Local Estimation for an Integral Equation of First Kind with Analytic Kernel

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August 25, 1998

1991 Mathematics Subject Classification: 45A05, 45H05, 45M10

Keywords: First kind integral equation, Riesz kernel, analytic kernel, pointwise estimate for the solution severly ill-posed, Cauchy problem, Laplace equation.

*Supported partially by the National Natural Science Foundation of China (19501001), Research funds from Fudan University at Shanghai, China and the fellowship of Monbusho of Japan Government.

†Supported partially by JSPS.

‡Supported partially by Sanwa Systems Development Co. Ltd(Tokyo, Japan).

Abstract

In this paper, an integral equation of the first kind with Riesz kernel is discussed. Since the kernel of this integral equation is analytic, this problem is severe ill-posed. We prove that, for solutions of the integral equation, a local conditional pointwise estimate holds at a point if the solution has some additional smoothness properties in a neighbourhood of this point.

1 Introduction

From many applications such as local tomography, geophysics and problems of detection (e.g. [3], [4], [8], [14], [15]), the following integral equation with Riesz kernel arises:

$$\int_D \frac{1}{r_{xy}^2}\mu(y)dy = f(x), \qquad x \in D_1 \tag{1.1}$$

where $r_{xy} = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}$, D and D_1 are simply connected domains in R^3 and $D\bigcap D_1 = \emptyset$.

Since $D\bigcap D_1 = \emptyset$, the kernel $\frac{1}{r_{xy}^2}$ is analytic with respect to $x \in D_1$ and $y \in D$. From the theory of ill-posed problems ([10], [16]), it is well known that this integral equation is severe ill-posed in Hadamard's sense. Since the singular values of the integral operator decrease to 0 very fast, it is rather difficult to get approximation solutions for this integral equation. The discussions on the stability of the problem (1.1) are important for numerical analysis, but there are very few such results.

In [1], the equation (1.1) is transformed into a Cauchy problem for the Laplace equation in R^4. Under the assumption that the solution is extra smooth over the whole domain D, some L^2-norm estimate is established for the solution of the integral equation. However, in applications, it is more natural that the solution of the integral equation is only a piecewise smooth function, that is, the solution has no global smoothness properties. Then only some local stability estimate can be expected.

In this paper, we study the local stability estimate for (1.1). By the complex extension method ([6], [7]) and the maximum principle for holomorphic functions ([2]), we can obtain a local stabilizing estimate for the integral equation. We prove that, if the solution of the integral equation (1.1) has some additional smoothness properties in a neighbourhood of a point, then a local conditional estimation of logarithmic type holds at this point.

This paper is organized as follows. In Section 2, we state our main result. In Section 3, we prove the main result and give some comments.

2 Notations and Main Results

Let D_1 and D be bounded domains in R^3 . Without loss of generality, we can assume that D and D_1 are simply connected domains in R^3 and D is compactly contained in the ball $B_R \equiv \{x \in R^3 || x| < R\}$. Henceforth the intersection of B_R and D_1 is assumed to be empty, that is, $D_1 \bigcap B_R = \emptyset$. Moreover, the spaces $L^p(\Omega)$, $C^n(\Omega)$,

$C_0^n(\Omega)$, $C^{n,\alpha}(\Omega)$, $0 < \alpha < 1$ are defined as usual.

We discuss the following integral equation with analytic kernel:

$$\int_D \frac{1}{r_{xy}^2} \mu(y) dy = f(x), \qquad x \in D_1. \tag{2.1}$$

We note that since x, y are in the disjoint domains, the kernel is an analytic function.

Our problem is: Given f in D_1, we want to establish a conditional pointwise estimate for the solution of the equation (2.1).

Remark 2.1. *This kind of integral equation comes from practical problems such as identification of steel reinforcement bars in concrete [3] and geophysics [14].*

Henceforth we set $O_\delta(x_0) = \{x \in R^3 || x - x_0| < \delta\}$ for $\delta > 0$ and $\epsilon = (\int_{D_1}(| \bigtriangledown f(x)|^2 + |f(x)|^2)dx)^{\frac{1}{2}}$.

