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PRICING VIA ANTICIPATIVE STOCHASTIC CALCULUS

ECKHARD PLATEN AND ROLANDO REBOLLEDO

ABSTRACT. The paper proposes a general model for pricing of derivative securities with different maturity. The underlying dynamics follows stochastic equations involving anticipative stochastic integrals. These equations are solved explicitly and structural properties of solutions are studied.

1. INTRODUCTION

Term structures of interest rates play an important role in finance and are used for pricing interest rate dependent securities as bonds with different maturities. Furthermore, derivative securities written on other assets depend on bonds. Therefore it is extremely important to have a simple and reliable model for bonds based on a flexible stochastic structure of interest rates. Within this paper we will describe a general model of bond price processes. Further we will discuss conditions under which assets discounted by bonds have a martingale behaviour. This structural property is of crucial importance in derivative security pricing.

Black and Scholes [2] described in their fundamental paper a simple and robust model for pricing options on given risky assets. Up to now there seems to be no equivalent simple model for pricing bonds. One reason may have been that a bond $P(t,T)$ at time t with maturity T must have at expiration the fixed value $P(T,T) = 1$ almost surely. Obviously it is difficult in the framework of nonanticipative stochastic

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analysis to model such a stochastic process which is driven by a Brownian motion and reaches a fixed value at a future time T almost surely. Another reason might have been that bonds discounted with respect to certain reference processes and also assets discounted by bonds have to show a martingale property in a good security model. Nevertheless there are several interesting and important approaches which deal with the modelling of bond price dynamics. For an impression of the diversity of bond pricing models we like to refer the reader e.g. to Heath, Jarrow and Morton [9], Harrison and Pliska [8], El Karoui, Myneni and Vishwanathan [12], Hull and White [11], Black, Derman and Toy [1], Ho and Lee [10], Brennan and Schwartz [3], Cox, Ingersoll and Ross [6], Vasicek [17].

Within this paper our first aim is to use anticipative stochastic equations (see [4], [14], [13]) to model a price dynamics which includes also bonds. Secondly, we will analyze the volatility dynamics in terms of maturity times. Consistently, the evolution of prices with different maturities in a risk neutral situation will be derived in terms of few very well interpretable parameters.

2. The price dynamics

2.1. Description of the model. Let us start from a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, fulfilling the usual conditions. We will introduce later the probability space required by anticipative stochastic differential equations.

We first consider a positive continuous process $\mu(T) = (\mu(t,T); t \geq 0)$ which could be anticipative. This represents the *average return process.* In some cases we will take $\mu(\cdot, T) = r$, where $r = (r(t); t \geq 0)$ is an adapted process which describes the *term structure* of interest rates. We call $r(t)$ the *spot rate* at time $t \geq 0$.

We also introduce the driving Wiener process $W = (W(t); t \geq 0)$ assumed to be defined over our filtered probability space, $\mathbb{F}$-adapted and $W(0) = 0$. To keep our notations simple we write our formulae just for a one–dimensional standard Wiener process but our results hold analogously in the multidimensional case.

It is now our aim to model the evolution of the *price* $P(t,T)$ *at time* $t \leq T$ of a T–*maturity contingent claim* H which is a random variable on our probability space. That is a security that claims H for certain at a specified maturity date T. Thus we have to fulfill a terminal condition for the price at maturity:

$$P(T,T) = H. \tag{2.1}$$

For instance, in the case of a bond we have $H = 1$. We introduce the following stochastic equation for the price with maturity T:

(2.2)

$$P(t,T) = P(T,T) - \int_t^T \mu(s,T)P(s,T)ds - \int_t^T \sigma(s,T)P(s,T)\partial W(s), \ (0 \leq t \leq T)$$

where the second stochastic integral, the Skorokhod integral, has an anticipative integrand (see [14]) involving the volatility process $\sigma(T) = (\sigma(t,T); 0 \leq t \leq T)$ which is not adapted and will be specified in the section below. We use $\int(\dots)\partial W(s)$ to denote the Skorokhod integral instead of the symbol $\int(\dots)dW(s)$ which is reserved to write the customary stochastic integral with an adapted process as integrand.

To continue with the description of main features of the model, let us analyze how to overcome the difficulty of having a *final* and not *initial* condition for our equation. The idea we exploit is very simple: take the maturity time T not as a final but as an initial time of a *backward evolution.* In order to preserve Wiener measure by time reversal we introduce the operator R over processes:

$$RV(t) = V(T-t) - V(T), \ (t \in [0,T]), \tag{2.3}$$

where V is any process. This operator is called the *time reversal operator.* Notice that $RR = identity$ and $\hat{W} = RW$ is a Brownian motion with respect to the filtration $\mathbb{B} = (\mathcal{B}_t; t \in [0,T])$ it generates.

