Finite time extinction of super-Brownian motions with catalysts

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(WIAS preprint No. 431 of 1998)

AMS Subject Classification Primary 60J80; Secondary 60J55, 60G57

Keywords catalytic super-Brownian motion, historical superprocess, critical branching, finite time extinction, measure-valued branching, random medium, good and bad paths, stopped measures, collision local time, comparison, coupling, stopped historical superprocess, branching rate functional, super-random walk, interacting Feller's branching diffusion

Running head: Extinction in catalytic branching

August 26, 1998 exti.tex typeset by LaTeX

* Supported by an NSERC grant and a Max Planck award.

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¶ Supported by an NSA grant.

Abstract

Consider a catalytic super-Brownian motion $X = X^\Gamma$ with finite variance branching. Here "catalytic" means that branching of the reactant X is only possible in the presence of some catalyst. Our intrinsic example of a catalyst is a stable random measure Γ on R of index $0 < \gamma < 1$. Consequently, here the catalyst is located in a countable dense subset of R. Starting with a finite reactant mass X_0 supported by a compact set, X is shown to die in finite time. Our probabilistic argument uses the idea of good and bad historical paths of reactant "particles" during time periods $[T_n, T_{n+1})$. Good paths have a significant collision local time with the catalyst, and extinction can be shown by individual time change according to the collision local time and a comparison with Feller's branching diffusion. On the other hand, the remaining bad paths are shown to have a small expected mass at time T_{n+1} which can be controlled by the hitting probability of point catalysts and the collision local time spent on them.

Contents

1 Introduction

Recently a number of papers have dealt with branching in catalytic media. These are models of chemical or biological reaction-diffusion systems of two substances or species, respectively. One we call the catalyst, and the other the reactant. The latter we model by a super-Brownian motion (SBM) with "critical binary" branching, and its branching rate is given by the catalyst.

In this paper we verify *finite time extinction* of the reactant for *three different types of catalysts,* provided the reactant was started with a finite mass. We begin with explaining the most interesting of these catalysts.

1.1 Model 1: stable catalyst Γ on R

Let $X^\Gamma = \{X_t^\Gamma : t \geq 0\}$ denote a *continuous super-Brownian motion (SBM)* with branching rate (catalyst) given by a *stable* random measure

$$\Gamma = \sum_i \alpha_i \, \delta_{b_i} \tag{1}$$

on the real line R with index $0 < \gamma < 1$.

At an *intuitive level,* this model can be explained as follows. A huge number of small reactant "particles" move independently according to Brownian motions in R. But if such a Brownian particle meets one of the point catalysts $\alpha_i \, \delta_{b_i}$ it may branch in a critical binary way. More precisely, branching is governed by the *collision local time*

$$L_{[W,\Gamma]}(\mathrm{d}s) := \mathrm{d}s \sum_i \alpha_i \, \delta_{b_i}(W_s) \tag{2}$$

in the sense of Barlow et al. [BEP91] of the Brownian reactant particle with path W and the stable random measure $\Gamma(\mathrm{d}b)$ describing the catalyst.

This process X^Γ was introduced as a Markov process by Dawson and Fleischmann [DF91, Lemma 2.3.5 and its application in Subsections 2.4 – 2.5]. The existence of a continuous version follows from [DF97, Theorem 1 b)]. The clumping features of X^Γ had been exhibited in [DF91] by a *time-space-mass scaling limit theorem.* In [DFR91] the states X_t^Γ of X^Γ had been shown to be *absolutely continuous* measures. Finally, in [DLM95], the so-called *compact support property* has been verified: If the finite initial measure X_0^Γ has compact support, then the range of X^Γ is compact, too.

Starting with a finite measure X_0^Γ, and given Γ, the total mass process $t \mapsto X_t^\Gamma(\mathsf{R})$ is a continuous martingale, hence has a.s. a limit as $t \uparrow \infty$ ([DF97, Proposition 3]). The *main purpose* of the present paper is to show that if X_0^Γ is of compact support, the process X^Γ *dies in finite time* (Theorem 6 on page 10), just as in the constant medium case (the formal $\gamma = 1$ boundary case).

To illustrate the problems we encounter in the proof, we consider the following. Given the catalyst Γ and starting X^Γ with a unit mass concentrated at a, that is $X_0^\Gamma = \delta_a$, the probability of extinction of X^Γ at time t is given by

$$P_{0,\delta_a}^\Gamma(X_t^\Gamma = 0) \;=\; \exp\left[-v_\infty\left(0, a \,\middle|\, t, \Gamma\right)\right] \tag{3}$$

where for θ, t, Γ fixed, $v_\theta \;=\; v_\theta(\cdot,\cdot \,|\, t, \Gamma) = \left\{v_\theta(s, a \,|\, t, \Gamma) : (s, a) \in [0, t] \times \mathsf{R}\right\}$ solves (formally) the following *reaction-diffusion equation in the stable catalytic medium* Γ :

$$-\frac{\partial}{\partial s}\, v_\theta \;=\; \frac{1}{2}\,\Delta v_\theta - \Gamma\, v_\theta^2, \qquad v_\theta(s, a \,|\, t, \Gamma)\big|_{s=t} \;\equiv\; \theta \geq 0, \tag{4}$$

and $v_\infty := \lim_{\theta\uparrow\infty} v_\theta$. Then, by Borel-Cantelli, it would suffice to show the following extinction property of solutions to (4):

$$\lim_{t\uparrow\infty} \lim_{\theta\uparrow\infty} v_\theta(0, a \,|\, t, \Gamma) \;=\; 0. \tag{5}$$

But we do not know how to attack this problem analytically. Recall that the coefficient Γ of the reaction term (reaction rate) in (4) is the generalized derivative of a (random) measure supported by a countable dense set in R, hence is highly singular.

Instead, to prove finite time extinction will use some *probabilistic* arguments concerning the stochastic process X^Γ, inspired by Fleischmann and Mueller [FM97].

At the same time, via the log-Laplace connection of X^Γ to the partial differential equation (4), our approach can be regarded as a probabilistic contribution to the study of asymptotic properties [such as (5)] of solutions to the reaction-diffusion equation (4) in the (random) heterogeneous singular medium Γ.

Equations as (4) have attracted some attention and are relevant in particular from an *applied* point of view; see e.g. Ortoleva and Ross [OR72], Pagliaro and Taylor [PT88]. For reaction-diffusion equations in heterogenous media with different species and where reaction may be concentrated on bounded interfaces, see Glitzky et al. [GGH96]. Note that reaction-diffusion equations arise in many branches of technology, e.g. in microelectronics.

The *main ideas* of our approach are as follows. First of all, since we start with an initial measure X_0 of compact support, and X^Γ has the compact support property ([DLM95]), we may "essentially" restrict our attention to a finite (space) interval $\mathsf{K} \subset \mathsf{R}$. Hence, by a coupling technique, the catalyst may be extended periodically outside K. Next, the probability of extinction can be estimated below by using a smaller branching rate. Therefore, we remove all atoms $\alpha_i \delta_{b_i}$ of the catalyst with large "action weights" α_i. Moreover, the action weights α_i belonging to $[2^{-n}, 2^{-n+1})$ are replaced by 2^{-n}, so that the corresponding atoms form Poisson point processes in K of intensity $c\,2^{\gamma n}$ (with c an appropriate constant). Finally, "large" distances between neighboring points of this Poisson point process are exceptional. Therefore we may restrict to the situation where the empty intervals are at most of a size Δ_n (to be specified later). Altogether, we then want to verify finite time extinction of $X^{\underline{\Gamma}}$ where the catalyst $\underline{\Gamma}$ is of the form $\sum_{n\geq 0} 2^{-n}\underline{\pi}_n$ where $\underline{\pi}_n$ is a periodic point measure with gaps between neighboring catalysts bounded by Δ_n.

The *central idea* is to look for a sequence of times $T_1 < T_2 < \cdots$ with *finite* accumulation point T_∞ with the following property. At time T_n, we distinguish between *"good and bad"* historical paths of Brownian reactant particles, starting from the state $X^{\underline{\Gamma}}_{T_n}$ at time T_n, as we now explain.

The *good* paths are those which have a "significant" collision local time with $2^{-n}\underline{\pi}_n$ on the time interval $[T_n, T_{n+1})$ (so we take into account only that part $2^{-n}\underline{\pi}_n$ of $\underline{\Gamma}$). Consider the total mass of the good paths. For the continuous SBM with a *uniform* branching rate, the total mass process would have the same distribution as the standard *Feller branching diffusion* which satisfies the one-dimensional stochastic equation

$$\mathrm{d}Z_r = \sqrt{2\,Z_r}\,\mathrm{d}B_r\,, \quad Z_0 \geq 0, \tag{6}$$

(with B a standard Brownian motion). It is well-known that this diffusion is absorbed at 0 in finite time. In our catalytic case, the total mass of the good paths can in law be *compared* with Feller's branching diffusion. But now its time scale during $[T_n, T_{n+1})$ is, roughly speaking, individually given by the collision local times of the good paths with the catalytic medium $2^{-n}\underline{\pi}_n$. Since these collision local times are "significant" on the good paths, it follows that the total mass of the good paths dies out by time T_{n+1} with high probability.

The remaining *bad* paths may *not* die out by time T_{n+1}, but we can estimate the probability that this mass is larger than a certain size at time T_{n+1}, by using Markov's inequality and the simple but powerful expectation formula for (historical) superprocesses. Then we need to derive some estimates concerning hitting probabilities of a neighboring point catalyst from $2^{-n}\underline{\pi}_n$, and the Brownian (collision) local time spent on it.

1.2 Model 2: i.i.d. uniform catalysts on the lattice Z^d

In the other two models we discuss, the basic ideas of distinguishing between good and bad historical paths, and how to handle them, are the same. So here we only introduce the models, and indicate how to classify the paths.

For the second model, we replace the phase space R by the lattice Z^d, and Brownian motion by a continuous time *simple random walk* in Z^d. The catalysts

$\varrho = \{\varrho_b : b \in \mathsf{Z}^d\}$ are *i.i.d.* random variables, *uniform* in the interval $(0,1)$. So here the catalysts are present everywhere but again their action weights fluctuate randomly. By the discreteness of Z^d, and since masses can be arbitrarily small in superprocesses, one does not expect that the compact support property holds. Therefore, as opposed to Model 1, the super-random walk X^{ϱ} with catalyst ϱ may be influenced by large regions, where the catalysts are small. However, calling paths *bad* which reach such a region, these paths should have a small expected mass, and we will be able to show the finite time extinction property for X^{ϱ} along the lines indicated.

1.3 Model 3: a deterministic "parabolic" catalyst χ_q

For the moment, consider the continuous SBM with phase space R and uniform branching rate, except on $(-1,1)$. More precisely, we consider the branching rate $\chi = \mathbf{1}_{\mathsf{R}\setminus(-1,1)}$. As we will see in the next Subsection, if $X_0\big((-1,1)\big) > 0$, then this superprocess does not die in finite time.

Motivated by this, for a fixed constant $q > 0$, we consider the truncated *"parabolic"* branching rate

$$\chi_q(b) := |b|^q \wedge 1, \qquad b \in \mathsf{R}, \tag{7}$$

(see the figure). We show that starting with a finite initial mass, the SBM X^{χ} with parabolic catalyst χ_q *dies in finite time,* just as in the constant branching rate case, despite the "depression" of branching rate close to the origin, even if q is very large. Here the good historical paths are those which do not spent too much time near 0, where the catalytic mass is small.

$\chi_{1/3}(b) = |b|^{1/3} \wedge 1$ $\qquad$ $\chi_1(b) = |b| \wedge 1$ $\qquad$ $\chi_5(b) = |b|^5 \wedge 1$

Variants of the "parabolic" catalyst

1.4 Non-extinction in finite time

If we change Model 1 so that the catalysts are not dense, then the mass fails to die out in finite time. In fact, if $I \neq \emptyset$ is an open interval without catalysts, then a corresponding catalytic SBM X is bounded below by the heat flow in I with absorption at the boundary ∂I, starting with $X_0(\cdot \cap I)$. If now $X_0(I) > 0$, then the L^1–norm of that heat solution decays according to $\langle X_0\,, \varphi_\lambda\rangle\,\mathrm{e}^{-\lambda t}$ with $\lambda > 0$ the first eigenvalue of $\frac{1}{2}\Delta$ on I, and φ_λ is the corresponding eigenfunction, hence is (strictly) positive at any time t, that is $X_t(I) > 0$ for all t.

Note that catalytic SBMs with a gap cover the single point catalytic SBM, where survival for all finite times was known from Fleischmann and Le Gall [FL95].

It would be interesting to establish conditions on the catalytic medium which are necessary *and* sufficient for extinction in finite time. Unfortunately, our methods seem to be too crude for this.

Remark 1 (decomposition of initial measures) Suppose a decomposition $\mu = \sum_i \mu_i$ of the initial measure is given. If we can show finite time extinction for each initial measure $X_0 = \mu_i$ then the branching property implies finite time extinction for $X_0 = \mu$. $\diamond$

1.5 Outline

To give a precise meaning to the above ideas, some technical problems have to be overcome. For instance, to have access to reactant particle paths, we will work with the *historical* catalytic SBM $\widetilde{X}^\Gamma$ instead of X^Γ. Or, since we want to use time scales of individual reactant particles, we will exploit Dynkin's [Dyn91a] framework of "stopped" historical superprocesses.

The *outline* of the paper is as follows. In the next section we recall the model of *continuous SBM* X in R^d with branching rate functional K as provided in Dawson and Fleischmann [DF97] (this goes back to Dynkin [Dyn91a]). Then K is specialized to be the Brownian collision local time $K = L_{[W,\psi]}$ of a (deterministic) locally finite measure ψ (catalyst) on R, also taken from [DF97]. Further specialized to $\psi = \Gamma$, our main result, Theorem 6 at p.10, can be formulated.

In Section 3 we first recall the *historical* SBM $\widetilde{X}$ in R^d with branching rate functional K. For this model, we give an *abstract sufficiency criterion* (Theorem 10 at p.15) for finite time extinction based on the idea of good and bad paths. For the extinction of good paths, a *comparison* with Feller's branching diffusion is provided (Proposition 12 at p.17), as a refinement of an argument in [FM97]. This makes use of Dynkin's concept of (individually) *"stopped"* historical superprocesses.