Now we state our main result.

Theorem 2.1. *Let μ be a solution of (2.1). Suppose $\mu \in L^\infty(D)$ and for $x_0 \in D$, there exists a positive constant δ such that $\mu \in C^{2,\alpha}(O_\delta(x_0))$. If $||\mu||_{L^\infty(D)} \leq M$ and $||\mu||_{C^{2,\alpha}(O_\delta)} \leq M$, then there exists a constant C depending on M and δ such that*

$$|\mu(x_0)| \leq C\frac{1}{log\frac{1}{\epsilon}}$$

where $\epsilon < 1$.

Remark 2.2. *Theorem 2.1 indicates that, if some smoothness assumption is added in a neighbourhood of x_0, we can obtain a pointwise conditional estimate. Such an additional assumption is essential for restoring stability because the original integral equation is an ill-posed problem. However we do not know whether the assumption can be further weakened.*

In applications, one frequently assumes that μ is piecewise constant. In this case, from Theorem 2.1, we can directly deduce

Corollary 2.1. *Suppose $\mu = \sum_{j=1}^k c_j \chi_{\Omega_j}$ where c_j is a constant and χ_{Ω_j} is the characteristic function of the domains Ω_j, $j = 1, \dots, k$, $\bigcup_{j=1}^k \overline{\Omega}_j = \overline{D}$, $\Omega_j \cap \Omega_i = \emptyset$ for $j \neq i$.*

Then, for any $x \in \Omega_j$, there exists a constant C which depends on $d(x, \partial\Omega_j)$ and $\max_{1 \leq i \leq k}\{c_i\}$ such that

$$|\mu(x)| \leq C\frac{1}{\log\frac{1}{\epsilon}}$$

where $\epsilon < 1$ and $d(x, \partial\Omega_j)$ is the distance from x to $\partial\Omega_j$.

We should notice that the corollary does not assert stability for $x \in \partial\Omega_j$, $1 \le j \le k$.

Remark 2.3. *In [4], under some stronger a-priori information about μ, the Lipschitz stability estimation is established. Corollary 2.1 asserts weaker stability estimation under more general a-priori information.*

3 Proof of the Main Result

3.1 Transform to a Cauchy Problem for Laplace Equation

We define a new function

$$G(x,\xi) = \int_D \frac{1}{r_{xy}^2 + \xi^2}\mu(y)dy, \qquad \xi \in R^1. \tag{3.1}$$

In [1], the following properties for $G(x,\xi)$ are proved. We will state these properties without proofs.

Proposition 3.1. *The function $G(x,\xi)$ satisfies*

$$(\Delta_x + \frac{\partial^2}{\partial\xi^2})G(x,\xi) = 0, \qquad (x,\xi) \in \widehat{\Omega} \tag{3.2}$$

$$G(x,0) = \int_D \frac{1}{r_{xy}^2}\mu(y)dy = f(x), \quad x \in D_1 \tag{3.3}$$

$$\frac{\partial G}{\partial \xi}(x,0) = 0, \qquad x \in D_1, \tag{3.4}$$

where $\widehat{\Omega} = R^4 \setminus (D \times \{\xi = 0\})$.

Proposition 3.2. *If $\mu \in L^p(D)$ with $p > 1$, then*

$$\frac{\partial G(\cdot,\xi)}{\partial\xi} \longrightarrow -\omega_4\mu(\cdot), \qquad \xi \to +0 \qquad in \quad L^p(D)$$

where ω_4 is the area of the unit sphere in R^4.

Moreover, for $x_0 \in D$ and $0 < \alpha \leq 1$, if $\mu \in C^\alpha(O_\delta(x_0))$, then

$$\frac{\partial G(x_0, \xi)}{\partial \xi} \longrightarrow -\omega_4 \mu(x_0), \qquad \xi \to +0. \tag{3.5}$$

For the second part of the Proposition 3.2, we refer to [10] or [11].