Define $X(t) = RP(t,T) + P(T,T) = P(T-t,T)$ and $\alpha(t) = -\mu(T-t,T)$, $\beta(t) = \sigma(T-t,T)$, $0 \leq t \leq T$. We assume β satisfies regularity conditions, which we will specify below,under which Skorokhod indefinite integrals exist. Equation (2.2) is then transformed into

$$X(t) = P(T,T) + \int_0^t \alpha(s)X(s)ds + \int_0^t \beta(s)X(s)\partial\hat{W}(s). \tag{2.4}$$

Notice that now we have $X(0) = P(T,T)$, which is equivalent to the terminal condition (2.1).

We take (2.4) as our fundamental equation to describe the price dynamics. To give an explicit solution we need to introduce first some additional notations and tools from Anticipative Stochastic Calculus: that is the aim of the next section. Before we go into those technical details, let us give the expression we obtain for P by solving (2.4) and reversing time afterwards.

2.2. Explicit expression for price and yield. The solution to (2.2) obtained by time reversal on the solution of (2.4) is given by

(2.5)

$$P(t,T) = P(T,T)\exp\left[-\left(\int_t^T \sigma(s,T)\partial W(s) - \frac{1}{2}\int_t^T \sigma^2(s,T)ds + \int_t^T \mu(s,T)ds\right)\right],$$

for all $0 \leq t \leq T$ and follows from Corollary 1 which we are going to prove in section 3.

We recall that the *yield* associated to a price at maturity T is a process Y defined by

$$Y(t) = \frac{\log P(T,T) - \log P(t,T)}{T-t}, \ (0 \leq t \leq T). \tag{2.6}$$

In our case, the explicit expression for the yield is

$$Y(t) = \frac{1}{T-t}\left(\int_t^T \sigma(s,T)\partial W(s) - \frac{1}{2}\int_t^T \sigma^2(s,T)ds + \int_t^T \mu(s,T)ds\right). \tag{2.7}$$

2.3. Observed processes. Coefficients are in general anticipating in the expression above and cannot be known by an observation at time t before the maturity time T. In general one will try also in pricing to deal with observed quantities. For this reason we study projections of such coefficients with respect to the filtration $\mathbb{F}$ which is a model of available information. This leads us to consider the so called *optional projections* of our coefficents. They coincide with *predictable projections* because of continuity assumptions. These projections have been introduced in the development of the General Theory of Processes. Given a process V we recall that its optional projection ${}^{o}V$ with respect to $\mathbb{F}$ is the unique process (up to indistinguishability) given as the conditional expectation

$$ {}^{o}V(\tau) = \mathbb{E}(V(\tau)/\mathcal{F}_{\tau}), $$

for all stopping times τ of the filtration $\mathbb{F}$ (see [7]).

To construct a convenient link to dynamics with nonanticipative integrands we interpret the optional projections ${}^{o}\mu(t,T)$ and ${}^{o}\sigma(t,T)$ as the observed quantities at time t and we can introduce another equation for an *observed price* $\tilde{P}$ inspired by (2.2):

(2.8)
$$ \tilde{P}(t,T) = \tilde{P}(T,T) - \int_t^T {}^{o}\mu(s,T)\tilde{P}(s,T)ds - \int_t^T {}^{o}\sigma(s,T)dW(s), \ (t \in [0,T]). $$

Notice that $\tilde{P}$ is not the optional projection of P. Indeed $\tilde{P}$ is even not an $\mathbb{F}$-adapted process though ${}^{o}\mu(\cdot,T)$ and ${}^{o}\sigma(\cdot,T)$ do. By a similar technique as will be applied for the case of anticipative coefficients we obtain

(2.9)
$$ \tilde{P}(t,T) = \tilde{P}(T,T)\exp\left[-\left(\int_t^T {}^{o}\sigma(s,T)dW(s) - \frac{1}{2}\int_t^T {}^{o}\sigma^2(s,T)ds + \int_t^T {}^{o}\mu(s,T)ds\right)\right], $$

for all $0 \le t \le T$.

We also associate to the price $\tilde{P}$ an *observed yield* which is given by

$$(2.10)\, \tilde{Y}(t) = \frac{1}{T-t}\left(\int_t^T {}^o\sigma(s,T)dW(s) - \frac{1}{2}\int_t^T {}^o\sigma^2(s,T)ds + \int_t^T {}^o\mu(s,T)ds\right).$$

We remark that the difference between P and $\tilde{P}$ (respectively between Y and $\tilde{Y}$) represents for the observer a measure of the *risk* due to incomplete information about the future.

The following section is rather technical and we refer the interested reader to [14] and [5] for more details. First we will sketch an intuitive description of what we need from anticipative stochastic calculus which we will specify afterwards. Despite the fact that this calculus is still rather technical we will see that it provides a powerful approach to basic problems which are easy to solve in deterministic but very complex in stochastic evolution.

