Electro–reaction–diffusion systems including cluster reactions of higher order

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Dedicated to Professor Arno Langenbach on his seventieth birthday

Abstract. In this paper we consider electro–reaction–diffusion systems modelling the transport of charged species in two-dimensional heterostructures. Our aim is to investigate the case that besides of reactions with source terms of at most second order so called cluster reactions of higher order are involved. We prove the unique solvability of the model equations and show the global boundedness and asymptotic properties of the solution. In order to get necessary a priori estimates we apply an anisotropic iteration scheme followed by usual Moser iterations. Then existence is obtained by cutting off the reaction terms.

1. Introduction

We investigate differential equations modelling the transport of electrically charged species in heterostructures. The redistribution of the species results from reactions, diffusion processes and their electric interaction. Concrete model equations which we are interested in are mainly motivated from applications to semiconductor technology.

In [9, 10] we studied a basic model leading to a system of partial differential equations consisting of continuity equations for the densities of all present mobile species (including electrons and holes) and a linear Poisson equation for the electrostatic potential. For the Poisson equation one has to assume mixed boundary conditions.

In many applications to semiconductor technology this basic model can be somewhat simplified. Here the kinetic coefficients of electrons and holes are very large compared with those of the other species. Thus at least approximately we can determine the electron and hole densities by the relations between the concentration and the corresponding chemical potential such that the continuity equations for the electrons and holes can be omitted. This reduced model leads to continuity equations for the remaining species coupled with a nonlocal nonlinear Poisson equation for the chemical potential of the electrons. For a detailed derivation of these equations we refer to [10].

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Both problems can be treated in a unified way. We consider m species X_i with charge numbers q_i. After homogenization of the boundary data for the Poisson equation we get an initial boundary value problem involving the concentrations u_i, the charge density u_0, the chemical potentials v_i, and some additional potentials v_0, ζ_0:

$$(1.1)\quad \left\{\begin{aligned} \frac{\partial u_i}{\partial t} - \nabla\cdot(D_i u_i \nabla(v_i + q_i v_0)) + R_i^\Omega &= 0 && \text{on } (0,\infty)\times\Omega,\\ \nu\cdot(D_i u_i \nabla(v_i + q_i v_0)) + R_i^\Gamma &= 0 && \text{on } (0,\infty)\times\Gamma,\\ u_i &= \overline{u}_i\,\mathrm{e}^{v_i} && \text{on } (0,\infty)\times\Omega,\\ u_i(0) &= U_i && \text{on } \Omega,\ i=1,\dots,m;\\ -\nabla\cdot(\varepsilon\,\nabla v_0) + e_0(\cdot,v_0) &= u_0 && \text{on } (0,\infty)\times\Omega,\\ v_0 &= \zeta_0 && \text{on } (0,\infty)\times\Gamma_D,\\ \nu\cdot(\varepsilon\,\nabla v_0) + \tau v_0 &= \tau\zeta_0 && \text{on } (0,\infty)\times\Gamma_N,\\ u_0 &= \sum_{i=1}^m q_i u_i && \text{on } (0,\infty)\times\Omega,\\ \int_\Omega e_0(\cdot,v_0)\,\mathrm{d}x &= \int_\Omega u_0\,\mathrm{d}x && \text{on } (0,\infty). \end{aligned}\right.$$

Here R_i^Ω and R_i^Γ are reaction rates in the domain and at the boundary given by

$$R_i^\Sigma = \sum_{(\alpha,\,\beta)\in\mathcal{R}^\Sigma} k_{\alpha\beta}^\Sigma\Big(\prod_{k=1}^m \mathrm{e}^{\alpha_k(v_k+q_k v_0)} - \prod_{k=1}^m \mathrm{e}^{\beta_k(v_k+q_k v_0)}\Big)(\alpha_i - \beta_i), \quad \Sigma = \Omega,\Gamma,$$

where each pair (α,β) from the finite sets $\mathcal{R}^\Omega$, $\mathcal{R}^\Gamma$ defines a reaction of mass action type

$$\alpha_1 X_1 + \cdots + \alpha_m X_m \rightleftharpoons \beta_1 X_1 + \cdots + \beta_m X_m.$$

The function e_0 has the form

$$(1.2)\quad e_0(x,v_0) = \sum_{i=1}^m q_i U_i(x) + f_0(x) + f_1(x)\,\mathrm{e}^{v_0} - f_2(x)\,\mathrm{e}^{-v_0}, \text{ where } f_1,\ f_2 \ge 0$$

and the last equation in (1.1) represents the global charge conservation. The quantities $\overline{u}_i$, D_i are a suitable reference density and the diffusivity of the i–th species, ε is the dielectric permittivity and τ is the surface capacity. Mainly we are interested in the investigation of heterostructures. Then all physical parameters $\overline{u}_i$, D_i, $k_{\alpha\beta}^\Sigma$, ε, τ, f_1, f_2 as well as the functions U_i, f_0 depend on the space variable x in a nonsmooth way. In general the kinetic coefficients D_i and $k_{\alpha\beta}^\Sigma$ depend on the state variables. But in this paper such a dependency will be considered only for the coefficients $k_{\alpha\beta}^\Sigma$.

Remark 1.1. In order to get the basic model equations from (1.1) we set $\zeta_0 = 0$ and $f_j = 0$, $j = 0,1,2$. Furthermore we assume that each reaction conserves the electric charge what means that $\sum_{i=1}^m q_i(\alpha_i - \beta_i) = 0$ for all $(\alpha,\beta) \in \mathcal{R}^\Omega \cup \mathcal{R}^\Gamma$. Then v_0 has the meaning of the electrostatic potential and the charge conservation relation in (1.1) follows immediately from the continuity equations. In the reduced model after omitting the continuity equations for the electrons and holes we have to assume that $f_1, f_2 > 0$ and the charge conservation relation must be added to the model equations. Here v_0 corresponds to the chemical potential of the electrons and their electrochemical potential (or Fermi level) ζ_0 occurs as an additional unknown quantity. Then the electrostatic potential is given by $v_0 - \zeta_0$.

Existence results related to a weak formulation of (1.1) can be found in [5, 9, 10]. Global properties, especially the asymptotic behaviour of solutions are investigated in [6, 8, 9]. The methods used in [5] and [9, 10] in order to prove a priori estimates allow only volume reactions with source terms of maximal order 2 and boundary reactions with source terms of maximal order 1 to be included. In this paper we focus our attention to the treatment of electro–reaction–diffusion problems where a broad class of higher order reactions is involved. Again motivated from semiconductor technology among other volume and boundary reactions we consider reactions which describe the formation and disintegration of clusters (see [2, 13, 14, 15])

$$\alpha_1 X_1 + \cdots + \alpha_l X_l \rightleftharpoons X_j, \quad j \in \{l+1, \ldots, m\}$$

where $\sum_{i=1}^{l} \alpha_i$ can be a high value. The set of volume reactions $\mathcal{R}^\Omega$ splits up into two disjoint parts, one only contains such cluster reactions and the other one contains the usual volume reactions with source terms of at most second order. Precise assumptions are given in Section 2. In order to get a priori estimates in this situation we had to modify usual iteration techniques in such a way that the test functions for the different continuity equations contain simultaneously different powers of normed concentrations (see the proof of Lemma 4.1). In [7] we applied a similar anisotropic iteration procedure to a very special reaction–diffusion system including boundary reactions of second order.

The paper is organized as follows. In the next section we give the weak formulation of the initial boundary value problem (1.1) and summarize the assumptions which our considerations are based on. In Section 3 we collect known results for such problems and derive some conclusions which will be of importance for the treatment of cluster reactions. The main part of the paper, Section 4, contains the proof of global a priori estimates. It is divided into three steps: anisotropic start iteration for upper bounds, Moser iteration for upper and finally, for lower bounds. Results concerning the asymptotic behaviour are summarized in Section 5. In Section 6 we prove the solvability of our problem by some regularization technique. In the Appendix we collect some auxiliary results used in the paper.

2. Formulation of the problem

We shall state a general evolution problem which involves the weak formulation of the concrete model problems introduced in Section 1. We use the variables

$$\begin{aligned} v &= (v_0, v_1, \ldots, v_m) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{m+1} \quad \text{(potentials)}, \\ u &= (u_0, u_1, \ldots, u_m) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{m+1} \quad \text{(densities)}. \end{aligned}$$

Analogously we set $U = (U_0, U_1, \dots, U_m)$ where $U_0 = \sum_{i=1}^m q_i U_i$ (cf. (2.2)). Since we want to take into account heterostructures the potentials must belong to a space of sufficiently smooth functions while the densities are regarded as elements of the corresponding dual space. We work with the function spaces

$$X := H \times H^1(\Omega, \mathbb{R}^m),\ Y := L^2(\Omega, \mathbb{R}^{m+1})$$

and their duals X^*, $Y^* = Y$ where H is a suitable subspace of $H^1(\Omega)$ (cf. (2.3)). In addition, let

$$W := X \cap L^\infty(\Omega, \mathbb{R}^{m+1}).$$

We define the operators $A\colon W \longrightarrow X^*$, $E_0\colon H \longrightarrow H^*$ and $E\colon X \longrightarrow X^*$ by

$$\begin{aligned}\langle Av, \overline{v}\rangle := & \int_\Omega \Big\{ \sum_{i=1}^m D_i \overline{u}_i \mathrm{e}^{v_i} \nabla\zeta_i \cdot \nabla\overline{\zeta}_i \\ & + \sum_{(\alpha,\, \beta)\in\mathcal{R}^\Omega} R^\Omega_{\alpha\beta}(\cdot, v, \pi(v_0))\,(\alpha - \beta)\cdot\overline{\zeta} \Big\}\,\mathrm{d}x \\ & + \int_\Gamma \sum_{(\alpha,\, \beta)\in\mathcal{R}^\Gamma} R^\Gamma_{\alpha\beta}(\cdot, v, \pi(v_0))\,(\alpha - \beta)\cdot\overline{\zeta}\,\mathrm{d}\Gamma, \quad \overline{v} \in X,\end{aligned}$$

where $\zeta_i = v_i + q_i v_0,\ \overline{\zeta}_i = \overline{v}_i + q_i \overline{v}_0,\ i = 1, \dots, m$, and $\pi \in \mathcal{L}(H^1(\Omega), \mathbb{R})$ (cf. (2.3)),

$$\begin{aligned}\langle E_0 v_0, \overline{v}_0\rangle := & \int_\Omega \Big\{ \varepsilon \nabla v_0 \cdot \nabla \overline{v}_0 + e_0(\cdot, v_0)\overline{v}_0 \Big\}\,\mathrm{d}x \\ & + \int_{\Gamma_N} \tau(v_0 - \pi(v_0))(\overline{v}_0 - \pi(\overline{v}_0))\,\mathrm{d}\Gamma, \quad \overline{v}_0 \in H, \\ \langle Ev, \overline{v}\rangle := & \ \langle E_0 v_0, \overline{v}_0\rangle + \int_\Omega \sum_{i=1}^m \overline{u}_i \mathrm{e}^{v_i} \overline{v}_i\,\mathrm{d}x, \quad \overline{v} \in X.\end{aligned}$$