On the basis of the above result, our problem can be reformulated as a Cauchy problem (3.2) - (3.4) for the Laplace equation. Thus our problem can be stated as

Problem: Given a function f in D_1, we want to find a harmonic function $G(x, \xi)$ which satisfies (3.2)-(3.4). Then by (3.5), the solution of the original integral equation (2.1) can be obtained from $\frac{-1}{\omega_4} \lim_{\xi \to 0} \frac{\partial G(x,\xi)}{\partial \xi}$, $x \in D$.

Henceforth we simply write

$$\frac{\partial G}{\partial \xi}(x, 0) = \lim_{\xi \to 0} \frac{\partial G(x, \xi)}{\partial \xi} \qquad for \quad x \in D.$$

3.2 Auxiliary Lemmas

We first show a result on conditional stability of a Cauchy problem for the Laplace equation which will be used for our estimate below. The readers can find the proof in Payne [12].

Lemma 3.1. *Let $\Omega \subset R^n$ be a domain which is bounded by a closed surface S, Σ a part of S, and $W(z)$ satisfy*

$$\Delta W(z) = 0, \qquad z = (z_1, ..., z_n) \in \Omega$$

and

$$|W(z)| \leq M_1, \qquad z \in \Omega$$

with a constant $M_1 > 0$. Then, for a point $\hat{z}$ inside Ω, the following inequality holds:

$$Max\{|W(\hat{z})|, |\nabla W(\hat{z})|\} \leq K_0 M_1^{2(1-\alpha_0)}[\epsilon_1 + \epsilon_2]^{\alpha_0}$$

where $\alpha_0 \in (0,1)$ and K_0 are constants which depend on Σ and $d(\hat{z}, S)$, the distance between $\hat{z}$ and S. We set $\epsilon_1 = \int_\Sigma W^2 d\sigma$, $\epsilon_2 = \int_\Sigma (\frac{\partial W}{\partial z_1}\frac{\partial W}{\partial z_1} + \frac{\partial W}{\partial z_2}\frac{\partial W}{\partial z_2} + \frac{\partial W}{\partial z_3}\frac{\partial W}{\partial z_3}) d\sigma = \int_\Sigma |\nabla W|^2 d\sigma$.

It should be remarked that, if $d(\hat{z}, S)$ tends to zero, then α_0 may tend to zero and the constant K_0 may tend to ∞.

Remark 3.1. *The same estimation holds for $\frac{\partial^2 W(\hat{z})}{\partial z_i \partial z_j}$, $1 \leq i, j \leq n$ and higher derivatives of W at $\hat{z}$ ([12], P.43).*

Since μ will be obtained as the boundary value of $\frac{\partial G(x,\xi)}{\partial \xi}$, we need a sharper result concerning conditional stability estimation.

To this end, we first show some results about the conditional stability estimation for holomorphic functions.

We will first prove some estimate for harmonic measure in the complex plane C which is similar to one in [6], [7].

Let $\Omega_1 = \{\zeta \in C | |\zeta - 2| < 2, \frac{\pi}{2} < \arg(\zeta - 2) < \frac{3\pi}{2}\}$ and $[\rho_1, \rho_2] \subset \Omega_1$ where $0 < \rho_1 < \rho_2 < 2$.

Definition 3.1. *A function $\phi(\zeta)$ defined in Ω_1 is called a harmonic measure for Ω_1 and $[\rho_1, \rho_2]$, if $\phi(\zeta)$ satisfies the following equation and boundary conditions:*

$$\begin{aligned} \Delta\phi(\zeta) &= 0 \quad in \quad \Omega_1 \setminus [\rho_1, \rho_2] \\ \phi(\zeta) &= 0 \quad on \quad \partial\Omega_1 \\ \phi(\zeta) &= 1 \quad on \quad [\rho_1, \rho_2] \end{aligned}$$

For the unique existence of the harmonic measure $\phi(\zeta)$, we refer to [5] and Chapter X in [9]. In particular, $\phi(\zeta) \in C(\Omega \setminus (\rho_1, \rho_2))$.

