.

External criterion There exists a point $z \in \partial Z$ and a sequence $\{w_n; n \in \mathbb{N}\}$ in $X \setminus T_Z(z)$ such that $|w_n| = 1$, $\lim_{n\to\infty} w_n = w$, $z + rw \in \partial Z$.

The terminology is justified by the fact that we always have $(x+y)/2 \in Z$ for $x, y \in Z$ and $z + rw \notin Z$ for $z \in \partial Z$, $w \in X \setminus T_Z(z)$ and $r > 0$.

According to these criteria we introduce the functions $\alpha, \delta : [0,\infty[\to [0,\infty[$ by the formulae

$$\tag{2.10} \begin{cases} \delta(r) := \inf\left\{\text{dist}\,\left(\frac{1}{2}(x+y), \partial Z\right);\ x, y \in Z,\ |x-y| = 2r\right\}, \\ \alpha(r) := \inf\left\{|P(z+rw)|;\ z \in \partial Z,\ w \in X \setminus T_Z(z),\ |w| = 1\right\}, \end{cases}$$

where P is defined by (2.3). We naturally have $\delta(r) = +\infty$ if $2r > \text{diam}\, Z := \sup\{|x-y|;\ x, y \in Z\}$ (the *diameter* of Z) and $\delta(r) \le |(x+y)/2 - x| = r$ for $0 \le r < 1/2\ \text{diam}\, Z$. Choosing an arbitrary $x \in X \setminus Z$ we obtain $\alpha(r) \le \big|P(Qx + r\, Px/|Px|)\big| = r$ by Lemma 2.2.

The case $\dim X = 1$ is trivial (then $\delta(r) = r$ for $r < 1/2\ \text{diam}\, Z$, $\alpha(r) = r$ for all $r \ge 0$), as well as the case $\text{Int}\, Z = \emptyset$ (then $\delta(r) = \alpha(r) = 0$ for $r < 1/2\ \text{diam}\, Z$).

Proposition 2.11 *Let $Z \subset X$ be a convex closed set,* $\text{Int}\, Z \neq \emptyset$. *Then for all $0 \le p < r$ we have*

(i) $\alpha(p)/p \le \alpha(r)/r$,

(ii) $\delta(p)/p \le \delta(r)/r$,

(iii) $\alpha(r) \le \delta(r)$.

Proof.
(i) Let $0 \le p < r$ and $\varepsilon > 0$ be given. Put $\gamma := p/r$. We fix $z \in \partial Z$ and $w \in X \setminus T_Z(z)$, $|w| = 1$ such that $|P(z+rw)| < \alpha(r) + \varepsilon$. For $v := (1-\gamma)z + \gamma Q(z+rw) \in Z$ we have

$$\alpha(p) \le |P(z+pw)| \le |z + pw - v| = \gamma |P(z+rw)| < \frac{p}{r}(\alpha(r) + \varepsilon)$$

hence (i) holds.

(ii) It suffices to assume $\delta(r) < \infty$. We find $x, y \in Z$ and $z \in \partial Z$ such that $|x - y| = 2r$ and

$$\tag{2.11} \left|\frac{x+y}{2} - z\right| - \frac{\varepsilon}{2} \le \text{dist}\left(\frac{x+y}{2}, \partial Z\right) \le \delta(r) + \frac{\varepsilon}{2}.$$

Put $\hat{x} := \gamma x + (1-\gamma)z$, $\hat{y} := \gamma y + (1-\gamma)z$ with γ as above. Then $\hat{x}, \hat{y} \in Z$, $|\hat{x} - \hat{y}| = 2p$ and $\delta(p) \le \big|(\hat{x}+\hat{y})/2 - z\big| = \gamma\big|(x+y)/2 - z\big| \le (\delta(r) + \varepsilon)\, p/r$.

(iii) Let x, y, z, ε be as in (ii). We fix an arbitrary $\psi \in N_Z(z)$, $|\psi| = 1$ and assume $\langle \psi, x - y\rangle \ge 0$ (otherwise we interchange x and y). Put $v_\varepsilon := (x-y)/2 + \varepsilon\psi \in X \setminus T_Z(z)$. Then $\alpha(r) \le \big|P\big(z + r\, v_\varepsilon/|v_\varepsilon|\big)\big| \le \big|z + r\, v_\varepsilon/|v_\varepsilon| - x\big| \le \big|z - (x+y)/2\big| + \big|\varepsilon\psi - \big(1 - r/|v_\varepsilon|\big)v_\varepsilon\big| \le \big|z - (x+y)/2\big| + \varepsilon$. Letting ε tend to 0 we obtain (iii) from (2.11). ❒

We see that both α, δ are nondecreasing in their domains. One can derive by elementary means further interesting properties of these functions. Details are left to the reader as an exercise.

Exercise 2.12 Let $Z \subset X$ be a closed convex domain with a nonempty interior. Prove that

(i) $\delta(r) \leq \alpha(2r + 2\delta(r))/2$ for $r \in [0, 1/2 \text{ diam } Z[$,

(ii) $\alpha(r) - \alpha(p) \leq r - p$ for $0 \leq p < r$,

(iii) if $\dim X \geq 2$, then for every $x \in \text{Int } Z$, $c := \text{dist}(x, \partial Z)$ and $r \in [0, c]$ we have $2c\delta(r) \leq r^2 + \delta^2(r)$;

(iv) if $\delta(r) > 0$ for some $r \in]0, 1/2 \text{ diam } Z[$, then

$$\text{diam } Z \leq \frac{r}{\delta^2(r)}(r^2 + \delta^2(r)).$$

Hint. (i) Assume $\alpha(2r + 2\delta(r)) < 2\delta(r) - \varepsilon$ for some $r > 0$, $\varepsilon > 0$. Find $x \in \partial Z$, $w \in \partial B_1(0) \cap (X \setminus T_Z(x))$ such that $\left|P(x + (2r + 2\delta(r))w)\right| < 2\delta(r)$ and put $z := Q(x + (2r + 2\delta(r))w)$. Then $z \in Z$, $|x - z| > 2r$, $x + (r + \delta(r))w \notin Z$, hence $\left|x + (r + \delta(r))w - (x + z)/2\right| > \delta(r)$ which is a contradiction.

(ii) Use the Lipschitz continuity of P which follows from Lemma 2.2 (ii).

(iii) Let $z_\varepsilon \in \partial Z$ be such that $|z_\varepsilon - x| \leq c + \varepsilon$. Find $w_\varepsilon \in B_1(0)$ such that $\langle w_\varepsilon, z_\varepsilon - x\rangle = 0$ and put $u_\pm := x + \sqrt{c^2 - r^2}(z_\varepsilon - x)/|z_\varepsilon - x| \pm r w_\varepsilon$. Then $u_\pm \in B_c(x) \subset Z$, $|u_+ - u_-| = 2r$ and $\delta(r) \leq \left|z_\varepsilon - (u_+ + u_-)/2\right|$.

(iv) Assume $s := |x - y|/2 > r/(2\delta^2(r))\left(r^2 + \delta^2(r)\right)$ for some $x, y \in Z$. Then $s > r$, hence $\delta(s) \geq s\,\delta(r)/r > \left(r^2 + \delta^2(r)\right)/(2\delta(r)) \geq r$. By (iii) we have $2\delta(s)\delta(r) \leq r^2 + \delta^2(r)$ which is a contradiction.

The upper bound for diam Z in Exercise 2.12 (iv) does not seem to be optimal. If Z is a ball, then we obtain for instance $\text{diam } Z = (r^2 + \delta^2(r))/\delta(r)$. We can nevertheless conclude that Z *is unbounded if and only if* $\alpha(r) = 0$ *for all* $r \geq 0$. Let us consider now the opposite situation.

Definition 2.13

(i) A convex closed set $Z \subset X$ is said to be strictly convex, *if $(x + y)/2 \in \text{Int } Z$ for all $x, y \in Z$, $x \neq y$.*

(ii) A convex closed set $Z \subset X$ is said to be uniformly strictly convex, *if $\alpha(r) > 0$ for all $r > 0$.*

Proposition 2.14 *Let Z be a uniformly strictly convex subset of X, $\dim X \geq 2$, $B_m(x) \subset Z$ for some $x \in \text{Int } Z$. Then $\alpha^{-1} : [0, \infty[\to [0, \infty[$ is locally Lipschitz in $]0, \infty[$, $\lim\limits_{s \to \infty} \alpha^{-1}(s)/s = 1$, $\alpha^{-1}(s) \geq \sqrt{ms}$ for all $s \geq 0$.*

Proof. Proposition 2.11 (i) entails $\alpha(r) - \alpha(p) \geq (r - p)\,\alpha(p)/p$ for all $r > p > 0$, hence α^{-1} is locally Lipschitz in $]0, \infty[$. We obviously have $r \geq \alpha(r) \geq r - \text{diam } Z$, hence $\lim_{s \to \infty}(\alpha^{-1}(s))/s = 1$. To conclude, notice that Exercise 2.12 (iii) and Proposition 2.11 (iii) yield $m\alpha(r) \leq r^2$ for $r \in [0, m]$ and the trivial inequality $\alpha(r) \leq r < r^2/m$ for $r > m$ completes the proof. ❐

2.4 The Minkowski functional

Definition 2.15 *Let $A \subset X$ be a given set. Then*

$$A^* := \{y \in X;\ \langle y, x\rangle \le 1 \quad \forall x \in A\} \tag{2.12}$$

is called the polar *of A.*

We immediately see that A^* is convex and closed, $0 \in A^*$. The following duality statement holds.

Lemma 2.16 *Let A be as in Definition 2.15, let A^{**} be the polar of A^* and let $\overline{\mathrm{conv}}$ denote the closure of the convex hull. Then*

$$A^{**} = \overline{\mathrm{conv}}\,(A \cup \{0\})\,.$$

Proof. Put $\hat{A} := \overline{\mathrm{conv}}\,(A \cup \{0\})$. We have by definition

$$A^{**} = \{z \in X;\ \langle y, z\rangle \le 1 \quad \forall y \in A^*\}\,, \tag{2.13}$$

hence $0 \in A^{**}$ and $A \subset A^{**}$. Since A^{**} is convex and closed, we necessarily have $\hat{A} \subset A^{**}$. To prove the inclusion $A^{**} \subset \hat{A}$, we fix an arbitrary $z \in A^{**}$ and apply Lemma 2.1 with the projection pair $(\hat{P}, \hat{Q})$ associated to $\hat{A}$. This yields

$$\langle \hat{P}z\,,\, z - \hat{P}z - x\rangle \ge 0 \quad \forall x \in \hat{A}\,. \tag{2.14}$$

For every k>0 we have in particular

$$\langle k\,\hat{P}z\,,\, z\rangle \ge k\,|\hat{P}z|^2 + \sup\Big\{\langle k\,\hat{P}z\,,\, x\rangle\,;\ x \in A\Big\}. \tag{2.15}$$

Put

$$\kappa := \inf\Big\{k > 0\,;\ k\,\hat{P}z \notin A^*\Big\}\,. \tag{2.16}$$

From (2.15) it follows $\kappa > 0$ and we distinguish two cases.

(i) $\kappa = +\infty$: Putting $x := 0$ in (2.14), we obtain

$$k\,|\hat{P}z|^2 \le \langle k\,\hat{P}z\,,\, z\rangle \le 1 \qquad \forall\, k > 0\,. \tag{2.17}$$

(ii) $\kappa < +\infty$: Then $\kappa\,\hat{P}z \in \partial A^*$, hence $\sup\Big\{\langle \kappa\,\hat{P}z, x\rangle\,;\ x \in A\Big\} = 1$ and (2.15) yields

$$1 + \kappa\,|\hat{P}z|^2 \le \langle \kappa\,\hat{P}z\,,\, z\rangle \le 1\,. \tag{2.18}$$

In both cases (2.17) and (2.18), we conclude $\hat{P}z = 0$, hence $z \in \hat{A}$. Lemma 2.16 is proved. ❐

Lemma 2.17 *Let A, A^* be as in Definition 2.15 and let $R > 0$ be given. Then*

$$A \subset B_R(0) \iff B_{1/R}(0) \subset A^* . \tag{2.19}$$

Proof. Assume $A \subset B_R(0)$ and fix $y \in B_{1/R}(0)$. Then for $x \in A$ we have $\langle y, x \rangle \leq |y|\,|x| \leq 1$, hence $y \in A^*$. Conversely, let $B_{1/R}(0) \subset A^*$ and fix $x \in A$. Then $|x| = \sup\{\langle x, w\rangle\,;\, w \in B_1(0)\} = R\,\sup\{\langle x, y\rangle\,;\, y \in B_{1/R}(0)\} \leq R$. ❒

Definition 2.18 *Let $Z \subset X$ be a convex closed set, $0 \in Z$. The functional $M : X \to \mathbb{R}^+ \cup \{+\infty\}$ defined by the formula*

$$M(x) := \inf\left\{ s > 0\,;\, \frac{1}{s}x \in Z \right\} \quad \text{for } x \in X . \tag{2.20}$$

is called the Minkowski functional *of Z.*

The functional M is sometimes called *gauge*, cf. [Ro]. We list without proof some of its basic properties.

Proposition 2.19 *In the situation of Definition 2.18, we have*

$$\begin{array}{rl}
\text{(i)} & Z = \{x \in X;\, M(x) \leq 1\}, \\
\text{(ii)} & C_Z := \{x \in X;\, M(x) = 0\}, \\
\text{(iii)} & M(tx) = t\,M(x) \quad \forall\, x \in X, \quad \forall\, t \geq 0, \\
\text{(iv)} & M(x+y) \leq M(x) + M(y) \quad \forall\, x, y \in X .
\end{array}$$

As an immediate consequence of the above considerations, we have the following

Proposition 2.20 *Let $Z \subset X$ be a convex closed set and let $R > r > 0$ be given numbers such that*

$$B_r(0) \subset Z \subset B_R(0) . \tag{2.21}$$

Then

$$B_{1/R}(0) \subset Z^* \subset B_{1/r}(0) , \tag{2.22}$$

$$\frac{1}{R}|x| \leq M(x) \leq \frac{1}{r}|x| \quad \forall\, x \in X , \tag{2.23}$$

where Z^* is the polar and M is the Minkowski functional of Z.

According to (2.23) and Proposition 2.19, the Minkowski functional of a convex set Z satisfying the hypotheses of Proposition 2.20 is convex and Lipschitz continuous. Its subdifferential has the following properties.

Lemma 2.21 *Let Z satisfy the hypotheses of Proposition 2.20, let M, M^* be the Minkowski functionals of Z, Z^*, respectively, and let ∂M be the subdifferential of M. Then*

(i) $\partial M(x) \neq \emptyset \quad \forall\, x \in X$,

(ii) $\partial M(tx) = \partial M(x) \quad \forall\, x \in X, \forall\, t > 0$,

(iii) $\langle w, x\rangle = M(x), \quad \langle w, y\rangle \le M(y) \quad \forall\, x, y \in X, \quad \forall\, w \in \partial M(x)$.

(iv) $M^*(w) = 1 \quad \forall\, w \in \partial M(x), \; \forall\, x \neq 0$.

Proof. (i) We have for all $x \in X$

$$w \in \partial M(x) \iff \langle w, x - y\rangle \ge M(x) - M(y) \quad \forall\, y \in X, \tag{2.24}$$

hence $0 \in \partial M(0)$. For $x \neq 0$, we choose a sequence $0 < t_n \nearrow M(x)$, $n = 1, 2, \ldots$, and put $x_n := x/t_n$, $x_0 := x/M(x)$. Let (P, Q) be the projection pair associated to Z by (2.3). Then $x_n \notin Z$ for $n \ge 1$, hence $Px_n \neq 0$ and

$$\langle Px_n, Qx_n - z\rangle \ge 0 \quad \forall\, z \in Z. \tag{2.25}$$

On the other hand, we have $Qx_0 = x_0$, and $|Qx_n - x_0| \le |x_n - x_0| \to 0$ as $n \to \infty$. Selecting a subsequence, if necessary, we may assume that $Px_n/|Px_n|$ converge weakly to some $w_0 \in B_1(0)$. Then (2.25) yields

$$\langle w_0, x_0 - z\rangle \ge 0 \quad \forall\, z \in Z. \tag{2.26}$$

Putting $z := r\, Px_n/|Px_n|$ in (2.25) and passing to the limit as $n \to \infty$, we obtain

$$\langle w_0, x_0\rangle \ge r > 0. \tag{2.27}$$

Inequality (2.26) implies

$$\left\langle w_0, \frac{x}{M(x)} - \frac{y}{M(y)}\right\rangle \ge 0 \quad \forall\, y \in X \setminus \{0\}, \tag{2.28}$$

or equivalently,

$$\langle w_0, x - y\rangle \ge (M(x) - M(y))\, \langle w_0, x_0\rangle \quad \forall\, y \in X. \tag{2.29}$$

According to (2.24) and (2.27), we have $w := w_0/\langle w_0, x_0\rangle \in \partial M(x)$ and (i) is proved. Using Proposition 2.19 (iii) we obtain (ii) trivially from (2.24), part (iii) follows from (2.24) by putting successively $y := 0$ and $y := 2x$ and part (iv) follows from (iii). ❐

Remark 2.22 Lemma 2.21 does not hold for general convex closed sets Z. To see this, we first notice that by (2.24), for every x with $M(x) > 0$ and every $w \in \partial M(x)$ we have

$$w \neq 0, \tag{2.30}$$

$$\left\langle w, \frac{x}{M(x)} - y\right\rangle \ge 0 \quad \forall\, y \in Z. \tag{2.31}$$

As an example, we choose $X := L^2(0,1)$, $Z := \{z \in X; -1 \le z(t) \le 1 \text{ a.e. }\}$, $x(t) := t$ for $t \in [0,1]$. Then Z is convex and closed, $0 \in Z$, $M(x) = 1$. Assume that $\partial M(x)$ is nonempty and let $w \in \partial M(x)$ be arbitrary. By (2.31), we have

$$\begin{aligned}\int_0^1 w(t)\, t\, dt \ &\ge \ \sup\left\{\int_0^1 w(t)\, y(t)\, dt\,;\ y \in X\,, -1 \le y(t) \le 1 \text{ a.e.}\right\}\\ &= \ \int_0^1 |w(t)|\, dt\end{aligned}$$

hence $w = 0$, which contradicts (2.30).

The main result of this section reads as follows.

Theorem 2.23 *Let Z satisfy the hypotheses of Proposition 2.20. Let Z^* be the polar of Z and let M, M^* be the Minkowski functionals of Z, Z^*, respectively. For $x \in X$ put $J(x) := M(x)\,\partial M(x)$, $J^*(x) := M^*(x)\,\partial M^*(x)$. Then*

$$\begin{aligned}
&\text{(i)} \quad \langle w - z, x - y\rangle \ \ge\ (M(x) - M(y))^2\\
&\qquad\qquad \forall\, x, y \in X\,,\ w \in J(x)\,,\ z \in J(y)\,,\\
&\text{(ii)} \quad \langle w^* - z^*, x - y\rangle \ \ge\ (M^*(x) - M^*(y))^2\\
&\qquad\qquad \forall\, x, y \in X\,,\ w^* \in J^*(x)\,,\ z^* \in J^*(y)\,,\\
&\text{(iii)} \quad y \in J(x) \iff x \in J^*(y) \quad \forall\, x, y \in X\,,\\
&\text{(iv)} \quad Z^* = J(Z)\,,\ Z = J^*(Z^*)\,,
\end{aligned}$$

where $J(Z) := \bigcup_{x \in Z} J(x)$, $J^*(Z^*) := \bigcup_{y \in Z^*} J^*(y)$.

Before proving Theorem 2.23, we state an auxiliary Lemma.