Then the problem which we are interested in reads as

$$\text{(P)} \qquad \begin{cases} u'(t) + Av(t) = 0,\ u(t) = Ev(t) \text{ f.a.a. } t \in \mathbb{R}_+,\ u(0) = U, \\ u \in H^1_{\mathrm{loc}}(\mathbb{R}_+, X^*),\ v \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^{m+1})). \end{cases}$$

We summarize the assumptions which our further considerations are based on:

$$\text{(2.1)} \quad \begin{cases} \Omega \text{ is a bounded Lipschitzian domain in } \mathbb{R}^2,\ \Gamma := \partial\Omega, \\ \Gamma_D, \Gamma_N \text{ are disjoint open subsets of } \Gamma,\ \Gamma = \overline{\Gamma_D} \cup \overline{\Gamma_N}, \\ \overline{\Gamma_D} \cap \overline{\Gamma_N} \text{ consists of finitely many points;} \end{cases}$$

$$\text{(2.2)} \quad \begin{cases} q_i \in \mathbb{Z},\ \overline{u}_i, U_i \in L^\infty(\Omega),\ \overline{u}_i, U_i \ge c > 0, \\ D_i \in L^\infty(\Omega),\ D_i \ge c > 0,\ i = 1, \dots, m; \\ U_0 := \sum_{i=1}^m q_i U_i; \\ \varepsilon \in L^\infty(\Omega),\ \varepsilon \ge c > 0,\ \tau \in L^\infty_+(\Gamma_N),\ |\Gamma_D| + \|\tau\|_{L^1(\Gamma_N)} > 0; \end{cases}$$

$$(2.3)\quad \begin{cases} H \text{ is a linear closed subspace of } H^1(\Omega),\ H^1_0(\Omega\cup\Gamma_N)\subset H;\\ \pi\in\mathcal{L}(H^1(\Omega),\mathbb{R}),\\ v-\pi(v)\in H^1_0(\Omega\cup\Gamma_N)\ \forall v\in H,\\ \pi(h)\int_{\Gamma_N}\tau(v-\pi(v))\,\mathrm{d}\Gamma=0\ \forall h\in H^1_0(\Omega\cup\Gamma_N),\ \forall v\in H; \end{cases}$$

$$(2.4)\quad \begin{cases} e_0\colon\Omega\times\mathbb{R}\to\mathbb{R} \text{ satisfies the Carathéodory conditions},\\ |e_0(x,y)|\le c\,\mathrm{e}^{c|y|} \text{ f.a.a. } x\in\Omega,\ \forall y\in\mathbb{R},\ c>0,\\ e_0(x,y)-e_0(x,z)\ge b_0(x)\,(y-z) \text{ f.a.a. } x\in\Omega,\ \forall y,z\in\mathbb{R} \text{ with } y\ge z,\\ b_0\in L^\infty_+(\Omega),\ \|b_0\|_{L^1}\ge c\,\|\pi\|,\ c>0; \end{cases}$$

$$(2.5)\quad \begin{cases} \mathcal{R}^\Omega,\ \mathcal{R}^\Gamma \text{ are finite subsets of } \mathbb{Z}^m_+\times\mathbb{Z}^m_+;\\ \text{for } \Sigma=\Omega,\ \Gamma \text{ and } (\alpha,\beta)\in\mathcal{R}^\Sigma \text{ we define}\\ R^\Sigma_{\alpha\beta}:=k^\Sigma_{\alpha\beta}(x,y,z)\,(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta}),\ x\in\Sigma,\ y=(y_0,y_1,\cdots,y_m)\in\mathbb{R}^{m+1},\\ \zeta_i:=y_i+q_iy_0,\ i=1,\dots,m,\ z\in\mathbb{R},\ \text{where}\\ k^\Sigma_{\alpha\beta}\colon\Sigma\times\mathbb{R}^{m+1}\times\mathbb{R}\to\mathbb{R}_+ \text{ satisfies the Carathéodory conditions},\\ k^\Sigma_{\alpha\beta}(x,\cdot,\cdot) \text{ is locally Lipschitz continuous uniformly with respect to } x,\\ k^\Sigma_{\alpha\beta}(\cdot,y,z)\le c\mathrm{e}^{c\,(|y_0|+|z|)} \text{ a.e. on } \Sigma,\ \forall(y,z)\in\mathbb{R}^{m+2},\\ k^\Sigma_{\alpha\beta}(\cdot,y,z)\ge b^\Sigma_{\alpha\beta,R}(\cdot) \text{ a.e. on } \Sigma,\ \forall(y,z)\in\mathbb{R}^{m+2} \text{ with } y_0,z\in[-R,R],\\ b^\Sigma_{\alpha\beta,R}\in L^\infty_+(\Sigma),\ \|b^\Sigma_{\alpha\beta,R}\|_{L^1(\Sigma)}>0; \end{cases}$$

$$(2.6)\quad \begin{cases} \mathcal{R}^\Omega=\mathcal{R}^\Omega_1\cup\mathcal{R}^\Omega_2,\ \mathcal{R}^\Omega_1\cap\mathcal{R}^\Omega_2=\emptyset \text{ and there exist integers}\\ L>2,\ l\in\{1,\dots,m\} \text{ and a constant } c>0 \text{ such that } \forall\zeta\in\mathbb{R}^m\\ \bullet\quad 2<\sum_{i=1}^l\alpha_i\le L,\ \sum_{i=1}^l\beta_i=0,\ \sum_{i=l+1}^m\alpha_i=0,\ \sum_{i=l+1}^m\beta_i=1\ \ \forall(\alpha,\beta)\in\mathcal{R}^\Omega_1,\\ \bullet\quad \max_{i=1,\dots,l}(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta})\,(\beta_i-\alpha_i)\le c\Big(\sum_{k=1}^m\mathrm{e}^{\zeta_k}\Big(\sum_{j=1}^l\mathrm{e}^{\zeta_j}+1\Big)+1\Big),\\ \qquad \max_{i=l+1,\dots,m}(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta})\,(\beta_i-\alpha_i)\le c\Big(\sum_{k=1}^m\mathrm{e}^{2\zeta_k}+1\Big)\quad\forall(\alpha,\beta)\in\mathcal{R}^\Omega_2,\\ \bullet\quad \max_{i=1,\dots,l}(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta})\,(\beta_i-\alpha_i)\le c\Big(\sum_{j=1}^l\mathrm{e}^{\zeta_j}+1\Big),\\ \qquad \max_{i=l+1,\dots,m}(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta})\,(\beta_i-\alpha_i)\le c\Big(\sum_{k=l+1}^m\mathrm{e}^{\zeta_k}+1\Big)\quad\forall(\alpha,\beta)\in\mathcal{R}^\Gamma. \end{cases}$$

The assumptions (2.1)–(2.5) are supposed to be fulfilled up to the end of the paper without any citation. For the proof of solvability we will additionally assume (2.6).

Finally, in order to get global bounds and further asymptotic properties we need an additional assumption concerning the structure of the reaction system which will be introduced later on (see (3.6)).

Remark 2.1. If (u,v) is a solution to (P) then u, v have following regularity properties: $u \in C(\mathbb{R}_+, Y)$, $u \in C_{w^*}(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^{m+1}))$, $v_0 \in C(\mathbb{R}_+, H)$, $v_i \in C(\mathbb{R}_+, L^2)$, $i = 1, \dots, m$, $v \in C_{w^*}(\mathbb{R}_+, L^\infty(\Omega, \mathbb{R}^{m+1}))$. These properties imply the relations

$$\begin{aligned} &u_0(t) = \sum_{i=1}^m q_i u_i(t) \text{ in } L^2(\Omega) \ \forall t \in \mathbb{R}_+, \\ &E_0 v_0(t) = u_0(t) \text{ in } H^* \ \forall t \in \mathbb{R}_+, \\ &v_i(t) = \ln(u_i(t)/\overline{u}_i) \text{ in } L^\infty(\Omega) \ \forall t \in \mathbb{R}_+, \ i = 1, \dots, m. \end{aligned} \tag{2.7}$$

Remark 2.2. In order to explain the meaning of the assumptions (2.3), (2.4) let us consider once more the concrete model equations (1.1) (cf. Remark 1.1, too). Here the function e_0 is given by (1.2). In the basic model we assume $f_j = 0$, $j = 0, 1, 2$, and set

$$H = H_0^1(\Omega \cup \Gamma_N), \quad \pi = 0.$$

Then (2.3), (2.4) are obviously fulfilled. Testing the equation $u'(t) + Av(t) = 0$ with $\overline{v} = (0, q_1, \cdots, q_m) \in X$ the global charge conservation is obtained. In the reduced model we assume $f_j \in L^\infty(\Omega)$, $j = 0, 1, 2$, $f_1, f_2 \geq c > 0$ a.e. on Ω and set (see [9, 10])

$$H = H_0^1(\Omega \cup \Gamma_N) + \mathbb{R}, \quad \pi(w) = \begin{cases} |\Gamma_D|^{-1} \displaystyle\int_{\Gamma_D} w \, \mathrm{d}\Gamma & \text{if } |\Gamma_D| \neq 0, \\ \|\tau\|_{L^1(\Gamma_N)}^{-1} \displaystyle\int_{\Gamma_N} \tau \, w \, \mathrm{d}\Gamma & \text{if } |\Gamma_D| = 0, \end{cases} \quad w \in H^1(\Omega).$$

Then (2.3), (2.4) are fulfilled, too. Now the global charge conservation relation is obtained by testing the equation $E_0 v_0(t) = u_0(t)$ with $\overline{v}_0 = 1 \in H$. The unknown Fermi level ζ_0 is evaluated as $\zeta_0 = \pi(v_0)$.