Section 4 is devoted to two one-dimensional *applications* of the abstract criterion:

(i) the parabolic catalyst χ_q of Model 3, and

(ii) a (deterministic) point catalyst $\underline{\Gamma} = \sum_{n \geq N} 2^{-n} \underline{\pi}_n$ with dense locations and gaps between neighboring catalysts in $\underline{\pi}_n$ bounded by some Δ_n.

In Section 5 we prove our main theorem, the finite time extinction for the SBM X^Γ with a stable catalytic rate Γ (Model 1). In fact, by a coupling and comparison argument, we reduce the problem to the case (ii) above.

Finally, in Section 6, finite time extinction for the super-random walk X^ϱ with i.i.d. uniform catalysts is derived.

2 Stable catalysts – main result

Here we carefully introduce the continuous SBM X in R^d with a sufficiently nice branching rate functional K. After specializations to Model 1, we will formulate our main result, Theorem 6 at p.10.

2.1 Preliminaries: some spaces

Measurability is always meant with respect to the related Borel fields. The lower index $+$ refers to the subset of all non-negative members of a set.

Let $\mathcal{B}[E_1, E_2]$ denote the set of all *measurable* mappings $f: E_1 \to E_2$ where E_1, E_2 are topological spaces. Write $\mathcal{B}[E_1]$ instead of $\mathcal{B}[E_1, E_2]$ if $E_2 = \mathsf{R}$, the real line, and only $\mathcal{B}$ if additionally $E_1 = \mathsf{R}^d$, $d \geq 1$.

If we restrict our consideration to *continuous* functions f, the letter $\mathcal{B}$ is replaced by $\mathcal{C}$ in the respective cases. If we restrict to *bounded* functions, we write $b\mathcal{B}$ and $b\mathcal{C}$, etc.

Fix a dimension $d \geq 1$, and a constant $p > d$, and introduce the *reference function*

$$\phi_p(b) := (1 + |b|^2)^{-p/2}, \qquad b \in \mathsf{R}^d, \tag{8}$$

of p–potential decay at infinity. Denote by $\mathcal{B}^p$ the set of all those $\varphi \in \mathcal{B}$ such that $|\varphi| \leq c_\varphi \phi_p$ for some constant c_φ.

Write $\langle \mu, f \rangle$ for the integral $\int \mu(\mathrm{d}b)\, f(b)$. Let $\mathcal{M}_p = \mathcal{M}_p[\mathsf{R}^d]$ denote the set of all (non-negative) measures μ defined on R^d satisfying $\langle \mu, \phi_p \rangle < \infty$. We endow this set $\mathcal{M}_p$ of p–tempered measures with the weakest topology such that all the maps $\mu \mapsto \langle \mu, \varphi \rangle$ are continuous, where $\varphi \geq 0$ is continuous and of compact support, or $\varphi = \phi_p$. The set of all finite measures on a Polish space E is denoted by $\mathcal{M}_\mathrm{f}[E]$ and equipped with the topology of weak convergence. Write $\|\mu\|$ for the total mass $\mu(E) = \langle \mu, 1 \rangle$ of $\mu \in \mathcal{M}_\mathrm{f}[E]$.

Set $\mathcal{M}_\mathrm{f} = \mathcal{M}_\mathrm{f}\left[\mathsf{R}^d\right]$, and denote by $\mathcal{M}\left[\mathsf{R}_+ \times E\right]$ the set of all measures η defined on $\mathsf{R}_+ \times E$ such that $\eta\big([0,t] \times E\big) < \infty$ for all $t > 0$.

With c we always denote a positive constant which may be different at various places. On the other hand, constants c_i are fixed within each subsection.

2.2 Branching rate functional K and BCLT $L_{[W,\psi]}$

Let $W = [W, \Pi_{s,a},\ s \geq 0,\ a \in \mathsf{R}^d]$ denote the *standard Brownian motion* in R^d, on canonical path space $\mathcal{C}\left[\mathsf{R}_+, \mathsf{R}^d\right]$ of continuous functions w.

Remark 2 (inhomogeneous setting) Although Brownian motion is time-homogeneous, we use this inhomogeneous setting, and we read $\Pi_{s,a}\, \varphi(W_t)$ as 0 if $s > t$. This formalism looks artificial since changing the paths before time s does not change the laws $\Pi_{s,a}$. The advantage becomes clear when we work with historical SBM. Note that the measure $\Pi_{s,a}$ is concentrated on the set of paths $\left\{w \in \mathcal{C}[\mathsf{R}_+, \mathsf{R}^d] : w_s = a\right\}$. $\diamond$

Write p for the continuous *transition density* function of W,

$$\mathrm{p}_t(a,b) \;=\; \mathrm{p}_t(b-a) \;=\; (2\pi t)^{-d/2}\,\exp\left[-\frac{(b-a)^2}{2t}\right], \qquad t>0,\;\; a,b\in\mathsf{R}^d, \quad (9)$$

and

$$\Pi_\eta \;:=\; \int\!\eta(\mathrm{d}s,\mathrm{d}a)\,\Pi_{s,a}\,, \qquad \eta\in\mathcal{M}\left[\mathsf{R}_+\times\mathsf{R}^d\right], \qquad (10)$$

for the "law" of W starting at time s in a point a where (s,a) is "distributed" according to the measure η. Put

$$\Pi_{s,\mu} \;:=\; \Pi_{\delta_s\times\mu}\,, \qquad s\ge 0, \quad \mu\in\mathcal{M}_{\mathrm{f}}\,. \qquad (11)$$

For convenience, we introduce the following definition.

Definition 3 (branching rate functional) A continuous additive functional $K=K_{[W]}$ of Brownian motion W is called a *branching rate functional* in $\mathbf{K}^\nu$, for some $\nu>0$, if the following two conditions hold:

(a) It is *locally admissible,*[1] i.e.

$$\sup_{a\in\mathsf{R}^d}\,\Pi_{s,a}\int_s^t K(\mathrm{d}r)\,\phi_p(W_r) \;\underset{s,t\to r_0}{\longrightarrow}\; 0, \qquad r_0\ge 0.$$

(b) For each N there is a constant $c_N>0$ such that

$$\Pi_{s,a}\int_s^t K(\mathrm{d}r)\,\phi_p^2(W_r) \;\le\; c_N\,|t-s|^{\xi}\,\phi_p(a),$$

$0\le s\le t\le N,\;\; a\in\mathsf{R}^d.$ ◇

To come to our *main example* of a branching rate functional, consider for the moment $d=1$ and fix $\psi\in\mathcal{M}_p$. Intuitively,

$$L_{[W,\psi]}(\mathrm{d}r) \;:=\; \mathrm{d}r\int\!\psi(\mathrm{d}b)\,\delta_b(W_r) \qquad (12)$$

is the *Brownian collision local time (BCLT)* of ψ. From [DF97, Corollary 2, p.257] we immediately get the following statement.[2]

Lemma 4 (Brownian collision local time of ϱ) *Fix $d=1$ and $\psi\in\mathcal{M}_p$. The Brownian collision local time $L_{[W,\psi]}$ of ψ makes sense non-trivially as a continuous additive functional of (one-dimensional) Brownian motion W, and it is a branching rate functional in $\mathbf{K}^\nu$ with $\xi=\frac{1}{2}$.*

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[1] For non-admissible functionals, we refer to [DFL98].

[2] For the more general case if ϱ is also time-dependent or, in particular a path of ordinary SBM, we refer to [FK98] and references therein.

2.3 SBM X with branching rate functional K

A slight modification of Proposition 12 (p.230) and Theorem 1 (p.234) in [DF97] gives the following lemma.

Lemma 5 (continuous SBM with branching rate functional K) *Fix a dimension* $d \geq 1$, *and* $K \in \mathbf{K}^{\nu}$ *for some* $\nu > 0$.

(a) (existence) *There exists a continuous* $\mathcal{M}_{\mathrm{f}}$*–valued (time-inhomogeneous) Markov process* $X = \left[X,\, P_{s,\mu}\,,\, s \geq 0,\, \mu \in \mathcal{M}_{\mathrm{f}}\right]$ *with Laplace functional*

$$P_{s,\mu} \exp\left\langle X_t\,, -\varphi\right\rangle \;=\; \exp\left\langle \mu\,, -v(s,\cdot\,|\,t)\right\rangle, \tag{13}$$

$0 \leq s \leq t,\ \mu \in \mathcal{M}_{\mathrm{f}},\ \varphi \in b\mathcal{B}_+\,,$ *where for* t, φ *fixed,* $v = v(\cdot\,,\cdot\,|\,t) \geq 0$ *is uniquely determined by the log-Laplace equation*

$$v(s,a) \;=\; \Pi_{s,a}\Big[\varphi(W_t) - \int_s^t K(\mathrm{d}r)\, v^2(r, W_r)\Big], \tag{14}$$

$0 \leq s \leq t,\ a \in \mathsf{R}^d.$

(b) (modification) *To each* $\eta \in \mathcal{M}\left[\mathsf{R}_+ \times \mathsf{R}^d\right]$, *there is an* $\mathcal{M}_{\mathrm{f}}$*–valued Markov process* $[X, P_\eta]$ *such that*

$$P_\eta \exp\left\langle X_t\,, -\varphi\right\rangle \;=\; \exp\left\langle \eta\,, -v(\cdot,\cdot\,|\,t)\right\rangle, \qquad t \geq 0, \tag{15}$$

with $v(s,\cdot\,|\,t)$ *from* (a) *if* $0 \leq s \leq t$, *and* $v(s,\cdot\,|\,t) = 0$ *otherwise.*

(c) (moments) $[X, P_{s,\mu}]$ *has finite moments of all orders. In particular, for* $\eta \in \mathcal{M}\left[\mathsf{R}_+ \times \mathsf{R}^d\right]$ *and* $\varphi_1, \varphi_2 \in b\mathcal{B}_+\,,$ *as well as* $t_1, t_2 \geq 0$

$$P_\eta \left\langle X_{t_1}\,, \varphi_1\right\rangle \;=\; \Pi_\eta\, \varphi_1(W_{t_1}), \tag{16}$$

$$\left.\begin{aligned} &\mathrm{Cov}_\eta\left[\langle X_{t_1}\,,\varphi_1\rangle, \langle X_{t_2}\,,\varphi_2\rangle\right] \\ &= 2\,\Pi_\eta \int K(\mathrm{d}r)\,\left[\Pi_{r,W_r}\varphi_1(W_{t_1})\right]\left[\Pi_{r,W_r}\varphi_2(W_{t_2})\right]. \end{aligned}\right\} \tag{17}$$

This superprocess X is said to be the *continuous SBM with branching rate functional* K. Note that the lemma in particular applies in the case of a BCLT $K = L_{[W,\psi]}$ according to Lemma 4, resulting in a time-homogeneous Markov process.

2.4 Main result: finite time extinction of X^Γ

Let $d = 1$. Fix a constant $0 < \gamma < 1$, and a (not necessarily normalized) Lebesgue measure ℓ on R. The *stable catalyst* $(\Gamma, \mathbb{P})$ is by definition the stable random measure on R with Laplace functional

$$\mathbb{P} \exp\langle \Gamma, -\varphi\rangle \;=\; \exp\left[-\int \ell(\mathrm{d}b)\, \varphi^\gamma(b)\right], \qquad \varphi \in \mathcal{B}_+\,. \tag{18}$$

Recall that Γ has independent increments, and that it allows a *representation*

$$\Gamma = \sum_i \alpha_i \, \delta_{b_i} \tag{19}$$

where the set of locations b_i is *dense* in R, with $\mathbb{P}$–probability one.

We now additionally require $p > \frac{1}{\gamma}$ [for the exponent p of potential decay occurring in the reference function (8)]. Then by (18), the realizations of the catalyst Γ belong $\mathbb{P}$–almost surely to $\mathcal{M}_p$. Hence we may apply the constructions of the previous two subsections to introduce the *continuous SBM* $X^\Gamma = \left[X^\Gamma, \, P^\Gamma_{s,\mu}, \, s \geq 0, \, \mu \in \mathcal{M}_{\mathrm{f}}\right]$ *with stable catalyst* Γ. More precisely, we use the so-called *quenched* approach: First a realization Γ of the catalytic medium is selected according to $\mathbb{P}$, and then, given Γ, the continuous time-homogeneous Markov process X^Γ evolves, governed by the BCLT $L_{[W,\Gamma]}$.

Note that by a formal differentiation of the log-Laplace equation (14) with $K = L_{[W,\Gamma]}$ of (12), using the semigroup of W, and replacing φ by the constant function θ, we get back the reaction-diffusion equation (4) in the catalytic medium Γ.

Now we are in a position to formulate our *main result.* Recall that $d = 1$.

Theorem 6 (finite time extinction of X^Γ) *Fix $\mu \in \mathcal{M}_{\mathrm{f}}$ with compact support. For $\mathbb{P}$–almost all Γ the following holds:*

$$P^\Gamma_{0,\mu}\Big(X^\Gamma_t = 0 \ \text{for some} \ t\Big) \; = \; 1. \tag{20}$$

The proof of this theorem needs some preparation and is postponed until Section 5.

We mention that it is an *open problem* whether finite time extinction holds also in some $\gamma = 0$ boundary cases.

3 An abstract finite time extinction criterion

The purpose of this section is to establish a general sufficient criterion for extinction in finite time for a SBM X in R^d with branching rate functional K. The central idea is to divide a finite time interval into an infinite number of stages in such a way that all of the mass will be dead at the end of all these stages. For this purpose, at each stage we distinguish between good and bad historical paths. The good paths accumulate a “significant” rate of branching, so that they die by the next stage, with high probability. The remaining bad paths may not die, but by assumption they carry a small expected mass at the beginning of the next stage.