Lemma 2.24 *Let the hypotheses of Theorem 2.23 hold. Then for all $x, y \in X \setminus \{0\}$ we have*

$$\langle y, x\rangle \le M(x)\, M^*(y)\,, \tag{2.32}$$

$$\langle y, x\rangle = M^*(y)\, M(x) \iff \frac{x}{M(x)} \in \partial M^*(y) \iff \frac{y}{M^*(y)} \in \partial M(x)\,. \tag{2.33}$$

Proof of Lemma 2.24. Inequality (2.32) follows immediately from the definition of Z^* and Lemma 2.21 (iii) yields the implications

$$\begin{aligned}
\frac{x}{M(x)} \in \partial M^*(y) \ &\Rightarrow\ \langle y, x\rangle = M^*(y)\, M(x)\,,\\
\frac{y}{M^*(y)} \in \partial M(x) \ &\Rightarrow\ \langle y, x\rangle = M^*(y)\, M(x)\,.
\end{aligned}$$

Assume now

$$\langle x, y\rangle = M(x)\, M^*(y) \quad \text{for some } x, y \in X \setminus \{0\}\,. \tag{2.34}$$

Then, by (2.32) we have

$$\begin{aligned} \left\langle \frac{x}{M(x)}, y - z \right\rangle &\geq M^*(y) - M^*(z) \quad \forall\, z \in X\,, \\ \left\langle \frac{y}{M^*(y)}, x - z \right\rangle &\geq M(x) - M(z) \quad \forall\, z \in X \end{aligned}$$

and the assertion follows. ❐

Proof of Theorem 2.23. Inequalities (i), (ii) follow from (2.24) (and the corresponding inequality for M^*). To prove (iii), it suffices to fix $x \in X$ and $y \in J(x)$ and prove that $x \in J^*(y)$. The other implication then follows from the duality $Z = Z^{**}$ and $J = J^{**}$. The definition of J immediately entails $J(0) = \{0\}$, $J^*(0) = \{0\}$, hence it suffices to assume $x \neq 0$. By Lemma 2.21 (iii), (iv) we have

$$\langle y, x\rangle = M^2(x)\,, \quad M^*(y) = M(x)\,. \tag{2.35}$$

and Lemma 2.24 (ii) yields the assertion. To prove (iv), it suffices to use (iii) and (2.35). ❐

We call J the *duality mapping* induced by Z. It can be interpreted geometrically by means of the normal cone $N_Z(x)$ in the following way.

Proposition 2.25 *Let the hypotheses of Theorem 2.23 hold. Then for every $x \in \partial Z$, we have $J(x) \subset N_Z(x)$. Conversely, for each $y \in N_Z(x)$, $y \neq 0$, we have $\langle y, x\rangle = M^*(y)$ and $y/\langle y, x\rangle \in J(x)$.*

Proof. The inclusion $J(x) \subset N_Z(x)$ follows immediately from the definition. Let now $y \in N_Z(x)$, $y \neq 0$ be given. Then $\langle y, x\rangle \geq \langle y, z\rangle$ for all $z \in Z$, hence $y/\langle y, x\rangle \in Z^*$. We have in particular $M^*(y) \leq \langle y, x\rangle$ and from (2.32) (note that $M(x) = 1$) we obtain $\langle y, x\rangle = M^*(y)$. Lemma 2.24 then completes the proof. ❐

Exercise 2.26 Prove that $M^{*2}/2$ is the *conjugate function* to $M^2/2$ in the sense of [AE], that is,

$$\frac{1}{2}M^{*2}(y) = \sup\left\{\langle y, x\rangle - \frac{1}{2}M^2(x)\,;\ x \in X\right\} \quad \text{for every } \ y \in X\,. \tag{2.36}$$

'Smooth' convex domains $Z \subset X$ are those where $N_Z(x)$ reduces to a half-line for each $x \in \partial Z$. By Proposition 2.25, this is equivalent to saying that J is a single-valued mapping. We have the following dual characterization of such domains.

Theorem 2.27 *In the situation of Theorem 2.23, the following conditions are equivalent.*

(i) *J is single-valued,*

(ii) *Z^* is strictly convex according to Definition 2.9.*

Proof.

(ii) $\Rightarrow$ (i): Let $x \in X$ and $y_0, y_1 \in J(x)$ be given. For $x = 0$ we have $y_0 = y_1 = 0$, otherwise we put $y := (y_0 + y_1)/2$. Then $y \in J(x)$ and $M^*(y) = M^*(y_0) = M^*(y_1) = M(x)$. Consequently, all $y_0/M(x)$, $y_1/M(x)$, $y/M(x)$ belong to ∂Z^*, hence $y_0 = y_1$.

non (ii) $\Rightarrow$ non (i): Assume that there exist $y_0 \neq y_1 \in Z^*$ such that $y := (y_0 + y_1)/2 \in \partial Z^*$. Let $x \in J^*(y)$ be arbitrarily chosen. Then $M(x) = M^*(y) = 1$ and

$$1 = \langle x, y \rangle = \frac{1}{2}\left(\langle x, y_0 \rangle + \langle x, y_1 \rangle\right) \leq 1 .$$

This yields $\langle x, y_0 \rangle = \langle x, y_1 \rangle = 1 = M^*(y_0) = M^*(y_1)$ and from Lemma 2.24 (ii), we conclude $y_0, y_1 \in J(x)$ and Theorem 2.27 is proved. ❐

Example 2.28 If $Z = \{x \in X\,;\, \langle x, n_i \rangle \leq \beta_i\,,\, i = 1, \ldots, p\}$ is a polyhedron with a system $\{n_i\,;\, i = 1, \ldots, p\}$ of unit vectors and with $\beta_i > 0$, then Z^* is the polyhedron $Z^* = \operatorname{conv}(\{0, n_1/\beta_1, \ldots, n_p/\beta_p\})$.

3 The play and stop operators

The elementary hysteresis operators called *stop* and *play* have already been introduced in Section 1. The rigorous construction presented here follows the exposition in [K] and is slightly different from the approach of [KP] and [V]. We admit the infinitely dimensional case and start with nonsmooth input functions. More precisely, we define the inputs and outputs in the space $CBV(0, T\,; X)$ of continuous functions of bounded variation with values in a Hilbert space X. We further prove that the restriction of the play and stop operators to Sobolev spaces $W^{1,p}(0, T\,; X)$ is continuous and bounded if $1 \leq p < \infty$ and discontinuous for $p = +\infty$. If the convex constraint Z has nonempty interior, the extension of these operators is shown to be continuous (but not necessarily bounded) from $C([0,T]; X)$ to $C([0,T]; X)$, together with an interesting smoothening property of the play, namely that it maps $C([0,T]; X)$ into $CBV(0, T\,; X)$. A brief survey of the functional framework used here can be found in Section 8. The first step consists in proving the following generalization of Theorem 1.7.

Theorem 3.1 *Let a real separable Hilbert space X, a convex closed set $Z \subset X$ with $0 \in Z$, an element $x_0 \in Z$ and a function $u \in CBV(0, T\,; X)$ be given. Then there exist uniquely determined $\xi \in CBV(0, T\,; X)$, $x \in CBV(0, T\,; Z)$ such that*

(3.1)
$$\begin{aligned}
&\text{(i)} && x(t) + \xi(t) = u(t) \quad \forall t \in [0, T]\,,\\
&\text{(ii)} && x(0) = x_0\,,\\
&\text{(iii)} && \int_0^T \big\langle x(t) - \varphi(t), d\xi(t) \big\rangle \geq 0 \quad \forall \varphi \in C([0,T]; Z)\,.
\end{aligned}$$

We rewrite the Riemann-Stieltjes integral in (iii) in an equivalent, but more convenient form.

Lemma 3.2 *Let $x \in C([0,T];Z)$ and $\xi \in NBV(0,T;X)$ satisfy (3.1) (iii). Then*

$$\int_s^t \big\langle x(\tau) - \psi(\tau), d\xi(\tau)\big\rangle \geq 0 \quad \forall \psi \in C([s,t];Z) \qquad \text{for all } \ 0 \leq s < t \leq T\,. \tag{3.2}$$

Proof. Let $0 < s < t \leq T$ and $\psi \in C([s,t];Z)$ be given (the case $s = 0$ is analogous). For $0 < \delta < \min\{s, t-s\}$ put

$$\varphi_\delta(\tau) \ := \ \begin{cases} x(\tau) & \text{for} \quad \tau \in [0, s-\delta[\, \cup\,]t, T]\,, \\ x(s-\delta) + \frac{\tau - s + \delta}{\delta}(\psi(s) - x(s-\delta)) & \text{for} \quad \tau \in [s-\delta, s[\ , \\ \psi(\tau) & \text{for} \quad \tau \in [s, t-\delta]\,, \\ x(t) + \frac{t-\tau}{\delta}(\psi(t-\delta) - x(t)) & \text{for} \quad \tau \in\,]t-\delta, t]\,. \end{cases}$$

Then (3.1) (iii) and (8.26) yield

$$\begin{aligned} 0 \ &\leq \ \int_0^T \big\langle x(\tau) - \varphi_\delta(\tau), d\xi(\tau)\big\rangle \\ &= \ \int_s^t \big\langle x(\tau) - \psi(\tau), d\xi(\tau)\big\rangle \ + \ \frac{1}{\delta}\int_{s-\delta}^s \big\langle \xi(s) - \xi(\tau), x(s-\delta) - \psi(s)\big\rangle\, d\tau \\ &+ \ \int_{s-\delta}^s \big\langle x(\tau) - x(s-\delta), d\xi(\tau)\big\rangle \ + \ \int_{t-\delta}^t \big\langle \psi(\tau) - \psi(t-\delta), d\xi(\tau)\big\rangle \\ &+ \ \frac{1}{\delta}\int_{t-\delta}^t \big\langle \xi(t) - \xi(\tau), \psi(t-\delta) - x(t)\big\rangle\, d\tau\,. \end{aligned}$$

Using (8.22) and (8.7) we can pass to the limit as $\delta \to 0$ and the proof is complete. ❒

Let us note that if two variational inequalities of the form (3.2) are satisfied, that is,

$$\int_s^t \big\langle x_i(\tau) - \psi(\tau), d\xi_i(\tau)\big\rangle \geq 0 \quad \forall \psi \in C([s,t];Z)\,, \quad i = 1,2\,, \tag{3.3}$$

with $u_i = x_i + \xi_i$, $x_i \in C([s,t];Z)$, $\xi_i \in CBV(s,t;X)$, then putting $\psi := (x_1 + x_2)/2$, we obtain from (3.2), (8.24) and (8.22)

$$|\xi_1(t) - \xi_2(t)|^2 \ \leq \ |\xi_1(s) - \xi_2(s)|^2 + 2|u_1 - u_2|_\infty \left(\operatorname*{Var}_{[s,t]} \xi_1 + \operatorname*{Var}_{[s,t]} \xi_2 \right)\,. \tag{3.4}$$

Proof of Theorem 3.1. Uniqueness follows immediately from the inequality (3.4). The existence proof is carried out by a simple time-discretization scheme. For a fixed $n \in \mathbb{N}$ we define

$$u_j := u\left(\frac{jT}{n}\right), \quad j = 0, \ldots, n\,. \tag{3.5}$$

Let (P, Q) be the projection pair defined by formula (2.3). We construct the sequences

$$\begin{cases} x_j := Q(x_{j-1} + u_j - u_{j-1}), \quad j = 1, \ldots, n, \\ \xi_j := u_j - x_j, \quad j = 0, \ldots, n. \end{cases} \tag{3.6}$$

We have $\xi_j - \xi_{j-1} = P(x_{j-1} + u_j - u_{j-1})$ and Lemma 2.2 (i) yields

$$\langle \xi_j - \xi_{j-1}, x_j - z \rangle \geq 0 \qquad \forall z \in Z, \quad \forall j \in \{1, \dots, n\}. \tag{3.7}$$

Putting $z := x_{j-1}$ and $V_u := \operatorname*{Var}_{[0,T]} u$, we immediately obtain from (3.7)

$$\sum_{j=1}^{n} |\xi_j - \xi_{j-1}| \leq V_u . \tag{3.8}$$

We now define piecewise linear functions $u^{(n)}, \xi^{(n)}, x^{(n)} \in W^{1,1}(0, T\ X)$ by the formula

$$\begin{cases} u^{(n)}(t) := u_{j-1} + n\left(\frac{t}{T} - \frac{j-1}{n}\right)(u_j - u_{j-1}), \\ \xi^{(n)}(t) := \xi_{j-1} + n\left(\frac{t}{T} - \frac{j-1}{n}\right)(\xi_j - \xi_{j-1}), \\ x^{(n)}(t) := x_{j-1} + n\left(\frac{t}{T} - \frac{j-1}{n}\right)(x_j - x_{j-1}) \end{cases} \tag{3.9}$$

for $t \in [(j-1)T/n, jT/n[$ and $j = 1, \dots, n$, continuously extended to $t = T$.
Let $\mu_u : \mathbb{R}^+ \to \mathbb{R}^+$ be the continuity modulus of u, that is

$$\mu_u(\delta) := \sup\{|u(t) - u(s)|\,;\ |t - s| \leq \delta\} \qquad \text{for} \quad \delta > 0. \tag{3.10}$$

For every $\tau \in](j-1)T/n, jT/n[$ and $z \in Z$ we have by (3.7) and Lemma 2.2 (i)

$$\begin{aligned} \langle \dot{\xi}^{(n)}(\tau), x^{(n)}(\tau) - z \rangle &\geq -\frac{n}{T}\langle \xi_j - \xi_{j-1}, x_j - x_{j-1} \rangle \\ &\geq -\frac{n}{T}\langle \xi_j - \xi_{j-1}, u_j - u_{j-1} \rangle \\ &\geq -\frac{n}{T}\mu_u\left(\frac{T}{n}\right)|\xi_j - \xi_{j-1}| \end{aligned}$$

and estimate (3.8) yields

$$\int_0^t \langle x^{(n)}(\tau) - \varphi(\tau), d\xi^{(n)}(\tau) \rangle \geq -V_u\, \mu_u\left(\frac{T}{n}\right) \tag{3.11}$$

for all $n \in \mathbb{N}$, $t \in [0, T]$ and $\varphi \in C([0, T]; Z)$.
The proof of Theorem 3.1 will be complete if we prove that

$$\{\xi^{(n)}; n \in \mathbb{N}\} \quad \text{is a uniformly convergent sequence.} \tag{3.12}$$

Indeed, in this case it suffices to use inequality (3.11) and Theorem 8.16, since the sequence $\{u^{(n)}\}$ is uniformly convergent and $\operatorname*{Var}_{[0,T]} \xi^{(n)} \leq V_u$ by (3.8).

To prove (3.12), we put $\varphi(\tau) := \left(x^{(n)}(\tau) + x^{(\ell)}(\tau)\right)/2$ for two different values of n in (3.11), say n, ℓ. Then

$$\int_0^t \langle \dot{\xi}^{(n)}(\tau) - \dot{\xi}^{(\ell)}(\tau), x^{(n)}(\tau) - x^{(\ell)}(\tau) \rangle d\tau \geq -V_u\left(\mu_u\left(\frac{T}{n}\right) + \mu_u\left(\frac{T}{\ell}\right)\right), \tag{3.13}$$

hence, by inequality (8.22),

$$\frac{1}{2}\left|\xi^{(n)}-\xi^{(\ell)}\right|_\infty^2 \le \left|u^{(n)}-u^{(\ell)}\right|_\infty \left(\operatorname*{Var}_{[0,T]}\xi^{(n)}+\operatorname*{Var}_{[0,T]}\xi^{(\ell)}\right)+V_u\left(\mu_u(\frac{T}{n})+\mu_u(\frac{T}{\ell})\right).$$

The sequence $\{\xi^{(n)}\}$ is therefore fundamental in $C([0,T];X)$, hence (3.12) holds and Theorem 3.1 is proved. ❐

Definition 3.3 *Let $Z\subset X$ be a convex closed set, $0\in Z$ and let $u\in CBV(0,T\,;X)$, $x^0\in Z$ be given. Let (x,ξ) be the solution of (3.1). We define the value $\mathcal{P}(x_0,u),\mathcal{S}(x_0,u)$ of the* play *and* stop *operators $\mathcal{P},\mathcal{S}: Z\times CBV(0,T\,;X)\to CBV(0,T\,;X)$, respectively, by the formula*

$$\mathcal{P}(x_0,u):=\xi,\qquad \mathcal{S}(x_0,u):=x. \tag{3.14}$$

Remark 3.4 The initially unperturbed state is characterized by the choice $x_0=Qu(0)$ of the initial condition (3.1) (ii). In this case we use the simplified notation

$$\mathcal{P}(u):=\mathcal{P}(Qu(0),u),\quad \mathcal{S}(u):=\mathcal{S}(Qu(0),u). \tag{3.15}$$

3.1 Absolutely continuous inputs

It is natural to expect that play and stop operators act in Sobolev spaces $W^{1,p}(0,T\,;X)$. Before passing to the continuity statement, we give in Proposition 3.5 below a precise meaning to the normality rule mentioned in Section 1. It also yields the unique orthogonal decomposition of $\dot u(t)$ into the components $\dot\xi(t)\in N_Z(x(t))$ and $\dot x(t)\in T_Z(x(t))$, see Subsection 2.2. This can be used as an alternative definition of the play and stop operators, see [KP].

Proposition 3.5 *Let $Z\subset X$ be a convex closed set with $0\in Z$, let $x_0\in Z$ be a given initial value and let $u\in W^{1,1}(0,T\,;X)$ be given. Then $\xi:=\mathcal{P}(x_0,u)$, $x:=\mathcal{S}(x_0,u)$ belong to $W^{1,1}(0,T\,;X)$ and satisfy*

$$\begin{aligned} &\text{(i)} &&\left\langle\dot\xi(t),\,x(t)-z\right\rangle\ge 0\quad\text{a.e.}\quad\forall z\in Z,\\ &\text{(ii)} &&\left\langle\dot\xi(t),\,\dot x(t)\right\rangle=0\quad\text{a.e.}\end{aligned} \tag{3.16}$$

Proof. For arbitrary $0\le s<t\le T$ and $\tau\in[s,t]$ put $\psi(\tau):=x(s)$ in (3.2). Then (8.24) and (8.26) yield

$$\begin{aligned}\frac{1}{2}|\xi(t)-\xi(s)|^2 &\le \int_s^t\langle u(\tau)-u(s),\,d\xi(\tau)\rangle = \int_s^t\langle\xi(t)-\xi(\tau),\,\dot u(\tau)\rangle\,d\tau\\ &\le \max_{s\le\tau\le t}\{|\xi(t)-\xi(\tau)|\}\int_s^t|\dot u(\tau)|\,d\tau\,,\end{aligned} \tag{3.17}$$

hence

$$|\xi(t)-\xi(s)|\le 2\int_s^t|\dot u(\tau)|\,d\tau\qquad\forall\,0\le s<t\le T. \tag{3.18}$$

This implies $\xi \in W^{1,1}(0,T\,;X)$ and according to (8.25), we have

$$\int_s^t \langle \dot{\xi}(\tau),\, x(\tau) - \psi(\tau)\rangle \, d\tau \;\geq\; 0 \quad \forall\, \psi \in C([s,t];Z)\,,\ \forall\, 0 \leq s < t \leq T \tag{3.19}$$

which is equivalent to (3.16) (i). To prove (3.16) (ii), it suffices to put $z := x(t \pm h)$ in (3.16) (i) and let h tend to $0+$ using Theorem 8.14. ❐

It is easy to see that $\mathcal{P}, \mathcal{S}$ are Lipschitz continuous as operators from $Z \times W^{1,1}(0,T\,;X)$ to $C([0,T];X)$. Indeed, putting $x := \mathcal{S}(x_0,u)$, $y := \mathcal{S}(y_0,v)$ for given $x_0, y_0 \in Z$, $u,v \in W^{1,1}(0,T\,;X)$ we immediately obtain from (3.16)(i)

$$\frac{1}{2}\frac{d}{dt}\big|x(t) - y(t)\big|^2 \leq \big\langle x(t) - y(t), \dot{u}(t) - \dot{v}(t)\big\rangle \quad \text{a.e.}, \tag{3.20}$$

consequently

$$|x(t) - y(t)| \leq |x_0 - y_0| + \int_0^t |\dot{u}(\tau) - \dot{v}(\tau)|\, d\tau \qquad \forall t \in [0,T]\,. \tag{3.21}$$

The continuity of $\mathcal{P}, \mathcal{S}$ in $Z \times W^{1,p}(0,T\,;X) \to W^{1,p}(0,T\,;X)$ for $1 \leq p < \infty$ with respect to the norm $|u|_{1,p} := |u(0)| + |\dot{u}|_p$ is established in Theorem 3.6 below.