Lemma 3.2. *Assume that ϕ is a harmonic measure for Ω_1 and $[\rho_1, \rho_2]$. Then there exists a positive constant C_1 which depends on ρ_1, ρ_2 and Ω_1 such that*

$$\phi(x) \geq C_1 x, \qquad x \in [0, \rho_1]. \tag{3.6}$$

Proof. From the definition of $\phi(z)$, by using the maximum principle for the Laplace equation (Theorems 6 and 7 (Page 64,65) in [13]), we know that

$$0 < \phi(z) < 1, \qquad z \in \Omega_1 \setminus [\rho_1, \rho_2]$$

and

$$\frac{\partial \phi(z)}{\partial x_1}|_{z=0} \neq 0$$

where $z = x_1 + i x_2$.

If the conclusion (3.6) is not true, there exist $\{y_n\}_{n=0}^{\infty} \subset [0, \rho_1]$ such that

$$\frac{\phi(y_n)}{y_n} \longrightarrow 0, \qquad n \to \infty. \tag{3.7}$$

Since $[0, \rho_1]$ is a compact set, there exists a point $\tilde{y} \in [0, \rho_1]$ such that

$$y_n \longrightarrow \tilde{y}, \qquad n \to \infty.$$

If $\tilde{y} = 0$, from (3.7) and $\phi(0) = 0$, we have

$$\frac{\partial \phi(z)}{\partial x_1}|_{z=0} = 0.$$

This is a contradiction. If $\tilde{y} \neq 0$, from (3.7) and $\phi \in C[0, \rho_1]$, we have

$$\phi(\tilde{y}) = 0, \qquad for \quad \tilde{y} \in (0, \rho_1].$$

This is also a contradiction. Thus the proof is complete. □

For a holomorphic function in Ω_1, we have

Lemma 3.3. *Suppose $u = u(\zeta)$ is holomorphic in Ω_1 and continuous in $\overline{\Omega}_1$. Suppose $|u(\zeta)| \leq M_2, \quad \zeta \in \Omega_1$. Then there exists a positive constant C_2 which is dependent on Ω_1, but independent of x and u, such that*

$$|u(x)| \leq M_2^{1-C_2x}\epsilon_1^{C_2x}, \qquad x \in [0, \frac{3}{4}]$$

where $\epsilon_1 = \max_{x\in[\frac{3}{4},1]} |u(x)|$.

Proof. By taking $\rho_1 = \frac{3}{4} < \rho_2 = 1 < 2$ in Lemma 3.2, the argument on p.121 in [2] yields

$$|u(x)| \leq M_2(\frac{\epsilon_1}{M_2})^{\phi(x)}, \qquad x \in [0, \frac{3}{4}].$$

Applying Lemma 3.2, we have the conclusion of this lemma. □

Henceforth without loss of generality, we may assume $x_0 = 0$. Let us recall that $\epsilon = (\int_{D_1}(|\bigtriangledown f(x)|^2 + |f(x)|^2)dx)^{\frac{1}{2}}$.

Applying Lemma 3.1, we obtain

Lemma 3.4. *Suppose $\mu \in L^\infty(D)$ and $\|\mu\|_{L^\infty(D)} \leq M$. Then there exist constants C_3, $\alpha_1 \in (0,1)$ such that*

$$|G(0,\xi)| \leq C_3\epsilon^{\alpha_1} \tag{3.8}$$

$$|\frac{\partial G(0,\xi)}{\partial\xi}| \leq C_3\epsilon^{\alpha_1} \tag{3.9}$$

$$|\frac{\partial^2 G(0,\xi)}{\partial\xi^2}| \leq C_3\epsilon^{\alpha_1} \tag{3.10}$$

for $\xi \in [\frac{3}{4}, 1]$.