3. Anticipative Stochastic Calculus techniques

To state the model we fix the maturity time T throughout this section. Our probability space Ω is the space of continuous real functions $C([0,T],\mathbb{R})$, endowed with the uniform topology; the canonical process on Ω is defined to be $\hat{W}_t(\omega) = \omega(t)$; $\mathcal{B}_t$ is the sigma–field generated by $\hat{W}_s, s \leq t$; $\mathcal{B}(I)$ denotes the sigma field generated by $\hat{W}_t$ for all $t \in I$, where I is a subset of $[0,T]$; the probability we consider over Ω is the Wiener measure $\mathbb{P}$ under which the process $\hat{W}$ becomes a Brownian motion. The reversed process $W = R\hat{W}$ is also a Brownian motion under this probability measure. The filtration associated to W is denoted by $\mathbb{F} = (\mathcal{F}_t; t \in [0,T])$. We complete all σ–fields by $\mathbb{P}$ and keep the same notations for them. Our starting point here will be equation (2.4) which we are going to solve explicitly. By a time reversal argument we will come to equation (2.2) with anticipative integrands and deduce its solution P.

3.1. Introducing the Skorokhod integral. In some cases the Skorokhod integral can be understood as a limit of a special class of Riemann sums. Indeed, assume $u \in L^2([0,T] \times \Omega)$. Given a partition $\pi : 0 = t_0 < t_1 < \ldots < t_n = T$ consider over $[0,T] \times \Omega$ the σ–field $\mathcal{B}^\pi$ generated by all sets of the form $]t_k, t_{k+1}] \times F_k$, where F_k runs over the σ–field $\mathcal{B}(]t_k, t_{k+1}]^c)$. We construct the conditional expectation $E_\nu(u/\mathcal{B}^\pi)$ of u with respect to $\mathcal{B}^\pi$ and the finite measure $d\nu = d\mathbb{P}dt$ over $[0,T] \times \Omega$:

$$(3.1)\, E_\nu(u/\mathcal{B}^\pi)(s,\omega) = \sum_{k=0}^{n-1} \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \mathbb{E}(u(r)/\mathcal{B}(]t_k, t_{k+1}]^c))(\omega) dr \mathbb{1}_{]t_k, t_{k+1}]}(s),$$

ν–almost surely on $[0,T] \times \Omega$.

It can be observed that if π is refined then the σ–field $\mathcal{B}^\pi$ grows up to the product of the Borel σ field of $[0,T]$ with $\mathcal{B}([0,T])$. Therefore, by the Martingale Limit Theorem, we have

$$E_\nu(u/\mathcal{B}^\pi) \to u, \tag{3.2}$$

ν-almost surely and in $L^2([0,T] \times \Omega)$, as π is refined.

Now, we can integrate the above process $E_\nu(u/\mathcal{B}^\pi)$ with respect to the Wiener process in a simple form:

$$\int_0^T E_\nu(u/\mathcal{B}^\pi)(s) \partial \hat{W}(s) = \sum_{k=0}^{n-1} E_\nu(u/\mathcal{B}^\pi)(t_{k+1})(\hat{W}(t_{k+1}) - \hat{W}(t_k)). \tag{3.3}$$

We let again π vary by taking finer partitions. If the sum (3.3) has a limit in $L^2(\Omega)$, then we can define its limit as $\int_0^1 u(s) \partial \hat{W}(s)$, since u is approached by the integrands in (3.3) according to (3.2).

However, this is not the general Skorokhod integral and the construction presented above does not allow to understand the integral as the adjoint of a *derivative*, which is also a useful notion when dealing with anticipative processes. To go further, we resume below very brieﬂy the connections between the Skorokhod integral and the Malliavin derivative.

3.2. Malliavin derivative and Skorokhod integral. First, we let $C_p^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions f from $\mathbb{R}^n$ into $\mathbb{R}$, such that f together with all its derivatives have polynomial growth order. Let $\mathcal{S}$ denote the set of all *smooth* real random variables; that is the class of all $F : \Omega \to \mathbb{R}$, for which there exists a finite collection $t_1, \ldots, t_m \in [0, T]$, and a function $f \in C_p^\infty(\mathbb{R}^n)$, such that

$$F = f(\hat{W}(t_1), \ldots, \hat{W}(t_m)) \tag{3.4}$$

The set $\mathcal{S}$ is dense in $L^2(\Omega)$ with the norm $\|\cdot\|_2$. For every F in $\mathcal{S}$, the derivative $D_t F$ is defined to be the process

$$D_t F = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(\hat{W}(t_1), \ldots, \hat{W}(t_m)) 1_{[0,t_i]}(t) \tag{3.5}$$

for all $0 \leq t \leq T$.