Theorem 3.6 *Let $Z \subset X$ be a convex closed set with $0 \in Z$, let $\{u_n\ ;\ n \in \mathbb{N} \cup \{0\}\}$ be a given sequence in $W^{1,p}(0,T\,;X)$ for some $p \in [1,\infty[$ such that $\lim_{n\to\infty} |u_n - u_0|_{1,p} = 0$ and let $x_n^0 \in Z$ be given initial values, $\lim_{n\to\infty} |x_n^0 - x_0^0| = 0$. Put $\xi_n := \mathcal{P}(x_n^0, u_n)$ for $n \in \mathbb{N} \cup \{0\}$. Then $\lim_{n\to\infty} |\xi_n - \xi_0|_{1,p} = 0$.*

Proof. For $n \in \mathbb{N} \cup \{0\}$ put $x_n := u_n - \xi_n$, $y_n := x_n - \xi_n$. From (3.21) we infer $|\xi_n - \xi_0|_\infty \to 0$, $|x_n - x_0|_\infty \to 0$, $|y_n - y_0|_\infty \to 0$. By (3.16)(ii) we also have

$$|\dot{y}_n| \;=\; |\dot{u}_n| \qquad \text{a.e.} \qquad \forall\, n \in \mathbb{N} \cup \{0\}\,. \tag{3.22}$$

Theorem 8.7 for $v_n := \dot{y}_n$, $g_n := |\dot{u}_n|$ yields $\lim_{n\to\infty} |y_n - y_0|_{1,1} = 0$. There exists therefore a subsequence $\{y_{n_k}\}$ such that $\lim_{k\to\infty} |\dot{y}_{n_k}(t) - \dot{y}_0(t)| = 0$ a.e. and from Theorem 8.5 we conclude

$$\lim_{k\to\infty} |y_{n_k} - y_0|_{1,p} = 0\,. \tag{3.23}$$

Since every subsequence of $\{y_n\}$ contains a subsequence satisfying (3.23), the proof is complete if we take into account the relations $x_n = (u_n + y_n)/2$, $\xi_n = (u_n - y_n)/2$. ❐

In [K] it is proved that the play operator depends continuously also on its characteristic Z in terms of the Hausdorff distance $\mathcal{H}(A,B)$ of two sets $A, B \subset X$ defined as

$$\mathcal{H}(A,B) := \max\{\sup\{\mathrm{dist}\,(y,A); y \in B\}, \sup\{\mathrm{dist}\,(x,B); x \in A\}\}\,. \tag{3.24}$$

The result reads as follows.

Theorem 3.7 *Let $\{Z_n\,; n \in \mathbb{N} \cup \{0\}\}$ be a sequence of convex closed sets in X such that $0 \in \bigcap_{n=0}^\infty Z_n$, $\lim_{n\to\infty} \mathcal{H}(Z_0, Z_n) = 0$ and let $\{x_n^0\}$ be a sequence of initial values such that $\lim_{n\to\infty} |x_n^0 - x_0^0| = 0$. Let $\{u_n\,; n \in \mathbb{N} \cup \{0\}\}$ be a sequence in $W^{1,p}(0,T\,;X)$ such that $\lim_{n\to\infty} |u_n - u_0|_{1,p} = 0$ for some $p \in [1,+\infty[$. Put $\xi_n := \mathcal{P}_n(x_n^0, u_n)$ for $n \in \mathbb{N} \cup \{0\}$, where $\mathcal{P}_n$ is the play with characteristic Z_n. Then $\lim_{n\to\infty} |\xi_n - \xi_0|_{1,p} = 0$.*

Remark 3.8 A counterpart of Theorem 3.6 does not hold for $p = +\infty$ even if $\dim X = 1$. It suffices to consider $Z = [-1, 1]$, $T = 1$ and the sequence $u_n(t) := (1 + 1/n)t$ for $t \in [0, 1]$, $n \in \mathbb{N}$ with $u_0(t) := t$, $x_n^0 := 0$. We then have

$$\xi_0(t) \equiv 0, \quad \xi_n(t) := \begin{cases} 0 \quad \text{for} \quad t \in \left[0, \frac{n}{n+1}\right], \\ \left(1 + \frac{1}{n}\right) t - 1 \quad \text{for} \quad t \in \left]\frac{n}{n+1}, 1\right] \end{cases} \qquad \text{for } n \in \mathbb{N},$$

hence $|u_n - u_0|_{1,\infty} \to 0$, $|\xi_n - \xi_0|_{1,\infty} \geq 1$.

Remark 3.9 *rm On smooth characteristics, i.e. those, where the unit outward normal $n(x)$ is defined for every $x \in \partial Z$, we can derive explicit differential equations for the output values of the stop and play operators. Denoting as usual $\xi = \mathcal{P}(x_0, u)$, $x = \mathcal{S}(x_0, u)$, we have*

$$\dot{x}(t) = \begin{cases} \dot{u}(t) - \big\langle \dot{u}(t), n(x(t)) \big\rangle n(x(t)) & \textit{if } x(t) \in \partial Z, \ \big\langle \dot{u}(t), n(x(t)) \big\rangle > 0, \\ \dot{u}(t) & \textit{otherwise}. \end{cases} \tag{3.25}$$

3.2 Continuous inputs

Theorem 3.12 below enables us to extend the stop and play to the space $C([0,T]; X)$. The construction in [KP] has originally been designed for bounded convex sets Z with nonempty interior in a finite-dimensional space X. Using the concept of complementary function (see Definition 2.4), we apply the same idea to the general case of recession sets (Definition 2.6) in a Hilbert space. The argument relies on the following Lemma.

Lemma 3.10 *Let $\mathcal{B} \subset C([0,T]; X)$ be a compact set, let $Z \subset X$ be a recession set with $B_m(0) \subset Z$ and let $r > 0$ be given. Then there exists a constant $C > 0$ such that for every $u \in \mathcal{B} \cap BV(0, T; X)$ and every $x_0 \in Z \cap B_r(0)$ we have*

$$\operatorname*{Var}_{[0,T]} \mathcal{P}(x_0, u) \leq C, \tag{3.26}$$

where $\mathcal{P}$ is the play operator corresponding to Z.

Proof. Put $\gamma := m/6$. We find $u_1, \ldots, u_N \in \mathcal{B}$ such that $\mathcal{B} \subset \cup_{k=1}^N \{u \in C([0,T]; X)\,;\, |u - u_k|_\infty < \gamma\}$, and fix $\delta > 0$ such that $\max\{\mu_{u_k}(\delta)\,;\, k = 1, \ldots, N\} < \gamma$. We first prove that for every $u \in \mathcal{B} \cap BV(0, T; X)$, $x_0 \in Z$ and $0 \leq s < t \leq T$ such that $|t - s| < \delta$ we have

$$\operatorname*{Var}_{[s,t]} \mathcal{P}(x_0, u) \leq \frac{1}{m} K_Z^2(|\mathcal{S}(x_0, u)(s)|), \tag{3.27}$$

where K_Z is the complementary function. Put $\xi := \mathcal{P}(x_0, u)$, $x := \mathcal{S}(x_0, u)$. We find $\hat{x} \in C_Z$ such that $|x(s) - \hat{x}| \leq K_Z(|x(s)|)$ and put for $\tau \in [s, t]$

$$\psi(\tau) := \hat{x} + u(\tau) - u(s) + \frac{m}{2}\varphi(\tau)$$

for some $\varphi \in C([s,t]; X)$, $|\varphi|_\infty \leq 1$. We have $|\psi(\tau) - \hat{x}| \leq m$ for all $\tau \in [s, t]$, hence $\psi \in C([s,t]; Z)$ by Proposition 2.5. Inequality (3.2) and identity (8.24) then entail

$$\frac{m}{4} \int_s^t \langle \varphi(\tau), d\xi(\tau) \rangle \leq \int_s^t \big\langle u(s) - \hat{x} - \xi(\tau), d\xi(\tau) \big\rangle = \frac{1}{2}|x(s) - \hat{x}|^2 - \frac{1}{2}|u(s) - \hat{x} - \xi(t)|^2,$$

and inequality (3.27) follows from (8.23).

Putting $R := 1 + T/\delta$, we obtain from (3.27)

$$\operatorname*{Var}_{[0,T]} \xi \le \frac{R}{m} K_Z^2(|x|_\infty). \tag{3.28}$$

Inequality (3.4) for $u_2 = \xi_2 = 0$, $s = 0$, $u_1 = u$, $\xi_1 = \xi$ yields $|\xi|_\infty^2 \le |u(0) - x_0|^2 + 2|u|_\infty \operatorname*{Var}_{[0,T]} \xi$, hence

$$|x|_\infty^2 \le 4\,|u|_\infty^2 + r^2 + \frac{2R}{m}|u|_\infty\,(K_Z(|x|_\infty))^2$$

The set $\mathcal{B}$ is bounded, hence the last inequality and property (2.7) of recession sets provide an upper bound for $|x|_\infty$ independent of $u \in \mathcal{B}$. Inequality (3.26) then follows from (3.28). ❐

We now use Lemma 3.10 to extend the operators $\mathcal{P}, \mathcal{S}$ to arbitrary continuous inputs in the following way.

Theorem 3.11 *Let $Z \subset X$ be a recession set and let $x_0 \in Z$, $u \in C([0,T];X)$ be given. Then there exist uniquely determined $\xi \in CBV(0,T;X)$, $x \in C([0,T];Z)$ such that (3.1) holds.*

Proof. Let $\{u_n; n \in \mathbb{N}\}$ be a sequence in $CBV(0,T;X)$ such that $\lim_{n\to\infty} |u - u_n|_\infty = 0$. From Lemma 3.10 we obtain

$$\exists\, C > 0\ \forall n \in \mathbb{N}: \quad \operatorname*{Var}_{[0,T]} \mathcal{P}(x_0, u_n) \le C \tag{3.29}$$

and (3.4) yields

$$|\mathcal{P}(x_0, u_n) - \mathcal{P}(x_0, u_k)|_\infty^2 \le |u_n(0) - u_k(0)|^2 + 4C|u_n - u_k|_\infty \tag{3.30}$$

for all $k, n \in \mathbb{N}$. The sequence $\{\mathcal{P}(x_0, u_n)\}$ therefore admits a uniform limit in the space $C([0,T];X)$. This limit is independent of the concrete choice of the sequence $\{u_n\}$ and we denote it by $\mathcal{P}(x_0, u)$. By Proposition 8.10 (ii) we have

$$\operatorname*{Var}_{[0,T]} \mathcal{P}(x_0, u) \le C\,, \tag{3.31}$$

and using Theorem 8.16 we can pass to the limit in (3.1). ❐

As a consequence of Theorem 3.11, we see that inequality (3.4) holds whenever $u_1, u_2 \in C([0,T];X)$. This immediately yields the following result which states that $\mathcal{P} : Z \times C([0,T];X) \to C([0,T];X)$ is 1/2-Hölder continuous on compact sets.

Theorem 3.12 *Let the hypotheses of Lemma 3.10 be satisfied. Then there exists a constant $C > 0$ such that for all $u, v \in \mathcal{B}$ and $x^0, y^0 \in Z \cap B_r(0)$ we have*

$$|\mathcal{P}(x^0, u) - \mathcal{P}(y^0, v)|_\infty \le C\,(|u - v|_\infty)^{1/2} + |u - v|_\infty + |x^0 - y^0|\,. \tag{3.32}$$

Corollary 3.13 *Let $\{u_n\,; n \in \mathbb{N} \cup \{0\}\}$ be a given sequence in $C([0,T];X)$ such that $\lim_{n\to\infty} |u_n - u_0|_\infty = 0$ and let $x_n^0 \in Z$ be given initial values, $x_0^0 = \lim_{n\to\infty} x_n^0$. Put $\xi_n := \mathcal{P}(x_n^0, u_n)$ for $n \in \mathbb{N} \cup \{0\}$. Then $\lim_{n\to\infty} |\xi_n - \xi_0|_\infty = 0$.*

Similarly as in Theorem 3.7, the play operator depends continuously on the set Z also in $C([0,T];X)$, see [K].

Theorem 3.14 *Let $\{u_n\}_{n=0}^\infty$ be a sequence in $C([0,T];X)$, let $\{Z_n\}_{n=0}^\infty$ be a sequence of recession sets such that $\lim_{n\to\infty} |u_n - u_0|_\infty = 0$, $\lim_{n\to\infty} \mathcal{H}(Z_n, Z_0) = 0$, and let $x_n^0 \in Z_n$ be given initial values, $\lim_{n\to\infty} |x_n^0 - x_0^0| = 0$. Put $\xi_n := \mathcal{P}_n(x_n^0, u_n)$ for $n \in \mathbb{N} \cup \{0\}$, where $\mathcal{P}_n$ is the play with characteristic Z_n. Then $\lim_{n\to\infty} |\xi_n - \xi_0|_\infty = 0$.*

By Proposition 3.5, we have $|\dot{x}(t)| \le |\dot{u}(t)|$ almost everywhere for every $u \in W^{1,p}(0,T\,;X)$ and $x_0 \in Z$, hence the stop operator $\mathcal{S} : Z \times W^{1,p}(0,T\,;X) \to W^{1,p}(0,T\,;X)$ is not only continuous, but also bounded. Example 3.15 below shows that this is not true in $C([0,T];X)$ in general. In fact, such behavior arises typically in isotropic hardening models (cf. Example 1.5). Combining this result with identity (1.32) we obtain an elegant example of general interest in functional analysis of an operator which is continuous together with its inverse, but neither the operator itself, nor its inverse are bounded.

Example 3.15 Consider a set $Z := \{(a,b) \in \mathbb{R}^2\,; -f(a) \le b \le f(a)\} \subset X = \mathbb{R}^2$, where $f : [-1,\infty[\to [-1,1[$ is a concave increasing smooth function, $f(-1) = 0$, $f'(-1+) = +\infty$. Let w be an arbitrary continuously differentiable function in $[0,1]$ and put $u(t) := (0, w(t))$, $x(t) = (a(t), b(t)) := \mathcal{S}(0,u)(t)$ for $t \in [0,1]$. By (3.25) we have

$$\dot{a}(t) = \begin{cases} |\dot{w}(t)| \dfrac{f'(a(t))}{1 + f'^2(a(t))} & \text{if } \dot{w}(t) \ne 0\,,\ \mathrm{sign}\,(\dot{w}(t))\, b(t) = f(a(t))\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{3.33}$$

hence a is nondecreasing and nonnegative. Assume that w increases in an interval $[s,t]$, $w(s) = -1$, $w(t) = 1$. Then $a(\tau) = a(s)$ as long as $b(\tau) = b(s) + w(\tau) - w(s)$ stays below $f(a(s))$, i.e. for $\tau \in [s, \tau_0]$ with $w(\tau_0) - w(s) = f(a(s)) - b(s)$, while in $]\tau_0, t]$ we find $a(\tau)$ as solution of the equation (3.33) with initial condition $a(\tau_0) = a(s)$. We have in particular

$$\begin{aligned} a(t) - a(s) \;&=\; \int_{\tau_0}^t \dot{w}(\tau) \frac{f'(a(\tau))}{1 + f'^2(a(\tau))}\, d\tau \;\ge\; (w(t) - w(\tau_0)) \frac{f'(a(t))}{1 + f'^2(a(t))} \\ &\ge\; 2\,(1 - f(a(t)))\, \frac{f'(a(t))}{1 + f'^2(a(t))}\,. \end{aligned} \tag{3.34}$$

The same estimate holds if we assume that w decreases in $[s,t]$, $w(s) = 1$, $w(t) = -1$. The above considerations show that $\mathcal{S}(0,\cdot)$ does not map the bounded set $\mathcal{M} = \{u_n \in C([0,1];\mathbb{R}^2)\,; u_n(t) = (0, \cos n\pi t)\}$ into a bounded set, since by (3.34) we have $\sup\{a_n(1)\,; n \in \mathbb{N}\} = \infty$, where $(a_n, b_n) := \mathcal{S}(0, u_n)$.

Remark 3.16 We have seen in (1.13), (1.14) that in classical models of plasticity, the characteristic Z has often the form of a convex cylinder $Z = \tilde{Z} + Y^\perp$ as in Definition 2.8.

Let $\mathcal{P},\mathcal{S} : Z \times C([0,T];X) \to C([0,T];X)$, $\tilde{\mathcal{P}},\tilde{\mathcal{S}} : \tilde{Z} \times C([0,T];Y) \to C([0,T];Y)$ be the play and stop operators with characteristics $Z, \tilde{Z}$, respectively, let $\tilde{x}^0 \in \tilde{Z}$, $y \in Y$, $x^0 = \tilde{x}^0 + y$ be given vectors and let $v \in C([0,T];Y)$, $w \in C([0,T];Y^\perp)$, $u = v + w$ be given functions. The time-discrete construction in the proof of Theorem 3.1 and Remark 2.10 then yield

$$\mathcal{P}(x^0,u) = \tilde{\mathcal{P}}(\tilde{x}^0,v)\,, \quad \mathcal{S}(x^0,u) = \tilde{\mathcal{S}}(\tilde{x}^0,v) + w\,. \tag{3.35}$$

4 Uniformly strictly convex characteristics

The concept of uniform strict convexity introduced in Definition 2.13 enables us to prove a uniform continuity result for the play operator in $Z \times C([0,T];X) \to C([0,T];X)$ following Section 17.1 of [KP].

Theorem 4.1 *Let $Z \subset X$ be uniformly strictly convex and let α be the function associated to Z by formula (2.10). Then for all $u,v \in C([0,T];X)$, $x^0, y^0 \in Z$ we have*

$$|\mathcal{P}(x^0,u) - \mathcal{P}(y^0,v)|_\infty \le \max\left\{|x^0 - y^0 - u(0) + v(0)|\,;\, \alpha^{-1}\left(|u-v|_\infty\right)\right\}. \tag{4.1}$$

Proof. By density and continuity, it suffices to assume $u,v \in W^{1,1}(0,T;X)$. Put $\xi := \mathcal{P}(x^0,u)$, $\eta := \mathcal{P}(y^0,v)$, $x := u - \xi$, $y := v - \eta$ and

$$V(t) := \max\left\{|\xi(t) - \eta(t)|\,;\, \alpha^{-1}\left(|u-v|_\infty\right)\right\} \qquad \text{for } \ t \in [0,T]\,.$$

The function V is absolutely continuous. Assume that for some $t \in]0,T[$ we have $\dot{V}(t) > 0$. Then

$$|\xi(t) - \eta(t)| > \alpha^{-1}\left(|u-v|_\infty\right) \tag{4.2}$$

and $d/dt\ \left(|\xi(t)-\eta(t)|^2\right) = 2\big\langle\dot{\xi}(t) - \dot{\eta}(t), \xi(t) - \eta(t)\big\rangle > 0$.