Proof. It is sufficient to verify that $G(x,\xi)$ is bounded in the domain $R^4 \setminus (B_R \times \{|\xi| \leq \frac{1}{2}\})$. Since D is contained in B_R compactly, we have that $d(\partial B_R, D) > 0$.

From the expression of $G(x,\xi)$, we can obtain

$$|G(x,\xi)| \le (\min\{\frac{1}{2}, d(\partial B_R, D)\})^{-2}\int_D |\mu(y)|dy \le (\min\{\frac{1}{2}, d(B_{\partial R}, D)\})^{-2} M|D|$$

for $(x,\xi) \in R^4 \setminus (B_R \times \{|\xi| \le \frac{1}{2}\})$. So Lemma 3.1 yields (3.8). Similarly we can show (3.9) and (3.10). $\square$

In the following, we extend $\mu(x)$ outside D by $\mu(x) = 0, \quad x \in R^3 \setminus D$.

For $\eta \in C$, we define a function $H = H(\eta)$ with respect to the complex variable $\eta \in C$ by

$$H(\eta) = \int_{B_R} \frac{1}{r^2+\eta^2}\mu(y)dy$$

where $r^2 = y_1^2 + y_2^2 + y_3^2$.

Henceforth we set $\Omega = \{\eta \in C||\eta - 2| < 2\}$.

Then we have

Lemma 3.5. *Suppose $\mu \in L^\infty(D)$. Then H is holomorphic with respect to η in the domain Ω.*

Proof. Introducing the polar coordinates, we set

$$\beta(r) = \int_{S_2} \mu(y)d\omega = \int_{S_2} \mu(r,\omega)d\omega$$

where S_2 is the unit sphere in R^3.

Then we have

$$\begin{aligned}\int_{B_R} \frac{1}{r^2+\eta^2}\mu(y)dy &= \int_0^R \frac{r^2\beta(r)}{r^2+\eta^2}dr \\ &= \frac{1}{2}\int_0^R [\frac{1}{r+i\eta} + \frac{1}{r-i\eta}] r\beta(r)dr\end{aligned}$$

where $i = \sqrt{-1}$.

Changing variables in the second term, we have

$$\frac{1}{2}\int_0^R \frac{1}{r-i\eta} r\beta(r)dr = \frac{1}{2}\int_{-R}^0 \frac{1}{s+i\eta} s\beta(-s)ds,$$

so that

$$H(\eta) = \int_{B_R} \frac{1}{r^2+\eta^2}\mu(y)dy = \frac{1}{2}\int_{-R}^{R} \frac{1}{s+i\eta} s\widehat{\beta}(s)ds \tag{3.11}$$

where we set $\widehat{\beta}(s) = \beta(s), s \geq 0$ and $\widehat{\beta}(s) = \beta(-s), s < 0$.

Since $\mu \in L^\infty(R^3)$, from the expression for $\beta(r)$, it follows that $\widehat{\beta} \in L^\infty(R^1)$. Therefore $H(\eta)$ is a holomorphic function in Ω. The proof is complete. □

We recall that $O_\delta(0) = O_\delta = \{x \in R^3 \| x| = \sqrt{x_1^2+x_2^2+x_3^2} < \delta\}$ and $C^\alpha(O_\delta)$ is the usual space of Hölder continuous functions in O_δ. We have

Lemma 3.6. *Suppose $\mu \in C^\alpha(O_\delta)$. Then we have $\widehat{\beta} \in C^\alpha(-\delta,\delta)$ and*

$$|\widehat{\beta}|_{C^\alpha(-\delta,\delta)} \leq C_4|\mu|_{C^\alpha(O_\delta)}, \qquad |\widehat{\beta}|_{L^\infty(-R,R)} \leq C_4|\mu|_{L^\infty(D)}$$

where a constant C_4 is independent of μ, but dependent on δ.