3.1 Refinement: historical SBM $\widetilde{X}$

To realize this concept, we have to pass to a *"historical"* setting. That is, the measures $X_t(\mathrm{d}b)$ on R^d are thought of to be projections of measures $\widetilde{X}_t(\mathrm{d}w)$ where w is a continuous function on the interval $[0,t]$. Heuristically, a particle in X_t with position b is additionally equipped with a path $w:[0,t]\to\mathsf{R}^d$ with terminal point $w_t=b$, which gives the spatial past history of the particle and its ancestors. (For a more detailed exposition, we refer e.g. to [FM97].)

Equip $\mathbf{C}:=\mathcal{C}[\mathsf{R}_+,\mathsf{R}^d]$ with the topology of uniform convergence on all compact subsets of R_+. To each $w\in\mathbf{C}$ and $t\geq 0$, we associate the *stopped path* $w^t\in\mathbf{C}$ by setting $w^t_s:=w_{t\wedge s}$, $s\geq 0$. Write $\mathbf{C}^t$ for the closed subspace of $\mathbf{C}$ of all these paths stopped at time t. Note that $\mathbf{C}^t$ could be regarded as $\mathcal{C}\big[[0,t],\mathsf{R}^d\big]$ (as we did in the previous paragraph), and $\mathbf{C}^0$ as R^d.

To every $w\in\mathbf{C}$ we associate the corresponding *stopped path trajectory* $\widetilde{w}$ by setting $\widetilde{w}_t:=w^t$, $t\geq 0$. Writing $\|\cdot\|_\infty$ for the supremum norm, for $0\leq s\leq t$ we get

$$\|\widetilde{w}_t-\widetilde{w}_s\|_\infty=\|w^t-w^s\|_\infty=\sup_{s\leq r\leq t}|w_r-w_s|\longrightarrow 0\quad\text{as}\quad t-s\downarrow 0.$$

Hence, $\widetilde{w}$ belongs to $\mathcal{C}[\mathsf{R}_+,\mathbf{C}]$.

The image of the Brownian motion W under the map $w\mapsto\widetilde{w}$ is called the *Brownian path process*

$$\widetilde{W}=\Big[\widetilde{W},\widetilde{\Pi}_{s,w}\,,s\geq 0,\ w\in\mathbf{C}^s\Big].$$

That is, at time s we start with a path $w=\widetilde{W}_s$ stopped at time s, and let a path trajectory $\big\{\widetilde{W}_t:t\geq s\big\}$ evolve with law $\widetilde{\Pi}_{s,w}$ determined by the path $\{W_t:t\geq s\}$ starting at time s from w_s.

Note that if K belongs to $\mathbf{K}^\nu$ for some $\nu>0$, then K is also a continuous additive functional with respect to the Brownian path process $\widetilde{W}$.

Set

$$\mathsf{R}_+\widehat{\times}\mathbf{C}^\bullet:=\Big\{(s,w):\ s\in\mathsf{R}_+,\ w\in\mathbf{C}^s\Big\}\tag{21}$$

and write $\mathcal{M}\big[\mathsf{R}_+\widehat{\times}\mathbf{C}^\bullet\big]$ for the set of all measures η on $\mathsf{R}_+\widehat{\times}\mathbf{C}^\bullet$ which are finite if restricted to a finite time interval. Moreover, let

$$\widetilde{\Pi}_\eta:=\int\eta(\mathrm{d}s,\mathrm{d}w)\,\widetilde{\Pi}_{s,w}\,,\qquad s\geq 0,\quad \eta\in\mathcal{M}\big[\mathsf{R}_+\widehat{\times}\mathbf{C}^\bullet\big],\tag{22}$$

and

$$\widetilde{\Pi}_{s,\mu}:=\widetilde{\Pi}_{\delta_s\times\mu}\,,\qquad s\geq 0,\quad \mu\in\mathcal{M}_\mathrm{f}[\mathbf{C}^s].\tag{23}$$

W can be reconstructed from $\widetilde{W}$ by *projection:* $W_t:=(\widetilde{W}_t)_t$. This will often be used in the sequel.

Now we give the following historical version of Lemma 5, which follows from a modification of Propositions 1 (p.225), 12 (p.230), and Lemma 4 (p.232) in [DF97].

Proposition 7 (historical SBM with branching rate functional K) *Let $d \geq 1$, and fix $K \in \mathbf{K}^{\nu}$ for some $\nu > 0$.*

(a) (existence) *There exists a (time-inhomogeneous) Markov process*

$$\widetilde{X} = \left[\widetilde{X},\, \widetilde{P}_{s,\mu}\,,\ s \geq 0,\ \mu \in \mathcal{M}_{\mathrm{f}}[\mathbf{C}^s]\right]$$

with states $\widetilde{X}_t \in \mathcal{M}_{\mathrm{f}}[\mathbf{C}^t]$, $t \geq s$, and with Laplace functional

$$\widetilde{P}_{s,\mu} \exp\left\langle \widetilde{X}_t\,, -\varphi \right\rangle \;=\; \exp\left\langle \mu\,, -v(s,\cdot\,|\,t) \right\rangle, \tag{24}$$

$0 \leq s \leq t$, $\mu \in \mathcal{M}_{\mathrm{f}}[\mathbf{C}^s]$, $\varphi \in b\mathcal{B}_+[\mathbf{C}]$, where for t, φ fixed, $v = v(\cdot\,,\cdot\,|\,t) \geq 0$ is uniquely determined by the log-Laplace equation

$$v(s,\omega_s) \;=\; \widetilde{\Pi}_{s,\omega_s}\left[\varphi(\widetilde{W}_t) - \int_s^t K(\mathrm{d}r)\, v^2(r,\widetilde{W}_r)\right], \tag{25}$$

$0 \leq s \leq t$, $\omega_s \in \mathbf{C}^s$.

(b) (modification) *To each $\eta \in \mathcal{M}\left[\mathsf{R}_+ \widehat{\times} \mathbf{C}^{\bullet}\right]$ there is a Markov process $\left[\widetilde{X},\, \widetilde{P}_{\eta}\right]$ with states $\widetilde{X}_t \in \mathcal{M}_{\mathrm{f}}[\mathbf{C}^t]$ and such that*

$$\widetilde{P}_{\eta} \exp\left\langle \widetilde{X}_t\,, -\varphi \right\rangle \;=\; \exp\left\langle \eta\,, -v(\cdot,\cdot\,|\,t) \right\rangle, \qquad t \geq 0, \tag{26}$$

with $v(s,\cdot\,|\,t)$ from (a) *if $0 \leq s \leq t$, and $v(s,\cdot\,|\,t) = 0$ otherwise.*

(c) (moments) *$(\widetilde{X}, \widetilde{P}_{s,\mu})$ has finite moments of all orders. In particular, for $\eta \in \mathcal{M}\left[\mathsf{R}_+ \widehat{\times} \mathbf{C}^{\bullet}\right]$ and $\varphi_1, \varphi_2 \in b\mathcal{B}_+[\mathbf{C}]$, as well as $t_1, t_2 \geq 0$,*

$$\widetilde{P}_{\eta}\left\langle \widetilde{X}_{t_1}\,, \varphi_1 \right\rangle \;=\; \widetilde{\Pi}_{\eta}\, \varphi_1(\widetilde{W}_{t_1}), \tag{27}$$

$$\left.\begin{aligned} &\widetilde{C}\mathrm{ov}_{\eta}\left[\left\langle \widetilde{X}_{t_1}\,, \varphi_1 \right\rangle, \left\langle \widetilde{X}_{t_2}\,, \varphi_2 \right\rangle\right] \\ &= 2\,\widetilde{\Pi}_{\eta} \int K(\mathrm{d}r) \left[\widetilde{\Pi}_{r,\widetilde{W}_r} \varphi_1(\widetilde{W}_{t_1})\right]\left[\widetilde{\Pi}_{r,\widetilde{W}_r} \varphi_2(\widetilde{W}_{t_2})\right]. \end{aligned}\right\} \tag{28}$$

We call this superprocess $\widetilde{X}$ the *historical SBM with branching rate functional K.*

Of course, X can be gained back from $\widetilde{X}$ by *projection:*

$$X_t \;=\; \widetilde{X}_t\Big(\left\{ w \in \mathbf{C}^t :\ w_t \in (\cdot) \right\}\Big). \tag{29}$$

3.2 Dynkin's "stopped" measures $\widetilde{X}_\tau$

We also have to recall Dynkin's [Dyn91a, Dyn91b] concept of *"stopped"* historical superprocesses. We have two reasons for this. First, to handle also the lattice Model 2, we have to allow the times $T_1 < T_2 < \cdots$ mentioned in Subsection 1.1 to be Brownian stopping times. The second reason is that we intend to scale the SBM along individual particles' trajectories according to their accumulated rate of branching.

Roughly speaking, if τ is a *Brownian* stopping time, Dynkin's stopped measure $\widetilde{X}_\tau$ describes the population one gets, if each (individual) reactant path is stopped in the moment τ. For a detailed development, we refer to [Dyn91a, Subsection 1.5] and [Dyn91b, Subsection 1.10]. For convenience, here we collect only the following facts.

Let τ_t, $t \geq 0$, be *stopping times* with respect to the (natural) filtration of Brownian motion W, satisfying $\tau_s \leq \tau_t$ if $s \leq t$. Then there is a family

$$\left\{ \widetilde{X}_{\tau_t} : \ t \geq 0 \right\},$$

of random measures in $\mathcal{M}\left[\mathsf{R}_+ \widehat{\times} \mathbf{C}^\bullet\right]$, the so-called *"stopped" historical SBM* related to the family of Brownian stopping times $\{\tau_t : t \geq 0\}$. This family satisfies the so-called *special Markov property,* which roughly says the following. For $s \geq 0$, let $\mathcal{G}_{\tau_s}$ denote the pre-τ_s σ–field (concerning the historical superprocess $\widetilde{X}$). Given $\mathcal{G}_{\tau_s}$, hence in particular $\widetilde{X}_{\tau_s} =: \vartheta$, the stopped process $\{\widetilde{X}_{\tau_t} : t \geq s\}$ starts anew ([Dyn91a, Theorem 1.6] and [Dyn91b, Theorem 1.5]), namely based on the law $\widetilde{P}_\vartheta$.

Similarly, the notation of a sequence $\{\widetilde{X}_{\tau_n} : n \geq 1\}$ of stopped measures related to Brownian stopping times $\tau_1 \leq \tau_2 \leq \cdots$ can be introduced.

In formal analogy with Proposition 7 (c), the following first two *moment formulas* hold ([Dyn91a, (1.50a)]). For $t \geq 0$ and $\eta \in \mathcal{M}\left[\mathsf{R}_+ \widehat{\times} \mathbf{C}^\bullet\right]$ as well as φ in $b\mathcal{B}_+[\mathbf{C}]$,

$$\widetilde{P}_\eta \langle \widetilde{X}_{\tau_t}, \varphi \rangle = \widetilde{\Pi}_\eta \, \varphi\big(\widetilde{W}_{\tau_t}\big), \tag{30}$$

$$\widetilde{\mathrm{V}}\mathrm{ar}_\eta \langle \widetilde{X}_{\tau_t}, \varphi \rangle = 2 \, \widetilde{\Pi}_\eta \int K(\mathrm{d}r) \Big[\widetilde{\Pi}_{r, \widetilde{W}_r} \varphi\big(\widetilde{W}_{\tau_t}\big) \Big]^2. \tag{31}$$

3.3 The method of good and bad historical paths

Fix $K \in \mathbf{K}^\nu$, for some $\nu > 0$, and a finite measure μ on R^d. Consider the historical SBM $\widetilde{X}$ of Proposition 7 starting from $\widetilde{X}_0 = \mu$. First we introduce some Brownian stopping times and small constants.

Hypothesis 8 (stage quantities) Let $0 < \varepsilon < 1$ and $N = N(\varepsilon) \geq 0$.

(a) (stage duration) Consider *Brownian stopping times*

$$0 \leq T_N^\varepsilon < T_{N+1}^\varepsilon < \cdots < T_\infty^\varepsilon < \infty, \tag{32}$$

where the bound T_∞^ε is *non*-random.

(b) (constants) For $n \geq N = N(\varepsilon)$, let $M_n^\varepsilon, \delta_n^\varepsilon, \lambda_n^\varepsilon, \xi_n^\varepsilon$ and, in addition, δ_{N-1}^ε be (strictly) positive constants with the following properties:

(b1) $$M_n^\varepsilon \downarrow 0 \quad \text{as} \quad n \uparrow \infty.$$

(b2) $$\lim_{\varepsilon \downarrow 0} \left(\delta_{N-1}^\varepsilon + \sum_{n \geq N} (\delta_n^\varepsilon + \lambda_n^\varepsilon) \right) = 0.$$ ◇

Introduce the set E_n^ε of so-called *good historical paths* (*during* $[T_n^\varepsilon, T_{n+1}^\varepsilon]$),

$$E_n^\varepsilon := \left\{ w \in \mathbf{C} : \int_{T_n^\varepsilon}^{T_{n+1}^\varepsilon} K(\mathrm{d}r) \geq \xi_n^\varepsilon \right\}, \tag{33}$$

that is, paths with at least the amount ξ_n^ε of accumulated branching over the time interval $[T_n^\varepsilon, T_{n+1}^\varepsilon)$. We call $(E_n^\varepsilon)^{\mathrm{c}} = \mathbf{C} \backslash E_n^\varepsilon$ the set of *bad* paths. On the good and bad paths we impose the following hypothesis.

Hypothesis 9 (good and bad paths) Fix $\varepsilon \in (0,1)$. First of all,

$$\widetilde{P}_{0,\mu} \left(\left\| \widetilde{X}_{T_N^\varepsilon} \right\| > M_N^\varepsilon \right) \leq \delta_{N-1}^\varepsilon, \tag{34}$$

and for all $n \geq N = N(\varepsilon)$,

$$\widetilde{P}_{0,\mu} \left\{ \widetilde{X}_{T_{n+1}^\varepsilon}(E_n^\varepsilon) > 0 \;\middle|\; \left\| \widetilde{X}_{T_n^\varepsilon} \right\| \leq M_n^\varepsilon \right\} \leq \delta_n^\varepsilon, \tag{35}$$

$$\widetilde{P}_{0,\mu} \left\{ \widetilde{X}_{T_{n+1}^\varepsilon} \left((E_n^\varepsilon)^{\mathrm{c}} \right) \;\middle|\; \left\| \widetilde{X}_{T_n^\varepsilon} \right\| \leq M_n^\varepsilon \right\} \leq \lambda_n^\varepsilon \, M_{n+1}^\varepsilon. \tag{36}$$

◇

Here is our *interpretation* of Hypothesis 9. Recall that by Hypothesis 8 (b) the numbers $M_n^\varepsilon, \delta_{N-1}^\varepsilon, \delta_n^\varepsilon$, and λ_n^ε are small. So at the beginning of the N^{th} stage the total mass $\left\| \widetilde{X}_{T_N^\varepsilon} \right\|$ is already small with a high $\widetilde{P}_{0,\mu}$–probability. Then starting with a small mass at the beginning of the n^{th} stage, our condition (35) says that *good paths survive* only with a small (conditional) probability in the present stage, whereas (36) means that the (conditional) *expected mass of bad paths* is small.