At least one of the expressions $\big\langle\dot{\xi}(t), \xi(t) - \eta(t)\big\rangle$, $\big\langle\dot{\eta}(t), \eta(t) - \xi(t)\big\rangle$ must therefore be positive. Let us choose for instance $\big\langle\dot{\xi}(t), \xi(t) - \eta(t)\big\rangle > 0$. This implies $\dot{\xi}(t) \neq 0$, hence by Proposition 3.5 we have $x(t) := u(t) - \xi(t) \in \partial Z$ and $\xi(t) - \eta(t) \in X \setminus T_Z\big(x(t)\big)$. From the definition of the function α we infer

$$\alpha(|\xi(t)-\eta(t)|) \le |P\big(x(t) + \xi(t) - \eta(t)\big)| \le |x(t) + \xi(t) - \eta(t) - y(t)| = |u(t) - v(t)|$$

which contradicts (4.2). We conclude $\dot{V}(t) \le 0$ a.e. and the assertion follows. ❒

Proposition 2.14 implies that Theorem 4.1 cannot give better results than a global 1/2-Hölder estimate. Example 4.3 of [K] shows that for $Z = B_r(0)$, the uniform bound (4.1) is optimal.

5 Smooth characteristics

In this section we derive further regularity properties of the play operator $\mathcal{P}$ under the following hypothesis.

Hypothesis 5.1 *$Z \subset X$ is a convex closed set such that*

$$(5.1)\qquad \begin{array}{ll} \text{(i)} & B_r(0) \subset Z \subset B_R(0)\,, \\ \text{(ii)} & \textit{for each } x \in \partial Z \textit{ there exists a unique outward normal vector } n(x)\,, \\ & |n(x)| = 1\,, \\ \text{(iii)} & \textit{the mapping } n : \partial Z \to \partial B_1(0) \textit{ is continuous}\,. \end{array}$$

Indeed, (iii) follows from (ii) if $\dim X < \infty$. This need not be true for $\dim X = \infty$, cf. the example on p. 46 of [K].

5.1 Strict continuity

We already know that the play maps in general $Z \times C([0,T];X)$ into $CBV(0,T;X)$. This mapping is discontinuous with respect to the strong topologies of $C([0,T];X)$ and $BV(0,T;X)$ even in the simplest case $\dim X = 1$. This can easily be verified by the following construction.

Example 5.2 Put $X := \mathbb{R}^1$, $Z = [-1,1]$, $u_0(t) := 1 + t$, $u_n(t) := 1 + t + (1/n)\sin nt$ for $n \in \mathbb{N}$ and $t \in [0, 2\pi]$, $\xi_n := \mathcal{P}(1, u_n)$, $x_n := u_n - \xi_n$ for $n \in \mathbb{N} \cup \{0\}$. The functions u_n are nondecreasing, $x_n(0) = 1$. Proposition 3.5 yields $x_n(t) = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t \in [0, 2\pi]$, hence $\xi_0(t) = t$, $\xi_n(t) = t + (1/n)\sin nt$ for $n \in \mathbb{N}$, and we easily check that $\lim_{n\to\infty} |u_n - u_0|_\infty = 0$, $\operatorname*{Var}_{[0,2\pi]}(\xi_n - \xi_0) = 4$.

We however prove here that the play operator is continuous with respect to the strict metric (8.11).

Proposition 5.3 *Let Hypothesis 5.1 hold. Then for every sequence $\big\{(x_k^0, u_k)\,;\ k \in \mathbb{N} \cup \{0\}\big\}$ in $Z \times C([0,T];X)$ such that $\lim_{k\to\infty} |u_k - u_0|_\infty = 0$, $\lim_{k\to\infty} |x_k^0 - x_0^0| = 0$ we have $\operatorname*{Var}_{[0,T]} \mathcal{P}(u_0) = \lim_{k\to\infty} \operatorname*{Var}_{[0,T]} \mathcal{P}(u_k)$.*

Proposition 5.3 is an easy consequence of Lemma 3.10, Corollary 3.13, Theorem 8.16 and of the following Lemma.

Lemma 5.4 *Let the assumptions of Proposition 5.3 be satisfied. Let $\nu : Z \to B_1(0)$ be defined by the formula $\nu(0) := 0$, $\nu(x) := M(x)\, n(x/M(x))$ for $x \in Z \setminus \{0\}$, where M is the Minkowski functional associated to Z by formula (2.9). Then for every $u \in C([0,T];X)$ and $x^0 \in Z$ we have*

$$(5.2)\qquad \operatorname*{Var}_{[0,T]} \xi = \int_0^T \big\langle \nu(x(t)), d\xi(t) \big\rangle,$$

where $\xi = \mathcal{P}(x^0, u)$, $x = u - \xi$.

Proof of Lemma 5.4. Let us first assume $u \in W^{1,1}(0,T\,;X)$. Then $\dot\xi(t) = 0$ if $x(t) \in \operatorname{Int} Z$, $\dot\xi(t) = |\dot\xi(t)|\, n(x(t))$ if $x(t) \in \partial Z$, hence, $|\dot\xi(t)| = \langle \nu(x(t)), \dot\xi(t) \rangle$ a.e. and (5.2) holds.

Let now $u \in C([0,T];X)$ be arbitrary and let $\{u_k\,;\, k \in \mathbb{N}\}$ be a sequence in $W^{1,1}(0,T\,;X)$ such that $\lim_{k\to\infty} |u_k - u|_\infty = 0$, and put $\xi_k := \mathcal{P}(x^0, u_k)$, $x_k := u_k - \xi_k$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be an arbitrary partition of $[0,T]$. The mapping ν is continuous; by Lemma 3.10, Corollary 3.13 and Theorem 8.16 we therefore have $\operatorname*{Var}_{[0,T]} \xi_k \le$ const., $\lim_{k\to\infty} \operatorname*{Var}_{[0,T]} \xi_k = \int_0^T \langle \nu(x(t)), d\xi(t) \rangle$ and

$$\sum_{j=1}^{N} |\xi(t_j) - \xi(t_{j-1})| = \lim_{k\to\infty} \sum_{j=1}^{N} |\xi_k(t_j) - \xi_k(t_{j-1})| \le \int_0^T \langle \nu(x(t)), d\xi(t) \rangle \le \operatorname*{Var}_{[0,T]} \xi,$$

hence (5.2) holds. ❐

A.A. Vladimirov's example below shows that the smoothness assumption in Proposition 5.3 cannot be omitted.

Example 5.5 Assume that there exists $\bar x \in \partial Z$ and $n_1, n_2 \in N_Z(\bar x)$ such that $n_1 \ne n_2$, $|n_1| = |n_2| = 1$. We define a sequence $\{u_k\,; k \in \mathbb{N} \cup \{0\}\}$ in $W^{1,1}(0,1\,;X)$ by the formula

$$\begin{aligned} u_k(0) &= 0 \qquad \text{for } k \ge 0\,, \\ \dot u_0(t) &= \frac{1}{2}(n_1 + n_2) \qquad \text{for } t \in\,]0,1[\,, \\ \dot u_k(t) &= \begin{cases} n_1 & \text{for } t \in\,](j-1)2^{1-k}, (2j-1)2^{-k}[\,, \\ n_2 & \text{for } t \in\,](2j-1)2^{-k}, j\,2^{1-k}[\,, \end{cases} \qquad j = 1, \dots, 2^{k-1}\,,\ k > 0\,. \end{aligned}$$

For $k \ge 1$ put $\xi_k := \mathcal{P}(\bar x, u_k)$. Then $\dot u_k(t) \in N_Z(\bar x)$ a.e., hence $\dot\xi_k(t) = \dot u_k(t)$ a.e. By construction we have $\lim_{k\to\infty} |u_k - u_0|_\infty = 0$ and $\operatorname*{Var}_{[0,1]} \xi_k = \int_0^1 |\dot u_k(t)|\, dt$, hence

$$\operatorname*{Var}_{[0,1]} \xi_0 = \frac{1}{2}|n_1 + n_2| < 1 = \lim_{k\to\infty} \operatorname*{Var}_{[0,1]} \xi_k\,.$$

5.2 Local Lipschitz continuity in $W^{1,1}(0,T\,;X)$

In this subsection (Theorem 5.6 and Corollary 5.9 below) we derive a local Lipschitz estimate which improves the result of [D] mentioned without proof in [KP], in the sense that we give an explicit upper bound for the Lipschitz constant. For the ball $Z = B_r(0)$ we fill the gap between inequality (A.34) and Example A.8 of [BK] and show that our estimate (5.6) is optimal.

We start by introducing an auxiliary functional $\mu : X \to \mathbb{R}^+$ by the formula

$$\mu(z) := \frac{|z|^2}{M^*(z)} \quad \text{for } z \in X \setminus \{0\}\,, \quad \mu(0) = 0\,. \tag{5.3}$$

Then, by Proposition 2.20, we have for all $z \in X$,

$$\frac{1}{R}|z| \le \mu(z) \le M(z) \le \frac{1}{r}|z|. \tag{5.4}$$

The functional μ is not necessarily convex. It suffices to consider the complex plane $\mathbb{C}$ with $\mathbb{C} \supset Z := \{a+bi\,;\ |a|^p+|b|^p \le 1\}$ for some $p > 2$. Then $Z^* = \{a+bi\,;\ |a|^{p'}+|b|^{p'} \le 1\}$ with $1/p+1/p' = 1$ and the set $\tilde{Z} := \{z \in \mathbb{C};\ \mu(z) \le 1\}$ has the form $\tilde{Z} := \{a+bi\,;\ a^2+b^2 \le (|a|^{p'}+|b|^{p'})^{1/p'}\}$. To check that $\tilde{Z}$ is nonconvex, we put $z = a+bi$, $\hat{z} = a-bi$ with $a = (1+\varepsilon^{p'})^{1/p'}/(1+\varepsilon^2)$, $b = \varepsilon\, a$. Then z, $\hat{z}$ belong to $\tilde{Z}$, but $(z+\hat{z})/2 \notin \tilde{Z}$ for $\varepsilon > 0$ sufficiently small.

Theorem 5.6 *Let Z satisfy Hypothesis 5.1. Then for every $x_0, y_0 \in Z$ and $u, v \in W^{1,1}(0,T;X)$, we have*

$$\begin{aligned} \mu(\dot{\xi} - \dot{\eta}) \ &+\ \frac{1}{2}\frac{d}{dt}\left|M^2(x) - M^2(y)\right| \\ &\le\ \frac{R}{r}\left(|\dot{u}|\,|J(x) - J(y)| + M(\dot{u} - \dot{v})\right) \quad a.e., \end{aligned} \tag{5.5}$$

where $\xi = \mathcal{P}(x_0, u)$, $\eta = \mathcal{P}(y_0, v)$, $x = u - \xi$, $y = v - \eta$, M is the Minkowski functional of Z and J is the duality mapping from Theorem 2.23. Note that J is single-valued by Proposition 2.25 and Hypothesis 5.1.

Example 5.7 Estimate (5.5) is optimal for $Z = B_r(0)$. In this case we have $\mu(z) = M(z) = |z|/r$, $J(z) = z/r^2$ and (5.5) reads

$$|\dot{\xi} - \dot{\eta}| + \frac{1}{2r}\frac{d}{dt}\left||x|^2 - |y|^2\right| \le \frac{1}{r}|\dot{u}|\,|x-y| + |\dot{u} - \dot{v}| \quad \text{a.e.} \tag{5.6}$$

To see that (5.6) cannot be improved, it suffices to consider the disk $Z = B_r(0)$ in the complex plane $\mathbb{C}$. For some $\alpha > 0$ and $h > 0$ we define the functions $u(t) := r\, e^{i\alpha t}$, $v(t) := (r+h)\, e^{i\alpha t}$ for $t \in [0,T]$. Let $\phi \in]0, \pi/2[$ be the solution of the equation $\cos\phi = r/(r+h)$. We easily check using formula (3.25) that the functions $x(t) := u(t)$, $y(t) := r\, e^{i(\alpha t + \phi)}$ satisfy $x = \mathcal{S}(r, u)$, $y = \mathcal{S}(r\, e^{i\phi}, v)$. We therefore have $|\dot{\xi} - \dot{\eta}| = |\dot{y} - \dot{v}| = \alpha\left|r + h - r\, e^{i\phi}\right| = \alpha\left(h^2 + 2rh\right)^{1/2}$, $|x - y| = r\left|1 - e^{i\phi}\right| = r\left(2h/(r+h)\right)^{1/2}$, $|\dot{u} - \dot{v}| = h\alpha$, $|\dot{u}| = r\alpha$, $|x| = |y| = r$. The quantity $C > 0$ for which the inequality

$$|\dot{\xi} - \dot{\eta}| + \frac{1}{2r}\frac{d}{dt}\left||x|^2 - |y|^2\right| \le C\,|x - y| + |\dot{u} - \dot{v}|$$

holds independently of α and h, must satisfy

$$C \ge \frac{\alpha\sqrt{2r+2h}}{\sqrt{2r+h} + \sqrt{h}},$$

hence $C = \alpha = |\dot{u}|/r$ is the best possible.

Proof of Theorem 5.6. Let t be a Lebesgue point of all functions $\dot{u}$, $\dot{v}$, $\dot{\xi}$, $\dot{\eta}$, $d/dt\, M(x)$, $d/dt\, M(y)$ and $d/dt\,|M^2(x) - M^2(y)|$. Using Remark 3.9, we distinguish the following cases (omitting the argument t which is the same everywhere).

A. $\dot\xi = \dot\eta = 0$. Then $\dot x = \dot u$, $\dot y = \dot v$ and

$$\begin{aligned}\mu(\dot\xi - \dot\eta) + \frac{1}{2}\frac{d}{dt}\Big|M^2(x) - M^2(y)\Big| &\le \Big|\langle J(x), \dot u\rangle - \langle J(y), \dot v\rangle\Big| \\ &\le |\dot u||J(x) - J(y)| + \Big|\langle J(y), \dot u - \dot v\rangle\Big| \\ &\le |\dot u||J(x) - J(y)| + M^*(J(y))\, M(\dot u - \dot v),\end{aligned}$$

where $M^*(J(y)) = M(y) \le 1$.

B. $\dot\xi \ne 0$, $\dot\eta = 0$. Then $M(x) = 1$, $M(y) \le 1$, $\dot y = \dot v$, $d/dt\, M(x) = 0$, $\dot\xi = \langle n(x), \dot u\rangle\, n(x)$, $\langle n(x), \dot u\rangle > 0$, hence

$$\begin{aligned}\mu(\dot\xi - \dot\eta) &+ \frac{1}{2}\frac{d}{dt}\Big|M^2(x) - M^2(y)\Big| \\ &= \mu(\dot\xi) + \frac{1}{2}\frac{d}{dt}\left(M^2(x) - M^2(y)\right) = \mu(\dot\xi) - \langle J(y), \dot v\rangle .\end{aligned}$$

Here, we have by Proposition 2.25 $n(x) = J(x)/|J(x)|$ and

$$\mu(\dot\xi) = \frac{\langle n(x), \dot u\rangle^2}{M^*(n(x))\langle n(x), \dot u\rangle} = \langle J(x), \dot u\rangle$$

and we obtain the same conclusion as in case A.

C. $\dot\xi = 0$, $\dot\eta \ne 0$. We proceed analogously as in B. with the same result.

D. $\dot\xi \ne 0$, $\dot\eta \ne 0$. Then $M(x) = M(y) = 1$, $d/dt\, M(x) = d/dt\, M(y) = 0$, $\langle n(x), \dot u\rangle > 0$, $\langle n(y), \dot v\rangle > 0$, $\dot\xi = \langle n(x), \dot u\rangle\, n(x)$, $\dot\eta = \langle n(y), \dot v\rangle\, n(y)$, hence, by (5.4),

$$\begin{aligned}\mu(\dot\xi - \dot\eta) + \frac{1}{2}\frac{d}{dt}\Big|M^2(x) - M^2(y)\Big| &\le M\left(\langle n(x), \dot u\rangle\, n(x) - \langle n(y), \dot v\rangle\, n(y)\right) \\ &\le M\left(\langle n(y), \dot u - \dot v\rangle\, n(y)\right) + M\left(\langle n(x), \dot u\rangle\, n(x) - \langle n(y), \dot u\rangle\, n(y)\right),\end{aligned}$$

where

$$\begin{aligned}M\left(\langle n(y), \dot u - \dot v\rangle\, n(y)\right) &\le \frac{1}{r}\Big|\langle n(y), \dot u - \dot v\rangle\Big| \\ &\le \frac{1}{r} M^*(n(y))\, M(\dot u - \dot v) \le \frac{R}{r} M(\dot u - \dot v),\end{aligned}$$

$$\begin{aligned}M\left(\langle n(x), \dot u\rangle\; n(x) - \langle n(y), \dot u\rangle\; n(y)\right) &\le \frac{1}{r}\Big|\langle n(x), \dot u\rangle\, n(x) - \langle n(y), \dot u\rangle\, n(y)\Big| \\ \le \frac{1}{r}\left(|\dot u|\, |n(x) - n(y)|\right) &\le \frac{R}{r}\, |\dot u|\, |J(x) - J(y)| .\end{aligned}$$

The last two inequalities follow from Lemma 5.8 below and from the fact that for $x, y \in \partial Z$ we have $|J(x)| \ge |M^*(J(x))|/R = 1/R$, $|J(y)| \ge 1/R$. Theorem 5.6 is proved. ❐

Lemma 5.8

(i) *For all* $e, f, w \in X$ *with* $|e| = |f| = 1$ *we have*

$$\big|\langle e,w\rangle\, e \,-\, \langle f,w\rangle\, f\big| \,\le\, \frac{1}{2}\,|w|\,|e\,-\,f|\,|e\,+\,f|\,,$$

(ii) *for all* $u,\, v \in X$, $|u|, |v| \ge 1/R$, *we have*

$$\left|\frac{u}{|u|} \,-\, \frac{v}{|v|}\right| \,\le\, R\,|u\,-\,v|\,.$$

Proof.

(i) The case $e = \pm f$ is trivial. For $e \neq \pm f$ we have

$$\begin{aligned}
\big|\langle e,w\rangle\, e - \langle f,w\rangle\, f\big|^2 \;&=\; \frac{1}{4}\,|e-f|^2\,|e+f|^2\left(\left\langle \frac{e-f}{|e-f|}\,,\,w\right\rangle^2 + \left\langle \frac{e+f}{|e+f|}\,,\,w\right\rangle^2\right)\\
&\le\; \frac{1}{4}\,|e\,-\,f|^2\,|e\,+\,f|^2\,|w|^2\,,
\end{aligned}$$

since $\langle e-f\,,\,e+f\rangle \;=\; 0$.

(ii) The inequality follows from the elementary computation

$$\begin{aligned}
\left|\frac{u}{|u|} \,-\, \frac{v}{|v|}\right|^2 \;&=\; 2\,-\,\frac{2\,\langle u,v\rangle}{|u|\,|v|}\\
&=\; 2\,+\,2\langle u,v\rangle\left(R^2 \,-\, \frac{1}{|u|\,|v|}\right) \,-\, 2\,R^2\,\langle u,v\rangle\\
&\le\; 2R^2\,(|u|\,|v| \,-\, \langle u,v\rangle) \;\le\; R^2\,|u\,-\,v|^2\,.
\end{aligned}$$

❐

Corollary 5.9 *Under the hypotheses of Theorem 5.6 we have*

$$\left|\dot\xi \,-\, \dot\eta\right| \,+\, \frac{R}{2}\frac{d}{dt}\left|M^2(x) - M^2(y)\right| \,\le\, \frac{R^2}{r}\,|\dot u|\,\,|J(x) \,-\, J(y)| \,+\, \frac{R^2}{r^2}\,|\dot u\,-\,\dot v| \qquad a.e.$$

If moreover there exists a constant $k > 0$ *such that*

$$|n(x) \,-\, n(y)| \,\le\, k\,|x\,-\,y| \qquad \forall\, x,y \,\in\, \partial Z\,, \tag{5.7}$$

then

$$\int_0^T \left|\dot\xi \,-\, \dot\eta\right|\,dt \;\le\; \left(\frac{R^2}{r^2} + L\,\frac{R^2}{r}\int_0^T |\dot u|\,\,dt\right)\left(\left|x^0 - y^0\right| + \int_0^T |\dot u\,-\,\dot v|\,\,dt\right)\,,$$

where

$$L \;:=\; \left(\frac{1}{r^2} \,+\, \frac{k}{r}\left(1 \,+\, \frac{R}{r}\right)^2\right). \tag{5.8}$$

Corollary 5.9 is an immediate consequence of inequalities (5.5), (5.4), (3.21) and of the following lemma.