Proof. From the expression of β, we can see the conclusion easily. □

Next we will estimate $H(\eta)$ for $\eta \in \Omega$.

Lemma 3.7. *Suppose $\mu \in L^\infty(D)$ and $\mu \in C^\alpha(O_\delta)$. Then*

$$|H(\eta)| \leq C_5(|\mu|_{C^\alpha(O_\delta)} + |\mu|_{L^\infty(D)}), \qquad \eta \in \Omega$$

where the constant C_5 is independent of μ, but dependent on $\delta > 0$.

Proof. We rewrite (3.11) as

$$\begin{aligned} H(\eta) &= \frac{1}{2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{s\widehat{\beta}(s)}{s+i\eta}ds + \frac{1}{2}\int_{-R}^{-\frac{\delta}{2}} \frac{s\widehat{\beta}(s)}{s+i\eta}ds + \frac{1}{2}\int_{\frac{\delta}{2}}^{R} \frac{s\widehat{\beta}(s)}{s+i\eta}ds \\ &\equiv A_1 + A_2 + A_3. \end{aligned}$$

First we have

$$\begin{aligned}A_1 &= \frac{1}{2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{s\widehat{\beta}(s)}{s+i\eta}ds \\ &= \frac{1}{2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{s(\widehat{\beta}(s)-\widehat{\beta}(\frac{\delta}{2}))}{s+i\eta}ds + \frac{\widehat{\beta}(\frac{\delta}{2})}{2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \frac{s}{s+i\eta}ds.\end{aligned}$$

By Lemma 3.6 and the theory of one dimensional singular integral equations (e.g. [11]), we know that, for $\zeta \in \Omega$

$$|A_1| \leq C_6|\widehat{\beta}|_{C^\alpha(-\delta,\delta)} \leq C_7(|\mu|_{C^\alpha(O_\delta)} + |\mu|_{L^\infty(D)}) \tag{3.12}$$

where C_6 and C_7 are positive constants which depend on δ. For $\eta \in \Omega$ and $s \in [-R, -\frac{\delta}{2}] \cup [\frac{\delta}{2}, R]$, it easy to verify that

$$|\frac{1}{s+i\eta}| \leq \frac{2}{\delta}.$$

Therefore we find

$$|A_3| \leq \frac{R}{\delta}\int_{\frac{\delta}{2}}^{R} |\beta(s)|ds,$$

so that

$$|A_3| \leq \frac{C_8}{\delta}|\mu|_{L^\infty(D)}. \tag{3.13}$$

By Lemma 3.6, the constant C_8 is independent of μ. Similarly we have

$$|A_2| \leq \frac{C_8}{\delta}|\mu|_{L^\infty(D)}. \tag{3.14}$$

Therefore, from (3.12), (3.13), (3.14), we obtain the conclusion. Thus the proof is complete. □

3.3 Completion of Proof of Theorem 2.1

By Proposition 3.2, we have

$$-\omega_4\mu(0) = \frac{\partial G}{\partial \xi}(0, \frac{3}{4}) + \int_{\frac{3}{4}}^{0} \frac{\partial^2 G(0,t)}{\partial t^2}dt. \tag{3.15}$$

Lemma 3.4 implies

$$|\frac{\partial G}{\partial \xi}(0, \frac{3}{4})| \leq C_3 \epsilon^{\alpha_1}. \tag{3.16}$$