Our abstract criterion now reads as follows. Recall that $d = 1$, $\mu \in \mathcal{M}_{\mathrm{f}}$, and that $K \in \mathbf{K}^\nu$ for some $\nu > 0$.

Theorem 10 (abstract criterion for finite time extinction) *Impose Hypotheses* 8 *and* 9*. Then with* $\widetilde{P}_{0,\mu}$*–probability one,* $\widetilde{X}_t = 0$ *for some* t*.*

We mention that under additional conditions, estimate (35) can be obtained by a comparison with Feller's branching diffusion, see Subsection 3.5 below, whereas the expectation formula for stopped historical SBM is available to reduce assertion (36) to a statement on the probability of a path to be bad, i.e. to have a small accumulated rate of branching. In fact, by the special Markov property and the expectation formula (30) applied to the indicator function $\varphi = \mathbf{1}_{(E_n^\varepsilon)^c}$ and the starting measure $\eta = \delta_{T_n^\varepsilon} \times \widetilde{X}_{T_n^\varepsilon}$, we have

$$\widetilde{P}_{0,\mu}\left\{\widetilde{X}_{T_{n+1}^\varepsilon}\left((E_n^\varepsilon)^c\right)\;\Big|\;\mathcal{G}_{T_n^\varepsilon}\right\} \;=\; \int \widetilde{X}_{T_n^\varepsilon}(\mathrm{d}w)\;\widetilde{\Pi}_{T_n^\varepsilon,w}\left(\widetilde{W}_{T_{n+1}^\varepsilon} \in (E_n^\varepsilon)^c\right).$$

Since E_n^ε only depends on $\{w_s : s \geq T_n^\varepsilon\}$, we can write

$$\widetilde{P}_{0,\mu}\left\{\widetilde{X}_{T_{n+1}^\varepsilon}\left((E_n^\varepsilon)^c\right)\;\Big|\;\mathcal{G}_{T_n^\varepsilon}\right\} \;=\; \int X_{T_n^\varepsilon}(\mathrm{d}a)\;\Pi_{T_n^\varepsilon,a}\left(W \in (E_n^\varepsilon)^c\right) \tag{37}$$

(recall Remark 2). Then (37) implies the following result.

Lemma 11 (sufficient condition) *If the estimate*

$$\Pi_{T_n^\varepsilon,a}\left(W \in (E_n^\varepsilon)^c\right) \;\leq\; \lambda_n\,\frac{M_{n+1}^\varepsilon}{M_n^\varepsilon}, \qquad a \in \operatorname{supp} X_{T_n^\varepsilon}\,, \tag{38}$$

holds, then the conditional expectation estimate (36) *is true.*

3.4 Proof of the abstract criterion

Here we want to prove Theorem 10. For $0 < \varepsilon < 1$ and $N = N(\varepsilon) \geq 0$, set

$$A_n^\varepsilon \;:=\; \left\{\left\|\widetilde{X}_{T_n^\varepsilon}\right\| \leq M_n^\varepsilon\right\}, \quad n \geq N, \quad \text{and} \quad A^\varepsilon \;:=\; \bigcap_{n\geq N} A_n^\varepsilon\,, \tag{39}$$

as well as

$$\overline{T}_\infty^\varepsilon \;:=\; \lim_{n\uparrow\infty} T_n^\varepsilon\,. \tag{40}$$

Note that this limiting Brownian stopping time satisfies $\overline{T}_\infty^\varepsilon \leq T_\infty^\varepsilon < \infty$.

1° (*extinction on* A^ε) First of all we show that for all $\varepsilon \in (0,1)$,

$$\left\|\widetilde{X}_{\overline{T}_\infty^\varepsilon}\right\| \;=\; 0 \quad \text{on } A^\varepsilon, \quad \widetilde{P}_{0,\mu}\text{–a.s.} \tag{41}$$

Indeed, fix $0 < \varepsilon < 1$ and a $\zeta > 0$. Then by Markov's inequality, for each $n \geq N$,

$$\widetilde{P}_{0,\mu}\left(\left\{\left\|\widetilde{X}_{\overline{T}_\infty^\varepsilon}\right\| > \zeta\right\} \cap A^\varepsilon\right) \;\leq\; \zeta^{-1}\,\widetilde{P}_{0,\mu}\,\mathbf{1}_{A_n^\varepsilon}\left\|\widetilde{X}_{\overline{T}_\infty^\varepsilon}\right\|. \tag{42}$$

But by the special Markov property, the expectation formula (30), and the definition of A_n^ε,

$$\widetilde{P}_{0,\mu}\left\{\mathbf{1}_{A_n^\varepsilon}\,\|\widetilde{X}_{\overline{T}_\infty^\varepsilon}\|\;\Big|\;\mathcal{G}_{T_n^\varepsilon}\right\} \;=\; \mathbf{1}_{A_n^\varepsilon}\,\widetilde{P}_{\widetilde{X}_{T_n^\varepsilon}}\|\widetilde{X}_{\overline{T}_\infty^\varepsilon}\| \;=\; \mathbf{1}_{A_n^\varepsilon}\,\|\widetilde{X}_{T_n^\varepsilon}\| \;\leq\; M_n^\varepsilon\,.$$

Hence, estimate (42) can be continued with

$$\leq\; \zeta^{-1}M_n^\varepsilon \;\underset{n\uparrow\infty}{\longrightarrow}\; 0,$$

by Hypothesis 8 (b1). Thus,

$$\widetilde{P}_{0,\mu}\left(\left\{\|\widetilde{X}_{\overline{T}_\infty^\varepsilon}\| > \zeta\right\}\cap A^\varepsilon\right) \;=\; 0 \quad \forall\,\zeta > 0,$$

and (41) follows.

2° (A_{n+1}^ε *fails with small conditional probability*) From Markov's inequality and (36) we get, for $0<\varepsilon<1$ and $n\geq N$,

$$\widetilde{P}_{0,\mu}\left\{\widetilde{X}_{T_{n+1}^\varepsilon}\left((E_n^\varepsilon)^{\mathrm{c}}\right) \;>\; M_{n+1}^\varepsilon \;\Big|\; A_n^\varepsilon\right\} \;\leq\; \lambda_n^\varepsilon\,.$$

Together with (35), we conclude for

$$\widetilde{P}_{0,\mu}\left\{\left(A_{n+1}^\varepsilon\right)^{\mathrm{c}} \;\big|\; A_n^\varepsilon\right\} \;\leq\; \delta_n^\varepsilon + \lambda_n^\varepsilon\,.$$

3° (A^ε *fails with small probability*) Next we show that

$$\lim_{\varepsilon\downarrow 0}\widetilde{P}_{0,\mu}\left((A^\varepsilon)^{\mathrm{c}}\right) \;=\; 0. \tag{43}$$

We decompose the complement $(A^\varepsilon)^{\mathrm{c}}$ of A^ε according to the smallest natural number $n\geq N$ such that A_n^ε fails:

$$\begin{aligned}\widetilde{P}_{0,\mu}\left((A^\varepsilon)^{\mathrm{c}}\right) &= \widetilde{P}_{0,\mu}\left((A_N^\varepsilon)^{\mathrm{c}}\right) + \sum_{n\geq N}\widetilde{P}_{0,\mu}\left(A_N^\varepsilon\cap\cdots\cap A_n^\varepsilon\cap\left(A_{n+1}^\varepsilon\right)^{\mathrm{c}}\right)\\ &\leq \widetilde{P}_{0,\mu}\left((A_N^\varepsilon)^{\mathrm{c}}\right) + \sum_{n\geq N}\widetilde{P}_{0,\mu}\left\{\left(A_{n+1}^\varepsilon\right)^{\mathrm{c}} \;\big|\; A_n^\varepsilon\right\}.\end{aligned}$$

Then (43) follows from (34), step 2° and Hypothesis 8 (b2).

4° (*conclusion*) Let $\varepsilon=k^{-1}$, $k>1$. From (43) and the monotonicity property of measures, we learn that the event that $A^{1/k}$ fails for all k, has $\widetilde{P}_{0,\mu}$–probability 0. In other words,

$$\widetilde{P}_{0,\mu}\Big(\bigcup_{k>1}A^{1/k}\Big) \;=\; 1.$$

Then step 1° implies that

$$\exists\, k>1 \text{ such that } \widetilde{X}_{\overline{T}_\infty^{1/k}}=0, \quad \widetilde{P}_{0,\mu}\text{–a.s.}$$

Again applying the special Markov property, we obtain

$$\exists\, k \text{ such that } \widetilde{X}_{T_\infty^{1/k}}=0, \quad \widetilde{P}_{0,\mu}\text{–a.s.}$$

Since $T_\infty^{1/k}$ is non-random, the proof of our abstract Theorem 10 is finished. ■

3.5 Scaled comparison with Feller's branching diffusion

Consider a pair of Brownian stopping times $0 \le T_0 < T_1$, a constant $\xi > 0$, and define

$$E := \left\{ w \in \mathbf{C} : \int_{T_0}^{T_1} K(\mathrm{d}r) \ge \xi \right\}.$$

We want to estimate a conditional probability as in (35) in Hypothesis 9 under a mild additional assumption on the branching rate functional K. For this purpose, we will compare with the survival probability in Feller's branching diffusion.

Recall that $d \ge 1$, $\mu \in \mathcal{M}_{\mathrm{f}}$, and that $\mathcal{G}_{T_0}$ denotes the pre-T_0 σ–field.

Proposition 12 (comparison with Feller's branching diffusion) *Assume that the branching rate functional $K \in \mathbf{K}^\nu$ $(\nu > 0)$ is homogeneous and satisfies*

$$\int_0^t K(\mathrm{d}r) \underset{t\uparrow\infty}{\longrightarrow} \infty, \quad \Pi_{0,a}\text{–}a.s., \quad a \in \mathsf{R}^d. \tag{44}$$

Then $\widetilde{P}_{0,\mu}$–almost surely,

$$\widetilde{P}_{0,\mu}\left\{ \widetilde{X}_{T_1}(E) > 0 \;\middle|\; \mathcal{G}_{T_0} \right\} \;\le\; \frac{1}{\xi}\,\|\widetilde{X}_{T_0}\|. \tag{45}$$

We will prove Proposition 12 in the next subsection, using an idea from [FM97], which was in turn inspired by a modulus of continuity technique of [DP91]. In fact, since the paths in E have a "significant" accumulated rate of branching over the time interval $[T_0, T_1)$ [recall (33)], we can *compare* (in law) $\widetilde{X}_{T_1}(E)$ with the mass in Feller's branching diffusion after an appropriate individual time change. (Recall that the total mass of the classical super-Brownian motion is equal in distribution to Feller's branching diffusion.) But for Feller's branching diffusion, there is a well-known estimate for the probability that the process survives in the given time.

Remark 13 We usually apply Proposition 12 for $T_0 = T_n^\varepsilon$, $T_1 = T_{n+1}^\varepsilon$, $\xi = \xi_n^\varepsilon$, and $E = E_n^\varepsilon$, with fixed ε and n. ◇

3.6 Proof of the comparison argument

Consider the process $\{\widetilde{X}_t : t \geq T_0\}$, given $\mathcal{G}_{T_0}$. In particular, the starting measure $\widetilde{X}_{T_0} =: \vartheta$ is given. Note that this (conditional) process has the law $\widetilde{P}_\vartheta$, by the special Markov property. In order to prove Proposition 12, we first intend to define a *new time scale* denoted by r, dictated by the additive functional K. Given for the moment a path $w \in \mathbf{C}$, set

$$R(t) := \int_{T_0}^{T_0+t} K(\mathrm{d}s), \qquad t \geq 0, \tag{46}$$

(recall that K is a continuous additive functional of Brownian motion W). Note that $R(t) \uparrow \infty$ as $t \uparrow \infty$, $\widetilde{\Pi}_\vartheta$–almost everywhere [by assumption (44)], and that $R(t)$ depends continuously on t [by the continuity of K]. Define finite (Brownian) stopping times $\tau(r)$ (converging to infinity as $r \uparrow \infty$) by

$$\tau(r) := \inf\Big\{t > 0 : R(t) \geq r\Big\}, \qquad r \geq 0. \tag{47}$$

Consider the stopped historical SBM $r \mapsto \widetilde{X}_{T_0+\tau(r)}$. Put

$$Z_r := \big\|\widetilde{X}_{T_0+\tau(r)}\big\|, \qquad r \geq 0, \tag{48}$$

for its total mass process. Assume for the moment that $\widetilde{P}_{0,\mu}$–a.s. under the probability laws $\widetilde{P}_\vartheta$ the following two statements hold:

(i) If $Z_\xi = 0$ then $\widetilde{X}_{T_1}(E) = 0$ (extinction of good paths).

(ii) The process $r \mapsto Z_r$ satisfies equation (6) at p.4 with $Z_0 = \|\vartheta\|$.

Then, from the well-know survival probability formula for solutions Z of equation (6), that is of Feller's branching diffusion, we have

$$\widetilde{P}_\vartheta\Big(\widetilde{X}_{T_1}(E) > 0\Big) \leq \widetilde{P}_\vartheta\left(Z_\xi > 0\right) = 1 - \exp\Big[-\xi^{-1}\,\|\vartheta\|\Big] \leq \xi^{-1}\,\|\vartheta\|, \tag{49}$$

which would imply (45).