Lemma 5.10 *Let Hypothesis 5.1 and inequality (5.7) hold. Then for every $x, y \in X$ we have $|J(x) - J(y)| \leq L\,|x - y|$ with Q given by (5.8).*

Proof. Assume first $x, y \in \partial Z$. Then

$$\begin{aligned} |J(x) - J(y)| &= \left|\frac{n(x)}{M^*(n(x))} - \frac{n(y)}{M^*(n(y))}\right| \\ &\leq \frac{1}{M^*(n(x))}\left[|n(x) - n(y)| + \frac{1}{M^*(n(y))}\, M^*\,(n(x) - n(y))\right] \\ &\leq \frac{1}{r}\left(1 + \frac{R}{r}\right)|n(x) - n(y)| \;\leq\; K\,|x - y|\,, \end{aligned}$$

where $K := (1 + R/r)\,k/r$.

The assertion is trivial if $x = 0$ or $y = 0$. For arbitrary $x, y \in X \setminus \{0\}$ we have

$$\begin{aligned} |J(x) - J(y)| &= \left|M(x)\,J\left(\frac{x}{M(x)}\right) - M(y)\,J\left(\frac{y}{M(y)}\right)\right| \\ &\leq \left|J\left(\frac{x}{M(x)}\right)\right| M(x - y) + M(y)\left|J\left(\frac{x}{M(x)}\right) - J\left(\frac{y}{M(y)}\right)\right| \\ &\leq \frac{1}{r^2}\,|x - y| + K\,M(y)\left|\frac{x}{M(x)} - \frac{y}{M(y)}\right| \\ &\leq \left(\frac{1}{r^2} + K\right)|x - y| + K\frac{|x|}{M(x)}\,M(x - y) \\ &\leq \left(\frac{1}{r^2} + K\left(1 + \frac{r}{R}\right)\right)|x - y| \end{aligned}$$

and Lemma 5.10 is proved. ❐

Example 5.11 The smoothness of Z is substantial for the local Lipschitz continuity of the play in $W^{1,1}(0, T\,; X)$. We show here a counterexample motivated by an idea of [D]. Let $Z \in \mathbb{R}^3$ be a cone of the form

$$Z = \left\{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3\,;\; c \geq \sqrt{a^2 + b^2}\right\}$$

We define a two-parameter family $\{u_{\alpha,\phi}\,;\ \alpha > 1\,,\ \phi \in [0, \pi/2]\}$ of functions $[0,1] \to \mathbb{R}^3$ by the formula

$$u_{\alpha,\phi}(t) := \frac{1}{\alpha}\begin{pmatrix} \cos \alpha t \\ \sin \alpha t \\ -\alpha t\, \sin \phi \end{pmatrix} \quad , \qquad t \in [0,1]\,.$$

Putting

$$x_{\alpha,\phi}(t) := \frac{1}{\alpha}\cos\phi \begin{pmatrix} \cos(\alpha t+\phi) \\ \sin(\alpha t+\phi) \\ 1 \end{pmatrix}, \quad \xi_{\alpha,\phi}(t) := \frac{1}{\alpha}\begin{pmatrix} \sin\phi\,\sin(\alpha t+\phi) \\ -\sin\phi\,\cos(\alpha t+\phi) \\ -\alpha t\,\sin\phi - \cos\phi \end{pmatrix},$$

we check similarly as in Example 5.7 above that we have $x_{\alpha,\phi} = \mathcal{S}\left(x_{\alpha,\phi}(0), u_{\alpha,\phi}\right)$, $\xi_{\alpha,\phi} = \mathcal{P}\left(x_{\alpha,\phi}(0), u_{\alpha,\phi}\right)$ for all α and ϕ.

Assume that the operator $\mathcal{P}$ is locally Lipschitz in $W^{1,1}(0,1;\mathbb{R}^3)$. The system $\{u_{\alpha,\phi}\}$ is bounded in $W^{1,1}(0,1;\mathbb{R}^3)$, there must therefore exist a constant $C>0$ independent of α and ϕ such that

$$\text{(5.9)} \qquad \int_0^1 \left|\dot\xi_{\alpha,\phi}(t) - \dot\xi_{\alpha,\frac{\pi}{2}}(t)\right| dt \;\le\; C\left(\left|x_{\alpha,\phi}(0) - x_{\alpha,\frac{\pi}{2}}(0)\right| + \left|u_{\alpha,\phi}(0) - u_{\alpha,\frac{\pi}{2}}(0)\right| + \int_0^1 \left|\dot u_{\alpha,\phi}(t) - \dot u_{\alpha,\frac{\pi}{2}}(t)\right| dt\right).$$

We have $\left|\dot u_{\alpha,\phi}(t) - \dot u_{\alpha,\frac{\pi}{2}}(t)\right| = 1-\sin\phi$, $\left|\dot\xi_{\alpha,\phi}(t) - \dot\xi_{\alpha,\frac{\pi}{2}}(t)\right| = \sqrt{2(1-\sin\phi)}$ for all $t \in [0,1]$, $\left|u_{\alpha,\phi}(0) - u_{\alpha,\frac{\pi}{2}}(0)\right| = 0$, $\left|x_{\alpha,\phi}(0) - x_{\alpha,\frac{\pi}{2}}(0)\right| = (\sqrt{2}/\alpha)\cos\phi$, hence (5.9) reads $\sqrt{2(1-\sin\phi)} \le C\left((\sqrt{2}/\alpha)\cos\phi + 1 - \sin\phi\right)$ independently of $\alpha\to\infty$ and $\phi\to\pi/2$, which is a contradiction.

6 Polyhedral characteristics

In this section we investigate continuity properties of the play with a polyhedral characteristic of the form

$$\text{(6.1)} \qquad Z := \{z \in X\,;\ \langle z, n_i\rangle \le \beta_i\,,\ i = 1,\dots,p\}$$

with given unit vectors $n_1,\dots,n_p$ and given positive numbers $\beta_1,\beta_2,\dots,\beta_p$ as in Example 2.28. According to Remark 3.16 we may assume $X = \operatorname{span}\{n_1,\dots,n_p\}$, $\dim X = N < \infty$.

Notation 6.1 For an arbitrary subspace $X' \subset X$ we denote by $P_{X'}$ the orthogonal projection onto X'. In particular, $P_X = I$ is the identity operator and the orthogonal projection onto $\operatorname{span}\{n_i\}$ is denoted by P_i, i.e.

$$\text{(6.2)} \qquad P_i z := \langle z, n_i\rangle\, n_i\,, \quad z \in X\,, \quad i = 1,\dots,p\,.$$

We further denote by $\mathcal{D}_k$, $0 \le k \le N$, the system of all k-dimensional subspaces of X generated by the vectors $n_1,\dots,n_p$, that is, $\mathcal{D}_0 = \{\{0\}\}$, $\mathcal{D}_N = \{X\}$ and

$$\mathcal{D}_k := \{X' \subset X\,;\ X' = \operatorname{span}\{n_{i_1},\dots,n_{i_r}\}\,,\ i_j \in \{1,\dots,p\} \text{ for } j = 1,\dots,r\,,\ \dim X' = k\}\,, \quad k = 1,\dots,N-1\,.$$

We need in the sequel the following elementary properties of projections.

Lemma 6.2 Let $X'' \subset X' \subset X$ be subspaces of X. Then

(i) $\quad P_{X''}P_{X'} \;=\; P_{X'}P_{X''} \;=\; P_{X''}\,,$

(ii) $\quad |\langle z, v\rangle| \;\le\; |P_{X'}z| \;\le\; |z| \quad \forall\, z \in X\,,\ \forall\, v \in X'\,, |v| \le 1\,.$

Our main objective is to prove that on a polyhedron, $\mathcal{P}$ is globally Lipschitz continuous in both $Z \times C([0,T];X) \to C([0,T];X)$ and $Z \times W^{1,1}(0,T\,;X) \to W^{1,1}(0,T\,;X)$. Theorem 6.3 goes back to [KP]. Independently, a more general result in the space of regulated functions (i.e. functions which have both one-sided limits at each point) endowed with the sup-norm was obtained in [DI] for a larger class of problems. We show here a different (and simpler) proof based on the approach proposed in [P]. Theorem 6.5 was conjectured without proof in [KP]. It has recently been proved by Desch and Turi in [DT] and we repeat here their elegant argument.

Theorem 6.3 *For every $u, v \in C([0,T];X)$, $x^0, y^0 \in Z$ we have under the above hypotheses*

$$|\mathcal{P}(x^0,u) - \mathcal{P}(y^0,v)|_\infty \le M_N\left(|x^0 - y^0| + |u - v|_\infty\right)\,, \tag{6.3}$$

where M_N is defined recurrently by the formula

$$M_0 \;:=\; 0\,, \quad M_{k+1} \;:=\; \left(\frac{1}{1-\varepsilon_k^2}\left(1 + M_k^2 + 2\,\varepsilon_k\, M_k\right)\right)^{1/2}\,, \tag{6.4}$$

$$\varepsilon_k \;:=\; \max\{|P_{X'}n_j|\,;\ X' \in \mathcal{D}_k\,,\, n_j \notin X'\} \tag{6.5}$$

for $k = 0, 1, \ldots, N-1$.

Remark 6.4 For $N \le 2$, the Lipschitz constant M_N in Theorem 6.3 is the best possible, for $N \ge 3$ this question is open. The optimality of $M_1 = 1$ is obvious. In the case $N = 2$ we can identify X with the complex plane $\mathbb{C}$ and for a given $\alpha \in\,]0, \pi/2[$ put

$$Z := \left\{\varrho\, e^{i\varphi} \in \mathbb{C}\,;\ \varrho \ge 0,\ \varphi \in [-\alpha, \alpha]\right\}\,. \tag{6.6}$$

In fact, the condition $0 \in \text{Int } Z$ does not hold here, but it can be satisfied by shifting simply the figure to the left. We have here

$$M_2 \;=\; \begin{cases} 1/\sin\alpha & \text{for} \quad \alpha \in\,]0, \pi/4]\,, \\ 1/\cos\alpha & \text{for} \quad \alpha \in\,]\pi/4, \pi/2[\,. \end{cases} \tag{6.7}$$

To check that these values are optimal, we fix an arbitrary $r > 0$ and an arbitrary partition $0 = t_0 < t_1 < \cdots < t_K = T$, and construct continuous functions u, v to be affine in each interval $[t_{i-1}, t_i]$, $|u - v|_\infty = r$, with the intention to get $|\mathcal{P}(x^0,u) - \mathcal{P}(x^0,v)|_\infty$ arbitrarily close to $M_2\, r$ for a suitably chosen $x^0 \in Z$ and for K sufficiently large. The argument is different in each of the two cases distinguished in (6.7). Technical details are left to the reader, cf. also [KP], [K] and [DI].

A. $\alpha \in\,]0, \pi/4]$: For $k \in \mathbb{N}$ put $u(t_{4k}) = u(t_{4k+2}) := 0$, $u(t_{4k+1}) := r\, i\, e^{i\alpha}$, $u(t_{4k+3}) := -r\, i\, e^{-i\alpha}$, $v(t) \equiv 0$, $x^0 := 0$. Then $|\mathcal{P}(x^0,u) - \mathcal{P}(x^0,v)|(t_{4k})$ converge to $M_2\, r$ as $k \to \infty$.

B. $\alpha \in]\pi/4, \pi/2[$: Put $x^0 := 0$, $u(0) = v(0) := 0$ and assume that u, v are defined in an interval $[0, t_{8k}]$, $u(t_{8k}) = v(t_{8k}) = U_k$. We introduce the sequences

$$
\begin{aligned}
\varrho_m &:= r \tan\alpha\,(1 - |\cos 2\alpha|^m)\,,\\
\mu_m &:= -\varrho_m \frac{\cos 2\alpha}{\sin\alpha}\,,
\end{aligned}
$$

for $m \in \mathbb{N} \cup \{0\}$ and define the functions u, v in $[t_{8k}, t_{8k+8}]$ recurrently as

$$
\begin{array}{ll}
u(t_{8k+1}) := U_k - r\,i\,e^{-i\alpha}\,, & v(t_{8k+1}) := U_k\,,\\
u(t_{8k+2}) := U_k\,, & v(t_{8k+2}) := U_k\,,\\
u(t_{8k+3}) := U_k + i\,\mu_{2k}\,, & v(t_{8k+3}) := U_k + i\,\mu_{2k}\,,\\
u(t_{8k+4}) := U_k + i\,\mu_{2k} + r\,i\,e^{i\alpha}\,, & v(t_{8k+4}) := U_k + i\,\mu_{2k} + r\,i\,e^{i\alpha}\,,\\
u(t_{8k+5}) := u(t_{8k+4})\,, & v(t_{8k+5}) := v(t_{8k+4}) + r\,i\,e^{i\alpha}\,,\\
u(t_{8k+6}) := u(t_{8k+4})\,, & v(t_{8k+6}) := v(t_{8k+4})\,,\\
u(t_{8k+7}) := u(t_{8k+4}) - i\,\mu_{2k+1}\,, & v(t_{8k+7}) := v(t_{8k+4}) - i\,\mu_{2k+1}\,,\\
u(t_{8k+8}) := u(t_{8k+7}) - r\,i\,e^{-i\alpha}\,, & v(t_{8k+8}) := v(t_{8k+7}) - r\,i\,e^{-i\alpha}\,.
\end{array}
$$

Putting $U_{k+1} := u(t_{8k+8}) = v(t_{8k+8})$ we continue by induction and show that for all k, the outputs $x := \mathcal{S}(x^0, u)$, $y := \mathcal{S}(x^0, v)$ of the stop operator $\mathcal{S}$ take on the values $x(t_{8k}) = 0$, $y(t_{8k}) = \varrho_{2k}\, e^{-i\alpha}$, $x(t_{8k+4}) = \varrho_{2k+1}\, e^{i\alpha}$, $y(t_{8k+4}) = 0$, and $|\mathcal{P}(x^0, u)(t_{8k+2}) - \mathcal{P}(x^0, v)(t_{8k+2})| = |x(t_{8k+2}) - y(t_{8k+2})|$ converge to $M_2\, r$ as $k \to \infty$.

The problem of optimality of the Lipschitz constant L_N in Theorem 6.5 below is completely open, except for the trivial case $N = 1$, indeed.

Theorem 6.5 *For every $u, v \in W^{1,1}(0, T; X)$, $x^0, y^0 \in Z$ we have under the above hypotheses*

$$
\int_0^T \left| \frac{d}{dt}\mathcal{P}(x^0, u) - \frac{d}{dt}\mathcal{P}(y^0, v)\right|(t)\,dt \le L_N \left(|x^0 - y^0| + \int_0^T |\dot u(t) - \dot v(t)|\,dt \right), \tag{6.8}
$$

where L_N is defined recurrently by the formula

$$
L_1 := 1\,, \quad L_k := \frac{1 + L_{k-1}}{1 - \delta_k}\,, \tag{6.9}
$$

$$
\begin{aligned}
\delta_1 := 0\,, \quad \delta_k := \max\Big\{ &|(I - P_j)\,(\delta_{k-1} P_{X'} w + (I - P_{X'})w)|\,;\\
&j = 1, \dots, p\,,\ X' \in \mathcal{D}_{k-1}\,,\ w \in X' \oplus \operatorname{span}\{n_j\}\,,\ |w| = 1\Big\}
\end{aligned} \tag{6.10}
$$

for $k = 2, \dots, N$.

The definition of δ_k, L_k is meaningful, since the set $\mathcal{D}_{k-1}$ is finite, the unit ball in X is compact and the following Lemma holds.

Lemma 6.6 $\delta_k < 1$ *for all* $k \in \{1, \dots, N\}$.

Proof. We proceed by induction over k. Assume that $\delta_{k-1} < 1$. We obviously have $\delta_k \le 1$. Assume that for some w, j and X' we have

$$(6.11) \qquad \Big|(I - P_j)\,(\delta_{k-1}P_{X'}w + (I - P_{X'})w)\Big| = 1\,.$$

Then

$$\begin{aligned} |P_{X'}w|^2 + |(I - P_{X'}w)|^2 &= 1 = |(I - P_j)\,(\delta_{k-1}P_{X'}w + (I - P_{X'})w)|^2 \\ = \; \delta_{k-1}^2|P_{X'}w|^2 &+ |(I - P_{X'})w|^2 - |P_j\,(\delta_{k-1}P_{X'}w + (I - P_{X'})\,w)|^2\,, \end{aligned}$$

hence $P_{X'}w = 0$, $P_jw = 0$ and consequently $w = 0$, which contradicts (6.11). ❐

6.1 Lipschitz continuity in $C([0,T];X)$

To prove Theorem 6.3, we start with two auxiliary Lemmas which are due to V. Lovicar, see [P].

Lemma 6.7 *Let Z be a polyhedron (6.1). For $z \in Z$ put $\Gamma(z) := \{k \in \{1,\ldots,p\}\,;\ \langle z, n_k\rangle = \beta_k\}$, $C(z) := \{w \in X\,;\ w = \sum_{k\in\Gamma(z)} a_k n_k\,,\ a_k \ge 0\}$. Then $C(z) = N_Z(z)$, where $N_Z(z)$ is the normal cone (2.8).*

Proof. We obviously have $C(z) \subset N_Z(z)$. The set $C(z)$ is a convex closed cone and we can associate to it the projection pair (P_z, Q_z) according to formula (2.3). Let $w \in N_Z(z)$ be arbitrary. We have by definition

$$(6.12) \qquad \langle P_z w, Q_z w - \varphi\rangle \;\ge\; 0 \qquad \forall \varphi \in C(z)\,,$$

$$(6.13) \qquad \langle w, z - \psi\rangle \;\ge\; 0 \qquad \forall \psi \in Z.$$

For $k \in \Gamma(z)$ we have $Q_z w + n_k \in C(z)$, and (6.12) yields $\langle P_z w, n_k\rangle \le 0$. For $k \in \{1,\ldots,p\}\setminus\Gamma(z)$ we have $\langle z, n_k\rangle < \beta_k$. In both cases we obtain $z + \delta P_z w \in Z$ for some sufficiently small $\delta > 0$. Putting $\psi := z + \delta P_z w$ we infer from (6.13) and Lemma 2.2 (iii) that $|P_z w|^2 \le \langle P_z w, w\rangle \le 0$, hence $w \in C(z)$. ❐

Lemma 6.8 *Let Z be as above and let $u, v \in W^{1,1}(0,T\,;X)$ be given. For $t \in [0,T]$ put $\xi(t) := \mathcal{P}(x^0,u)(t)$, $\eta(t) := \mathcal{P}(y^0,v)(t)$, $x(t) := u(t) - \xi(t)$, $y(t) := v(t) - \eta(t)$, $g(t) := \xi(t) - \eta(t)$. Then for every $j \in \Gamma(x(t))$ we have $\langle n_j, g(t)\rangle \le |u(t) - v(t)|$ and for every $i \in \Gamma(y(t))$ we have $\langle n_i, g(t)\rangle \ge -|u(t) - v(t)|$.*

Proof. For $j \in \Gamma(x(t))$ we have $n_j \in N_Z(x(t))$, hence $\langle n_j, g(t)\rangle \le \langle n_j, u(t) - v(t)\rangle \le |u(t) - v(t)|$ and similarly for $i \in \Gamma(y(t))$. ❐

We now pass to the proof of Theorem 6.3.