D is a closed operator, its domain is denoted $\mathbb{D}^{1,2}$, with norm

$$\|F\|_{1,2} = \|F\|_2 + \|DF\|_{L^2([0,T]\times\Omega)}, \tag{3.6}$$

where $\|\cdot\|_2$ denotes the $L^2(\Omega)$ norm. $\mathbb{D}^{1,2}$ is the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{1,2}$. We also denote by $\mathbb{L}^{1,2}$ the class of processes $u \in L^2([0,T] \times \Omega)$ such that $u(t) \in \mathbb{D}^{1,2}$ for almost all t, and there exists a measurable version of the two parameter process $D_s u(t)$ which satisfies $E(\int_0^T \int_0^T (D_s u(t))^2 ds dt) < \infty$. For $u \in \mathbb{L}^{1,2}$ we extend (3.6) by defining

$$\|u\|_{1,2} = \|u\|_{L^2([0,T]\times\Omega)} + (\mathbb{E}\int_0^T \int_0^T |D_s u(t)|^2 ds dt))^{1/2}. \tag{3.7}$$

The adjoint D^* of D is the Skorokhod integral. D^* is a closed and unbounded operator defined over a domain $Dom(D^*)$ included in $L^2([0,T] \times \Omega)$, taking values in $L^2(\Omega)$. The couple $(Dom(D^*), D^*)$ is characterized as follows:

(i) $Dom(D^*)$ is defined as the set of all $u \in L^2([0,T] \times \Omega)$, such that

$$|E(\int_0^T D_t F u(t)dt)| \leq c\|F\|_2, \tag{3.8}$$

for all F in $\mathcal{S}$, where $c > 0$ is a constant which depends only on u.

(ii) If u belongs to $Dom(D^*)$ then $D^*(u)$ is the element of $L^2(\Omega)$ defined by

$$E(FD^*(u)) = E(\int_0^T D_t F u(t)dt), \tag{3.9}$$

for any $F \in \mathcal{S}$.

The *indefinite Skorokhod integral process* is constructed for processes u such that $u1_{[0,t]}$ belongs to $Dom(D^*)$ for all $t \in [0,T]$. So that $\int_0^t u(s)\partial\hat{W}(s) = D^*(u1_{[0,t]})$, and it holds

$$R(\int_0^{\cdot} u(s)\partial\hat{W}(s))(t) = \int_0^t u(T-s)\partial W(s),\ (t \in [0,1]), \tag{3.10}$$

where $W = R\hat{W}$. Indeed this follows from the definition of the Skorokhod integral: integration with respect to $\hat{W}$ is connected through duality with D_s; integration with respect to W is connected with D_{T-s}, so that for all $t \in [0,T]$ and all $F \in \mathcal{S}$ we have

$$\begin{aligned}
\mathbb{E}(F\int_{T-t}^T (-u(s))\partial\hat{W}(s)) &= \mathbb{E}(\int_0^T D_s F(-u(s))1_{[T-t,T]}(s)ds) \\
&= \mathbb{E}(\int_0^T D_{T-r} F u(T-r)1_{[0,t]}(r)dr) \\
&= \mathbb{E}(F\int_0^t u(T-r)\partial W(r)).
\end{aligned}$$

3.3. The linear stochastic differential equation. To introduce the equation we first make basic assumptions on the coefficients.

- The coefficient $\beta : [0,T] \times \Omega \to \mathbb{R}$ is considered to be continuous in time, measurable in ω and uniformly bounded. Furthermore we assume that $(s,\omega) \mapsto D_s\beta(t)$ exists and belongs to $L^\infty([0,T] \times \Omega)$, for all $t \in [0,T]$.
- The coefficient $\alpha : [0,T] \times \Omega \to \mathbb{R}$ is assumed to be continuous in the time variable, measurable in ω and uniformly bounded.

- H is assumed to be measurable and essentially bounded.

Theorem 1. *Under the above hypotheses and notations, define*

$$X(t,\omega) = H(\omega)\varepsilon_t(\omega)\exp(\int_0^t \alpha(s,\omega)ds), \tag{3.11}$$

$t \in [0,1]$, *where* ε_t *denotes Doléans exponential*

$$\varepsilon_t := \exp(\int_0^t \beta(s)\partial\hat{W}(s) \; - \frac{1}{2}\int_0^t \beta^2(s)ds), \; (t \in [0,T]). \tag{3.12}$$

Then $1_{[0,t]}\beta X$ *is Skorokhod–integrable for all* $t \in [0,T]$, $X \in L^2([0,T] \times \Omega)$, *and it is the unique solution to the equation:*

$$X(t) = H + \int_0^t \alpha(s)X(s)ds + \int_0^t \beta(s)X(s)\partial\hat{W}(s), \; (t \in [0,T]) \tag{3.13}$$