It remains to prove the statements **(i)** and **(ii)**. To show **(i)**, assume that

$$\big\|\widetilde{X}_{T_0+\tau(\xi)}\big\| = Z_\xi = 0.$$

That is, all paths which accumulated the rate of branching ξ have died [by time $T_0 + \tau(\xi)$]. Therefore we will not find paths with accumulated rate of branching greater or equal to ξ, in particular at time $T_1 \geq T_0 + \tau(\xi)$, which holds under E. Consequently, $\widetilde{X}_{T_1}(E) = 0$, and **(i)** is verified.

We are left with proving **(ii)**. The initial condition is trivially fulfilled. To simplify notation, we write $\mathcal{G}_r$ for the pre-$[T_0 + \tau(r)]$ σ–field. It is sufficient to show that $\widetilde{P}_{0,\mu}$–almost surely, Z is a $\left(\widetilde{P}_\vartheta, (\mathcal{G}_r)_{r\geq 0}\right)$–*martingale* with square variation

$$\langle\!\langle Z\rangle\!\rangle_r = 2\int_0^r \mathrm{d}s\ Z_s\,, \qquad r\geq 0.$$

This would be verified if we proved that for $0\leq r<r'$,

$$\widetilde{P}_\vartheta\left\{Z_{r'}\,\middle|\,\mathcal{G}_r\right\} = Z_r\,, \tag{50}$$

$$\widetilde{P}_\vartheta\left\{Z_{r'}^2 - 2\int_r^{r'}\mathrm{d}s\ Z_s\,\middle|\,\mathcal{G}_r\right\} = Z_r^2. \tag{51}$$

By the special Markov property, statement (50) follows from the expectation formula (30). But then (51) reduces to

$$\widetilde{\mathrm{V}}\mathrm{ar}_{\vartheta(r)}\, Z_{r'} = 2\,(r-r')\,Z_r \tag{52}$$

with $\vartheta(r):=\widetilde{X}_{T_0+\tau(r)}$. But from the variance formula (31), the left hand side of (52) equals

$$2\ \widetilde{\Pi}_{\vartheta(r)}\int_{\tau(r)}^{\tau(r')} K(\mathrm{d}s).$$

Using the definition (47) of $\tau(r)$, by a change of variables, see e.g. [RY91, Proposition (0.4.9)], we can continue with

$$= 2\ \widetilde{\Pi}_{\vartheta(r)}\int_r^{r'}\mathrm{d}s = 2\,(r'-r)\,\|\vartheta(r)\|,$$

getting the right hand side of (52).

This completes the proof of (45), that is of Proposition 12. ■

4 Two applications of the abstract criterion

Here we want to apply our abstract finite time extinction criterion, combined with the comparison with Feller's branching diffusion, to two *one-dimensional* models, namely SBM with the parabolic catalyst χ_q of Model 3, and with a certain point catalyst $\underline{\Gamma}$ with atoms whose locations are dense in R (we will need $\underline{\Gamma}$ later on).

4.1 Parabolic catalyst χ_q (Model 3)

In the parabolic catalyst model, the branching rate functional K is given by the BCLT $L_{[W,\psi]}$ (recall Lemma 4). Here the measure $\psi(\mathrm{d}b)\in\mathcal{M}_p\,,\ p>1$, has a density function $\chi_q(b)=|b|^q\wedge 1,\ b\in\mathsf{R}$, with respect to the (normalized)

Lebesgue measure $\mathrm{d}b$, as introduced in (7) at p.5 (where the exponent $q > 0$ is a fixed constant).

Since the catalyst is small only near the origin $b = 0$, the *bad* historical paths will be those which spend a large amount of time near 0. To estimate the probability of such paths, we need the following lemma. Let $L_t(b)$ denote the *local time* at $b \in \mathsf{R}$ of a (one-dimensional standard) Brownian path W, up to time t.

Lemma 14 (Brownian local time large deviations) *There exists a constant $c_0 > 0$ such that for all $\theta \in (0,1]$, and for all $a \in \mathsf{R}$, the following holds:*

$$\Pi_{0,a}\left(\int_{-\theta}^{\theta} \mathrm{d}b\ L_t(b) \geq \frac{t}{2}\right) \leq \exp\left[-\frac{c_0\, t}{\theta^2}\right]. \tag{53}$$

Proof By Brownian scaling, we have

$$\Pi_{0,a}\Big(\int_{-\theta}^{\theta} \mathrm{d}b\ L_t(b) \geq \frac{t}{2}\Big) = \Pi_{0,a/\theta}\Big(\int_{-1}^{1} \mathrm{d}b\ L_{t/\theta^2}(b) \geq \frac{t}{2\theta^2}\Big).$$

So it suffices to prove the claim for $\theta = 1$. But then we may apply Lemma 2.2 of Donsker and Varadhan [DV77] with A there the set of subprobability measures ν on R such that $\nu\big([-1,1]\big) \geq 1/2$. ∎

In order to specify the quantities entering in Hypotheses 8 and 9, fix an $\alpha > 0$ and a $\beta \in (0, 2\alpha)$. For $n \geq N(\varepsilon) \equiv 0$, and $0 < \varepsilon < 1$, put

$$\theta_n := \mathrm{e}^{-\alpha n}, \quad M_n^\varepsilon \equiv M_n := \mathrm{e}^{-(1+\beta+\alpha q)n}$$

$$t_n^\varepsilon := \varepsilon^{-1}\mathrm{e}^{-\beta n}, \quad \xi_n^\varepsilon := \frac{1}{2}\, t_n^\varepsilon\, \theta_n^q\,, \quad \delta_n^\varepsilon := M_n \,/\, \xi_n^\varepsilon\,, \quad \delta_{-1}^\varepsilon := \varepsilon,$$

and finally

$$\lambda_n^\varepsilon := \frac{M_n}{M_{n+1}} \exp\Big[-\frac{c_0}{\theta_n^2}\, t_n^\varepsilon\Big] \tag{54}$$

(with c_0 from Lemma 14). We will use the deterministic times

$$T_{n+1}^\varepsilon := T_n^\varepsilon + t_n^\varepsilon\,, \quad T_0^\varepsilon := 0. \tag{55}$$

By Remark 1 we may assume without loss of generality that $\mu(\mathsf{R}) \leq 1$. Then the "starting condition" (34) is trivially satisfied.

Note that these constants satisfy Hypothesis 8. In fact, the series in condition (b2) can be estimated from above by

$$c \sum_{n \geq 0} \left[\varepsilon\, \mathrm{e}^{-n} + \exp\left[-c_0\, \varepsilon^{-1} \mathrm{e}^{(2\alpha-\beta)n}\right]\right]$$

Since $\mathrm{e}^{(2\alpha-\beta)n} \geq c\,n$, this bound is of order ε, and property (b2) follows.

By the choice of δ_n^ε, inequality (35) concerning the good paths holds by the comparison Proposition 12.

It remains to verify the expectation estimate (36) for the mass of bad paths at time T_{n+1}^ε, for which we will use Lemma 11. By time-homogeneity and definition (33) of E_n^ε, for any $a \in \mathsf{R}$,

$$\Pi_{T_n^\varepsilon,a}\left(W \in (E_n^\varepsilon)^{\mathrm{c}}\right) \;=\; \Pi_{0,a}\Big(\int_0^{t_n^\varepsilon} K(\mathrm{d}r) \leq \xi_n^\varepsilon\Big) \tag{56}$$

and

$$\int_0^{t_n^\varepsilon} K(\mathrm{d}r) \;=\; \int_0^{t_n^\varepsilon} \mathrm{d}r\; \chi_q(W_r) \;=\; \int \mathrm{d}b\; \chi_q(b)\, L_{t_n^\varepsilon}(b) \;\geq\; \int_{|b|\,\geq\,\theta_n} \mathrm{d}b\;\, \theta_n^q\, L_{t_n^\varepsilon}(b).$$

Thus, the probability expression in (56) can be estimated from above by

$$\Pi_{0,a}\Big(\int_{|b|\,\geq\,\theta_n} \mathrm{d}b\;\, L_{t_n^\varepsilon}(b) \leq \frac{t_n^\varepsilon}{2}\Big) \;=\; \Pi_{0,a}\Big(\int_{|b|\,\leq\,\theta_n} \mathrm{d}b\;\, L_{t_n^\varepsilon}(b) \geq \frac{t_n^\varepsilon}{2}\Big).$$

Hence, by Lemma 14, we get

$$\Pi_{T_n^\varepsilon,a}\left(W \in (E_n^\varepsilon)^{\mathrm{c}}\right) \;\leq\; \exp\Big[-\frac{c_0}{\theta_n^2}\, t_n^\varepsilon\Big] \;=\; \frac{M_{n+1}}{M_n}\, \lambda_n^\varepsilon\,,$$

where we used (54). In other words, (38) in Lemma 11 holds, and (36) is valid.

Altogether, we showed that all requirements for the abstract criterion Theorem 10 are satisfied, hence *finite time extinction holds* for the SBM with parabolic catalyst χ_q for any finite initial measure μ on R. ■

4.2 A point catalyst $\underline{\Gamma}$ with dense locations

Now we consider the case

$$K \;=\; L_{[W,\underline{\Gamma}]} \quad \text{with} \quad \underline{\Gamma} \;=\; \sum_{n=N}^{\infty} 2^{-n}\, \underline{\pi}_n\,, \tag{57}$$

for a fixed $N \geq 0$, independent of ε. Here $\underline{\pi}_n$ is assumed to be a (locally finite, deterministic) point measure on R such that all neighboring points have a distance of at most Δ_n, where, for some $\beta \in (0,1)$,

$$\Delta_n \;:=\; \mathrm{e}^{-\beta n}, \qquad n \geq N. \tag{58}$$

We claim that the continuous SBM $X^{\underline{\Gamma}}$ with catalyst $\underline{\Gamma}$ *has the finite time extinction property.* This will follow from our abstract extinction criterion Theorem 10 once we have found the appropriate quantities $T_n^\varepsilon, M_n^\varepsilon, \delta_n^\varepsilon, \lambda_n^\varepsilon, \xi_n^\varepsilon$ entering into Hypotheses 8 and 9.

1° (*some constants*) Choose $\alpha \in (\frac{\beta}{2}, \beta)$. For $n \geq N$ and $0 < \varepsilon < 1$, set

$$m_n^\varepsilon := \left[\frac{\mathrm{e}^{\alpha n}}{\varepsilon}\right], \quad s_n^\varepsilon := \frac{\mathrm{e}^{-\beta n}}{\varepsilon^2}, \quad t_n^\varepsilon := 2\, m_n^\varepsilon s_n^\varepsilon, \quad M_n^\varepsilon \equiv M_n := 2^{-n} \tag{59}$$

(where $[z]$ denotes the integer part of z). We again use deterministic times $T_{n+1}^\varepsilon := T_n^\varepsilon + t_n^\varepsilon$, $T_N^\varepsilon := 0$. Note that by our choice of t_n they satisfy (32). The quantities $\xi_n^\varepsilon, \delta_n^\varepsilon, \lambda_n^\varepsilon$ will be defined in (64), (65), and (68), respectively.

Assume without loss of generality that $\mu(\mathsf{R}) \leq 2^{-N}$. Then, if we set $\delta_{N-1}^\varepsilon = \varepsilon$, the *starting condition* (34) is trivially satisfied.

2° (*partitioning*) For $n \geq N$ and $0 \leq \varepsilon < 1$ fixed, our next aim is to introduce a partition of the time period $[T_n^\varepsilon, T_{n+1}^\varepsilon)$ by means of some Brownian stopping times. This construction allows us to consider hitting times of neighboring points of $\underline{\pi}_n$ and local times spent on them.

Given $\vartheta_n := \widetilde{X}_{T_n^\varepsilon}$, and a path w "distributed" according to $\vartheta_n(\mathrm{d}w)$, we consider the Brownian path process $\widetilde{W}$ distributed according to $\widetilde{\Pi}_{T_n^\varepsilon, w}$, and its projection $t \mapsto (\widetilde{W}_t)_t = W_t$ with law

$$\Pi_{T_n^\varepsilon, w(T_n^\varepsilon)} =: \underline{\Pi}.$$

(For typographical simplicity, sometimes we write $w(t)$ instead of w_t, etc.)

Set $\kappa_0 := T_n^\varepsilon$. For $m \geq 1$, we inductively define (Brownian) *stopping times* $\overline{\kappa}_m = \overline{\kappa}_{m,n}^\varepsilon(W)$ and $\kappa_m = \kappa_{m,n}^\varepsilon(W)$ as follows. Given κ_{m-1}, let $\overline{\kappa}_m$ denote the first time point $t \geq \kappa_{m-1}$ such that W hits one of the atoms of $\underline{\pi}_n$. Given $\overline{\kappa}_m$, we simply define $\kappa_m = \overline{\kappa}_m + s_n^\varepsilon$ [with $s_n^\varepsilon > 0$ introduced in (59)].

Write $H_m = H_{m,n}^\varepsilon$ for the *hitting time* $\overline{\kappa}_m - \kappa_{m-1}$ of $\underline{\pi}_n$ (starting at time κ_{m-1}). Recall that by definition the distance between neighboring atoms of $\underline{\pi}_n$ is at most Δ_n. Using the eigenfunction representation of solutions to the heat equation, we have that for some constant $c_0 > 0$,

$$\underline{\Pi}\left(H_m \geq s_n^\varepsilon\right) \leq c_0^{-1} \exp\left[-\frac{c_0\, s_n^\varepsilon}{\Delta_n^2}\right], \qquad m \geq 0, \quad n \geq N.$$

Therefore,

$$\underline{\Pi}\left(\sum_{m=1}^{m_n^\varepsilon} H_m \geq m_n^\varepsilon s_n^\varepsilon\right) \leq c_0^{-1} m_n^\varepsilon \exp\left[-\frac{c_0\, s_n^\varepsilon}{\Delta_n^2}\right] =: \zeta_n^\varepsilon \tag{60}$$

[with $\Delta_n, s_n^\varepsilon, m_n^\varepsilon$ introduced in (58) and (59)].