Remark 2.3. By the assumptions (2.2)–(2.4) it follows that there exists a $c > 0$ such that

$$\|\nabla v_0\|_{L^2}^2 + \int_\Omega b_0 \, v_0^2 \, \mathrm{d}x + \int_{\Gamma_N} \tau (v_0 - \pi(v_0))^2 \, \mathrm{d}\Gamma \geq c \|v_0\|_{H^1}^2 \ \forall v_0 \in H. \tag{2.8}$$

Therefore the operator $E_0 \colon H \longrightarrow H^*$ is strongly monotone and there exists a constant $c > 0$ with

$$\|v_0(t)\|_{H^1}, \, |\pi(v_0(t))| \leq c \Big(1 + \sum_{i=1}^m \|u_i(t)\|_{L^2}\Big) \ \forall t \in \mathbb{R}_+ \tag{2.9}$$

if (u,v) is a solution to (P). Finally, let us note that the operator $E \colon X \longrightarrow X^*$ is strictly monotone.

Remark 2.4. The form of the reaction terms in (2.5) involves some important structural properties. First, it holds

$$R^{\Sigma}_{\alpha\beta}(x,y,z)\sum_{i=1}^{m}(\alpha_i-\beta_i)(y_i+q_iy_0)\geq 0 \quad \text{f.a.a. } x\in\Sigma,\ \forall(y,z)\in\mathbb{R}^{m+2}. \tag{2.10}$$

This relation will ensure the energy estimates in Section 3. Furthermore, for $i = 1,\dots,m$

$$\begin{aligned} \mathrm{e}^{-\zeta_i}\,(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta})(\alpha_i-\beta_i) &\leq \alpha_i\,\mathrm{e}^{\left\{(\alpha_i-1)\zeta_i+\sum_{j\neq i}\alpha_j\zeta_j\right\}} \text{ if } \alpha_i>\beta_i,\\ \mathrm{e}^{-\zeta_i}\,(\mathrm{e}^{\alpha\cdot\zeta}-\mathrm{e}^{\beta\cdot\zeta})(\alpha_i-\beta_i) &\leq \beta_i\,\mathrm{e}^{\left\{(\beta_i-1)\zeta_i+\sum_{j\neq i}\beta_j\zeta_j\right\}} \text{ if } \alpha_i<\beta_i. \end{aligned} \tag{2.11}$$

The growth conditions in (2.6) for the reactions $(\alpha,\beta)\in\mathcal{R}^{\Omega}_2$ exclude volume reactions of the form

$$X_h+X_i\rightleftharpoons X_j+X_k, \quad \text{where } h\in\{1,\dots,l\},\ i,\ j,\ k\in\{l+1,\dots,m\},$$

$$X_h+X_i\rightleftharpoons X_j+X_k, \quad \text{where } h,\ i\in\{1,\dots,l\},\ j,\ k\in\{l+1,\dots,m\}.$$

The method presented in this paper in order to prove the solvability of (P) is not able to handle such reactions which are quadratic in cluster species but where also non-cluster species are involved in the described way. Their additional treatment remains an open question.

3. Energy estimates

In this section we collect results established in [9] under assumptions which are fulfilled also for the problem discussed in the present paper. Furthermore we will make some conclusions which will be of importance for the treatment of the cluster reaction terms in Section 4 (cf. Lemma 4.1).

The following estimates of the solution to the Poisson equation can be found in [9, Lemma 3.1]. The proof is based on regularity and boundedness results for elliptic equations with mixed boundary conditions established in [11, 12].

Lemma 3.1. *There exist constants $c>0$, $q>2$ and a continuous increasing function d such that*

$$\|v_0\|_{L^\infty}\leq c\Big(\|u_0\ln|u_0|\|_{L^1}+d(\|v_0\|_{H^1})+1\Big), \tag{3.1}$$

$$\|v_0\|_{W^{1,q}}\leq c\Big(\|u_0\|_{L^{2q/(2+q)}}+d(\|v_0\|_{H^1})+1\Big) \tag{3.2}$$

if $v_0\in H$ and $E_0v_0=u_0\in L^2(\Omega)$.

Next, we collect results concerning the free energy (cf. [6, 8, 9]). We define the function ϕ_0 by

$$\phi_0(x,y):=e_0(x,y)y-\int_0^y e_0(x,\eta)\,\mathrm{d}\eta.$$

By (2.4) we find easily the following properties of e_0 and ϕ_0

$$
\begin{aligned}
&e_0(x,y)(y-\overline{y}) - \int_{\overline{y}}^{y} e_0(x,\eta)\,\mathrm{d}\eta \geq \frac{b_0(x)}{2}(y-\overline{y})^2, \;\; \phi_0(x,y) \geq \frac{b_0(x)}{2}\,y^2, \\
&\int_0^y e_0(x,\eta)\,\mathrm{d}\eta \geq \frac{b_0(x)}{2} y^2 + e_0(x,0)y \quad \text{f.a.a. } x\in\Omega,\ \forall y,\overline{y}\in\mathbb{R}.
\end{aligned} \tag{3.3}
$$

Because of (7.4) the functional $\Phi\colon X \longrightarrow \mathbb{R}$,

$$
\Phi(v) := \int_\Omega \Big\{ \frac{\varepsilon}{2}|\nabla v_0|^2 + \int_0^{v_0} e_0(\cdot,y)\,\mathrm{d}y + \sum_{i=1}^m \overline{u}_i\,(\mathrm{e}^{v_i}-1) \Big\}\,\mathrm{d}x + \int_{\Gamma_N} \frac{\tau}{2}(v_0-\pi(v_0))^2\,\mathrm{d}\Gamma,
$$

is continuous, strictly convex, Gâteaux differentiable and it holds $\partial\Phi = E$. Its conjugate functional $F\colon X^* \longrightarrow \overline{\mathbb{R}}$,

$$
F(u) := \sup_{v\in X} \Big\{ \langle u,v\rangle - \Phi(v) \Big\},
$$

is proper, lower semicontinuous and convex. It holds $u = Ev = \partial\Phi(v)$ if and only if $v \in \partial F(u)$. F may be interpreted as the free energy of the electro–reaction–diffusion system. If $u \in X^*$ and $u = Ev$ then $F(u)$ can be calculated as

$$
F(u) = \int_\Omega \Big\{ \frac{\varepsilon}{2}|\nabla v_0|^2 + \phi_0(\cdot,v_0) + \sum_{i=1}^m \Big(u_i(\ln\frac{u_i}{\overline{u}_i} - 1) + \overline{u}_i \Big) \Big\}\,\mathrm{d}x + \int_{\Gamma_N} \frac{\tau}{2}\,(v_0-\pi(v_0))^2\,\mathrm{d}\Gamma
$$

where v_0 fulfils the relation $E_0 v_0 = u_0$. Along any solution (u,v) to (P) the function $t \mapsto F(u(t))$ is absolutely continuous and it holds (see [1])

$$
\frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = \langle u'(t), v(t)\rangle = -D(v(t)) \quad \text{f.a.a. } t\in\mathbb{R}_+
$$

where the dissipation rate D is given by

$$
D(v) := \langle Av, v\rangle, \quad v \in W.
$$

By the definition of the operator A and by the property (2.10) of the reaction system the dissipation rate is nonnegative for all $v \in W$. Therefore Theorem 3.2 in [9] is valid in the setting of the present paper, too.

Theorem 3.2. *If problem* (P) *has a solution* (u,v) *then*

i) $F(u(t_2)) \leq F(u(t_1)) \leq F(U) \quad \textit{for} \quad t_2 \geq t_1 \geq 0,$

ii) $\|v_0(t)\|_{H^1} + \sum_{i=1}^m \Big(\|u_i(t)\ln u_i(t)\|_{L^1} + \|u_i(t)\|_{L^1} \Big) + \int_0^t D(v(s))\,\mathrm{d}s \leq c \quad \forall t\in\mathbb{R}_+,$

iii) $\|v_0(t)\|_{L^\infty},\ \|v_0(t)\|_{L^\infty(\Gamma)},\ |\pi(v_0(t))| \leq c \quad \forall t \in \mathbb{R}_+$

where c *depends only on the data.*

By $\mathcal{S} \subset \mathbb{R}^m$ we denote the stoichiometric subspace

$$\mathcal{S} := \operatorname{span}\{\alpha - \beta : \; (\alpha, \beta) \in \mathcal{R}^\Omega \cup \mathcal{R}^\Gamma\}.$$

We define the subspace

$$\mathcal{U} := \Big\{ u \in X^* \colon u_0 = \sum_{i=1}^m q_i u_i, \; (\langle u_1, 1\rangle, \ldots, \langle u_m, 1\rangle) \in \mathcal{S} \Big\}$$

and introduce its orthogonal complement

$$\mathcal{U}^\perp = \Big\{ v \in X \colon \nabla\zeta = 0, \; \zeta \in \mathcal{S}^\perp \text{ where } \zeta_i = v_i + q_i v_0, \; i = 1, \ldots, m \Big\}.$$

Having in mind Remark 2.1 and using the test function $(0, 1, \ldots, 1)$ we obtain for any solution (u, v) to (P) the following invariance property

$$u(t) \in \mathcal{U} + U \quad \forall t \in \mathbb{R}_+. \tag{3.4}$$

Therefore it makes sense to look for steady states (u^*, v^*) to (P) fulfilling the property $u^* \in \mathcal{U} + U$.

Theorem 3.3. *There exists a unique steady state*

$$(u^*, v^*)\colon Av^* = 0, \; u^* = Ev^*, \; u^* \in \mathcal{U} + U, v^* \in W \tag{3.5}$$

to (P). *The element u^* is the unique minimizer of F on $\mathcal{U} + U$, while the element v^* is the unique minimizer of $\Phi - \langle U, \cdot\rangle$ on $\mathcal{U}^\perp$. Furthermore*

$$u^*, \; v^* \in L^\infty(\Omega, \mathbb{R}^{m+1}), \;\; v^* \in L^\infty(\Gamma, \mathbb{R}^{m+1}),$$

$$u_i^* \geq c > 0 \text{ a.e. on } \Omega, \;\; a_i^* := \mathrm{e}^{v_i^* + q_i v_0^*} = \text{const} > 0, \; i = 1, \ldots, m.$$

For the proof we refer to [8, Theorem 3.1] or to [6, Theorem 3.2]. Because of (2.1) the assumption concerning the initial values required there is fulfilled.

As announced in Section 2 for our further investigations we will fix some additional assumption concerning the reaction system. We denote by $\mathcal{M}$ the set

$$\mathcal{M} := \Big\{ a \in \mathbb{R}^m_+, \; v_0 \in H \colon \prod_{i=1}^m a_i^{\alpha_i} = \prod_{i=1}^m a_i^{\beta_i} \; \forall (\alpha, \beta) \in \mathcal{R}^\Omega \cup \mathcal{R}^\Gamma, \\ (E_0 v_0, \overline{u}_1 \, a_1 \, \mathrm{e}^{-q_1 v_0}, \ldots, \overline{u}_m \, a_m \, \mathrm{e}^{-q_m v_0}) \in \mathcal{U} + U \Big\}$$

and suppose that

$$\mathcal{M} \subset \operatorname{int} \mathbb{R}^m_+ \times H. \tag{3.6}$$

The meaning of assumption (3.6) is explained in more detail in [8, 9]. Then by Theorem 3.3 it holds $\mathcal{M} = \{(a^*, v_0^*)\}$. Furthermore, in [9] under this additional assumption the following theorem and its first corollary are established.