Next we will give an estimate for $\frac{\partial^2 G(0,\xi)}{\partial \xi^2}$. From (3.2), we know that, for $\xi \neq 0$,

$$(\frac{\partial^2}{\partial \xi^2} + \Delta_y)\frac{1}{r^2+\xi^2} = 0,$$

where $\Delta_y = \sum_{j=1}^3 \frac{\partial^2}{\partial y_j^2}$. Therefore

$$\begin{aligned}
\frac{\partial^2 G}{\partial \xi^2}(0,\xi) &= \frac{\partial^2}{\partial \xi^2}\int_{O_\delta} \frac{\mu(y)}{r^2+\xi^2}dy + \frac{\partial^2}{\partial \xi^2}\int_{B_R \setminus O_\delta} \frac{\mu(y)}{r^2+\xi^2}dy \\
&= -\int_{O_\delta} \mu(y)\Delta_y(\frac{1}{r^2+\xi^2})dy + \frac{\partial^2}{\partial \xi^2}\int_{B_R \setminus O_\delta} \frac{\mu(y)}{r^2+\xi^2}dy.
\end{aligned}$$

Since $O_\delta(0)$ is a ball, $\frac{\partial}{\partial r}$ is the normal derivative on the sphere $\partial O_\delta(0)$, and we note that

$$\frac{\partial}{\partial r}\left(\frac{1}{r^2+\xi^2}\right) = -\frac{2r}{(r^2+\xi^2)^2}.$$

By integration by parts for the first term at the left side, we obtain

$$\begin{aligned}
\frac{\partial^2 G}{\partial \xi^2}(0,\xi) &= -\int_{O_{\frac{\delta}{2}}} \Delta_y\left(\frac{1}{r^2+\xi^2}\right)\mu(y)dy + \frac{\partial^2}{\partial \xi^2}\int_{B_R \setminus O_{\frac{\delta}{2}}} \frac{\mu(y)}{r^2+\xi^2}dy \\
&= -\int_{O_{\frac{\delta}{2}}} \frac{1}{r^2+\xi^2}(\Delta_y \mu)(y)dy + \int_{\partial O_{\frac{\delta}{2}}} \frac{\partial \mu}{\partial r}(y)\frac{1}{r^2+\xi^2}dS \\
&+ \int_{\partial O_{\frac{\delta}{2}}} \mu(y)\frac{2r}{(r^2+\xi^2)^2}dS + \frac{\partial^2}{\partial \xi^2}\int_{B_R \setminus O_{\frac{\delta}{2}}} \frac{\mu(y)}{r^2+\xi^2}dy \\
&= -\int_{O_{\frac{\delta}{2}}} \frac{1}{r^2+\xi^2}(\Delta_y \mu)(y)dy + \int_{\partial O_{\frac{\delta}{2}}} \frac{\partial \mu}{\partial r}(y)\frac{1}{r^2+\xi^2}dS \\
&+ \int_{\partial O_{\frac{\delta}{2}}} \mu(y)\frac{2r}{(r^2+\xi^2)^2}dS + \int_{B_R \setminus O_{\frac{\delta}{2}}} [\frac{8\xi^2}{(r^2+\xi^2)^3} - \frac{2}{(r^2+\xi^2)^2}]\mu(y)dy.
\end{aligned}$$

For $\eta \in C, \eta \neq 0$, we set

$$\begin{aligned}\Psi_1(\eta) &= \int_{\partial O_{\frac{\delta}{2}}} \frac{\partial \mu}{\partial r}(y)\frac{1}{r^2+\eta^2}dS \\ &+ \int_{\partial O_{\frac{\delta}{2}}} \mu(y)\frac{2r}{(r^2+\eta^2)^2}dS \\ &+ \int_{B_R\setminus O_{\frac{\delta}{2}}} [\frac{8\eta^2}{(r^2+\eta^2)^3} - \frac{2}{(r^2+\eta^2)^2}]\mu(y)dy.\end{aligned}$$

Then we have

$$\frac{\partial^2 G}{\partial \xi^2}(0,\xi) = -\int_{O_{\frac{\delta}{2}}} \frac{1}{r^2+\xi^2}(\Delta_y \mu)(y)dy + \Psi_1(\xi), \qquad \xi \in R, \quad \xi \neq 0. \quad (3.17)$$

It can be directly verified that $\Psi_1 = \Psi_1(\eta)$ is a holomorphic function in $\Omega = \{\eta \in C | |\eta - 2| < 2\}$ and

$$|\Psi_1(\eta)| \leq C_9(|\mu|_{C^1(O_\delta)} + |\mu|_{L^\infty(D)}), \qquad \eta \in \Omega$$

where C_9 is a constant which depends on δ.