Proof. We follow [5] to sketch the proof of this theorem. Our statement is a very rough particular case of Buckdahn's result. The interested reader is referred to his article for further details. The technique involves transformations on the Wiener space and an extension of Girsanov's Theorem. More precisely let T_t, $A_t : \Omega \to \Omega$, be defined by:

$$T_t\omega(s) := \omega(s) + \int_0^{t\wedge s} \beta(u)du; \tag{3.14}$$

$$A_t\omega(s) := \omega(s) - \int_0^{t\wedge s} \beta(u)du, \tag{3.15}$$

where $s, t \in [0,T]$. Notice that $T_tA_t = A_tT_t = identity$, and that these mappings are continuous on Ω endowed with the uniform topology.

Moreover, we denote by $U_t(\omega, x)$ the solution to the integral equation:

$$U_t(\omega,x) = x + \int_0^t \alpha(s, T_s(\omega))U_s(\omega,x)ds, \tag{3.16}$$

$t \in [0,T]$ which for $x = H(\omega)$ is given explicitly by

$$U_t(\omega, H) = H(T_t(\omega))\exp(\int_0^t \alpha(s, T_s(\omega))ds). \tag{3.17}$$

Finally following the arguments in [5], the solution to the original equation (3.13) is obtained in the form $X(t,\omega) = \varepsilon_t U_t(A_t(\omega), H(A_t(\omega)))$ which reduces to the expression (3.11). □

It is important to remark that (3.11) gives also the solution to the linear equation (3.13) in the case of adapted coefficients α and β with initial condition $H = constant$. Furthermore, in that case the assumptions on α and β can simply be reduced to continuity and adaptedness requirements (even less, see e.g. [16]). We will constantly assume the weaker conditions on α and β when we refer to the adapted case throughout the paper.

From the above theorem we can derive an explicit expression for prices.

Corollary 1. *Under the hypothesis of Theorem* 1, *prices with maturity* T *are given by the formula:*

(3.18)

$$P(t,T) = P(T,T)\exp\left[-\left(\int_t^T \sigma(s,T)\partial W(s) - \frac{1}{2}\int_t^T \sigma^2(s,T)ds + \int_t^T \mu(s,T)ds\right)\right],$$

for all $0 \le t \le T$.

Proof. It suffices to reverse time in formula (3.11). Indeed,

$$\begin{aligned} P(t,T) &= X(T-t) \\ &= X(0)\varepsilon_{T-t}\exp\left(-\int_0^{T-t}\mu(T-s,T)ds\right) \\ &= P(T,T)\exp\left[-\left(\int_t^T \sigma(s,T)\partial W(s) - \frac{1}{2}\int_t^T \sigma^2(s,T)ds + \int_t^T \mu(s,T)ds\right)\right]. \end{aligned}$$

□

Notice the above corollary can also be applied when coefficients μ and σ are replaced by their optional projections. This leads to the expression for observed prices in (2.9). In order to estimate differences between observed and anticipative processes we need to go further into anticipative calculus using L^2–estimates on Skorokhod integrals.

3.4. Estimating the gap between observed and anticipative terms. Our main result in this subsection is the following

Theorem 2. *Under the hypothesis of Theorem* 1, *for all* $t \in [0,T]$,

$$
\begin{aligned}
\|Y(t) - \tilde{Y}(t)\|_2 &\leq \frac{1}{T-t}[C\| \left(\int_t^T \int_0^T D_s(\sigma(u,T) - {}^o\sigma(u,T)) ds du \right)^{1/2} \|_2 \\
&+ \frac{1}{2} \int_t^T \|\sigma^2(s,T) - {}^o\sigma^2(s,T)\|_2 ds \qquad (3.19) \\
&+ \int_t^T \|\mu(s,T) - {}^o\mu(s,T)\|_2 ds],
\end{aligned}
$$

where $C > 0$ *is a constant.*

Proof. Fix any $t \in [0,T]$. Notice that we have

$$
\begin{aligned}
Y(t) - \tilde{Y}(t) &= \frac{1}{T-t}[\int_t^T (\sigma(s,T) - {}^o\sigma(s,T)) \partial W(s) \\
&- \frac{1}{2} \int_t^T (\sigma^2(s,T) - {}^o\sigma^2(s,T)) ds \\
&+ \int_t^T (\mu(s,T) - {}^o\mu(s,T)) ds].
\end{aligned}
$$

In what concerns Lebesgue integrals in the expression of $Y(t) - \tilde{Y}(t)$ straightforward estimates follow from the triangular inequality for the L^2–norm and Fubini's Theorem.