Theorem 3.4. *Let* (3.6) *be satisfied. Then there exists a* $\lambda > 0$ *depending only on the data such that*

$$F(u(t)) - F(u^*) \le \mathrm{e}^{-\lambda t}\,(F(U) - F(u^*)) \quad \forall t \ge 0 \tag{3.7}$$

if (u,v) *is a solution to* (P).

Corollary 3.5. *We suppose* (3.6). *Let* (u,v) *be a solution to* (P) *and*

$$a_i = \mathrm{e}^{v_i + q_i v_0} = \frac{u_i}{\overline{u}_i}\,\mathrm{e}^{q_i v_0},\ i = 1,\dots,m. \tag{3.8}$$

Then there exists a constant $c > 0$ *depending only on the data such that for* $i = 1,\dots,m$

$$\begin{aligned} &\|\sqrt{u_i(t)/u_i^*} - 1\|_{L^2},\ \|\sqrt{a_i(t)/a_i^*} - 1\|_{L^2} \le c\,\mathrm{e}^{-\lambda t/2}, \\ &\|v_0(t) - v_0^*\|_{H^1},\ \|u_i(t) - u_i^*\|_{L^1},\ \|a_i(t) - a_i^*\|_{L^1} \le c\,\mathrm{e}^{-\lambda t/2}\ \forall t \in \mathbb{R}_+. \end{aligned} \tag{3.9}$$

The next corollary supplies some inequalities which are useful for the start of global a priori estimates for the concentrations from above. In [9] we obtained such estimates without the use of assumption (3.6), but only for source terms in the volume reactions of maximal order 2. Higher order source terms as considered here forced us to assume (3.6) for this purpose.

Corollary 3.6. *We assume* (3.6). *Then for* $r \in [2,\infty)$ *and* r' *with* $1/r + 1/r' = 1$ *there exist constants* $c_r > 0$ *depending only on the data such that*

$$\|u_i/u_i^* - 1\|_{L^r(\mathbb{R}_+,L^{r'})} \le c_r, \quad i = 1,\dots,m, \tag{3.10}$$

if (u,v) *is a solution to* (P). *Especially* $\|u_i/u_i^* - 1\|_{L^2(\mathbb{R}_+,L^2)} \le c$.

Proof. Because of $\|v_0\|_{L^\infty(\mathbb{R}_+,L^\infty)} \le c$ we find the following estimate

$$D(v(t)) \ge c_0 \int_\Omega \sum_{i=1}^m |\nabla \sqrt{a_i(t)/a_i^*}|^2\,\mathrm{d}x \text{ f.a.a } t \in \mathbb{R}_+$$

where a_i is given by (3.8). Since $\|D(v)\|_{L^1(\mathbb{R}_+)} \le c$ we find together with (3.9)

$$\|\sqrt{a_i/a_i^*} - 1\|_{L^2(\mathbb{R}_+,H^1)} \le c. \tag{3.11}$$

Because of

$$|u_i/u_i^* - 1| \le c(|a_i/a_i^* - 1| + |v_0 - v_0^*|) \le c(|\sqrt{a_i/a_i^*} - 1|^2 + |\sqrt{a_i/a_i^*} - 1| + |v_0 - v_0^*|)$$

we obtain by Gagliardo–Nirenberg's inequality (7.3) and the continuous imbeddings of $L^2(\Omega)$ into $L^{r'}(\Omega)$ and $H^1(\Omega)$ into $L^{r'}(\Omega)$ that

$$\begin{aligned} &\|u_i/u_i^* - 1\|^r_{L^r(\mathbb{R}_+,L^{r'})} \\ &\le c\int_{\mathbb{R}_+} \Big\{\|\sqrt{a_i/a_i^*} - 1\|^{2r}_{L^{2r'}} + \|\sqrt{a_i/a_i^*} - 1\|^r_{L^{r'}} + \|v_0 - v_0^*\|^r_{L^{r'}}\Big\}\,\mathrm{d}s \\ &\le c\int_{\mathbb{R}_+} \Big\{\|\sqrt{a_i/a_i^*} - 1\|^{2r/r'}_{L^2}\|\sqrt{a_i/a_i^*} - 1\|^2_{H^1} + \|\sqrt{a_i/a_i^*} - 1\|^r_{L^2} + \|v_0 - v_0^*\|^r_{H^1}\Big\}\,\mathrm{d}s \end{aligned}$$

and because of (3.11), (3.9) the assertion follows. □

4. A priori estimates

4.1. Upper bounds: Anisotropic start iteration

We are going to find a priori estimates for solutions to (P) depending only on the data. At first we look for upper bounds for the concentrations. We intend to use the Moser technique. Because of the higher order of the cluster reactions (see (2.6)) we have to start with some preliminary estimates. These estimates seem to be the most difficult part in the treatment of the model equations containing cluster reactions.

Let $k_0 \in \mathbb{N}$ be given by

$$2^{k_0-1} \le (L-1) < 2^{k_0}. \tag{4.1}$$

Lemma 4.1. *Additionally we suppose* (2.6) *and* (3.6). *Then there exists a constant* $c > 0$ *depending only on the data such that*

$$\sum_{i=1}^{m} \|u_i(t)/\overline{u}_i\|_{L^{2^{k_0}}}^{2^{k_0}} \le c \quad \forall t \in \mathbb{R}_+$$

if (u, v) *is a solution to* (P).

Proof. The proof will be done iteratively. In every step we use estimates which have been established in the energy estimates (see Theorem 3.2 and Corollary 3.6) and assertions of the previous step. More precisely we show: For all $n \in \mathbb{N}$ with $1 \le n \le (2^{k_0}-1)L+1$ there exist constants c_n depending only on the data such that

$$\sum_{i=1}^{l} \|u_i(t)/\overline{u}_i\|_{L^n}^{n} + \sum_{i=l+1}^{m} \|u_i(t)/\overline{u}_i\|_{L^{(n+L-1)/L}}^{(n+L-1)/L} \le c_n \quad \forall t \in \mathbb{R}_+ \tag{4.2}$$

if (u, v) is a solution to (P). Note that for $n = 1$ this result is known from the energy estimates in Theorem 3.2.

Now let $n \in \mathbb{N} \cap [2, L(2^{k_0}-1)+1]$. Because of (2.1) we find constants γ, $\delta > 0$ not depending on n such that a.e. in Ω

$$\begin{aligned} &\overline{u}_i \frac{1}{n} \ge \gamma, && D_i\overline{u}_i \frac{4(n-1)}{n^2} \ge \delta, && i = 1, \dots, l, \\ &\overline{u}_i \frac{L}{n+L-1} \ge \gamma, && D_i\overline{u}_i \frac{4(n-1)}{(n+L-1)^2} \ge \delta, && i = l+1, \dots, m. \end{aligned} \tag{4.3}$$

Let $z_i := u_i/\overline{u}_i$, $i = 1, \dots, m$. In (P) we use the test function

$$\overline{v} := \mathrm{e}^t \Big(0, z_1^{n-1}, \dots, z_l^{n-1}, z_{l+1}^{(n-1)/L}, \dots, z_m^{(n-1)/L}\Big) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X)$$

and integrate over t. Note that the components of the cluster species are taken with sufficiently low power, adapted to the maximal order of the cluster reactions L whereas powers of the other components are chosen as usual. Thus we obtain the identity

$$I_1(t) + I_2(t) = -I_3(t) + I_4(t) + I_5(t) + I_6(t) \quad \forall t \in \mathbb{R}_+$$

where the terms $I_j(t)$ are defined in the following estimates. For convenience we introduce the functions

$$w_i := z_i^{n/2}, \quad i = 1, \ldots, l, \quad w_i := z_i^{(n+L-1)/(2L)}, \quad i = l+1, \ldots, m.$$

Note that these functions belong to $C(\mathbb{R}_+, L^2(\Omega))$.
1. We start with the parts from the time derivative. With (4.3) it follows

$$\begin{aligned} I_1(t) &:= \int_0^t \langle u'(s), \overline{v}(s)\rangle \,\mathrm{d}s \\ &\geq \sum_{i=1}^m \Big\{\gamma \|w_i(t)\|_{L^2}^2 - c\|w_i(0)\|_{L^2}^2 - c\int_0^t \mathrm{e}^s \|w_i\|_{L^2}^2 \,\mathrm{d}s\Big\}. \end{aligned}$$

2. Using once more the notation of (4.3) we get for the diffusion terms

$$\begin{aligned} I_2(t) &:= \int_0^t \mathrm{e}^s \int_\Omega \Big\{\sum_{i=1}^l D_i u_i \nabla v_i \nabla (z_i^{n-1}) + \sum_{i=l+1}^m D_i u_i \nabla v_i \nabla (z_i^{(n-1)/L})\Big\} \,\mathrm{d}x \,\mathrm{d}s \\ &\geq \int_0^t \mathrm{e}^s \sum_{i=1}^m \delta \|\nabla w_i\|_{L^2}^2 \,\mathrm{d}s \\ &\geq \int_0^t \mathrm{e}^s \sum_{i=1}^m \Big\{\delta \|w_i\|_{H^1}^2 - c\|w_i\|_{L^2}^2\Big\} \,\mathrm{d}s. \end{aligned}$$