Proof of Theorem 6.3. We may assume that $u, v \in W^{1,1}(0,T\,;X)$. We fix an arbitrary number $r > |x^0 - y^0| + |u - v|_\infty$ and introduce a Lyapunov function $V : X \to \mathbb{R}^1$ by the formula

$$(6.14) \qquad V(z) := \max\left\{M_k^2\, r^2 + |(I - P_{X'})z|^2\ ;\ X' \in \mathcal{D}_k\,,\ k = 0,\ldots,N\right\}\,.$$

The function V is convex, hence $t \mapsto V(g(t))$ is absolutely continuous. Keeping the notation from Lemma 6.8, we check that

$$\frac{d}{dt}V(g(t)) \le 0 \qquad \text{almost everywhere}. \tag{6.15}$$

Assume the contrary, namely that for some $t \in]0,T[$ the derivatives $\dot\xi(t), \dot\eta(t)$ exist and

$$\frac{d}{dt}V(g(t)) > 0\,. \tag{6.16}$$

Then there exist $k \in \{0,\dots,N-1\}$ and $X'' \in \mathcal{D}_k$ such that

$$V(g(t)) = M_k^2\, r^2 + |(I - P_{X''})g(t)|^2 \tag{6.17}$$

(note that for $k = N$ we have $I - P_{X''} = 0$). Inequality (6.16) yields $\big\langle \dot g(t), (I-P_{X''})g(t)\big\rangle > 0$. We can assume $\big\langle \dot\xi(t), (I-P_{X''})g(t)\big\rangle > 0$ (otherwise we interchange the roles of u and v). We have $\dot\xi(t) \in N_Z(x(t))$, hence by Lemmas 6.7, 6.8 there exists $j \in \Gamma(x(t))$ such that

$$r > \big\langle n_j, g(t)\big\rangle > \big\langle n_j, P_{X''}g(t)\big\rangle\,. \tag{6.18}$$

This implies in particular that $n_j \notin X''$. We put $X' := X'' \oplus \operatorname{span}\{n_j\}$ and find $v \in X''$, $|v| = 1$ and real numbers a, b such that

$$P_{X'}g(t) = a\, n_j + b\, v. \tag{6.19}$$

Put $\varepsilon := \big\langle n_j, v\big\rangle \in [-\varepsilon_k, \varepsilon_k]$. By Lemma 6.2 (ii) we have

$$|P_{X''}g(t)| \ge |\big\langle g(t), v\big\rangle| = |a\varepsilon + b|\,. \tag{6.20}$$

On the other hand, inequality (6.18) yields

$$r > a + b\varepsilon > a|P_{X''}n_j|^2 + b\varepsilon\,, \tag{6.21}$$

hence $a > 0$. From (6.20), (6.21) it follows $a(1-\varepsilon^2) < r - b\varepsilon - a\varepsilon^2 \le r + |\varepsilon|\,|P_{X''}g(t)|$ and

$$\begin{aligned} |P_{X'}g(t)|^2 &= a^2 + b^2 + 2ab\varepsilon = (a\varepsilon + b)^2 + a^2(1-\varepsilon^2) \\ &< |P_{X''}g(t)|^2 + \frac{1}{1-\varepsilon^2}\left(r + |\varepsilon|\,|P_{X''}g(t)|\right)^2 \\ &\le \frac{1}{1-\varepsilon_k^2}\left(|P_{X''}g(t)|^2 + r^2 + 2\, r\, \varepsilon_k\, |P_{X''}g(t)|\right)\,. \end{aligned} \tag{6.22}$$

We have $X' \in \mathcal{D}_{k+1}$. Assumption (6.17) then yields

$$M_{k+1}^2\, r^2 - |P_{X'}g(t)|^2 \ge M_k^2\, r^2 - |P_{X''}g(t)|^2\,, \tag{6.23}$$

and combining (6.23), (6.22) and the definition of M_{k+1} we obtain

$$\varepsilon_k\,\left(M_k^2\, r^2 - |P_{X''}g(t)|^2\right) \,+\, 2\,r\,\left(M_k\, r - |P_{X''}g(t)|\right) \,<\, 0, \tag{6.24}$$

hence

$$\begin{aligned} 0 \;&>\; M_k^2\, r^2 - |P_{X''}g(t)|^2 \\ &=\; \left(M_k^2\, r^2 + |(I-P_{X''})g(t)|^2\right) - \left(M_0^2\, r^2 + |(I - P_{\{0\}})g(t)|^2\right)\,, \end{aligned} \tag{6.25}$$

which contradicts the assumption (6.17). Consequently, (6.15) holds. By hypothesis, we have $|g(0)| < r$, hence $|g(t)|^2 \le V(g(t)) \le V(g(0)) \le M_N^2\, r^2$ for all t and the proof is complete. ❐

6.2 Lipschitz continuity in $W^{1,1}(0,T\,;X)$

Before passing to the proof of Theorem 6.5, we first define an equivalent norm in X based on the following construction.

Definition 6.9 *Let $m \in \mathbb{N}$, $z \in X$ and a sequence $\{i_j\}_{j=1}^m$ be given, $i_j \in \{1,\dots,p\}$ for $j = 1,\dots,m$. Then the sequence $\{z_j\}_{j=1}^m$ of elements of X defined by the formula*

$$z_1 := z\,, \quad z_{j+1} = (I - P_{i_j})\,z_j\,, \quad j = 1,\dots,m \tag{6.26}$$

is called a string. *We denote by $\mathcal{T}(z,m)$ the set of all strings of the form (6.26)*

$$\mathcal{T}(z,m) = \left\{\{z_j\}_{j=1}^{m+1}\ ;\ z_j \text{ given by (6.26)}\ ,\, i_j \in \{1,\dots,p\}\,,\ j = 1,\dots,m\right\}. \tag{6.27}$$

The main result of this subsection reads

Theorem 6.10 *For $z \in X$ put*

$$\|z\| := \sup\left\{\sum_{j=1}^{m} |P_{i_j} z_j|\,;\, m \in \mathbb{N}\,,\, \{z_j\}_{j=1}^{m+1} \in \mathcal{T}(z,m)\right\}. \tag{6.28}$$

Then $\|\cdot\|$ is a norm in X satisfying the inequalities

$$\|z\| \ge |P_i z| + \|(I - P_i)z\| \quad \forall\, z \in X\,, \quad \forall\, i \in \{1,\dots,p\}\,. \tag{6.29}$$

$$\|z\| \le L_N |z| \quad \forall\, z \in X\,, \tag{6.30}$$

with L_N defined in Theorem 6.5.

For the proof of Theorem 6.10 we need some auxiliary results which we state in the following form.

Lemma 6.11 *Let $i_1,\dots,i_r \in \{1,\dots,p\}$ be given, $X' = \operatorname{span}\{n_{i_1},\dots,n_{i_r}\}$, $\dim X' = k$. Then for every $z \in Z$ we have*

$$\left|(I - P_{i_r})(I - P_{i_{r-1}})\cdots(I - P_{i_1})P_{X'}z\right| \le \delta_k |P_{X'}z|\,. \tag{6.31}$$

Proof of Lemma 6.11. The statement is trivial for $k = 1$. For $k > 1$ we proved by induction. Assume that the assertion holds for $k-1$. We find $l \le r$ such that $X' = \operatorname{span}\{n_{i_1},\dots,n_{i_l}\}$, $X'' := \operatorname{span}\{n_{i_1},\dots,n_{i_{l-1}}\} \in \mathcal{D}_{k-1}$. Let $z \in X$ be given.

We define

$$\begin{aligned} w_1 &:= (I - P_{i_{l-1}})\cdots(I - P_{i_1})P_{X''}z \\ w_2 &:= (I - P_{X''})P_{X'}z \end{aligned}$$

By induction hypothesis, we have $|w_1| \leq \delta_{k-1}|P_{X''}z|$. Further, put

$$w := \begin{cases} w_2 & \text{if } \delta_{k-1} = 0\,, \\ w_2 + \frac{1}{\delta_{k-1}} w_1\,, & \text{if } \delta_{k-1} > 0\,. \end{cases}$$

Then $w_1 = \delta_{k-1} P_{X''} w$, $w_2 = (I - P_{X''})w$, and

$$|w|^2 = |P_{X''}w|^2 + \left|(I - P_{X''})w\right|^2 \leq |P_{X''}z|^2 + \left|(I - P_{X''})P_{X'}z\right|^2 = |P_{X'}z|^2\,.$$

By Lemma 6.2 we have

$$\begin{aligned} (6.32) \qquad & (I - P_{i_l})(I - P_{i_{l-1}}) \cdots (I - P_{i_1})P_{X'}z \\ & = (I - P_{i_l})\big[P_{X''}\big(I - P_{i_{l-1}}\big) \cdots (I - P_{i_1})P_{X'}z \\ & \quad + (I - P_{X''})(I - P_{i_{l-1}}) \cdots (I - P_{i_1})P_{X'}z\big] \\ & = (I - P_{i_l})(w_1 + w_2)\,, \end{aligned}$$

hence, by definition of δ_k, we have

$$\begin{aligned} (6.33) \qquad & \left|(I - P_{i_r})(I - P_{i_{r-1}}) \cdots (I - P_{i_1})P_{X'}z\right| \\ & \leq \left|(I - P_{i_l})(I - P_{i_l}) \cdots (I - P_{i_1})P_{X'}z\right| \\ & = \left|(I - P_{i_l})\left(\delta_{k-1}P_{X''}w + (I - P_{X''})w\right)\right| \leq \delta_k |w| \end{aligned}$$

and inequality (6.31) follows easily. ❐

Lemma 6.12 *Let $m \in \mathbb{N}$, $z \in X$ and a sequence $\{i_j\}_{j=1}^m$ be given, $i_j \in \{1, \ldots, p\}$ for $j = 1, \ldots, m$, $X := \operatorname{span}\{n_{i_1}, \ldots, n_{i_m}\} \in \mathcal{D}_k$ for some $k \in \{1, \ldots, N\}$. Let $\{z_j\}_{j=1}^{m+1}$ be the string defined by (6.26). Then*

$$(6.34) \qquad \sum_{j=1}^{m} \left|P_{i_j} z_j\right| \leq L_k |P_{X'}z|\,,$$

where L_k is given by (6.9).

Proof of Lemma 6.12. Here again, we use the induction over k. For $k = 1$, we obviously have $P_{i_j} z_j = 0$ for $j \geq 2$ and the assertion holds. Assume now that it holds for $1, \ldots, k-1$. For a given $z \in X$ we construct the sequence $j(0) < j(1) < \ldots < j(s) \leq m$ recurrently over $d = 0, 1, \ldots, s$ as follows

(i) $j(0) = 0$,

(ii) if $\operatorname{span}\{N_{i_{j(d)+1}}, n_{i_m}\} = X'$ then

$$j(d+1) := \min\left\{j > j(d); \quad \operatorname{span}\{n_{i_{j(d)+1}}, \ldots, n_{i_j}\} = X'\right\}.$$

We have

$$(6.35) \qquad \sum_{j=1}^{m} |P_{i_j} z_j| = \sum_{d=1}^{s} \left(\sum_{j=j(d-1)+1}^{j(d)} |P_{i_j} z_j| \right) + \sum_{j=j(s)+1}^{m} |P_{i_j} z_j|$$

where, according to the induction hypothesis, we have

$$(6.36) \qquad \sum_{j=j(d-1)+1}^{j(d)-1} |P_{i_j} z_j| \le L_{k-1} |P_{X'} z_{j(d-1)+1}|, \quad d = 1, \dots, s,$$

$$(6.37) \qquad \sum_{j=j(s)+1}^{m} |P_{i_j} z_j| \le L_{k-1} |P_{X'} z_{j(s)+1}|$$

and, obviously,

$$(6.38) \qquad |P_{i_{j(d)}} z_{j(d)}| \le |P_{X'} z_{j(d-1)+1}|, \quad d = 1, \dots, s.$$

Then (6.35–6.38) yield

$$(6.39) \qquad \sum_{j=1}^{m} |P_{i_j} z_j| \le \sum_{d=0}^{s} (1 + L_{k-1}) |P_{X'} z_{j(d)+1}|.$$

By Lemma 6.11, we have for all $d = 1, 2, \dots, s$

$$(6.40) \qquad |P_{X'} z_{j(d)+1}| \le \delta_k |P_{X'} z_{j(d-1)+1}|,$$

hence

$$(6.41) \qquad \sum_{j=1}^{m} |P_{i_j} z_j| \le (1 + L_{k-1}) \left(\sum_{d=0}^{\infty} \delta_k^d \right) |P_{X'} z| = \frac{1 + L_{k-1}}{1 - \delta_k} |P_{X'} z|$$

and Lemma 6.12 is proved. ❐

Proof of Theorem 6.10. Inequality (6.30) follows from Lemma 6.12 above. In particular, $\|z\|$ is finite for all $z \in Z$, $\|0\| = 0$. Conversely, $\|z\| = 0$ implies $|P_i z| = 0$ for all $i = 1, \dots, p$, hence $z = 0$. Furthermore, the triangle inequality and the identity $\|tz\| = |t|\,\|z\|$ for all $z \in X$ and $t \in \mathbb{R}$ follow automatically from (6.28). We thus proved that $\|\cdot\|$ is a norm. It remains to check that inequality (6.29) holds. Let $z \in X$, $i \in \{1, \dots, p\}$ and a sequence $\{i_j\}_{j=1}^{m}$, $i_j \in \{1, \dots, p\}$ be given and put $z_1 := (I - P_i) z$, $z_{j+1} := (I - P_{i_j}) z_j$, $j = 1, \dots, m$, $z_0 := z$, $i_0 := i$. Then

$$(6.42) \qquad \begin{aligned} \sum_{j=1}^{m} |P_{i_j} z_j| &= \sum_{j=0}^{m} |P_{i_j} z_j| - |P_i z| \\ &\le \|z\| - |P_i z| \end{aligned}$$

and passing to the supremum in the left-hand side of (6.42) we obtain (6.29). Theorem 6.10 is proved. ❐

Theorem 6.10 has the following consequences.

Corollary 6.13 *For every $z, w \in X$ such that w is of the form*

$$w = \sum_{i=1}^{p} \alpha_i n_i\,, \quad \alpha_i \in \mathbb{R}\,, \tag{6.43}$$

where the coefficients α_i satisfy the implication

$$\alpha_i \neq 0 \ \Rightarrow \ \alpha_i \langle z, n_i \rangle > 0\,, \tag{6.44}$$

there exists $\varepsilon_0 > 0$ such that

$$\|z - \varepsilon w\| + \varepsilon |w| \le \|z\| \qquad \forall\, \varepsilon \in\,]0, \varepsilon_0[. \tag{6.45}$$

Proof. Put $A_0 := \{i \in \{1, \dots, p\};\ \alpha_i = 0\}$, $A_0^C := \{1, \dots, p\} \setminus A_0$. Let $\varepsilon_0 > 0$ be chosen in such a way that

$$\sum_{i \in A_0^C} \frac{\alpha_i}{\langle z, n_i \rangle} \le \frac{1}{\varepsilon_0}. \tag{6.46}$$

For $\varepsilon \in]0, \varepsilon_0[$ and $i \in A_0^C$ put $\eta_i := \dfrac{\varepsilon \alpha_i}{\langle z, n_i \rangle}$. Then $\eta_i > 0$, $\sum_{i \in A_0^C} \eta_i < 1$ and

$$\begin{aligned} z - \varepsilon w &= z - \sum_{i \in A_0^C} \varepsilon\, \alpha_i\, n_i = z - \sum_{i \in A_0^C} \eta_i\, P_i z \\ &= z \left(1 - \sum_{i \in A_0^C} \eta_i \right) + \sum_{i \in A_0^C} \eta_i (I - P_i) z\,. \end{aligned}$$

From (6.29), we infer

$$\begin{aligned} \|z - \varepsilon w\| &\le \|z\| \left(1 - \sum_{i \in A_0^C} \eta_i \right) + \sum_{i \in A_0^C} \eta_i \left(\|z\| - |P_i z| \right) \\ &= \|z\| - \sum_{i \in A_0^C} \varepsilon |\alpha_i| \le \|z\| - \varepsilon |w| \end{aligned}$$

and Corollary 6.13 is proved. ❐

Corollary 6.14 *For every z, $w \in X$, such that w is of the form (6.43), with coefficients α_i satisfying the inequality*

$$\alpha_i \langle z, n_i \rangle \ge 0 \quad \forall\, i \in \{1, \dots, p\}\,, \tag{6.47}$$

we have

$$\|z + w\| \ge \|z\| + |w| \tag{6.48}$$

Proof. Put $A_1 := \{i \in \{1,\dots,p\}; \langle z, n_i\rangle = 0\}$, $A_1^C := \{1,\dots,p\}\backslash A_1$, $X_1 :=$ span $\{n_i; i \in A_1\}$. We fix a subset $A_2 \subset A_1$ in such a way that $\{n_i; i \in A_2\}$ are linearly independent, $X_1 =$ span $\{n_i; i \in A_2\}$. We then have

$$w \;=\; \sum_{i=1}^{p} \tilde{\alpha}_i\, n_i\,, \tag{6.49}$$

where $\tilde{\alpha}_i = 0$ for $i \in A_1 \setminus A_2$, $\tilde{\alpha}_i = \alpha_i$ for $i \in A_1^C$.

We now find the *dual basis* $\{n_i^*;\, i \in A_2\}$ of X_1 with the property

$$\langle n_i^*, n_j\rangle \;=\; \delta_{ij} \quad \text{for all } \; i,j \in A_2\,. \tag{6.50}$$

This can be done in the following way. Let $\{e_k;\, k \in A_2\}$ be an orthonormal basis in X_1. Then $n_j = \sum_{k\in A_2} n_{jk} e_k$, where $N = (n_{jk})_{j,k\in A_2}$ is a nonsingular matrix. We look for a matrix $N^* = (n_{jk}^*)_{j,k\in A_2}$ such that the vectors $n_j^* = \sum_{k\in A_2} n_{jk}^* e_k$ satisfy (6.50), that is

$$\sum_{k\in A_2} n_{jk} n_{ik}^* \;=\; \delta_{ij}\,. \tag{6.51}$$

In other words, it suffices to put $N^* = (N^{\perp})^{-1}$.

We further denote $h := \sum_{i\in A_2} \tilde{\alpha}_i n_i^*$ and claim that for $\delta > 0$ sufficiently small, the vectors $z + \delta h$, w satisfy the assumptions of Corollary 6.13. Indeed, let $\delta > 0$ be so small that

$$\operatorname{sign}\left(\langle z \,+\, \delta h, n_i\rangle\right) \;=\; \operatorname{sign}\left(\langle z, n_i\rangle\right) \quad \forall\, i \,\in\, A_1^C\,. \tag{6.52}$$

We want to prove that the implication

$$\tilde{\alpha}_i \,\neq\, 0 \,\Rightarrow\, \tilde{\alpha}_i\langle z \,+\, \delta h, n_i\rangle \,>0 \quad \forall\, i \,\in\, \{1,\dots,p\}\,. \tag{6.53}$$

holds.

For $i \in A_1^C$, (6.53) follows from (6.52) and (6.47), for $i \in A_1 \setminus A_2$, we have $\tilde{\alpha}_i = 0$, hence (6.53) holds. For $i \in A_2$, the definition of h entails

$$\langle z \,+\, \delta h, n_i\rangle \;=\; \delta\langle h, n_i\rangle \;=\; \delta\tilde{\alpha}_i \tag{6.54}$$

and (6.53) follows.

From Corollary 6.13 we infer that there exists some $\varepsilon_0(\delta)$ such that for all $\varepsilon \in]0, \varepsilon_0(\delta)[$ we have

$$\|z \,+\, \delta h \,-\, \varepsilon w\| \,+\, \varepsilon|w| \,\leq\, \|z \,+\, \delta h\|\,, \tag{6.55}$$

and, in particular,

$$\begin{aligned} (1 \,+\, \varepsilon)\|z \,+\, \delta h\| \;&=\; \|z \,+\, \delta h \,-\, \varepsilon w \,+\, \varepsilon(z \,+\, \delta h \,+\, w)\| \\ &\leq\; \|z \,+\, \delta h\| \,-\, \varepsilon|w| \,+\, \varepsilon\|z \,+\, \delta h \,+\, w\|\,, \end{aligned} \tag{6.56}$$

hence

$$\|z + \delta h\| + |w| \le \|z + \delta h + w\| . \tag{6.57}$$

Letting δ tend to 0, we obtain the assertion. ❒

Theorem 6.5 is an immediate consequence of Theorem 6.10 and of Theorem 6.15 below.