To handle the stochastic integral

$$
\int_t^T (\sigma(s,T) - {}^o\sigma(s,T)) \partial W(s),
$$

we recall an inequality due to Meyer (see e.g. [14]). If $u \in \mathbb{L}^{1,2}$, and $u(t) \in L^4(\Omega)$, for all $t \in [0,T]$ then

$$
\begin{aligned}
\|D^*(u)\|_2 &\leq C[(\int_0^T (\mathbb{E}u(t))^2 dt)^{1/2} \\
&+ \| \left(\int_0^T \int_0^T (D_s u(t))^2 ds dt \right)^{1/2} \|_2].
\end{aligned}
\tag{3.20}
$$

Notice that our assumptions imply that $\sigma(\cdot,T) \in \mathbb{L}^{1,2}$ and $\sigma(t,T) \in L^4(\Omega)$, for all $t \in [0,T]$. If we apply the above inequality to $u = (\sigma(\cdot,T) - {}^o\sigma(\cdot,T))1_{[t,T]}$, then we get the desired inequality since $\mathbb{E}u(s) = 0$ for all $s \in [0,T]$ in this case. □

To conclude this section we would like to point out that our method works in a more general framework than most others considered up to now. To compare with preceding models, one can discuss how the *observed price* $\tilde{P}$ is related to adapted prices derived before in the literature. This will be achieved for bonds in the next section. To simplify notations in that section we will assume our coefficients μ and σ to be adapted and continuous. Indeed the analysis we develop below is always applicable to $\tilde{P}$ since ${}^o\mu$ and ${}^o\sigma$ are adapted. Therefore structural properties we derive are valid for $\tilde{P}$ in the general case and the estimates obtained before give a precise idea about closeness to P.

4. No arbitrage conditions for bonds: comparison with adapted models

As we said before, to compare with other approaches throughout this section we place ourselves in the case of coefficients adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$. Let us first remark on some general facts in this framework: P coincides with the observed bond price $\tilde{P}$, we rewrite its explicit expression (see (3.18)) as follows:

(4.1)
$$
P(t,T) = P(0,T) \exp \left(\int_0^t \sigma(s,T) dW(s) - \frac{1}{2} \int_0^t \sigma^2(s,T) ds + \int_0^t \mu(s,T) ds \right),
$$

for all $t \in [0,T]$, (recall that $P(T,T) = 1$ in the case of bonds). Now we define

$$Z(t) = \frac{P(t,T)}{P(0,T)}, \quad (t \in [0,T]). \tag{4.2}$$

Then Z is expressed as

$$Z = \mathcal{E}(\sigma \cdot W + \mu \cdot \lambda), \tag{4.3}$$

where we use the symbol $\mathcal{E}$ for the Doléans exponential of a semimartingale which reduces to

$$\mathcal{E}(S) = \exp(S - \frac{1}{2}[S,S]), \tag{4.4}$$

for a continuous semimartingale S with quadratic variation $[S,S]$ (see e.g. [16]). In (4.3) the symbol '$\cdot$' is used for a customary adapted stochastic integral process as it is common in Stochastic Analysis and $\lambda(t) = t$ is the identity semimartingale. We also have simplified the notation for $\sigma(\cdot,T)$ and $\mu(\cdot,T)$ by writing σ and μ respectively.

Z defined in (4.2) is therefore a semimartingale with respect to $\mathbb{F}$. It is important to point out that (4.1) tells us that the non adapted process $P(\cdot,T)$ can be decomposed into a product of an anticipating term $(P(0,T))$ and an *adapted process* (Z), which turns out to be a semimartingale under the hypotheses of this section. We call Z the *normalized price*. In our general anticipative model, the explicit expression (3.18) of the price shows that $P(\cdot,T)$ cannot be adapted, even under adaptedness of μ and σ, unless it is a constant $\mathbb{P}$–almost surely. This means that we have to circumvent this technical difficulty to express *no arbitrage* conditions in a suitable manner. The key is given by (4.1): structural properties depending on adaptedness are carried in our model by the normalized price Z. Thus, to look for conditions which exclude arbitrage opportunities for bonds discounted by savings accounts or assets discounted by bonds, means for us to have a quotient of processes (Z/B or A/Z) which represent martingales.

To investigate general conditions under which no arbitrage conditions occur we first prove an auxiliary result. We recall Yor's formula for the product of the semimartingale exponentials (see e.g. [16]):

$$\mathcal{E}(S_1)\mathcal{E}(S_2) = \mathcal{E}(S_1 + S_2 + [S_1, S_2]). \tag{4.5}$$

Therefore we have the following lemma.