3. For the integral coming from the drift terms

$$I_3(t) := \int_0^t \mathrm{e}^s \int_\Omega \Big\{\sum_{i=1}^l D_i u_i q_i \nabla v_0 \nabla (z_i^{n-1}) + \sum_{i=l+1}^m D_i u_i q_i \nabla v_0 \nabla (z_i^{(n-1)/L})\Big\} \,\mathrm{d}x \,\mathrm{d}s$$

we estimate the absolute value by

$$\begin{aligned} |I_3(t)| &\leq 2\int_0^t \mathrm{e}^s \int_\Omega \sum_{i=1}^m D_i \overline{u}_i w_i \, |q_i \nabla v_0 \nabla w_i| \,\mathrm{d}x \,\mathrm{d}s \\ &\leq c\int_0^t \mathrm{e}^s \sum_{i=1}^m \|\nabla v_0\|_{L^q} \|w_i\|_{L^r} \|w_i\|_{H^1} \,\mathrm{d}s \end{aligned}$$

where q is given from Lemma 3.1, $r = 2q/(q-2)$ and $r' = 2q/(q+2)$. With (3.2), Gagliardo–Nirenberg's and Young's inequalities we continue our estimate by

$$\begin{aligned} |I_3(t)| &\leq c\int_0^t \mathrm{e}^s \Big(1 + \sum_{j=1}^m \|u_j\|_{L^{r'}}\Big) \sum_{i=1}^m \|w_i\|_{L^2}^{2/r} \|w_i\|_{H^1}^{2-2/r} \,\mathrm{d}s \\ &\leq c\int_0^t \mathrm{e}^s \Big(1 + \sum_{j=1}^m \Big(\|u_j - u_j^*\|_{L^{r'}} + \|u_j^*\|_{L^{r'}}\Big)\Big) \sum_{i=1}^m \|w_i\|_{L^2}^{2/r} \|w_i\|_{H^1}^{2-2/r} \,\mathrm{d}s \\ &\leq \int_0^t \mathrm{e}^s \sum_{i=1}^m \Big\{\frac{\delta}{5}\|w_i\|_{H^1}^2 + c\Big(1 + \sum_{j=1}^m \|u_j - u_j^*\|_{L^{r'}}^r\Big) \sum_{i=1}^m \|w_i\|_{L^2}^2\Big\} \,\mathrm{d}s. \end{aligned}$$

4. Using the estimate $\|v_0\|_{L^\infty(\mathbb{R}_+,L^\infty)} \le c$ (see iii) in Theorem 3.2) and the growth conditions for the source terms of reactions belonging to $(\alpha,\beta)\in\mathcal{R}_2^\Omega$ (see (2.6)) we find

$$\begin{aligned}
I_4(t) := &\int_0^t \mathrm{e}^s \int_\Omega \sum_{(\alpha,\beta)\in\mathcal{R}_2^\Omega} R^\Omega_{\alpha\beta}\Big(\sum_{i=1}^{l}(\beta_i-\alpha_i)z_i^{n-1} + \sum_{i=l+1}^{m}(\beta_i-\alpha_i)z_i^{(n-1)/L}\Big)\,\mathrm{d}x\,\mathrm{d}s\\
\le &\int_0^t c\,\mathrm{e}^s \int_\Omega \Big\{\Big(\sum_{k=1}^{m} z_k\Big(\sum_{i=1}^{l} z_i+1\Big)+1\Big)\sum_{i=1}^{l} z_i^{n-1}\\
&\qquad\qquad\qquad + \Big(\sum_{k=1}^{m} z_k^2+1\Big)\sum_{i=l+1}^{m} z_i^{(n-1)/L}\Big\}\,\mathrm{d}x\,\mathrm{d}s\\
\le &\int_0^t \hat{c}\,\mathrm{e}^s\Big\{1+\sum_{i=1}^{l}\|z_i\|^{n+1}_{L^{n+1}} + \sum_{k=l+1}^{m}\|z_k\|^{(2L+n-1)/L}_{L^{(2L+n-1)/L}}\\
&\qquad\qquad + \textstyle\sum_{i=1}^{l}\sum_{k=l+1}^{m}\int_\Omega\Big(z_i^2 z_k^{(n-1)/L} + w_i^2 z_k\Big)\,\mathrm{d}x\Big\}\,\mathrm{d}s.
\end{aligned}$$

By means of Hölder's, Gagliardo–Nirenberg's and Young's inequalities we estimate the several terms under the time integral as follows:

$$\begin{aligned}
\hat{c}\sum_{k=l+1}^{m}\int_\Omega z_i^2 z_k^{(n-1)/L}\,\mathrm{d}x &\le c\sum_{k=l+1}^{m}\int_\Omega z_i^2\Big(1+w_k^{2(n-1)/(n+1)}\Big)\,\mathrm{d}x\\
&\le \hat{c}\|z_i\|^{n+1}_{L^{n+1}} + c\Big(1+\sum_{k=l+1}^{m}\|w_k\|^2_{L^2}\Big), \qquad i=1,\dots,l,\\
2\hat{c}\|z_i\|^{n+1}_{L^{n+1}} &\le c\|z_i\|_{L^2}\|w_i\|^2_{L^4} \le c\|z_i\|_{L^2}\|w_i\|_{L^2}\|w_i\|_{H^1} \le \frac{\delta}{10}\|w_i\|^2_{H^1} + c\|z_i\|^2_{L^2}\|w_i\|^2_{L^2}\\
&\le \frac{\delta}{10}\|w_i\|^2_{H^1} + c(\|u_i-u_i^*\|^2_{L^2}+1)\|w_i\|^2_{L^2}, \qquad i=1,\dots,l,\\
\hat{c}\|z_k\|^{(2L+n-1)/L}_{L^{(2L+n-1)/L}} &\le c\|z_k\|_{L^2}\|w_k\|^2_{L^4} \le \frac{\delta}{10}\|w_k\|^2_{H^1} + c(\|u_k-u_k^*\|^2_{L^2}+1)\|w_k\|^2_{L^2},\\
&\qquad\qquad k=l+1,\dots,m,\\
\hat{c}\sum_{k=l+1}^{m}\int_\Omega w_i^2 z_k\,\mathrm{d}x &\le c\sum_{k=l+1}^{m}\|z_k\|_{L^2}\|w_i\|^2_{L^4}\\
&\le \frac{\delta}{10}\|w_i\|^2_{H^1} + c\Big(\sum_{k=l+1}^{m}\|u_k-u_k^*\|^2_{L^2}+1\Big)\|w_i\|^2_{L^2}, \qquad i=1,\dots,l.
\end{aligned}$$

Therefore we obtain for $I_4(t)$ the estimate

$$I_4(t) \le \int_0^t \mathrm{e}^s\sum_{i=1}^{m}\Big\{\frac{\delta}{5}\|w_i\|^2_{H^1} + c\Big(1+\sum_{i=1}^{m}\Big(\sum_{k=1}^{m}\|u_k-u_k^*\|^2_{L^2}+1\Big)\|w_i\|^2_{L^2}\Big)\Big\}\,\mathrm{d}s.$$

5. Since $\|v_0\|_{L^\infty(\mathbb{R}_+,L^\infty(\Gamma))} \le c$ and the boundary reactions for $(\alpha,\beta)\in\mathcal{R}^\Gamma$ have source

terms of maximal order one we can estimate by using the trace inequality (7.1)

$$
\begin{aligned}
I_5(t) &:= \int_0^t \mathrm{e}^s \int_\Gamma \sum_{(\alpha,\beta)\in\mathcal{R}^\Gamma} R^\Gamma_{\alpha\beta}\Big(\sum_{i=1}^l(\beta_i-\alpha_i)z_i^{n-1}+\sum_{i=l+1}^m(\beta_i-\alpha_i)z_i^{(n-1)/L}\Big)\,\mathrm{d}\Gamma\,\mathrm{d}s\\
&\le \int_0^t c\,\mathrm{e}^s\Big\{\sum_{i=1}^l\Big\{\|z_i\|^n_{L^n(\Gamma)}+\|z_i\|^{n-1}_{L^{n-1}(\Gamma)}\Big\}\\
&\qquad\qquad + \sum_{i=l+1}^m\Big\{\|z_i\|^{(L+n-1)/L}_{L^{(L+n-1)/L}(\Gamma)}+\|z_i\|^{(n-1)/L}_{L^{(n-1)/L}(\Gamma)}\Big\}\Big\}\,\mathrm{d}s\\
&\le \int_0^t c\,\mathrm{e}^s\sum_{i=1}^m\Big\{\|w_i\|^2_{L^2(\Gamma)}+1\Big\}\,\mathrm{d}s\\
&\le \int_0^t \mathrm{e}^s\sum_{i=1}^m\Big\{\frac{\delta}{5}\|w_i\|^2_{H^1}+c\Big(\|w_i\|^2_{L^2}+1\Big)\Big\}\,\mathrm{d}s.
\end{aligned}
$$

6. Now we handle the cluster reaction terms for $(\alpha,\beta)\in\mathcal{R}^\Omega_1$. Let $j_{(\alpha,\beta)}$ be the uniquely defined index such that $\beta_{j_{(\alpha,\beta)}}=1$. Then

$$
\begin{aligned}
I_6(t) &:= \int_0^t \mathrm{e}^s\int_\Omega \sum_{(\alpha,\beta)\in\mathcal{R}^\Omega_1} k^\Omega_{\alpha\beta}\Big[\prod_{i=1}^l(z_i\mathrm{e}^{q_iv_0})^{\alpha_i}-z_{j_{(\alpha,\beta)}}\mathrm{e}^{q_{j_{(\alpha,\beta)}}v_0}\Big]\times\\
&\qquad\qquad\Big[z_{j_{(\alpha,\beta)}}^{(n-1)/L}-\sum_{i=1}^l\alpha_iz_i^{n-1}\Big]\,\mathrm{d}x\,\mathrm{d}s\\
&\le \int_0^t \mathrm{e}^s\Big\{\int_\Omega \sum_{(\alpha,\beta)\in\mathcal{R}^\Omega_1} k^\Omega_{\alpha\beta}\prod_{i=1}^l(z_i\mathrm{e}^{q_iv_0})^{\alpha_i}\Big[z_{j_{(\alpha,\beta)}}^{(n-1)/L}-\sum_{i=1}^l\alpha_iz_i^{n-1}\Big]\,\mathrm{d}x\\
&\qquad\qquad +\overline{c}\int_\Omega\Big[\sum_{i=1}^l\sum_{j=l+1}^m z_j\,z_i^{n-1}\Big]\,\mathrm{d}x\Big\}\,\mathrm{d}s.
\end{aligned}
$$

For $t\in\mathbb{R}_+$ and every pair $(\alpha,\beta)\in\mathcal{R}^\Omega_1$ we define the sets

$$
\begin{aligned}
\Omega_+(t,\alpha,\beta) &:= \Big\{x\in\Omega\,:\, z_{j(\alpha,\beta)}(t)^{(n-1)/L}\le\sum_{i=1}^l\alpha_iz_i(t)^{n-1}\Big\},\\
\Omega_-(t,\alpha,\beta) &:= \Big\{x\in\Omega\,:\, z_{j_{(\alpha,\beta)}}(t)^{(n-1)/L}>\sum_{i=1}^l\alpha_iz_i(t)^{n-1}\Big\}.
\end{aligned}
$$