Now write $L_m = L_{m,n}^\varepsilon$ for the (Brownian) *local time* spent by W at the site $W(\overline{\kappa}_m)$ during the time interval $[\overline{\kappa}_m, \kappa_m)$ of length s_n^ε. That is, symbolically,

$$L_m := \int_{\overline{\kappa}_m}^{\kappa_m} \mathrm{d}r\, \delta_{W(\overline{\kappa}_m)}(W_r). \tag{61}$$

Recall that at this site $W(\overline{\kappa}_m)$ there is an atom of $\underline{\pi}_n$, and that the mass 2^{-n} is attached to it. Therefore, using the integers m_n^ε introduced in (59), for the BCLT $L_{[W,\underline{\Gamma}]}$ of $\underline{\Gamma} \geq 2^{-n}\underline{\pi}_n$ we get

$$\int_{T_n^\varepsilon}^{\kappa_{m_n^\varepsilon}} L_{[W,\underline{\Gamma}]}(\mathrm{d}r) \;\geq\; 2^{-n} \sum_{m=1}^{m_n^\varepsilon} L_m \,. \tag{62}$$

Clearly, the L_m are i.i.d. (with respect to $\underline{\Pi}$). Moreover, L_m is equal in law to

$$\sup_{0 \leq t \leq s_n^\varepsilon} W_t^0$$

where W^0 is distributed according to $\Pi_{0,0}$ (see e.g. [RY91, Theorem (6.2.3)]). Scaling time, we find that $(s_n^\varepsilon)^{-1/2} L_m$ is equal in law to

$$L^0 := \sup_{0 \leq t \leq 1} W_t^0$$

(which is independent of n and ε). Set $a := \frac{1}{2}\,\Pi_{0,0} L^0$. Since L^0 has finite exponential moments, by standard large deviation estimates there exists a constant $c_1 > 0$ such that

$$\underline{\Pi}\bigg((s_n^\varepsilon)^{-1/2} \sum_{m=1}^{k} L_m < a\,k \bigg) \leq \mathrm{e}^{-2c_1 k}, \qquad k \geq 1.$$

Combining with (62), we thus have

$$\underline{\Pi}\bigg(\int_{T_n^\varepsilon}^{\kappa_{m_n^\varepsilon}} L_{[W,\underline{\Gamma}]}(\mathrm{d}r) < \xi_n^\varepsilon \bigg) \leq \exp\left[-2c_1 m_n^\varepsilon\right], \tag{63}$$

where

$$\xi_n^\varepsilon \;:=\; a\, m_n^\varepsilon (s_n^\varepsilon)^{1/2} 2^{-n} \,. \tag{64}$$

3° (*good and bad historical paths*) Recall the set E_n^ε of *good paths* introduced in formula line (33) [based on t_n^ε defined in (59) and entering into (55), as well as ξ_n^ε from (64)]. Since the BCLT $L_{[W,\underline{\Gamma}]}$ satisfies (44), by Proposition 12 we get the survival probability estimate (35) for the good paths, if we set

$$\delta_n^\varepsilon \;:=\; M_n \,/\, \xi_n^\varepsilon \,. \tag{65}$$

On the other hand, in order to calculate the expected mass of bad paths as required in (36), we look at

$$\underline{\Pi}\left((E_n^\varepsilon)^{\mathrm{c}} \right). \tag{66}$$

In order to further estimate this, consider two cases. First let w have "large" hitting times, i.e.

$$\sum_{m=1}^{m_n^\varepsilon} H_m \;\geq\; m_n^\varepsilon s_n^\varepsilon \,.$$

By (60), this occurs with a $\underline{\Pi}$–probability bounded by ζ_n^ε. In the opposite case, by the definition of T_{n+1}^ε we have

$$\kappa_{m_n^\varepsilon} \;=\; T_n^\varepsilon + m_n^\varepsilon s_n^\varepsilon + \sum_{m=1}^{m_n^\varepsilon} H_m \;<\; T_{n+1}^\varepsilon$$

[recall (59) and (55)], hence here $(E_n^\varepsilon)^{\mathrm{c}}$ implies, by the definition (33) of E_n^ε, that

$$\int_{T_n^\varepsilon}^{\kappa_{m_n^\varepsilon}} L_{[W,\underline{\Gamma}]}(\mathrm{d}r) \;<\; \xi_n^\varepsilon .$$

The $\underline{\Pi}$–probability of this event is estimated in (63). Then, for (66) we get the bound

$$\zeta_n^\varepsilon + \exp\left[-2c_1 m_n^\varepsilon\right] \tag{67}$$

[with ζ_n^ε from (60)]. If we now set

$$\lambda_n^\varepsilon \;:=\; \frac{M_n}{M_{n+1}}\left[\zeta_n^\varepsilon + \exp\left[-2c_1 m_n^\varepsilon\right]\right], \tag{68}$$

we obtain (38) in Lemma 11 which gives (36).

4° (*verification of the stage quantities*) It remains to check that δ_n^ε and λ_n^ε introduced in (65) and (68), respectively, satisfy Hypothesis 8 (b2). First of all, by (65) and (64), δ_n^ε approximately equals

$$c\,\varepsilon^2 \exp\left[-\Big(\alpha - \frac{\beta}{2}\Big)n\right],$$

hence its sum over n is of order ε^2. Next, since $m_n^\varepsilon \geq c\,\varepsilon^{-1}n$, the second term of λ_n^ε is bounded by $\exp\left[-c\,\varepsilon^{-1}n\right]$, except a constant factor. Summing over n we arrive at a term of order ε. Finally, by (60), the first term of λ_n^ε is bounded from above by

$$c_0^{-1}\varepsilon^{-1}\exp\left[\alpha n - c_0\,\varepsilon^{-2}\mathrm{e}^{\beta n}\right].$$

But

$$\varepsilon^{-1}\int_1^\infty \mathrm{d}x\;\exp\left[\alpha x - c_0\,\varepsilon^{-2}\mathrm{e}^{\beta x}\right] \;\leq\; (\beta\varepsilon)^{-1}\int_1^\infty \mathrm{d}y\; y^{\frac{\alpha}{\beta}-1}\exp\left[-c_0\,\varepsilon^{-2}y\right],$$

and $r \mapsto r^z\mathrm{e}^{-r}$ is bounded for $r > 0$ bounded away from 0, for each fixed z. Hence, it suffices to consider

$$\varepsilon^{-1}\int_1^\infty \mathrm{d}y\;\exp\left[-\frac{c_0}{2}\varepsilon^{-2}y\right]$$

which is of order ε. Consequently, the series in Hypothesis 8 (b2) is bounded by $c\,\varepsilon$, hence this hypothesis is satisfied in the present case.

Summarizing, the *catalytic SBM* $X^{\underline{\Gamma}}$ *dies in finite time,* for any finite starting measure μ on R. ■

5 Proof of the main result

The proof of Theorem 6 (p.10) concerning finite time extinction of the one-dimensional SBM X^Γ with a stable catalyst proceeds in several steps. Since here we start with an initial measure μ of compact support, and X^Γ has the compact support property ([DLM95]), by some coupling technique we will pass to a periodic catalyst Γ^{K}. Moreover, because the survival probability is monotone in the catalyst, we will switch to a smaller catalyst, as already explained in Subsection 1.1. Altogether we will reduce to the case of a point catalyst $\underline{\Gamma}$ with dense locations as dealt with in Subsection 4.2.

5.1 A coupling of catalytic SBMs

Recall that the historical catalytic SBM $\widetilde{X}^\Gamma$ exists for $\mathbb{P}$–almost all Γ. Fix an initial measure $\mu \in \mathcal{M}_{\mathrm{f}}$ with compact support. We want to show that

$$\widetilde{P}^\Gamma_{0,\mu}\left(\widetilde{X}^\Gamma_t \neq 0,\ \forall t\right) \;=\; 0, \quad \mathbb{P}\text{–a.s.} \tag{69}$$

For $\mathsf{K} \geq 1$, let

$$E_{\mathsf{K}} \;:=\; \Big\{ w \in \mathbf{C}:\ |w_s| \leq \mathsf{K},\ \forall s \geq 0 \Big\}.$$

According to the compact support property of [DLM95],

$$\lim_{\mathsf{K}\uparrow\infty} \widetilde{P}^\Gamma_{0,\mu}\left(\mathrm{supp}\widetilde{X}^\Gamma_t \subseteq E_{\mathsf{K}},\ \forall t\right) \;=\; 1, \quad \mathbb{P}\text{–a.s.} \tag{70}$$

For the further proof, fix such a sample Γ. By (70), instead of (69) it suffices to show that

$$\widetilde{P}^\Gamma_{0,\mu}\left(\widetilde{X}^\Gamma_t \neq 0 \text{ and } \mathrm{supp}\widetilde{X}^\Gamma_t \subseteq E_{\mathsf{K}},\ \forall t\right) \;=\; 0, \quad \text{for all } \mathsf{K}. \tag{71}$$

But under this restriction to historical paths living in E_{K}, we may change the catalyst outside of $[-\mathsf{K},\mathsf{K}]$ without affecting the latter probability. This will be formalized in the following considerations establishing some coupling argument.

Fix $\mathsf{K} \geq 1$ such that the initial measure μ is supported by $(-\mathsf{K},\mathsf{K})$. Consider the hitting time τ^{K} of $\{-\mathsf{K},\mathsf{K}\}$, the boundary of the interval $(-\mathsf{K},\mathsf{K})$. Then replace the Brownian motion W of reactant particles by the stopped process $t \mapsto W_{t\wedge\tau^{\mathsf{K}}}$. This transfers $\widetilde{X}^\Gamma$ into the stopped historical catalytic SBM $t \mapsto \widetilde{X}^\Gamma_{t\wedge\tau^{\mathsf{K}}}$. Note that the paths of this stopped process live completely in the closed interval $[-\mathsf{K},\mathsf{K}]$.

Actually we *decompose* this stopped process,

$$\widetilde{X}^\Gamma_{t\wedge\tau^{\mathsf{K}}} \;=\; \mathrm{m}^{\mathsf{K}}_t + \mathrm{m}^{\circ}_t,$$

by distinguishing between paths

$$w \in {}^{\mathsf{K}}\mathbf{C}^t := \{w \in \mathbf{C}^t : |w_t| = \mathsf{K}\}$$

which end at the boundary $\{-\mathsf{K},\mathsf{K}\}$, and those which stay within $(-\mathsf{K},\mathsf{K})$:

$$w \in {}^{\circ}\mathbf{C}^t := \Big\{w \in \mathbf{C}^t : |w_s| < \mathsf{K},\ \forall s \le t\Big\}.$$

In other words,

$$\mathrm{m}_t^{\mathsf{K}} := \widetilde{X}^{\Gamma}_{t\wedge\tau^{\mathsf{K}}}\big((\cdot)\ \cap\ {}^{\mathsf{K}}\mathbf{C}^t\big), \qquad \mathrm{m}_t^{\circ} := \widetilde{X}^{\Gamma}_{t\wedge\tau^{\mathsf{K}}}\big((\cdot)\ \cap\ {}^{\circ}\mathbf{C}^t\big).$$

Note that the path $t \mapsto \mathrm{m}_t^{\mathsf{K}}$ of measures on $\mathbf{C}$ is monotonically non-decreasing. Thus, $\mathrm{m}^{\mathsf{K}}_{\mathrm{d}t}(\mathrm{d}\omega_t)$ can be considered as a measure in $\mathcal{M}\big[\mathsf{R}_+ \widehat{\times}\, \mathbf{C}^{\bullet}\big]$ [recall notation (21)]. Now we use the increments of this historical path m^{K} as an *immigration* process of a historical catalytic SBM starting from the zero measure, denoted by $\widetilde{Y} = \widetilde{Y}^{\Gamma,\mathrm{m}^{\mathsf{K}}}$. More precisely, (for the given Γ) given m^{K}, defining $\widetilde{Y} = \widetilde{Y}^{\Gamma,\mathrm{m}^{\mathsf{K}}}$ we use the modified process according to Proposition 7(b) with the collision local time $L_{[W,\Gamma]}$ as branching rate functional, and with η defined by

$$\eta(\mathrm{d}r, \mathrm{d}\omega_r) := \mathrm{m}^{\mathsf{K}}_{\mathrm{d}r}(\mathrm{d}\omega_r).$$

We need also another process. Let Γ^{K} denote the *periodic extension* of the restriction $\Gamma\left((\cdot)\cap(-\mathsf{K},\mathsf{K}]\right)$ of Γ to $(-\mathsf{K},\mathsf{K}]$ to all of R (for the fixed Γ). Now replace Γ by Γ^{K} in the definition of $\widetilde{Y} = \widetilde{Y}^{\Gamma,\mathrm{m}^{\mathsf{K}}}$, to obtain a historical catalytic SBM with *periodic* catalyst Γ^{K} and *immigration* controlled by m^{K}, which we denote by $\widetilde{Z} = \widetilde{Z}^{\Gamma^{\mathsf{K}},\mathrm{m}^{\mathsf{K}}}$.

Recall that both of our processes $\widetilde{Y}$ and $\widetilde{Z}$ are based on the same samples $\Gamma\left((\cdot)\cap(-\mathsf{K},\mathsf{K}]\right)$ and m^{K}. The reason we introduced these processes is the following obvious coupling result.

Lemma 15 (coupling of historical catalytic SBMs) *Fix* $\mathsf{K} \ge 1$ *such that the initial measure* μ *is supported by* $(-\mathsf{K},\mathsf{K})$. *Given* Γ, *the processes* $\widetilde{Y} + \mathrm{m}^{\circ}$ *and* $\widetilde{Z} + \mathrm{m}^{\circ}$ *coincide in law with the historical catalytic SBMs* $\widetilde{X}^{\Gamma}$ *and* $\widetilde{X}^{\Gamma^{\mathsf{K}}}$ *with catalysts* Γ *and* Γ^{K}, *respectively, and their restrictions to paths living in* E_{K} *are identical to* m°.