Lemma 1. *Assume Z be given by (4.3) and Z' to be a semimartingale of the same type:*

$$Z' = \mathcal{E}(\sigma' \cdot W' + \mu' \cdot \lambda),$$

where W' is another Brownian motion on the same probability space and μ', σ' are continuous adapted processes. Then the process Z/Z' is a martingale if and only if

$$(\mu - \mu' + \sigma'^2) \cdot \lambda - \sigma\sigma' \cdot [W, W'] = 0 \tag{4.6}$$

Proof. We write Z/Z' as the quotient of Doléans exponentials applying formula (4.5):

$$\begin{aligned} \frac{Z}{Z'} &= \frac{\mathcal{E}(\sigma \cdot W + \mu \cdot \lambda)}{\mathcal{E}(\sigma' \cdot W' + \mu' \cdot \lambda)} \\ &= \mathcal{E}(\sigma \cdot W + \mu \cdot \lambda)\mathcal{E}(-\sigma' \cdot W' - (\mu' - \sigma'^2) \cdot \lambda) \\ &= \mathcal{E}(\sigma \cdot W - \sigma' \cdot W' + (\mu - \mu' + \sigma'^2) \cdot \lambda - \sigma\sigma' \cdot [W, W']). \end{aligned}$$

To complete the proof it suffices to remark that for a continuous semimartingale S its exponential $\mathcal{E}(S)$ is a martingale if and only if S itself is a martingale. This is a straightforward consequence of Itô's formula. In our case the above condition is satisfied if and only if the drift term $(\mu - \mu' + \sigma'^2) \cdot \lambda - \sigma\sigma' \cdot [W, W']$ is zero. □

4.1. Bond price discounted by a savings account. We assume as before r to be an adapted process which describes the term structure of interest rates and we denote by

$$B(t) = \exp\left(\int_0^t r(u)du\right), \ (t \geq 0), \tag{4.7}$$

the *savings account* at time $t \geq 0$. The process $B = (B(t); t \geq 0)$, which is also called *continuously instantaneous interest paying savings account (or accumulation factor)*, is $\mathbb{F}$-adapted. Our aim is to study structural properties of the *discounted bond price* P/B. This is achieved through Z:

$$\frac{1}{P(0,T)} \frac{P(t,T)}{B(t)} = \frac{Z}{B}(t), \; (t \in [0,T]). \tag{4.8}$$

The quotient Z/B is a martingale if and only if

$$\int_0^t (\mu(s,T) - r(s))ds = 0, \; (t \in [0,T]).$$

Since both functions $\mu(\cdot, T)$ and r are positive and continuous, the above condition is equivalent to

$$\mu(t,T) = r(t), \; \text{for all } t \in [0,T]. \tag{4.9}$$

Thus we obtain in a very general setting that a bond price discounted by the accumulation factor forms a martingale if and only if the expected return is exactly the instantaneous interest rate. And one can immediately conclude from this martingale property:

$$\begin{aligned} \frac{P(t,T)}{P(0,T)} &= B(t)\mathbb{E}(\frac{P(T,T)}{P(0,T)B(T)}/\mathcal{F}_t) \\ &= \mathbb{E}\left(\frac{1}{P(0,T)}\exp(-\int_t^T r(s)ds)/\mathcal{F}_t\right), \; (t \in [0,T]). \end{aligned} \tag{4.10}$$

This formula is the analog to the well-known bond price used in many papers on security derivative pricing as e.g. [2], [9], [8], where the factor $1/P(0,T)$ can be cancelled out on both sides of (4.10) because of an assumption of adaptedness. In our case it is the normalized price which satisfies the relation:

$$Z(t) = \mathbb{E}\left(\exp(-\int_t^T r(s)ds)/\mathcal{F}_t\right), \; (t \in [0,T]) \tag{4.11}$$

Another approach was proposed by Cox, Ingersoll and Ross in [6]. They started from a stochastic differential equation for the instantaneous interest rate. Let us write this equation in the general form

$$dr(t) = g(t)dt + p(t)dW(t), \tag{4.12}$$

where g and p are adapted processes with continuous trajectories and time moves on $[0, T]$. In their bond price dynamics they choose

$$\mu(t,T) = r(t)\left(1 + \kappa\,\frac{\sigma(t,T)}{p(t)}\right), \; (t \in [0,T]), \tag{4.13}$$

where κ is called the *risk parameter* or *market price of risk.*

Now we change the probability measure $\mathbb{P}$ into another $\mathbb{P}_V$ by means of a Girsanov's transformation. It is well known that given a continuous adapted process ψ, the process

$$V(t) = W(t) - \int_0^t \psi(s)ds, \; (t \in [0,T]), \tag{4.14}$$

becomes a Brownian motion under $\mathbb{P}_V$ if and only if $\mathbb{P}_V$ is absolutely continuous with respect to $\mathbb{P}$ and its Radon–Nikodym derivative is given by

$$\frac{d\mathbb{P}_V|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \mathcal{E}(\psi \cdot W)(t), \; (t \in [0,T]), \tag{4.15}$$

where '$\cdot|_{\mathcal{F}_t}$' denotes the restriction of '$\cdot$' to the σ–algebra $\mathcal{F}_t$.