On $\Omega_+(t,\alpha,\beta)$ the terms $k^\Omega_{\alpha\beta}\prod_{i=1}^l(z_i\mathrm{e}^{q_iv_0})^{\alpha_i}\Big[z_{j_{(\alpha,\beta)}}^{(n-1)/L}-\sum_{i=1}^l\alpha_iz_i^{n-1}\Big]$ are non positive and can be omitted. Because of $\alpha_i\ge0$, $z_i\ge0$ we obtain

$$
z_{j_{(\alpha,\beta)}}^{(n-1)/L}>\alpha_iz_i^{n-1}\ \text{ on }\Omega_-(t,\alpha,\beta),\quad i=1,\dots,l,
$$

in particular, for i with $\alpha_i\ne0$ we get

$$
z_i<\alpha_i^{-1/(n-1)}z_{j_{(\alpha,\beta)}}^{1/L}\ \text{ on }\Omega_-(t,\alpha,\beta).
$$

Thus using the bounds for v_0 and the fact that $\sum_{i=1}^{l}\alpha_i \le L$ we can conclude as follows

$$\begin{aligned}
&\int_\Omega \sum_{(\alpha,\beta)\in\mathcal{R}_1^\Omega} k_{\alpha\beta}^\Omega \prod_{i=1}^{l}(z_i \mathrm{e}^{q_i v_0})^{\alpha_i}\Big[z_{j_{(\alpha,\beta)}}^{(n-1)/L} - \sum_{i=1}^{l}\alpha_i z_i^{n-1}\Big]\,\mathrm{d}x\\
\le\;& \sum_{(\alpha,\beta)\in\mathcal{R}_1^\Omega}\int_{\Omega_-(\alpha,\beta)} c\,k_{\alpha\beta}^\Omega \prod_{i=1,\alpha_i\neq 0}^{l} \alpha_i^{-\alpha_i/(n-1)} z_{j_{(\alpha,\beta)}}^{\alpha_i/L} z_{j_{(\alpha,\beta)}}^{(n-1)/L}\,\mathrm{d}x\\
\le\;& \sum_{(\alpha,\beta)\in\mathcal{R}_1^\Omega}\int_{\Omega_-(\alpha,\beta)} c(z_{j_{(\alpha,\beta)}}+1) z_{j_{(\alpha,\beta)}}^{(n-1)/L}\,\mathrm{d}x \le c\int_\Omega \Big(1+\sum_{j=l+1}^{m} z_{j_{(\alpha,\beta)}}^{(n+L-1)/L}\Big)\,\mathrm{d}x\\
\le\;& c\Big(1+\sum_{j=l+1}^{m}\|w_j\|_{L^2}^2\Big).
\end{aligned}$$

As last terms we have to estimate

$$\begin{aligned}
\overline{c}\int_\Omega z_j z_i^{n-1}\,\mathrm{d}x &\le c\int_\Omega z_j(z_i^n+1)\,\mathrm{d}x \le c\|z_j\|_{L^1} + c\|z_j\|_{L^2}\|w_i\|_{L^4}^2\\
&\le c\|z_j\|_{L^1} + c\|z_j\|_{L^2}\|w_i\|_{L^2}\|w_i\|_{H^1}\\
&\le \tfrac{\delta}{5m}\|w_i\|_{H^1}^2 + c\Big(\|z_j\|_{L^1} + \|z_j\|_{L^2}^2\|w_i\|_{L^2}^2\Big)\\
&\le \tfrac{\delta}{5m}\|w_i\|_{H^1}^2 + c\Big(\|z_j\|_{L^1} + \Big(\|u_j-u_j^*\|_{L^2}^2+1\Big)\|w_i\|_{L^2}^2\Big).
\end{aligned}$$

Since $\|z_j\|_{L^\infty(\mathbb{R}_+,L^1)} \le c$, this together with the previous estimates of step 6 yields

$$I_6(t) \le \int_0^t \mathrm{e}^s \sum_{i=1}^{m}\Big\{\frac{\delta}{5}\|w_i\|_{H^1}^2 + c\Big(\sum_{j=1}^{m}\Big(\|u_j-u_j^*\|_{L^2}^2+1\Big)\|w_i\|_{L^2}^2+1\Big)\Big\}\,\mathrm{d}s.$$

7. Finally, the estimates from step 1 up to step 6 imply

$$\begin{aligned}
\gamma \mathrm{e}^t\sum_{i=1}^{m}\|w_i(t)\|_{L^2}^2 \le\; & c\sum_{i=1}^{m}\|w_i(0)\|_{L^2}^2 + \int_0^t \mathrm{e}^s\Big\{1+\sum_{i=1}^{m}\Big\{-\frac{\delta}{5}\|w_i\|_{H^1}^2 + c\|w_i\|_{L^2}^2\\
&+c\sum_{j=1}^{m}\Big(\|u_j-u_j^*\|_{L^2}^2 + \|u_j-u_j^*\|_{L^{r'}}^r\Big)\|w_i\|_{L^2}^2\Big\}\,\mathrm{d}s.
\end{aligned}$$

From the properties of U_i and $\overline{u}_i$ in (2.2) the terms $\|w_i(0)\|_{L^2}^2$ are bounded by a constant depending only on the data. Moreover, by (7.3) and Young's inequality we obtain

$$\begin{aligned}
\|w_i\|_{L^2}^2 &\le c\|w_i\|_{L^{2(n-1)/n}}^{2(n-1)/n}\|w_i\|_{H^1}^{2/n} \le \tfrac{\delta}{5}\|w_i\|_{H^1}^2 + c\|z_i\|_{L^\infty(\mathbb{R}_+,L^{n-1})}^n, && i=1,\dots,l,\\
\|w_i\|_{L^2}^2 &\le c\|w_i\|_{L^{2(L+n-2)/(L+n-1)}}^{2(L+n-2)/(L+n-1)}\|w_i\|_{H^1}^{2/(L+n-1)}\\
&\le \tfrac{\delta}{5}\|w_i\|_{H^1}^2 + c\|z_i\|_{L^\infty(\mathbb{R}_+,L^{(L+n-2)/L})}^{(L+n-1)/L}, && i=l+1,\dots,m,
\end{aligned}$$

and using (4.2) for $n-1$ we arrive at

$$\mathrm{e}^t \sum_{i=1}^m \|w_i(t)\|_{L^2}^2 \le \; c(n, c_{n-1})\,\mathrm{e}^t + \int_0^t c\,\mathrm{e}^s \sum_{i=1}^m \sum_{j=1}^m \Big(\|u_j-u_j^*\|_{L^2}^2 + \|u_j-u_j^*\|_{L^{r'}}^r\Big)\|w_i\|_{L^2}^2\,\mathrm{d}s.$$

Since by Corollary 3.6 the time function $g := \sum_{j=1}^m \Big(\|u_j-u_j^*\|_{L^2}^2 + \|u_j-u_j^*\|_{L^{r'}}^r\Big)$ belongs to $L^1(\mathbb{R}_+)$ we can apply some special form of Gronwall's lemma (see Lemma 7.1) and obtain

$$\mathrm{e}^t \sum_{i=1}^m \|w_i(t)\|_{L^2}^2 \le c\mathrm{e}^t + \int_0^t c\,\mathrm{e}^s g(s)\,\mathrm{e}^{\|g\|_{L^1(\mathbb{R}_+)}}\,\mathrm{d}s \le c\,\mathrm{e}^t\,\|g\|_{L^1(\mathbb{R}_+)}\mathrm{e}^{\|g\|_{L^1(\mathbb{R}_+)}} \le c_n\,\mathrm{e}^t$$

for all $t \in \mathbb{R}_+$. Dividing this inequality by e^t and writing it in terms of u we verify inequality (4.2) for n and the assertion of the theorem is proved. □

Remark 4.2. Since $r' = 2q/(q+2) < 2 \le 2^{k_0}$, Lemma 4.1 and relation (3.2) give a constant $c_{4.4} > 0$ depending only on the data such that for a solution (u, v) to (P)

$$\|v_0(t)\|_{W^{1,q}} \le c_{4.4} \quad \forall t \in \mathbb{R}_+. \tag{4.4}$$

4.2. Upper bounds: Moser iteration

Theorem 4.3. *In addition to our standard assumptions we suppose* (2.6) *and* (3.6). *Then there exists a constant* $c > 0$ *depending only on the data such that*

$$\|u_i(t)/\overline{u}_i\|_{L^\infty} \le c, \quad i = 1, \dots, m, \quad \forall t \in \mathbb{R}_+$$

if (u, v) *is a solution to* (P). *The same estimate holds for the* $L^\infty(\Gamma)$*–norms of* $u_i(t)/\overline{u}_i$ *for a.a.* $t \in \mathbb{R}_+$.

Proof. The proof is based on Moser estimates. Let $z_i := (u_i/\overline{u}_i - K)^+$ with $K := \max\{1, \|U_1/\overline{u}_1\|_{L^\infty}, \dots, \|U_m/\overline{u}_m\|_{L^\infty}\}$, $w_i := z_i^{p/2}$ where $p > 2(L-1)$. We use the test function

$$\overline{v} := p\mathrm{e}^t\,(0, z_1^{p-1}, \dots, z_m^{p-1}) \in L^2_{\mathrm{loc}}(\mathbb{R}_+, X)$$

for (P). Additionally, we define κ by

$$\kappa := c_{4.4}^{2r} + 1 \text{ where } r = 2q/(q-2),\ q \text{ from (3.2)}. \tag{4.5}$$

Note that volume and boundary reaction terms satisfy the restrictions (2.6). Since K is a constant (defined by the data) and $u_i \le z_i + K$ for some suitable chosen $\delta > 0$ we have

$$
\begin{aligned}
\mathrm{e}^t &\sum_{i=1}^m \int_\Omega \overline{u}_i |w_i(t)|^2 \,\mathrm{d}x \\
&\le \int_0^t \mathrm{e}^s \sum_{i=1}^m \Big\{ \int_\Omega \Big\{ -\delta |\nabla w_i|^2 + cp\Big(|w_i|^2 + u_i |\nabla v_0| |\nabla z_i^{p-1}| + (u_i^L + 1) z_i^{p-1} \Big) \Big\} \,\mathrm{d}x \\
&\qquad\qquad + cp \int_\Gamma (u_i + 1) z_i^{p-1} \,\mathrm{d}\Gamma \Big\} \,\mathrm{d}s \\
&\le \int_0^t \mathrm{e}^s \sum_{i=1}^m \Big\{ -\delta \|w_i\|_{H^1}^2 + cp\Big(\|\nabla v_0\|_{L^q} \|\nabla w_i\|_{L^2} (\|w_i\|_{L^r} + 1) \\
&\qquad\qquad + \|w_i\|_{L^{2(p+L-1)/p}}^{2(p+L-1)/p} + \|w_i\|_{L^2(\Gamma)}^2 + 1 \Big) \Big\} \,\mathrm{d}s \\
&\le \int_0^t \mathrm{e}^s \sum_{i=1}^m \Big\{ -\frac{\delta}{2} \|w_i\|_{H^1}^2 + cp^{2r} (\|\nabla v_0\|_{L^q}^{2r} + 1)(\|w_i\|_{L^1}^2 + 1) \\
&\qquad\qquad + cp\Big(\|w_i\|_{H^1}^{(p+2L-2)/p} \|w_i\|_{L^1} + \|w_i\|_{H^1}^{3/2} \|w_i\|_{L^1}^{1/2} + 1 \Big) \Big\} \,\mathrm{d}s \\
&\le \int_0^t \mathrm{e}^s \sum_{i=1}^m \Big\{ c\Big(p^{2r} \kappa (\|w_i\|_{L^1}^2 + 1) + p^4 \|w_i\|_{L^1}^{2p/(p-2L+2)} + p^4 \|w_i\|_{L^1}^2 + 1 \Big) \Big\} \,\mathrm{d}s \\
&\le cp^{2r}\kappa \int_0^t \mathrm{e}^s \sum_{i=1}^m (\|w_i\|_{L^1}^{2p/(p-2L+2)} + 1) \,\mathrm{d}s \\
&\le cp^{2r}\kappa \mathrm{e}^t \sum_{i=1}^m (\sup_{s\in\mathbb{R}_+} \|z_i(s)\|_{L^{p/2}}^{p/2\; 2p/(p-2L+2)} + 1) \qquad \forall t \in \mathbb{R}_+.
\end{aligned}
$$