5.2 Completion of the proof of the main theorem

By Lemma 15, we may pass in (71) from Γ to the periodic Γ^{K}. Hence, instead of (71) it suffices to show

$$\widetilde{P}^{\Gamma^{\mathsf{K}}}_{0,\mu}\Big(\widetilde{X}^{\Gamma^{\mathsf{K}}}_t \ne 0,\ \forall t\Big) = 0, \tag{72}$$

for each fixed $\mathsf{K} \geq 1$ such that μ is supported by $(-\mathsf{K}, \mathsf{K})$. In other words, we want to show finite time extinction of the historical catalytic SBM $\widetilde{X}^{\Gamma^{\mathsf{K}}}$ with fixed periodic catalyst Γ^{K}.

In order to can apply later on the result of Subsection 4.2, we further use the fact that the collision local times $L_{[W,\psi]}$ are non-decreasing in $\psi \in \mathcal{M}_p$. That is, $\psi_1 \leq \psi_2$ implies $L_{[W,\psi_1]} \leq L_{[W,\psi_2]}$. Therefore the corresponding solutions v^ψ of the log-Laplace equation (24) are non-increasing: $v^{\psi_1} \geq v^{\psi_2}$. But this yields that the *extinction probability is non-decreasing* in ψ :

$$\psi_1 \leq \psi_2 \quad \text{implies} \quad P^{\psi_1}_{0,\mu}(X_t = 0) \leq P^{\psi_2}_{0,\mu}(X_t = 0)$$

[recall (3)]. Hence, for our purpose of verifying (72), we may replace the periodic catalyst Γ^{K} by a smaller measure.

To this end, as already mentioned in Subsection 1.1, we first drop all the "big" point catalysts: For the moment, fix $N \geq 0$ (independent of ε), and remove all those atoms $\alpha_i \delta_{b_i}$ of Γ^{K} [or Γ, recall the representation (19)] with action weight $\alpha_i \geq 2^{-N+1}$. Next, for each $n \geq N$, we replace the action weights $\alpha_i \in [2^{-n}, 2^{-n+1})$ by 2^{-n}. Note that with respect to $\mathbb{P}$, the positions $b_i \in (-\mathsf{K}, \mathsf{K}]$ of the related atoms are distributed as a *Poisson point process* with intensity measure $c_\gamma\, 2^{\gamma n}\, \mathbf{1}_{(-\mathsf{K},\mathsf{K}]}(b)\, \ell(\mathrm{d}b)$. Here the constant c_γ is given by

$$c_\gamma \;:=\; \gamma^{-1}(1-2^{-\gamma})\Big(\int_0^\infty \mathrm{d}r\;\, r^{-1-\gamma}\,(1-\mathrm{e}^{-r})\Big)^{-1}$$

(see e.g. [DF92]). Let $\underline{\pi}_n$ denote the periodic extension of this Poisson point process, extension from $(-\mathsf{K}, \mathsf{K}]$ to all of R.

What remains for the reduction to Subsection 4.2 is to show that $\mathbb{P}$–a.s. in $\underline{\pi}_n$ neighboring catalysts have a distance of at most $\Delta_n = \mathrm{e}^{-\beta n}$, for all $n \geq N$, for $N \geq 0$ appropriately chosen. For this purpose, we fix

$$\beta \in (0, \gamma \log 2)\,. \tag{73}$$

By Borel-Cantelli, it suffices to show that the quantities

$$\mathbb{P}\left(\exists \text{ two neighboring points in } \underline{\pi}_n \text{ with a distance larger than } \mathrm{e}^{-\beta n}\right) \tag{74}$$

are summable in $n \geq 1$. But each of these probabilities is bounded from above by

$$\mathbb{P}\left(\max_{1\,\leq\, i\,\leq\, J_n+1} \xi_i \;>\mathrm{e}^{-\beta n}\right) \tag{75}$$

where J_n is the Poissonian number of points in $(-\mathsf{K}, \mathsf{K}]$ with expectation $a_n := 2\mathsf{K}c_\gamma 2^{\gamma n}$, and the $\xi_1, \xi_2, \ldots$ are i.i.d. exponentials with parameter a_n. Now the J_n satisfy a standard large deviation principle as $n \uparrow \infty$, hence,

$$\mathbb{P}\left(J_n + 1 > 2a_n\right) \;\leq\; \exp\left[-\,c\, 2^{\gamma n}\right]$$

for all sufficiently large n. Since the right hand side is summable in n, in the probability expression (75) we may additionally restrict to $J_n + 1 \leq 2a_n$. Consequently, instead of (75) we look at

$$\mathbb{P}\left(\max_{1 \leq i \leq 2a_n} \xi_i > \mathrm{e}^{-\beta n}\right).$$

By scaling, we may switch to

$$\mathcal{P}\left(\max_{1 \leq i \leq 2a_n} \xi_i' > a_n \, \mathrm{e}^{-\beta n}\right) \tag{76}$$

where the ξ_i' are now i.i.d. standard exponentials (under the law denoted by $\mathcal{P}$).

Next we use the fact that for all $x \geq 0$ and $m \geq 2$,

$$\left|\mathcal{P}\left(\max_{1 \leq i \leq m} \xi_i' - \log m > x\right) - \left(1 - \exp\left[-\mathrm{e}^{-x}\right]\right)\right| \leq 2\,\mathrm{e}^{-2x} \tag{77}$$

(see Example 2.10.1 in [Gal78]; take $q = \frac{1}{2}$ there). Now, for all n sufficiently large, $m = [2a_n]$ and

$$x = a_n \, \mathrm{e}^{-\beta n} - \log(2a_n) = c\,\mathrm{e}^{n(\gamma \log 2 - \beta)} - \log(4\mathsf{K}c_\gamma) - n\gamma \log 2 \tag{78}$$

satisfy these conditions [recall (73)]. Thus, for (76) we get the bound

$$1 - \exp\left[-\mathrm{e}^{-x}\right] + 2\,\mathrm{e}^{-2x} \leq 3\,\mathrm{e}^{-x}$$

which for x from (78) is summable in n.

This finishes the proof of Theorem 6. ■

6 The lattice model

Now consider the model with random catalysts on the lattice Z^d. Recall that $\varrho = \{\varrho_b\}_{b \in \mathsf{Z}^d}$, the catalysts, are i.i.d. random variables which are uniformly distributed on $[0,1]$. Instead of Brownian motions, the motion process is now given by a continuous time simple random walk on Z^d, which moves to a neighboring site at rate 1. In other words, the times between jumps are i.i.d. exponential times with mean 1.

We use symbols analogous to the ones in earlier sections. In particular, $\mathbb{P}$ denotes the law of the catalyst, $W = \left[W, \Pi_{s,a}, s \geq 0, a \in \mathsf{Z}^d\right]$ the *simple random walk* in Z^d on canonical Skorohod path space $\mathbf{D} = \mathcal{D}\left[\mathsf{R}_+, \mathsf{Z}^d\right]$ of càdlàg functions, and

$$\widetilde{X}^\varrho = \left[\widetilde{X}^\varrho, \widetilde{P}^\varrho_{s,\mu}, s \geq 0, \mu \in \mathcal{M}_\mathrm{f}[\mathbf{D}^s]\right]$$

the *historical simple super-random walk* on Z^d (also called simple interacting Feller's branching diffusion) with catalyst ϱ. Note that Proposition 7, Theorem 10, Lemma 11, and Proposition 12 remain valid (with the obvious changes).

For simplicity, we now assume that $X_0 = \delta_0$. Our aim is to show the *finite time extinction* property for $\widetilde{X}^\varrho$, for $\mathbb{P}$–a.a. ϱ. In this case, the bad historical paths are those which spend a large amount of time at sites $b \in \mathsf{Z}^d$ where ϱ_b is small. We will choose time T_N so that, with high probability, most of the mass is dead by this time [in the sense of (34)]. This is the hardest part of the argument. Bounding the mass after this uses similar but easier ideas.

We also need the following crude estimate on the *distance traveled by the simple random walk* W in time t. Let J_t denote the number of jumps taken by W by time t. Since J_t is *Poisson* with parameter t, for $k \geq 0$ we have

$$\left.\begin{aligned} &\Pi_{0,0}\left(\sup_{0\leq s\leq t} |W_s| \geq k\right) \;\leq\; \Pi_{0,0}\left(J_t \geq k\right) \;=\; \mathrm{e}^{-1}\sum_{i=k}^{\infty}\frac{t^i}{i!} \\ &\leq\; \mathrm{e}^{-1}\frac{t^k}{k!}\sum_{i=0}^{\infty}\frac{t^i}{i!} \;=\; \frac{t^k}{k!} \;\leq\; \left(\frac{t\mathrm{e}}{k}\right)^k (2\pi k)^{-1/2}, \end{aligned}\right\} \tag{79}$$

the latter by Stirling's approximation.

For $n \geq 0$, let D_n denote the cube

$$D_n \;=\; \left\{(b_1,\ldots,b_d) \in \mathsf{Z}^d : \;\max(|b_1|,\ldots,|b_d|) \leq 2^n\right\} \tag{80}$$

in Z^d having $(2^{n+1}+1)^d$ sites. For a given path $w \in \mathbf{D}$, let $\tau_n = \tau_n(w)$ denote the first time $t \geq 0$ that w_t does not belong to D_n. We intend to use *Dynkin's special Markov property* to start the stopped historical super-random walk $\left\{\widetilde{X}^\varrho_{\tau_n} : n \geq 0\right\}$ afresh at the times τ_n.

We define in this proof that the quantities M_n^ε, λ_n^ε, ξ_n^ε, δ_n^ε, T_n^ε, and E_n^ε entering in Hypotheses 8 and 9 to be independent of ε, and therefore we omit the index ε. We will choose $N = N(\varepsilon) \geq 1$ later, such that $\lim_{\varepsilon\downarrow 0} N(\varepsilon) = \infty$. To be more specific, N must be so large that all of the statements involving the phrase "for N sufficiently large" are satisfied. Set

$$\delta_{N-1} \;:=\; 2^{-N/4}, \quad T_N \;:=\; \frac{2^N}{6} \wedge \tau_N\,,$$

and for $n \geq N = N(\varepsilon)$, let

$$M_n \;:=\; 2^{-n(d+3)}, \quad \lambda_n \;:=\; 2^{-2^n}, \quad \delta_n \;:=\; 2^{-n},$$

$$\xi_n \;:=\; 2^{-n(d+2)}, \quad T_{n+1} \;:=\; \left(T_n + 2^{-n}\right) \wedge \tau_{n+1}\,.$$

Note that δ_{N-1} and T_N implicitly depend on ε via $N(\varepsilon)$. One can easily show that $T_{n+1} > T_n$, $\Pi_{0,0}$–a.s. Clearly (b1) and (b2) of Hypothesis 8 are satisfied.

We need the following large deviations lemma on the simple random walk W. We say that a non-empty subset $S \subset \mathsf{Z}^d$ is *connected* if any two elements $a, b \in S$ are connected by a chain $a = z_0, \ldots, z_k = b$ of elements of S, such that for $1 \leq i \leq k$ the points z_{i-1}, z_i are nearest neighbors. That is, they are distance 1 apart.

Lemma 16 (large deviations) *Fix* $m \geq 1$. *Suppose that* $S \subset \mathsf{Z}^d$ *has the property that no connected subset of* S *has cardinality larger than* m. *Then there exist constants* $\alpha, c > 0$ (*depending on* m) *such that for all* $t \geq 1$,

$$\sup_{a \in \mathsf{Z}^d} \Pi_{0,a}\left(\int_0^t \mathrm{d}s\, \mathbf{1}\{W_s \in S^{\mathrm{c}}\} \leq \alpha t\right) \leq c^{-1}\mathrm{e}^{-ct}.$$

Proof By monotonicity in m, we may enlarge m if necessary, so we may assume that m is even. By our assumptions, if $a \in S$, then there exists a chain consisting of points $a = z_0, \ldots, z_m$ such that z_{i-1}, z_i are nearest neighbors for $1 \leq i \leq m$, and $z_m = z_m(a) \in S^{\mathrm{c}}$. If $a \in S^{\mathrm{c}}$, we construct such a chain as follows. Let b be one of the nearest neighbors of a, and let $z_{2k} := a$, $z_{2k+1} := b$.

Suppose that $W_0 = a$. Let η_a denote the first time t that $W_t = z_m(a)$. (If there is no such time, set $\eta_a = \infty$.) Let $F = F(a)$ denote the event that $\eta_a < 1/2$ and that $W_s = z_m(a)$ for $\eta_a \leq s \leq 1$. By the properties of our continuous-time simple random walk, using the constructed chain, there exists $\alpha > 0$ such that for all $a \in \mathsf{Z}^d$,

$$\Pi_{0,a}\left(F(a)\right) \geq 8\alpha. \tag{81}$$

Let $F_i := \theta_i F$, $i \geq 0$, where θ_s is the time-shift operator on paths. By the Markov property and (81), there exists a sequence of independent events $\overline{F}_i$ such that $\overline{F}_i \subset F_i$ and $\Pi_{0,a}(\overline{F}_i) = 8\alpha$, for each i. Set

$$G_k := \sum_{i=0}^{k-1} \mathbf{1}_{\overline{F}_i}.$$

By Chernoff's large deviations theorem (see [Bil86, Theorem 9.3]), there exist a constant $c > 0$ such that for all $a \in \mathsf{Z}^d$, we have

$$\Pi_{0,a}\left(\frac{G_k}{k} \leq 4\alpha\right) \leq c^{-1}\mathrm{e}^{-ck}, \qquad k \geq 0.$$

Note that

$$\int_i^{i+1} \mathrm{d}s\, \mathbf{1}\{W_s \in S^{\mathrm{c}}\} \geq 1/2 \quad \text{on } F_i. \tag{82}$$

Indeed, if F_i occurs, then $W_s \in S^{\mathrm{c}}$ for $s \in (i + 1/2, i + 1)$. Suppose that $G_k/k \geq 4\alpha$. Then there are at least $4\alpha k$ indices $i \leq k - 1$ such that $\overline{F}_i$

occurs, and hence F_i occurs. In that case, by (82),

$$\int_0^k \mathrm{d}s\, \mathbf{1}\{W_s \in S^{\mathrm{c}}\} \;\geq\; 2\alpha k.$$