It is easy to verify that $\int_{O_{\frac{\delta}{2}}} \frac{1}{r^2+\eta^2}(\Delta_y \mu)(y)dy$ is holomorphic with respect to $\eta \in \Omega$, because of $r^2 + \eta^2 \neq 0$ for $\eta \in \Omega$. Moreover, similarly to Lemma 3.7, we can see that there exists a constant C_{10} depending on $\delta > 0$ such that

$$|\int_{O_{\frac{\delta}{2}}} \frac{1}{r^2+\eta^2}(\Delta_y \mu)(y)dy| \leq C_{10}(|\mu|_{C^{2,\alpha}(O_\delta)} + |\mu|_{L^\infty(D)}), \quad \eta \in \Omega.$$

For $\eta \in C, \eta \neq 0$, we set

$$H_1(\eta) = \int_{B_R} [\frac{8\eta^2}{(r^2+\eta^2)^3} - \frac{2}{(r^2+\eta^2)^2}]\mu(y)dy.$$

Then, from (3.1), we find that

$$\frac{\partial^2 G}{\partial \xi^2}(0,\xi) = H_1(\xi), \qquad \xi \in R, \quad \xi \neq 0. \qquad (3.18)$$

Moreover, $H_1 = H_1(\eta)$ is a holomorphic function in $\Omega_1 \subset \Omega$. From (3.17), (3.18), the expression of $H_1(\eta)$ and the unicity of holomorphic functions, we can state that

$$H_1(\eta) = -\int_{O_{\frac{\delta}{2}}} \frac{1}{r^2+\eta^2}(\Delta_y \mu)(y)dy + \Psi_1(\eta), \quad \eta \in \Omega.$$

Consequently,

$$|H_1(\eta)| \le C_{11}(|\mu|_{C^{2,\alpha}(O_\delta)} + |\mu|_{L^\infty(D)}) \le 2C_{11}M, \qquad \eta \in \Omega_1$$

where the constant C_{11} depends on δ. Applying Lemma 3.3, we have

$$|H_1(t)| \le 2C_{11}M(\frac{\epsilon_1}{2C_{11}M})^{C_2 t}, \qquad t \in (0, \frac{3}{4}) \tag{3.19}$$

where $\epsilon_1 = \max_{t\in[\frac{3}{4},1]} |H_1(t)|$. Here we note that $C_2 > 0$ depends only on Ω_1. By Lemma 3.4 and (3.18), we have

$$\epsilon_1 \le C_3 \epsilon^{\alpha_1}. \tag{3.20}$$

Combining (3.15), (3.16), (3.18), (3.19) and (3.20), we obtain

$$|\mu(0)| \le C_3\epsilon^{\alpha_1} + \int_0^{\frac{3}{4}} 2C_{11}M(\frac{\epsilon_1}{2C_{11}M})^{C_2 t}dt.$$

It can be directly calculated that

$$\begin{aligned} |\mu(0)| &\le C_{12}(\epsilon^{\alpha_1} + \int_0^{\frac{3}{4}} \epsilon^{\alpha_1 C_2 t}dt) \\ &\le 2C_{12}\frac{1}{\log\frac{1}{\epsilon}} \end{aligned}$$

where $\epsilon < 1$ and C_{12} is a constant which depends on δ and M. Therefore the proof of Theorem 2.1 is complete.

Remark 3.2. *We can use the same method to treat the integral equation which models steel reinforcement bars in concrete ([3]) and is obtained by* *applying a partial differential operator to (2.1). That integral equation is discussed in [1] where L^2-conditional stability estimation is obtained. By using the method of Section 3, we can also get the local pointwise estimation if we assume some additional smoothness properties. We do not treat it here.*

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