Therefore we obtain the estimate

$$
(4.6) \quad \sum_{i=1}^m \|z_i(t)\|_{L^p}^p + 1 \le c_{4.6} p^{2r} \kappa \Big(\sum_{i=1}^m \sup_{s\in\mathbb{R}_+} \|z_i(s)\|_{L^{p/2}}^{p/2} + 1 \Big)^{2p/(p-2L+2)} \quad \forall t \in \mathbb{R}_+
$$

with $c_{4.6} > 1$ only depending on the data. We set $p = 2^k$, $k \in \mathbb{N}$, $k \ge k_0$ where k_0 is given by relation (4.1) and define

$$
b_k := \sum_{i=1}^m \sup_{s\in\mathbb{R}_+} \|z_i(s)\|_{L^{2^k}}^{2^k} + 1, \quad k \ge k_0.
$$

From (4.6) we conclude that

$$
\begin{aligned}
b_k &\le (2^{2r})^k \,(c_{4.6}\kappa)\, b_{k-1}^{\left\{\frac{2^k}{2^k - 2L + 2}\right\}} \\
&\le \left[(2^{2r})^{\left\{ \sum_{i=0}^{k-k_0-1} (k-i) 2^i \right\}} (\kappa\, c_{4.6})^{\left\{ \sum_{i=0}^{k-k_0-1} 2^i \right\}} b_{k_0}^{\left\{ 2^{k-k_0} \right\}} \right]^{\prod_{j=k_0}^{k-1} \frac{2^j}{2^j - L + 1}} .
\end{aligned}
$$

The last inequality can be proved by induction. Note that the product

$$c_\theta := \prod_{j=k_0}^{\infty} \frac{2^j}{2^j - L + 1}$$

is finite and all of its factors are greater than 1. Moreover

$$\sum_{i=0}^{k-k_0-1} 2^i \le 2^{k-k_0}, \qquad \sum_{i=0}^{k-k_0-1} (k-i)\, 2^i \le 2^{k+1}, \quad k > k_0,$$

such that

$$b_k \le (2^{4r} \kappa\, c_{4.6}\, b_{k_0})^{c_\theta 2^k}.$$

Thus we arrive at

$$\sum_{i=1}^{m} \|z_i(t)\|_{L^{2^k}} \le \sqrt{m}\, \Big(2^{4r} \kappa\, c_{4.6} \Big(\sum_{i=1}^{m} \sup_{s\in\mathbb{R}_+} \|z_i(s)\|_{L^{2^{k_0}}}^{2^{k_0}} + 1 \Big)\Big)^{c_\theta} \quad \forall t \in \mathbb{R}_+,\ k \ge k_0.$$

Passing to the limit $k \to \infty$ we obtain

$$\sum_{i=1}^{m} \|z_i(t)\|_{L^\infty} \le \sqrt{m}\, \Big(2^{4r} \kappa\, c_{4.6} \Big(\sum_{i=1}^{m} \sup_{s\in\mathbb{R}_+} \|z_i(s)\|_{L^{2^{k_0}}}^{2^{k_0}} + 1 \Big)\Big)^{c_\theta} \quad \forall t \in \mathbb{R}_+.$$

Applying the result of Lemma 4.1 we find the desired estimates in Ω. The estimates at the boundary follow from (7.2). □

4.3. Lower bounds

Having once obtained global upper bounds we find global lower bounds for the solution, too.

Theorem 4.4. *Let additionally* (2.6) *and* (3.6) *be fulfilled. Then there exist constants* $c_1, c_2 > 0$ *depending only on the data such that*

$$\|v_i^-(t)\|_{L^\infty} \le c_1,\ \operatorname{ess\,inf}_{x\in\Omega} u_i(t) \ge c_2,\ i = 1, \ldots, m,\ \forall t \in \mathbb{R}_+$$

if (u, v) *is a solution to* (P). *The same estimate holds for the* $L^\infty(\Gamma)$*–norms of* $v_i^-(t)$ *for a.a.* $t \in \mathbb{R}_+$.

Proof. If (u, v) is a solution to (P) Theorem 3.4 and Theorem 4.3 ensure the properties

$$F(u(t)) - F(u^*) \le \mathrm{e}^{-\lambda t}\, (F(U) - F(u^*)) \quad \forall t \ge 0$$
$$\|u_i\|_{L^\infty(\mathbb{R}_+, L^\infty(\Omega))},\ \|u_i/\overline{u}_i\|_{L^\infty(\mathbb{R}_+, L^\infty(\Gamma))} \le c, \quad i = 1, \ldots, m,$$

where c depends only on the data. With these estimates the desired global lower bounds for the solution follow from Theorem 5.4 in [9]. □

5. Asymptotic behaviour

Additionally to the result stated in Theorem 3.4 we find the following asymptotic estimates concerning the densities and potentials.

Theorem 5.1. *In addition to our standard assumptions we suppose* (2.6) *and* (3.6). *Then there exist constants* c, $\lambda_p > 0$ *depending only on the data such that*

$$\sum_{i=0}^{m} \Big(\|u_i(t) - u_i^*\|_{L^p} + \|v_i(t) - v_i^*\|_{L^p} \Big) \le c\, \mathrm{e}^{-\lambda_p t} \quad \forall\, t \ge 0, \qquad \text{where } p \in [1, +\infty)$$

if (u, v) *is a solution to* (P).

Proof. If (u, v) is a solution to (P) Theorem 4.3 and Theorem 4.4 ensure that

$$\|v_i(t)\|_{L^\infty} \le c, \quad i = 1, \dots, m, \quad \forall t \in \mathbb{R}_+.$$

Thus all assumptions of Theorem 5.5 in [9] are fulfilled and this theorem gives the desired estimates. □

6. Solvability

6.1. The regularized problem ($\mathbf{P_N}$)

In order to prove the existence of a solution to (P) we will use some regularization techniques. For any arbitrarily fixed time interval $S := [0, T]$ we consider a problem which arises from (P) by regularizing the volume and boundary reaction terms. Let, for $N \in \mathbb{R}_+$, $\rho_N\colon \mathbb{R}^{m+2} \to [0, 1]$ be a fixed Lipschitz continuous function such that

$$\rho_N(y, z) := \begin{cases} 0 & \text{if} \quad |(y, z)|_\infty \ge N, \\ 1 & \text{if} \quad |(y, z)|_\infty \le N/2 \end{cases}, \quad |(y, z)|_\infty := \max\{|y_0|, \dots, |y_m|, |z|\}$$

and let the functions $g^\Sigma_{Ni}\colon \Sigma \times \mathbb{R}^{m+1} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$g^\Sigma_{Ni}(x, y, z) := \rho_N(y, z) \sum_{(\alpha,\, \beta) \in \mathcal{R}^\Sigma} R^\Sigma_{\alpha\beta}(x, y, z)(\alpha_i - \beta_i), \ \Sigma = \Omega, \Gamma, \ i = 1, \dots, m.$$

We introduce the operator $A_N\colon W \longrightarrow X^*$ by

$$\langle A_N v, \overline{v} \rangle := \int_\Omega \sum_{i=1}^{m} \Big\{ D_i \overline{u}_i \mathrm{e}^{v_i} \nabla \zeta_i \cdot \nabla \overline{\zeta}_i + g^\Omega_{Ni}(\cdot, v, \pi(v_0)) \overline{\zeta}_i \Big\}\, \mathrm{d}x + \int_\Gamma \sum_{i=1}^{m} g^\Gamma_{Ni}(\cdot, v, \pi(v_0)) \overline{\zeta}_i \mathrm{d}\Gamma.$$

We are looking for solutions to the following regularized problem

$$(\mathrm{P_N})\qquad \begin{cases} u'(t) + A_N v(t) = 0,\ u(t) = Ev(t) \text{ f.a.a. } t \in S,\ u(0) = U, \\ u \in H^1(S, X^*),\ v \in L^2(S, X) \cap L^\infty(S, L^\infty(\Omega, \mathbb{R}^{m+1})). \end{cases}$$

Theorem 6.1. *For each $N \in \mathbb{R}_+$ there exists a unique solution to* $(\mathrm{P_N})$.