Theorem 6.15 *Let $u, v \in W^{1,1}(0,T)$ be given, let $\mathcal{P}_Z$ be the play with characteristic Z given by (6.1). Put $\xi := \mathcal{P}_Z(u)$, $\eta := \mathcal{P}_Z(v)$, $x := u - \xi$, $y := v - \eta$. Then*

$$\frac{d}{dt}\|x - y\| + |\dot{\xi} - \dot{\eta}| \le \|\dot{u} - \dot{v}\| \quad a.e. \tag{6.58}$$

Proof. Let $t \in]0,T[$ be an arbitrary Lebesgue point of all functions $\dot{u}$, $\dot{v}$, $\dot{\xi}$, $\dot{\eta}$, $\dot{x}$, $\dot{y}$, $d/dt\,\|x - y\|$ and let $\tau \in]0,t[$ be arbitrary. Put $z := x(t) - y(t)$, $w := \tau(\dot{\xi}(t) - \dot{\eta}(t))$. By Lemma 6.7, we have $\dot{\xi}(t) = \sum_{k\in\Gamma(x(t))} \gamma_k\, n_k$, $\dot{\eta}(t) = \sum_{j\in\Gamma(y(t))} \tilde{\gamma}_j\, n_j$ with $\gamma_k \ge 0$, $\tilde{\gamma}_j \ge 0$ for all $k \in \Gamma(x(t))$, $j \in \Gamma(y(t))$. We thus can write

$$w = \sum_{i=1}^{p} \alpha_i\, n_i \tag{6.59}$$

with

$$\left\{ \begin{array}{lll} \alpha_i = 0 & & \text{for } i \in \{1,\ldots,p\} \setminus (\Gamma(x(t)) \cup \Gamma(y(t))) \,, \\ \alpha_i = \gamma_i \,, & \langle z, n_i\rangle \ge 0 & \text{for } i \in \Gamma(x(t)) \setminus \Gamma(y(t)) \,, \\ \alpha_i = -\tilde{\gamma}_i \,, & \langle z, n_i\rangle \le 0 & \text{for } i \in \Gamma(y(t)) \setminus \Gamma(x(t)) \,, \\ \langle z, n_i\rangle = 0 & & \text{for } i \in \Gamma(x(t)) \cap \Gamma(y(t)) \,, \end{array} \right. \tag{6.60}$$

hence inequality (6.47) holds. From Corollary 6.14 it follows

$$\Big\| \big(x(t) - y(t)\big) + \tau\big(\dot{\xi}(t) - \dot{\eta}(t)\big) \Big\| \ge \Big\| x(t) - \eta(t) \Big\| + \tau \Big|\dot{\xi}(t) - \dot{\eta}(t)\Big| . \tag{6.61}$$

This yields

$$\begin{aligned} \Big\|(x - y)(t - \tau)\Big\| &= \Big\|(x - y)(t) - \int_{t-\tau}^{t} (\dot{x} - \dot{y})(s)\, ds\Big\| \\ &= \Big\|(x - y)(t) + \tau(\dot{\xi} - \dot{\eta})(t) - \tau(\dot{u} - \dot{v})(t) \\ &\qquad + \int_{t-\tau}^{t} ((\dot{x} - \dot{y})(t) - (\dot{x} - \dot{y})(s))\, ds\Big\| \\ &\ge \Big\|(x - y)(t)\Big\| + \tau\Big|(\dot{\xi} - \dot{\eta})(t)\Big| - \tau\Big\|(\dot{u} - \dot{v})(t)\Big\| \\ &\qquad - \tau\Big\|\frac{1}{\tau}\int_{t-\tau}^{t} ((\dot{x} - \dot{y})(t) - (\dot{x} - \dot{y})(s))\, ds\Big\| \end{aligned} \tag{6.62}$$

Dividing inequality (6.62) by τ and passing to the limit as $\tau \to 0+$ we obtain (6.58). ❒

7 Second order variation

In this short section we prove an additional regularity result for hysteresis operators, namely an upper bound for the total variation of the derivative of the output. It can be applied in particular to the play operator on smooth domains or polyhedrons, see Sections 5 and 6. Since the time derivative of the output is typically discontinuous, this is the maximal regularity we can expect. The result is formulated for the whole class of causal rate-independent operators (see (1.25), (1.26)) which are locally Lipschitz continuous in $W^{1,1}(0,T;X)$.

Lemma 7.1 *Let $\mathcal{F} : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ be a continuous, causal and rate-independent operator, let $u \in W^{1,1}(0,T;X)$ be given and let $\xi = \mathcal{F}(u)$. For a given $h \in]0,T[$ put*

$$u_0(t) := \begin{cases} u(0) & \text{for} \quad t \in [0,h]\,, \\ u(t-h) & \text{for} \quad t \in]h,T]\,. \end{cases}$$

Then

$$\mathcal{F}(u_0)(t) = \xi_0(t) := \begin{cases} \xi(0) & \text{for} \quad t \in [0,h]\,, \\ \xi(t-h) & \text{for} \quad t \in]h,T]\,. \end{cases}$$

Proof. For $\varepsilon \in]0, \frac{T-h}{2}[$ put

$$\alpha_\varepsilon(t) := \begin{cases} \frac{\varepsilon}{h+\varepsilon}\, t & \text{for} \quad t \in [0, h+\varepsilon]\,, \\ t-h & \text{for} \quad t \in]h+\varepsilon, T-\varepsilon]\,, \\ T + \frac{h+\varepsilon}{\varepsilon}(t-T) & \text{for} \quad t \in]T-\varepsilon, T]\,. \end{cases}$$

Then α_ε is an increasing homeomorphism for every ε and the rate-independence yields

$$\mathcal{F}(u \circ \alpha_\varepsilon)(t) = \xi(\alpha_\varepsilon(t)) \qquad \forall t \in [0,T]\,,\ \varepsilon \in \left]0, \frac{T-h}{2}\right[. \tag{7.1}$$

Let $0 < T^* < T$ be arbitrarily chosen. For $\varepsilon \in]0, T-T^*[$ put $u_\varepsilon^* := (u \circ \alpha_\varepsilon)\big|_{[0,T^*]}$, $\xi_\varepsilon^* := (\xi \circ \alpha_\varepsilon)\big|_{[0,T^*]}$, $u_0^* := u_0\big|_{[0,T^*]}$, $\xi_0^* := \xi_0\big|_{[0,T^*]}$. From the causality of $\mathcal{F}$ we infer $\xi_\varepsilon^* = \mathcal{F}^*(u_\varepsilon^*)$, where $\mathcal{F}^* : W^{1,1}(0,T^*;X) \to W^{1,1}(0,T^*;X)$ is the restriction of $\mathcal{F}$. We have $u_0^*(0) = u_\varepsilon^*(0)$ and

$$\begin{aligned} \int_0^{T^*} |\dot{u}_\varepsilon^*(t) - \dot{u}_0^*(t)|\, dt &\le \int_0^{h+\varepsilon} \frac{\varepsilon}{h+\varepsilon} \left| \dot{u}\left(\frac{\varepsilon}{h+\varepsilon}\, t\right)\right| dt + \int_h^{h+\varepsilon} |\dot{u}(t-h)|\, dt \\ &= 2\int_0^{\varepsilon} |\dot{u}(t)|\, dt\,, \end{aligned}$$

hence $u_\varepsilon^* \to u_0^*$ in $W^{1,1}(0,T^*;X)$ as $\varepsilon \to 0$. By construction, we analogously have $\xi_\varepsilon^* \to \xi_0^*$ in $W^{1,1}(0,T^*;X)$ as $\varepsilon \to 0$ and from the continuity of $\mathcal{F}$ it follows $\xi_0^* = \mathcal{F}^*(u_0^*)$. Since $T^* < T$ was arbitrary, the assertion follows. ❐

Theorem 7.2 *Assume that an operator* $\mathcal{F} : W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ *is causal, rate-independent and locally Lipschitz in the following sense: there exists a function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for every* $u, v \in W^{1,1}(0,T;X)$ *the functions* $\xi = \mathcal{F}(u)$, $\eta = \mathcal{F}(v)$ *satisfy*

$$\int_0^T |\dot{\xi}(t) - \dot{\eta}(t)|\, dt \leq f(\max\{|u|_{1,1}, |v|_{1,1}\})\, |u - v|_{1,1} . \tag{7.2}$$

Then for every $u \in W^{1,1}(0,T;X)$ *such that* $\dot{u} \in BV(0,T;X)$ *there exists a function* $w \in NBV(0,T;X)$ *such that* $w = \dot{\xi}$ *a.e. and*

$$\operatorname*{Var}_{[0,T]} w \;\leq\; f(|u|_{1,1}) \left(|\dot{u}(0+)| \,+\, \operatorname*{Var}_{[0,T]} \dot{u} \right) . \tag{7.3}$$

The hypotheses of Theorem 7.2 are fulfilled for the operator $\mathcal{F}(u) := \mathcal{P}(x^0, u)$ for a fixed $x^0 \in Z$ provided Z is a polyhedron (Theorem 6.5) or a smooth domain satisfyng the hypotheses of Lemma 5.10. In the latter case, estimate (7.2) follows from Lemma 5.10, Corollary 5.9 and inequality (3.21).

Proof of Theorem 7.2. Let $u \in W^{1,1}(0,T;X)$ with $\dot{u} \in BV(0,T;X)$. Put

$$v(t) := \begin{cases} u(t-h) & \text{for} \quad t \in [h,T] , \\ u(0) & \text{for} \quad t \in [0,h[. \end{cases}$$

Lemma 7.1 yields

$$\eta(t) := \begin{cases} \xi(t-h) & \text{for} \quad t \in [h,T] , \\ \xi(0) & \text{for} \quad t \in [0,h[. \end{cases}$$

From (7.2) we obtain

$$\int_h^T |\dot{\xi}(t) - \dot{\xi}(t-h)|\, dt \;\leq\; f(|u|_{1,1}) \left(\int_0^h |\dot{u}(t)|\, dt \,+\, \int_h^T |\dot{u}(t) - \dot{u}(t-h)|\, dt \right)$$

and the rest of the proof follows from Theorem 8.12. ❐

8 Integration of vector-valued functions

In this section we recall basic notions of the Bochner integral and of the theory of functions of bounded variation that are referred to in the text; for details, see [Bre], [Y], [HP], [K].

8.1 Bochner integral

Definition 8.1 *Let* B *be a real Banach space endowed with norm* $\|\cdot\|$ *and let* $[a,b] \subset \mathbb{R}^1$ *be a compact interval. A function* $u : [a,b] \to B$ *is called*

(i) simple, *if there exists a partition* $[a,b]=\bigcup_{k=1}^{N}E_k$ *of the interval* $[a,b]$ *into a finite union of pairwise disjoint Lebesgue measurable sets* $\{E_k;\ k=1,\dots,N\}$ *and a sequence* $\{x_k;\ k=1,\dots,N\}$ *in* B *such that for almost all* $t\in[a,b]$ *we have*

$$u(t)=\sum_{k=1}^{N}x_k\,\chi_{E_k}(t), \tag{8.1}$$

where χ_{E_k} *is the* characteristic function *of the set* E_k, *that is,*

$$\chi_{E_k}(t)=\begin{cases}0 & \text{if}\quad t\notin E_k,\\ 1 & \text{if}\quad t\in E_k;\end{cases}$$

(ii) strongly measurable, *if there exists a sequence* $\{u_n;\ n\in\mathbb{N}\}$ *of simple functions such that* $\lim\limits_{n\to\infty}\|u_n(t)-u(t)\|=0$ *for a.e.* $t\in[a,b]$.

It is easy to see that for a strongly measurable function, $u:[a,b]\to B$, the scalar-valued function $t\mapsto\|u(t)\|$ is Lebesgue measurable. The following characterization of strongly measurable functions is useful in applications.

Theorem 8.2 (Lusin) *A function* $u:[a,b]\to B$ *is strongly measurable if and only if for every* $\delta>0$ *there exist a closed set* $F_\delta\subset[a,b]$ *and a continuous function* $w:[a,b]\to B$ *such that* $\operatorname{meas}([a,b]\setminus F_\delta)<\delta$, $u(t)=w(t)$ *for all* $t\in F_\delta$ *and* $\sup\limits_{[a,b]}\|w(t)\|\le\sup\limits_{[a,b]}\|u(t)\|$.

For a simple function $u:[a,b]\to B$ of the form 8.1 we define its *Bochner integral* over a measurable set $A\subset[a,b]$, by the formula

$$\int_A u(t)\,dt:=\sum_{k=1}^{N}x_k\ \operatorname{meas}(E_k\cap A)\in B. \tag{8.2}$$

The general definition reads as follows.

Definition 8.3 *An arbitrary function* $u:[a,b]\to B$ *is said to be* Bochner integrable *in* $[a,b]$ *if there exists a sequence* $\{u_n;n\in\mathbb{N}\}$ *of simple functions* $[a,b]\to B$ *such that* $\lim\limits_{n\to\infty}\int_a^b\|u_n(t)-u(t)\|dt=0$ *and we define its Bochner integral over a measurable set* $A\subset[a,b]$ *as*

$$\int_A u(t)\,dt:=\lim_{n\to\infty}\int_A u_n(t)\,dt\in B. \tag{8.3}$$

Notice that the sequence $U_n:=\int_A u_n(t)\,dt$ in Definition 8.3 is fundamental in B and its limit (8.3) is independent of the choice of the sequence $\{u_n\}$. The definition immediately implies

$$\left\|\int_A u(t)\,dt\right\|\le\int_A\|u(t)\|\,dt<\infty \tag{8.4}$$

for each Bochner integrable function u and measurable set $A\subset[a,b]$.

Bochner's Theorem 8.4 below gives an elegant characterization of Bochner integrable functions.

Theorem 8.4 (Bochner) *A function $u : [a,b] \to B$ is Bochner integrable if and only if it is strongly measurable and $\int_a^b \|u(t)\| dt < \infty$.*

We define in a standard way in the set of strongly measurable functions an equivalence relation $u \sim v \Leftrightarrow u(t) = v(t)$ a.e. Identifying in an obvious sense functions with their equivalence classes we can define the normed linear spaces

(i) $L^1(a,b;B)$ of Bochner integrable functions $u : [a,b] \to B$ endowed with norm $|u|_1 := \int_a^b \|u(t)\| dt$,

(ii) $L^p(a,b;B)$ for $1 < p < \infty$ of functions $u \in L^1(a,b;B)$ such that $|u|_p := \left(\int_a^b \|u(t)\|^p dt\right)^{1/p} < \infty$, endowed with norm $|\cdot|_p$,

(iii) $L^\infty(a,b;B)$ of functions $u : [a,b] \to B$ which are essentially bounded and strongly measurable, endowed with norm $|u|_\infty := \inf\{\sup\{\|u(t)\|; t \in [a,b] \setminus M\};\ M \subset [a,b],\ \text{meas}\,(M) = 0\}$,

(iv) $C([a,b];B)$ of continuous functions $u : [a,b] \to B$ endowed with norm $|\cdot|_\infty$.

The fact that $L^p(a,b;B)$ for $p \in [1,\infty]$ and $C([a,b];B)$ are Banach spaces is well known ([Ad]). Let us also mention the following classical results.

Theorem 8.5 (Lebesgue Dominated Convergence Theorem) *Let $p \in [1,\infty[$ be given and let $v_n \in L^p(a,b;B)$, $g_n \in L^p(a,b;\mathbb{R}^1)$ be given sequences for $n \in \mathbb{N} \cup \{0\}$ such that*

(i) $\lim_{n\to\infty} \int_a^b |g_n(t) - g_0(t)|^p \, dt = 0$,

(ii) $\lim_{n\to\infty} \|v_n(t) - v_0(t)\| = 0$ *a.e.*,

(iii) $\|v_n(t)\| \le g_n(t)$ *a.e. for all* $n \in \mathbb{N} \cup \{0\}$.

Then $\lim_{n\to\infty} |v_n - v_0|_p = 0$.

Theorem 8.6 (Mean Continuity Theorem) *For every $p \in [1,\infty[$ and $u \in L^p(a,b;B)$, we have*

$$\lim_{h\to 0+} \int_{a+h}^b \big\|u(t) - u(t-h)\big\|^p dt = 0. \tag{8.5}$$

The following Theorem 8.7 in the context of Hilbert space - valued functions has been tailored especially for situations that occur in the theory of hysteresis operators, see [K]. Notice that it does not follow from Theorem 8.5, since *we do not assume the pointwise convergence* here.

Theorem 8.7 *Let X be a Hilbert space endowed with a scalar product $\langle\cdot,\cdot\rangle$ and the corresponding norm $|\cdot| = \langle\cdot,\cdot\rangle^{1/2}$. Let $v_n \in L^1(a,b;X)$, $g_n \in L^1(a,b;\mathbb{R}^1)$ be given sequences for $n \in \mathbb{N} \cup \{0\}$ such that*

(i) $\lim_{n\to\infty} \int_a^b \langle v_n(t), \varphi(t)\rangle\, dt = \int_a^b \langle v_0(t), \varphi(t)\rangle\, dt \quad \forall \varphi \in C([a,b];X)$,

(ii) $\lim_{n\to\infty} \int_a^b |g_n(t) - g_0(t)|\, dt = 0$,

(iii) $|v_n(t)| \le g_n(t)$ *a.e.* $\forall n \in \mathbb{N}$,

(iv) $|v_0(t)| = g_0(t)$ *a.e.*

Then $\lim_{n\to\infty} |v_n - v_0|_1 = 0$.

8.2 Functions of bounded variation

Definition 8.8 *A partition* $S := \{\{t_0, \dots, t_N\};\ a = t_0 < t_1 < \cdots < t_N = b\}$ *of the interval* $[a,b]$ *is said to be* δ-fine *for* $\delta > 0$, *if* $\max\{|t_i - t_{i-1}|;\ i = 1, \dots, N\} \le \delta$. *We denote by* $\Delta_\delta(a,b)$ *the set of* δ*-fine partitions of the interval* $[a,b]$, $\Delta_0(a,b) := \bigcup_{\delta>0} \Delta_\delta(a,b)$.

Definition 8.9 *Let* $S = \{t_0, \dots, t_N\} \in \Delta_0(a,b)$ *and a function,* $u : [a,b] \to B$ *be given. We define the* S*-variation* $\mathcal{V}_S(u)$ *of* u *and the total variation* $\operatorname*{Var}_{[a,b]} u$ *of* u *in* $[a,b]$ *by the formulae*

$$\begin{aligned} \mathcal{V}_S(u) &:= \sum_{i=1}^{N} \|u(t_i) - u(t_{i-1})\|, \\ \operatorname*{Var}_{[a,b]} u &:= \sup\{\mathcal{V}_S(u);\ S \in \Delta_0(a,b)\}. \end{aligned}$$

We denote by $BV(a,b;\,B) := \{u : [a,b] \to B;\ \operatorname*{Var}_{[a,b]} u < \infty\}$ the set of all functions of bounded total variation. For every $u, v \in BV(a,b;\,B)$ and every $c \in\,]a,b[$ we obviously have

$$\operatorname*{Var}_{[a,b]} u = \operatorname*{Var}_{[a,c]} u + \operatorname*{Var}_{[c,b]} u\,, \quad \operatorname*{Var}_{[a,b]} (u+v) \le \operatorname*{Var}_{[a,b]} u + \operatorname*{Var}_{[a,b]} v\,. \tag{8.6}$$

The definition entails that every function $u \in BV(a,b;\,B)$ is bounded, the one-sided limits $u(t+)\,(u(t-))$ exist for all $t \in [a,b[$ $(t \in\,]a,b]$, respectively) and the set $\{t \in [a,b];\ u(t+) \ne u(t)$ or $u(t-) \ne u(t)\}$ of discontinuity points is at most countable. Furthermore, for all $u \in BV(a,b;\,B)$ and $t \in\,]a,b]$ we have

$$u(t-) = u(t) \quad \Longleftrightarrow \quad \lim_{\delta\to 0+} \operatorname*{Var}_{[t-\delta,t]} u = 0\,. \tag{8.7}$$

Indeed, the implication '$\Leftarrow$' is straightforward. To prove the converse, we assume that $\varepsilon > 0$ is given and find $\delta > 0$ such that $\|u(t) - u(s)\| \le \varepsilon$ for $t - \delta < s < t$. There exists a partition $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = t$ such that $t - t_{N-1} < \delta$ and

$$\operatorname*{Var}_{[0,t]} u \le \varepsilon + \sum_{i=1}^{N} \|u(t_i) - u(t_{i-1})\| \le 2\varepsilon + \operatorname*{Var}_{[0,t_{N-1}]} u\,.$$

Relation (8.7) now follows from (8.6). We analogously have

$$(8.8) \qquad u(t+) = u(t) \iff \lim_{\delta \to 0+} \operatorname*{Var}_{[t,t+\delta]} u = 0$$

for every $u \in BV(a,b\,;\,B)$ and $t \in [a,b[$.