If P_V denotes the price obtained by an exchange of W by V and $Z_V = P_V(\cdot,T)/P_V(0,T)$, then Z_V/B is a $(\mathbb{P}_V, \mathbb{F})$–martingale if and only if $Z_V\mathcal{E}(\psi\cdot W)/B$ is a $(\mathbb{P}, \mathbb{F})$–martingale. Thus, an application of our Lemma 1 shows that ψ has to be chosen as

$$\psi(t) = \kappa\,\frac{r(t)}{p(t)}, \; (t \in [0,T]), \tag{4.16}$$

to have $P_V/(P(0,T)B)$ as a $(\mathbb{P}_V, \mathbb{F})$–martingale.

In [15] an alternative bond price is suggested which differs from both of the above. The average return process is taken as

$$\mu(t,T) = r(t) + \sigma^2(t,T), \; (t \in [0,T]). \tag{4.17}$$

This equation involves a certain *risk premium* represented by the square of the volatility of the bond price process. One obtains easily the following representation for this bond price:

$$P(t,T) = \left[\mathbb{E}(\exp(\int_t^T r(s)ds)/\mathcal{F}_t)\right]^{-1}, \ (t \in [0,T]). \tag{4.18}$$

After a transformation of measures similar to the above with the choice

$$\psi(t) = -\sigma(t,T), \ (t \in [0,T]), \tag{4.19}$$

the price (4.18) discounted by the savings account becomes a martingale with respect to the new measure.

We do not go further in the consideration of other examples of bond price models. We postpone to the next section the crucial question about which, among all these different models, could be useful in more general derivative security pricing.

4.2. Assets discounted by bonds. An important problem in derivative pricing is to describe the risk neutral measure under which a given asset, discounted by a bond, becomes a martingale. As an illustration we consider a continuous adapted asset A which is the solution of a linear stochastic differential equation

$$dA(t) = a(t)A(t)dt + b(t)A(t)dW'(t), \tag{4.20}$$

where W' is another Wiener process (possibly independent of W) and a, b are adapted processes. We may interpret A in many different ways, e.g. as a savings account, as a stock, also as a derivative price in an incomplete market (for example an option price on a stock with stochastic volatility).

Our aim is then to analyze structural properties of the quotient A/P. To apply Lemma 1 we identify Z' with $A/A(0)$, σ' with b and μ' with a. That is:

$$\frac{A}{A(0)} = Z' = \mathcal{E}(b \cdot W + a \cdot \lambda) \tag{4.21}$$

Now we exchange the role of Z and Z' in Lemma 1. Thus, Z'/Z is a martingale if and only if

$$(a - \mu(\cdot, T) + \sigma^2(\cdot, T)) \cdot \lambda - \sigma(\cdot, T)b \cdot [W', W] = 0. \tag{4.22}$$

The following particular applications of the above analysis cover in principle most practically important cases. We notice that they lead to *sufficient conditions* to obtain the martingale property for the discounted asset A/P:

(1) The case of independent Brownian motions W, W'. Then $[W, W'] = 0$ and (4.22) reduces to

$$\int_0^t (\mu(s, T) - a(s) - \sigma^2(s, T))ds = 0, \ (t \in [0, T]) \tag{4.23}$$

A sufficient condition to obtain the martingale property is then

$$\mu(t, T) - \sigma^2(t, T) = a(t), \ (t \in [0, T]). \tag{4.24}$$

The above condition is easily fulfilled for any maturity as follows from [15]: choose the bond as in (4.18) with μ given by (4.17) and $a = r$. This case is similar to the measure transformation proposed in [12], however our approach is more direct.

The choice of a bond price in a form different from (4.18) would consequently impose a to be maturity–dependent to fulfill condition (4.24). This would complicate the whole analysis. That is why we claim the more natural choice is given by (4.18).

(2) Assume W and W' to be dependent, then $[W, W'] \neq 0$ and (4.22) has to be solved with additional assumptions. In particular, when discounting bonds by bonds one may assume $W = W'$ so that $[W, W'] = \lambda$, (where $\lambda(t) = t$ for all

$t \in [0,T]$). Therefore, a sufficient condition to obtain the martingale property in this case is

$$\mu(t,T) - a(t) - \sigma^2(t,T) - b(t)\sigma(t,T) = 0, \ (t \in [0,T]). \tag{4.25}$$

When considering the bond price according to (4.18) the analysis remains simple in this case, though we have to perform a measure transformation: it is always a technique of semimartingale exponentials which is applied, as in the examples already studied.

As we mentioned at the beginning of the paper, our aim was to present a general approach to security derivative pricing by means of anticipative stochastic analysis. We hope our discussion in the last section illustrates the advantages of the proposed method. We will continue to consider further related issues in more detail in a forthcoming paper.

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