Proof. The functions g^Σ_{Ni} satisfy the Carathéodory conditions and easily one verifies the following properties where especially (2.10) and (2.11) have to be used:

$$\begin{aligned} &|g^\Sigma_{Ni}(x, y, z)| \le c_\Sigma \text{ f.a.a. } x \in \Sigma,\ \forall (y, z) \in \mathbb{R}^{m+2}, \\ &|g^\Sigma_{Ni}(x, y, z) - g^\Sigma_{Ni}(x, \overline{y}, \overline{z})| \le L_\Sigma\, |(y - \overline{y}, z - \overline{z})|_\infty \\ &\text{f.a.a. } x \in \Sigma,\ \forall (y, z),\ (\overline{y}, \overline{z}) \in \mathbb{R}^{m+2}, \\ &\sum_{i=1}^m g^\Sigma_{Ni}(x, y, z)(y_i + q_i y_0) \ge 0 \text{ f.a.a. } x \in \Sigma,\ \forall (y, z) \in \mathbb{R}^{m+2}, \\ &g^\Sigma_{Ni}(x, y, z) \le c_\Sigma\, \mathrm{e}^{y_i} \text{ f.a.a. } x \in \Sigma,\ \forall (y, z) \in \mathbb{R}^{m+2} \text{ with } y_i \le 0. \end{aligned}$$

Thus we can apply [10, Theorem 6.1] for electro–diffusion systems with weakly nonlinear volume and boundary source terms to obtain the assertion. □

6.2. Estimates for the solution to $(\mathrm{P_N})$

We are going to find estimates for solutions to $(\mathrm{P_N})$ which do not depend on N. At first note, that for the solution to $(\mathrm{P_N})$ the relations in (2.7) are valid. The corresponding dissipation rate $D_N(v) := \langle A_N v, v\rangle$ is nonnegative for all $v \in W$. Therefore the results of Theorem 3.2 and Lemma 3.1 remain true for the solution to $(\mathrm{P_N})$ and

$$(6.1)\qquad \begin{aligned} &\|D_N(v)\|_{L^1(S)} \le c,\ F(u(t)),\ \|u(t)\|_{L^1},\ \|v_0(t)\|_{H^1} \le c, \\ &\|v_0(t)\|_{L^\infty},\ \|v_0(t)\|_{L^\infty(\Gamma)},\ |\pi(v_0(t))| \le c_{6.1}, \\ &\|v_0(t)\|_{W^{1,q}} \le c\Big(\sum_{i=1}^m \|u_i(t)\|_{L^{2q/(2+q)}} + 1\Big) \quad \forall t \in S. \end{aligned}$$

All these estimates are independent of N and of the length of the time interval S. Next we look for upper bounds for the concentrations. We intend to adapt the estimates already done for (P). But let us note that the global assertions of Corollary 3.6 are not available for $(\mathrm{P_N})$. Using other arguments corresponding inequalities will be derived now with right hand sides depending on the length T of the time interval.

Theorem 6.2. *We assume* (2.6). *Then there exists a continuous increasing function $d_{6.2} > 0$ depending on the data, but not on N, such that for the solution (u, v) to* $(\mathrm{P_N})$

$$(6.2)\qquad \|u_i(t)/\overline{u}_i\|_{L^\infty} \le d_{6.2}(T),\ i = 1, \ldots, m,\ \forall t \in S.$$

The same estimate holds for the $L^\infty(\Gamma)$–norm of $u_i(t)/\overline{u}_i$ for a.a. $t \in S$.

Proof. We verify the validity of the assertions of Corollary 3.6 for $(\mathrm{P_N})$ on S as follows. With $u_i = \overline{u}_i a_i \mathrm{e}^{-q_i v_0}$, $i = 1, \dots, m$, we find from (6.1) that

$$\|\sqrt{a_i/a_i^*} - 1\|_{L^\infty(S,L^2)},\ \|v_0 - v_0^*\|_{L^\infty(S,H^1)} \le c$$

and since S is a finite interval

$$\|\sqrt{a_i/a_i^*} - 1\|_{L^r(S,L^2)},\ \|\sqrt{a_i/a_i^*} - 1\|_{L^2(S,H^1)} \le d(T)$$

where d does not depend on N. This leads (as in the proof of Corollary 3.6) to

$$\|u_i/u_i^* - 1\|_{L^r(S,L^{r'})} \le d(T) \quad \text{for } r \in [2,\infty),\ 1/r + 1/r' = 1.$$

Note that the absolute value of ρ_N can be estimated by 1. Thus we can now apply exactly the same procedure as in the proof of Lemma 4.1 and Remark 4.2 (but now only on the finite interval S) to obtain the estimate

$$\sum_{i=1}^m \|u_i(t)/\overline{u}_i\|_{L^{2^{k_0}}}^{2^{k_0}} \le d(T), \quad \|v_0(t)\|_{W^{1,q}}^{2r} + 1 \le \kappa(T) \quad \forall t \in S \tag{6.3}$$

with continuous increasing functions d and κ depending only on the data but not on N, k_0 was defined in (4.1). After this Moser estimates as done in the proof of Theorem 4.3 (now on S) supply the desired L^∞–estimates. □

Theorem 6.3. *We assume* (2.6). *Then there exists a continuous increasing function* $d_{6.4} > 0$ *depending on the data, but not on* N, *such that for the solution* (u, v) *to* $(\mathrm{P_N})$

$$\|v_i^-(t)\|_{L^\infty} \le d_{6.4}(T),\ i = 1, \dots, m,\ \forall t \in S. \tag{6.4}$$

The same estimate holds for the $L^\infty(\Gamma)$*–norm of* $v_i^-(t)$ *for a.a.* $t \in S$.

Proof. We start with (6.1), Theorem 6.2 and (6.3). By the ideas of the proof of Lemma 4.2 in [9] we find a continuous increasing function d, not depending on N and p such that for $p \ge 2$ and $i = 1, \dots, m$ the recursion formula

$$\mathrm{e}^t \|(v_i + K)^-(t)\|_{L^p}^p \le d(T) \int_0^t \mathrm{e}^s p^{2r} \kappa(T) \Big(\|(v_i + K)^-(s)\|_{L^{p/2}}^p + 1 \Big)\, \mathrm{d}s \tag{6.5}$$

holds for all $t \in S$ where $K := \max\{1, \ln \|\overline{u}_1/U_1\|_{L^\infty}, \dots, \ln \|\overline{u}_m/U_m\|_{L^\infty}\}$, $\kappa(T)$ from (6.3), $r = 2q/(q-2)$, q from (3.2). Continuing this estimate for $p = 2$ by

$$\mathrm{e}^t \|(v_i + K)^-(t)\|_{L^1}^2 \le c\mathrm{e}^t \|(v_i + K)^-(t)\|_{L^2}^2 \le d(T) \int_0^t \mathrm{e}^s \Big(\|(v_i + K)^-(s)\|_{L^1}^2 + 1 \Big)\, \mathrm{d}s$$

and applying Gronwall's Lemma we obtain that

$$\|(v_i + K)^-(t)\|_{L^1} \le d(T)\, \mathrm{e}^{d(T)} \le d(T) \quad \forall t \in S. \tag{6.6}$$

Similar as in the proof of Lemma 4.6 in [4] we find now from (6.5) that

$$\|(v_i + K)^-(t)\|_{L^\infty} \le d(T)\kappa(T) \Big(\sup_{s \in S} \|(v_i + K)^-(s)\|_{L^1} + 1 \Big) \quad \forall t \in S$$

which together with (6.6) supplies the estimate $\|v_i^-(t)\|_{L^\infty} \le d(T)$. The estimate for the boundary norm follows from (7.2). □

6.3. Existence and uniqueness result

Theorem 6.4. *Under the additional assumption* (2.6) *there exists a unique solution to* (P).

Proof. We define a mapping from $\mathbb{R}_+$ to $L^\infty(\Omega,\mathbb{R}^{m+1})\times L^\infty(\Omega,\mathbb{R}^{m+1})$ by

$$(u(t),v(t)) := (u_{\hat N(t)}(t), v_{\hat N(t)}(t)) \text{ for } t>0,$$
$$(u(0),v(0)) := (U, E_0^{-1}U_0, \ln[U_1/\overline{u}_1],\dots,\ln[U_m/\overline{u}_m])$$

where $(u_{\hat N(t)}, v_{\hat N(t)})$ is the solution to $(\mathrm{P}_{\hat N(t)})$ on $S:=[0,t]$ and

$$\hat N(t) := 2\max\Big\{c_{6.1}, \ln d_{6.2}(t), d_{6.4}(t)\Big\}.$$

Since $\hat N(t)\ge \hat N(s)$ for $t\ge s$ and since the solution to each problem $(\mathrm{P_N})$ is unique we get

$$(u_{\hat N(s)}(s), v_{\hat N(s)}(s)) = (u_{\hat N(t)}(s), v_{\hat N(t)}(s)),\ s\le t.$$

Thus we obtain that the pair of time functions $(u,v)|_{[0,t]}$ is a solution to $(\mathrm{P}_{\hat N(t)})$ on $[0,t]$. By the choice of $\hat N(t)$ we guarantee that the operators $A_{\hat N(t)}$ and A coincide on the solution to $(\mathrm{P}_{\hat N(t)})$. Therefore (u,v) defined here is a solution to (P). Uniqueness (even without using assumption (2.6)) has been proved in [9, Theorem 3.1]. □

7. Appendix

Here we collect some results which are relevant for the investigations of the paper. We assume that $\Omega\in\mathbb{R}^2$ is a bounded (strictly) Lipschitzian domain. We use the following imbedding result which can be derived from [16, p. 317, equ. (5)] by a modified application of the Hölder inequality

$$\|v\|^q_{L^q(\Gamma)} \le c_\Omega\, q\, \|v\|^{q-1}_{L^{2(q-1)}(\Omega)} \|v\|_{H^1(\Omega)}, \quad v\in H^1(\Omega), \quad q\ge 2. \tag{7.1}$$

By means of this trace inequality we get

$$\|w\|_{L^\infty(\Gamma)} \le \|w\|_{L^\infty(\Omega)} \quad \forall w\in L^\infty(\Omega)\cap H^1(\Omega). \tag{7.2}$$

As a special case of the Gagliardo–Nirenberg inequality (see [3, 17]) we use the estimate

$$\|w\|_{L^p} \le c_{p,k}\, \|w\|^{k/p}_{L^k} \|w\|^{1-k/p}_{H^1} \quad \forall w\in H^1(\Omega),\ 1\le k<p<\infty. \tag{7.3}$$

Especially, for p from compact intervals

$$\|w\|_{L^p} \le c\, \|w\|^{1/p}_{L^1} \|w\|^{1-1/p}_{H^1} \quad \forall w\in H^1(\Omega),\ p_1\le p\le p_2.$$

From Trudingers imbedding theorem (see [18]) we get

$$\|\mathrm{e}^{|w|}\|_{L^p} \le d_p(\|w\|_{H^1}) \quad \forall w\in H^1(\Omega),\ 1\le p<\infty. \tag{7.4}$$

Finally, we make use of an extended form of Gronwall's Lemma (see [19]).

Lemma 7.1. *Let $g \in L^1([0,T],\mathbb{R})$ with $g \geq 0$ a.e. in $[0,T]$ and let ϕ, $a \in C([0,T],\mathbb{R})$ with*

$$\phi(t) \leq a(t) + \int_0^t g(s)\phi(s)\,\mathrm{d}s \quad \forall t \in [0,T].$$

Then it holds

$$\phi(t) \leq a(t) + \int_0^t a(s)g(s)\,\mathrm{e}^{\int_s^t g(\tau)\mathrm{d}\tau}\mathrm{d}s \quad \forall t \in [0,T].$$

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