An important example of functions of bounded variation are the *step functions*

$$(8.9) \qquad \xi(t) := \sum_{j=1}^{N} x_j \chi_{]t_{j-1},t_j[}(t) + \sum_{j=0}^{N} y_j \chi_{\{t_j\}}(t)$$

as a special case of (8.1), where $S := \{t_0, \dots, t_N\} \in \Delta_0(a,b)$ is a given partition and $\{x_j\}, \{y_j\}$ are given sequences in B.

The following statement shows that functions of bounded variation are strongly measurable and that $BV(a,b\,;\,B)$ endowed with the norm

$$(8.10) \qquad |u|_{BV} := \sup\{\|u(t)\|\,;\, t \in [a,b]\} + \operatorname*{Var}_{[a,b]} u$$

is a Banach space.

Proposition 8.10

(i) *For every* $u \in BV(a,b\,;\,B)$ *there exists a sequence* $\{\xi_n;\, n \in \mathbb{N}\}$ *of step functions such that* $\lim_{n\to\infty} \sup_{[a,b]} \|u(t) - \xi_n(t)\| = 0$, $\operatorname*{Var}_{[a,b]} \xi_n \le \operatorname*{Var}_{[a,b]} u$.

(ii) *Let* $\{u_n;\, n \in \mathbb{N}\}$ *be a sequence in* $BV(a,b\,;\,B)$ *and let* $u : [a,b] \to B$ *be a function such that* $\lim_{n\to\infty} \|u_n(t) - u(t)\| = 0$ *for all* $t \in [a,b]$. *Then* $\operatorname*{Var}_{[a,b]} u \le \liminf_{n\to\infty} \operatorname*{Var}_{[a,b]} u_n$.

Proof.

(i) The function $V(t) := \operatorname*{Var}_{[a,t]} u$ is nondecreasing in $[a,b]$. For $n \in \mathbb{N}$ put $N(n) := \max(\mathbb{N} \cap [0, nV(b)])$ and $t_j^n := \sup\{t \in [a,b];\, V(t) \le \frac{j}{n}\}$ for $j = 1, \dots, N(n)$, $t_{N(n)+1}^n := b$, $t_0^n := a$. The assertion holds for $\xi_n(t_j^n) := u(t_j^n)$, $\xi_n(t) := u\big(\frac{1}{2}(t_j^n + t_{j+1}^n)\big)$ for $t \in]t_j^n, t_{j+1}^n[$, $j = 0, \dots, N(n)$, $\xi_n(b) := u(b)$, with the convention $]t_j^n, t_{j+1}^n[= \emptyset$ if $t_j^n = t_{j+1}^n$.

Part (ii) follows immediately from Definition 8.9. ❐

As a consequence of Proposition 8.10 we see that step functions form a dense subset of $BV(a,b\,;\,B)$ with respect to the so-called *strict metric* defined by the formula (see [V])

$$(8.11) \qquad d_s(u,v) := \sup\{\|u(t) - v(t)\|;\, t \in [a,b]\} + |\operatorname*{Var}_{[a,b]} u - \operatorname*{Var}_{[a,b]} v|.$$

Let us pass to another important concept.

Definition 8.11 *A function* $u : [a,b] \to B$ *is called* absolutely continuous, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that the implication*

$$(8.12) \qquad \sum_{k=1}^{n} (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} \big\|u(b_k) - u(a_k)\big\| < \varepsilon$$

holds for every sequence of intervals $]a_k, b_k[\subset [a,b]$ *such that* $]a_k, b_k[\cap]a_j, b_j[= \emptyset$ *for* $k \neq j$.

We introduce the spaces

(i) $AC(a,b;B)$ *of all absolutely continuous functions* $u : [a,b] \to B$,

(ii) $CBV(a,b;B) = BV(a,b;B) \cap C([a,b];B)$ *of continuous functions* $u : [a,b] \to B$ *of bounded variation,*

(iii) $NBV(a,b;B) = \{u \in BV(a,b;B)\,;\, u(a+) = u(a), u(t-) = u(t)\ \forall t \in]a,b]\}$ *of normalized functions* $u : [a,b] \to B$ *of bounded variation.*

It is easy to check the inclusions

$$AC(a,b;B) \subset CBV(a,b;B) \subset NBV(a,b;B) \subset BV(a,b;B) \tag{8.13}$$

as well as the implication

$$u \in BV(a,b;B) \Rightarrow \exists!\, u^* \in NBV(a,b;B)\,,\ u(t) = u^*(t) \quad \text{a.e.} \tag{8.14}$$

Functions of bounded variations can be characterized in terms of the mean continuity modulus (cf. Mean Continuity Theorem 8.6) in the following way.

Theorem 8.12 *Let* $v \in L^1(a,b;B)$ *be a given function satisfying*

$$\liminf_{h \to 0+} \frac{1}{h} \int_{a+h}^{b} \|v(t) - v(t-h)\|\, dt \;=\; C \;<\; \infty\,. \tag{8.15}$$

Then there exists $w \in NBV(a,b;B)$ *such that* $w(t) = v(t)$ *a.e.,* $\operatorname*{Var}_{[a,b]} w = C$.

Conversely, for each $v \in BV(a,b;B)$ *and* $h \in]0, b-a[$, *we have*

$$\frac{1}{h} \int_{a+h}^{b} \|v(t) - v(t-h)\|\, dt \;\leq\; \operatorname*{Var}_{[a,b]} v\,. \tag{8.16}$$

If moreover $v \in NBV(a,b;B)$, *then*

$$\operatorname*{Var}_{[a,b]} v \;=\; \lim_{h \to 0+} \frac{1}{h} \int_{a+h}^{b} \|v(t) - v(t-h)\|\, dt\,. \tag{8.17}$$

Theorem 8.12 is proved (in a slightly different form) in [Bre]. More precisely, it follows from Proposition A.5 of [Bre] and from Lemma 8.13 below.

Lemma 8.13 *For every* $v \in L^1(a,b;B)$ *and every* $h \in]0, b-a[$ *we have*

$$\frac{1}{h} \int_{a+h}^{b} \|v(t) - v(t-h)\|\, dt \;\leq\; \liminf_{h \to 0+} \frac{1}{h} \int_{a+h}^{b} \|v(t) - v(t-h)\|\, dt\,. \tag{8.18}$$

Proof of Lemma 8.13. Let $\lambda :]0, b-a[\to \mathbb{R}^+$ be the function defined by the formula

$$\lambda(h) := \frac{1}{h} \int_{a+h}^{b} \|v(t) - v(t-h)\| \, dt \,. \tag{8.19}$$

By the Mean Continuity Theorem, it is continuous in its domain of definition and it satisfies obviously for every $h > 0$, $k > 0$, $h + k < b - a$ the inequality

$$\lambda(h+k) \leq \frac{h}{h+k}\lambda(h) + \frac{k}{h+k}\lambda(k) \,.$$

For every $h > 0$ and every integer $p < (b-a)/h$ we obtain by induction

$$\lambda(p\,h) \leq \lambda(h). \tag{8.20}$$

Let $h_n \searrow 0$ be a sequence such that $\lim_{n\to\infty} \lambda(h_n) = \liminf_{h\to 0+} \lambda(h)$ and let $h \in]0, b-a[$ be arbitrarily chosen. For each n sufficiently large we find $p_n \in \mathbb{N}$ such that $h \leq p_n h_n < h + h_n$. Then (8.20) yields $\lambda(p_n h_n) \leq \lambda(h_n)$ and passing to the limit we obtain the assertion. ❒

In general, the problem of differentiability of absolutely continuous Banach space - valued functions is nontrivial (see [Bre]). For our purposes it is sufficient to consider in the sequel only functions with values in a *separable Hilbert space* X. We need the following representation theorem ([Bre]).

Theorem 8.14 *Let X be a separable Hilbert space. Then for every absolutely continuous function $u \in AC([a,b]\,; X)$ there exists an element $\dot{u} \in L^1(a,b\,; X)$ such that*

(i) $\dot{u}(t) = \lim_{h\to 0} \frac{1}{h}(u(t+h) - u(t))$ *a.e.,*

(ii) $u(t) - u(s) = \int_s^t \dot{u}(\tau)\, d\tau$ *for all* $a \leq s < t \leq b$.

According to Theorem 8.14, it is justified to denote similarly as in the scalar-valued case by $W^{1,1}(a,b\,;X)$ the space of absolutely continuous functions with values in a Hilbert space X and by $W^{1,p}(a,b\,;X)$ for $p \in]1,\infty]$ the space of all functions $u \in W^{1,1}(a,b\,;X)$ such that $\dot{u} \in L^p(a,b\,;X)$. The spaces $W^{1,p}$ are Banach spaces endowed with the norm $|u|_{1,p} := |u(0)| + |\dot{u}|_p$.

8.3 Riemann-Stieltjes integral

Let X be a separable Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$. For arbitrary functions $u \in C([a,b];X)$ and $\xi \in BV(a,b\,;X)$ and for an arbitrary partition $S = \{t_0, \ldots, t_N\} \in \Delta_\delta(a,b)$ we define the Riemann-Stieltjes sum

$$I_S(u,\xi) := \sum_{k=1}^{N} \langle u(t_k), \xi(t_k) - \xi(t_{k-1}) \rangle \tag{8.21}$$

with the intention to pass to the limit as $\delta \to 0$.

Below we list without proofs standard results on the Riemann-Stieltjes integral. Details can be found e.g. in [K].

Lemma 8.15 *Let $u \in C([a,b];X)$ and $\xi \in BV(a,b;X)$ be given. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary partitions $S, S' \in \Delta_\delta(a,b)$ we have*

$$|I_S(u,\xi) - I_{S'}(u,\xi)| < \varepsilon.$$

The limit $\lim\limits_{\delta\to 0+} I_S(u,\xi)$ therefore exists and is independent of the choice of $S \in \Delta_\delta(a,b)$. It is called the *Riemann-Stieltjes integral* and denoted by $\int_a^b \big\langle u(t), d\xi(t)\big\rangle$.

By construction, it is additive, that is $\int_a^b \big\langle u(t), d\xi(t)\big\rangle = \int_a^c \big\langle u(t), d\xi(t)\big\rangle + \int_c^b \big\langle u(t), d\xi(t)\big\rangle$ for every $c \in\,]a,b[$. Moreover, it is linear with respect to both u and ξ and that the estimate

$$\left|\int_a^b \big\langle u(t), d\xi(t)\big\rangle\right| \le |u|_\infty \operatorname*{Var}_{[a,b]} \xi \tag{8.22}$$

holds for all $u \in C([a,b];X)$ and $\xi \in BV(a,b;X)$. Conversely, for every function $\xi \in NBV(a,b;X)$ we have

$$\operatorname*{Var}_{[a,b]} \xi = \sup\Big\{\int_a^b \big\langle u(t), d\xi(t)\big\rangle;\ u \in C([a,b];X), |u|_\infty \le 1\Big\}. \tag{8.23}$$

For $u, \xi \in CBV(a,b;X)$ we have the integration-by-parts formula

$$\int_a^b \big\langle u(t), d\xi(t)\big\rangle + \int_a^b \big\langle \xi(t), du(t)\big\rangle = \big\langle u(b), \xi(b)\big\rangle - \big\langle u(a), \xi(a)\big\rangle, \tag{8.24}$$

and the following relations between Riemann-Stieltjes and Lebesgue integrals hold.

$$\begin{aligned}
&(8.25) && \int_a^b \big\langle u(t), d\xi(t)\big\rangle &=\ & \int_a^b \big\langle u(t), \dot\xi(t)\big\rangle dt \\
&&&&& \forall\, u \in C([a,b];X),\ \xi \in W^{1,1}(a,b;X), \\
&(8.26) && \int_a^b \big\langle u(t), d\xi(t)\big\rangle &=\ & \big\langle u(b), \xi(b)\big\rangle - \big\langle u(a), \xi(a)\big\rangle - \int_a^b \big\langle \xi(t), \dot u(t)\big\rangle dt \\
&&&&& \forall\, u \in W^{1,1}(a,b;X),\ \xi \in BV(a,b;X).
\end{aligned}$$

The Riemann-Stieltjes integral depends continuously on the functions u and ξ in the following sense.

Theorem 8.16 *Let $u, \xi : [a,b] \to X$ be given functions and let $\{u_n; n \in \mathbb{N}\}$, $\{\xi_n; n \in \mathbb{N}\}$ be given sequences in $C([a,b];X)$, $BV(a,b;X)$, respectively, such that*

(i) $\lim\limits_{n\to\infty} |u_n - u|_\infty = 0$,

(ii) $\lim_{n\to\infty} |\xi_n(t) - \xi(t)| = 0$ *for all* $t \in [a,b]$,

(iii) $\operatorname*{Var}_{[a,b]} \xi_n(t) \le c$, *where* $c > 0$ *is a constant independent of* n.

Then $\lim_{n\to\infty} \int_a^b \langle u_n(t), d\xi_n(t)\rangle = \int_a^b \langle u(t), d\xi(t)\rangle$.

Notice that the integral $\int_a^b \langle u(t), d\xi(t)\rangle$ is meaningful by Proposition 8.10. It is also worth mentioning that condition (8.15) for $v \in L^1(a,b;X)$ can equivalently be written in the form

(8.27)
$$\sup \left\{ \int_a^b \langle v(t), \dot\varphi(t)\rangle \, dt\,;\ \varphi \in W^{1,\infty}(a,b;X),\ |\varphi|_\infty \le 1,\ \varphi(a) = \varphi(b) = 0 \right\} = C.$$

References

[Ad] R.A. Adams, *Sobolev spaces.* Academic Press, New York - San Francisco - London, 1975.

[Al] H.-D. Alber, *Materials with memory.* Lecture Notes in Mathematics, Vol. 1682, Springer-Verlag, Berlin - Heidelberg, 1998.

[AGM] R.C. Anderssen, P.W. Gras, F. MacRitchie, *To what extent is the mixing of wheat flour dough rate-independent?* Preprint.

[AF] P.J. Armstrong, C.O. Frederick, *A mathematical representation of the multiaxial Bauschinger effect.* C.E.G.B., Report RD/B/N 731, 1966.

[AE] J.-P. Aubin, I. Ekeland, *Applied nonlinear analysis.* Wiley - Interscience, New York, 1984.

[Be] A. Bergqvist, *Magnetic vector hysteresis model with dry friction-like pinning* (to appear).

[Bre] H. Brézis, *Opérateurs maximaux monotones.* North-Holland Math. Studies, Amsterdam, 1973.

[Bro] M. Brokate, *Elastoplastic constitutive laws of nonlinear kinematic hardening type.* In: Functional analysis with current applications in science, technology and industry (Aligarh, 1996). Pitman Res. Notes Math. Ser., 377, Longman, Harlow, 1998, 238–272.

[BDK] M. Brokate, K. Dreßler, P. Krejčí, *Rainflow counting and energy dissipation for hysteresis models in elastoplasticity.* Euro. J. Mech. A/Solids **15** (1996), 473–497.

[BK] M. Brokate, P. Krejčí, *Wellposedness of kinematic hardening models in elastoplasticity.* Math. Model. Num. Anal. (M^2AN) **32** (1998), 177–209.

[BS] M. Brokate, J. Sprekels, *Hysteresis and phase transitions.* Springer-Verlag, New York, 1996.

[D] W. Desch, *Local Lipschitz continuity of the stop operator* (to appear).

[DT] W. Desch, J. Turi, *The stop operator related to a convex polyhedron* (to appear).

[DI] P. Dupuis, H. Ishii, *On Lipschitz continuity of the solution mapping to the Skorokhod problem*. Stochastics and Stochastic Reports **35** (1991), 31 – 62 .

[DN] P. Dupuis, A. Nagurney, *Dynamical systems and variational inequalities*. Ann. Oper. Res. **44** (1993), 9 – 42 .

[DL] G. Duvaut, J.-L. Lions, *Inequalities in mechanics and physics*. Springer-Verlag, Berlin 1976. French edition: Dunod, Paris 1972.

[HP] E. Hille, R. Phillips, *Functional analysis and semi-groups*. Publ. AMS, Vol. 31, Providence, 1957.

[KP] M.A. Krasnosel'skii, A.V. Pokrovskii, *Systems with hysteresis*. Nauka, Moscow, 1983 (English edition Springer 1989).

[K] P. Krejčí, *Hysteresis, convexity and dissipation in hyperbolic equations*. Gakuto Int. Ser. Math. Sci. Appl., Vol. 8, Gakkōtosho, Tokyo, 1996.

[KS1] P. Krejčí, J. Sprekels, *On a system of nonlinear PDE's with temperature-dependent hysteresis in one-dimensional thermoplasticity*. J. Math. Anal. Appl. **209** (1997), 25–46.

[KS2] P. Krejčí, J. Sprekels, *Temperature-dependent hysteresis in one-dimensional thermovisco-elastoplasticity.* Appl. Math. **43** (1998), 173–205.

[KS3] P. Krejčí, J. Sprekels, *A hysteresis approach to phase-field models* (to appear).

[KS4] P. Krejčí, J. Sprekels, *Hysteresis operators in phase-field models of Penrose-Fife type.* Appl. Math. **43** (1998), 207–222.

[KJF] A. Kufner, O. John, S. Fučík, *Function spaces.* Academia, Prague, 1977.

[LL] L.D. Landau, E.M. Lifschitz, *Continuum mechanics* (Russian). Second edition, Gostechizdat, Moscow, 1953.

[LC] J. Lemaitre, J.-L. Chaboche, *Mechanics of solid materials.* Cambridge University Press, 1985.

[LT] P. Le Tallec, *Numerical analysis of viscoelastic problems.* Masson, Paris, 1990.

[M] J.-J. Moreau, *Evolution problem associated with a moving convex set in a Hilbert space*. J. Diff. Eq. **26** (1977), 347 – 374.

[NH] J. Nečas, I. Hlaváček, *Mathematical theory of elastic and elastico-plastic bodies: an introduction.* Elsevier, Amsterdam, 1981.

[P] I. Picek, *Multidimensional hysteresis models* (Czech). Graduate Thesis. Charles University, Prague, 1991.

[Ra] Y.N. Rabotnov, *Mechanics of deformable solids* (Russian). Second edition, Nauka, Moscow, 1988.

[Ro] R.T. Rockafellar, *Convex analysis.* Princeton University Press, 1970.

[V] A. Visintin, *Differential models of hysteresis.* Springer, Berlin - Heidelberg, 1994.

[Y] K. Yosida, *Functional analysis.* Springer-Verlag, Berlin - Göttingen - Heidelberg, 1965.