Hence, for $a \in \mathsf{Z}^d$ and $k \geq 0$,

$$\Pi_{0,a}\left(\frac{G_k}{k} > 4\alpha\right) \;\leq\; \Pi_{0,a}\left(\int_0^k \mathrm{d}s\, \mathbf{1}\{W_s \in S^{\mathrm{c}}\} \;\geq\; 2\alpha k\right).$$

Interpolating, we have that for all $a \in \mathsf{Z}^d$ and $t \geq 1$,

$$\begin{aligned}\Pi_{0,a}\left(\int_0^t \mathrm{d}s\, \mathbf{1}\{W_s \in S^{\mathrm{c}}\} < \alpha t\right) \;&\leq\; \Pi_{0,a}\left(\int_0^{[t]} \mathrm{d}s\, \mathbf{1}\{W_s \in S^{\mathrm{c}}\} < \alpha\left([t]+1\right)\right)\\ &\leq\; c^{-1}\,\mathrm{e}^{-ct},\end{aligned}$$

finishing the proof of Lemma 16. ∎

For $m \geq 1$, $n \geq N = N(\varepsilon)$, and $0 \leq \zeta \leq 1$, let $A(m,n,\zeta)$ denote the (catalyst) event that there is no connected subset $S \subset D_n$ with cardinality m, on which all of the catalysts satisfy $\varrho_b \leq \zeta$. Note that there is a finite number $c(m,d)$ of connected sets of cardinality m, which contain a given point. Then we have

$$\left.\begin{aligned}\mathbb{P}\left(A^{\mathrm{c}}(m,n,\zeta)\right) \;&\leq\; \left(2^{n+1}+1\right)^d c(m,d)\left(\mathbb{P}\left(\varrho_b \leq \zeta\right)\right)^m\\ &=\; \left(2^{n+1}+1\right)^d c(m,d)\,\zeta^m.\end{aligned}\right\} \tag{83}$$

In particular, if

$$\zeta = \zeta_n = 2^{-(n-1)(d+1)}, \tag{84}$$

then

$$\mathbb{P}\left(A^{\mathrm{c}}(1,n,\zeta_n)\right) \;\leq\; c\,2^{nd}\,2^{-(n-1)(d+1)} \;=\; c\,2^{-n}. \tag{85}$$

For $m = 1$, all catalysts in D_n are greater than ζ on $A(1,n,\zeta)$. Put

$$A_1(n) \;:=\; \bigcap_{k=n}^{\infty} A(1,k,\zeta_k) \tag{86}$$

and note that

$$\mathbb{P}\left(A_1^{\mathrm{c}}(n)\right) \;\leq\; c_1\,2^{-n}. \tag{87}$$

From now on, let

$$m = 2(d+1), \tag{88}$$

and take

$$\overline{\zeta} = \overline{\zeta}_n = 2^{-n/2}. \tag{89}$$

Then, by (83), we have,

$$\mathbb{P}\left(A^{c}(m, n, \overline{\zeta}_n)\right) \ < \ c\, 2^{-n}.$$

Let

$$A_2(n) \ := \ \bigcap_{k=n}^{\infty} A\left(m, k, \overline{\zeta}_k\right) \tag{90}$$

and note that

$$\mathbb{P}\left(A_2^{c}(n)\right) \ \leq \ c_2\, 2^{-n}. \tag{91}$$

Fix $\overline{\varepsilon} > 0$. Using (87) and (91), we choose $\overline{n} = \overline{n}(\overline{\varepsilon})$ so large that

$$\mathbb{P}\left(A_1^{c}(\overline{n}) \cup A_2^{c}(\overline{n})\right) \ \leq \ (c_1 + c_2)\, 2^{-\overline{n}} \ < \ \overline{\varepsilon}. \tag{92}$$

We will apply our general Theorem 10 with N chosen to satisfy $N = N(\varepsilon, \overline{\varepsilon}) \geq \overline{n}$. We will conclude that for catalysts ϱ in $A_1(\overline{n}) \cap A_2(\overline{n})$, finite time extinction occurs with $\widetilde{P}^{\varrho}_{0,\mu}$–probability 1. Therefore, with $\mathbb{P}$–probability at least $1 - \overline{\varepsilon}$, finite time extinction occurs. Since $\overline{\varepsilon}$ is arbitrary, our proof will then be finished.

From now on we assume that ϱ belongs to the set $A_1(\overline{n}) \cap A_2(\overline{n})$. Extending the definition (33) of good historical paths, we write E_{N-1} for the set of paths w such that

$$\int_0^{T_N} L_{[w,\varrho]}(\mathrm{d}s) \ \geq \ \xi_{N-1} \ := \ \frac{\alpha\, 2^{N/2}}{6}\,. \tag{93}$$

Let $\overline{T}_N := 2^N/6$. Recall that $T_N = \overline{T}_N \wedge \tau_N$, and note that $\xi_{N-1} = \alpha\, \overline{T}_N\, \overline{\zeta}_N$. Then

$$\left.\begin{aligned}
\Pi_{0,0}\left(E^{c}_{N-1}\right) \ &\leq \ \Pi_{0,0}\bigg(\int_0^{T_N} L_{[W,\varrho]}(\mathrm{d}s) \ \leq \ \alpha\, \overline{T}_N \overline{\zeta}_N\bigg) \\
&\leq \ \Pi_{0,0}\left(\tau_N \leq \overline{T}_N\right) \\
&\quad + \Pi_{0,0}\bigg(\int_0^{T_N} L_{[W,\varrho]}(\mathrm{d}s) \ \leq \ \alpha\, \overline{T}_N\, \overline{\zeta}_N\,, \ \tau_N > \overline{T}_N\bigg) \\
&\leq \ \Pi_{0,0}\left(\tau_N \leq \overline{T}_N\right) \\
&\quad + \Pi_{0,0}\bigg(\int_0^{\overline{T}_N} L_{[W,\varrho]}(\mathrm{d}s) \ \leq \ \alpha\, \overline{T}_N\, \overline{\zeta}_N\bigg).
\end{aligned}\right\} \tag{94}$$

Let $\overline{\varrho} = \overline{\varrho}(N)$ be obtained from ϱ as follows. Let $\overline{\varrho}_b := \varrho_b$ if $b \in D_N$. Otherwise, set $\overline{\varrho}_b := 1$. Let S_N denote the collection of sites $b \in \mathsf{Z}^d$ such that $\overline{\varrho} \leq \overline{\zeta}_N$ [recall notation (89)]. Note that by the definition of $A\left(m, N, \overline{\zeta}_N\right) \supseteq$

$A_2(\overline{n})$, there is no connected subset of S_N with cardinality N. Thus, by Lemma 16,

$$\Pi_{0,0}\left(\int_0^{\overline{T}_N} \mathrm{d}s\, \mathbf{1}\{W_s \in S_N^{\mathrm{c}}\} \;\le\; \alpha \overline{T}_N\right) \;\le\; c^{-1} \exp\left[-c\,\overline{T}_N\right].$$

By the definition of S_N,

$$\int_0^{\overline{T}_N} \mathrm{d}s\, \mathbf{1}\{W_s \in S_N^{\mathrm{c}}\} \;>\; \alpha \overline{T}_N \quad \text{implies} \quad \int_0^{\overline{T}_N} L_{[W,\overline{\varrho}]}(\mathrm{d}s) \;>\; \alpha \overline{T}_N\, \overline{\zeta}_N,$$

and so

$$\begin{aligned} &\Pi_{0,0}\left(\int_0^{\overline{T}_N} L_{[W,\overline{\varrho}]}(\mathrm{d}s) \;\le\; \alpha \overline{T}_N\, \overline{\zeta}_N\right) \\ &\le\; \Pi_{0,0}\left(\int_0^{\overline{T}_N} \mathrm{d}s\, \mathbf{1}\{W_s \in S_N^{\mathrm{c}}\} \;\le\; \alpha \overline{T}_N\right) \end{aligned}$$

Therefore,

$$\Pi_{0,0}\left(\int_0^{\overline{T}_N} L_{[W,\overline{\varrho}]}(\mathrm{d}s) \;\le\; \alpha \overline{T}_N\, \overline{\zeta}_N\right) \;\le\; c^{-1} \exp\left[-c\,\overline{T}_N\right].$$

Now (94) and the previous estimate combined with (79) gives

$$\begin{aligned} \Pi_{0,0}\left(E_{N-1}^{\mathrm{c}}\right) \;&\le\; \Pi_{0,0}\left(\tau_N \le \overline{T}_N\right) + c^{-1} \exp\left[-c\,\overline{T}_N\right] \\ &\le\; \left(\frac{2^N \frac{\mathrm{e}}{6}}{2^N}\right)^{2^N} \left(2\pi 2^N\right)^{-\frac{1}{2}} + c^{-1} \exp\left[-c\;\frac{2^N}{6}\right] \\ &\le\; \exp\left[-2c_3\, 2^N\right] \end{aligned}$$

(increasing N if necessary). Therefore, by Markov's inequality, and the 'stopped expectation' formula (30),

$$\widetilde{P}^{\varrho}_{0,\delta_0}\left(\widetilde{X}^{\varrho}_{T_N}(E_{N-1}^{\mathrm{c}}) > \exp\left[-c_3\, 2^N\right]\right) \;\le\; \exp\left[-c_3\, 2^N\right]. \tag{95}$$

Next we consider E_{N-1}. We wish to show that in the case $K = L_{[W,\varrho]}$, condition (44) in Proposition 12 holds $\mathbb{P}$–a.s. By Fubini's theorem, it suffices to verify it $\Pi_{0,0} \times \mathbb{P}$–a.s. First note that with $\Pi_{0,0}$–probability 1 the range $\mathcal{R}(W)$ of the random walk W is infinite. For each site $b \in \mathcal{R}(W)$, let $Y_b := \sigma_b\, \varrho_b$, where σ_b is the amount of time which W spends at b between the time of first arrival at b and the first subsequent departure. Then the Y_b are i.i.d. with positive $\Pi_{0,0} \times \mathbb{P}$–expectation. Therefore, by the strong law,

$$\int_0^{\infty} L_{[W,\varrho]}(\mathrm{d}s) \;\ge\; \sum_{b \in \mathcal{R}(W)} Y_b \;=\; \infty, \quad \Pi_{0,0} \times \mathbb{P}\text{–a.s.},$$

giving (44).

By Proposition 12 with

$$T_0 = 0, \quad T_1 = T_{N-1}, \quad \xi = \xi_{N-1} = (\alpha/6)\, 2^{N/2}, \quad \text{and} \quad E = E_{N-1},$$

we obtain

$$\widetilde{P}^{\varrho}_{0,\delta_0}\left(\widetilde{X}^{\varrho}_{T_N}(E_{N-1}) > 0\right) \le \frac{1}{\xi_{N-1}} = \frac{6}{\alpha}\, 2^{-N/2}. \tag{96}$$

If N is large enough, we have $M_N \ge \exp\left[-c_3\, 2^N\right]$. Hence, by (95) and (96),

$$\begin{aligned}\widetilde{P}^{\varrho}_{0,\delta_0}\left(\|\widetilde{X}^{\varrho}_{T_N}\| > M_N\right) \;&\le\; \widetilde{P}^{\varrho}_{0,\delta_0}\left(\widetilde{X}^{\varrho}_{T_N}\left(E^{\mathrm{c}}_{N-1}\right) > \exp\left[-c_3\, 2^N\right]\right)\\ &\qquad + \widetilde{P}^{\varrho}_{0,\delta_0}\left(\widetilde{X}^{\varrho}_{T_N}\left(E_{N-1}\right) > 0\right)\\ &\le\; \exp\left[-c_3\, 2^N\right] + \frac{6}{\alpha}\, 2^{-N/2} \;\le\; 2^{-N/4} \;=\; \delta_{N-1}\,.\end{aligned}$$

This implies the starting condition (34) in Hypothesis 9.

Now we consider the other time intervals $[T_n\,, T_{n+1})$, $n \ge N$. To deal with these times, we no longer consider clusters of sites where the catalyst is small, but just consider single sites. Recall that we are on the set $A_1(\overline{n})$ and that $n \ge N \ge \overline{n}$. Note that on $A_1(n{+}1) \supseteq A_1(\overline{n})$ we have $\varrho_b > \zeta_{n+1}$, for $b \in D_{n+1}$. Therefore, if $W_s \in D_{n+1}$ for $T_n \le s \le T_n + 2^{-n}$, then $T_{n+1} = T_n + 2^{-n}$ and

$$\int_{T_n}^{T_{n+1}} L_{[W,\varrho]}(\mathrm{d}s) \;\ge\; 2^{-n}\zeta_{n+1} \;=\; \xi_n$$

[recall (84)]. Hence,

$$\Pi_{T_n,a}\left(E^{\mathrm{c}}_n\right) \;\le\; \Pi_{T_n,a}\left(W_s \notin D_{n+1} \text{ for some } s \in \left[T_n\,, T_n + 2^{-n}\right]\right), \qquad a \in D_n\,.$$

The strong Markov property applied to T_n gives

$$\Pi_{T_n,a}\left(E^{\mathrm{c}}_n\right) \;\le\; \Pi_{0,0}\bigg(\sup_{s \le 2^{-n}} |W_s| > 2^n\bigg).$$

From our "traveling estimate" (79), it follows that for N large enough, $n \ge N$ implies

$$\Pi_{T_n,a}\left(E^{\mathrm{c}}_n\right) \;\le\; \left(\frac{2^{-n}e}{2^n}\right)^{2^n} \left(2\pi 2^n\right)^{-1/2} \;\le\; c\, 2^{-2\cdot 2^n} \;\le\; \lambda_n \frac{M_{n+1}}{M_n}\,.$$

Thus, Lemma 11 gives the conditional expectation estimate (36).

Again, for $0 \le s \le 2^{-n}$, Proposition 12 yields

$$\widetilde{P}^{\varrho}_{0,\delta_0}\left\{\widetilde{X}^{\varrho}_{T_n}(E_n) > 0 \;\middle|\; \|\widetilde{X}^{\varrho}_{T^{\mathrm{c}}_n}\| \le M_n\right\} \le \frac{M_n}{\xi_n} = \delta_n\,. \tag{97}$$

This proves the good paths estimate (35).

So Hypothesis 9 is satisfied, and *finite time extinction for the lattice model* follows from the abstract Theorem 10. ■

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