Adaptation in Minimax Nonparametric Hypothesis Testing for Ellipsoids and Besov Bodies [*†‡]

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Abstract

We observe an infinitely dimensional Gaussian random vector $x = \xi + v$ where ξ is a sequence of standard Gaussian variables and $v \in l_2$ is an unknown mean. Let $V_\varepsilon(\tau, \rho_\varepsilon) \subset l_2$ be sets which correspond to l_q-ellipsoids of power semi-axes $a_i = i^{-s}R/\varepsilon$ with l_p-ellipsoid of semi-axes $b_i = i^{-r}\rho_\varepsilon/\varepsilon$ removed or to similar Besov bodies $B_{q,t;s}(R/\varepsilon)$ with Besov bodies $B_{p,h;r}(\rho_\varepsilon/\varepsilon)$ removed. Here $\tau = (\kappa, R)$ or $\tau = (\kappa, h, t, R)$; $\kappa = (p, q, r, s)$ are the parameters which define the sets V_ε for given radiuses $\rho_\varepsilon \to 0$, $0 < p, q, h, t \leq \infty$, $-\infty < r, s < \infty$, $R > 0$; $\varepsilon \to 0$ is asymptotical parameter.

For the case τ is known hypothesis testing problem $H_0 : v = 0$ versus alternatives $H_{\varepsilon,\tau} : v \in V_\varepsilon(\tau, \rho_\varepsilon)$ have been considered by Ingster and Suslina [11] in minimax setting. It was shown that there is a partition of the set of κ on to regions with different types of asymptotics: classical, trivial, degenerate and Gaussian (of two main and some "boundary" types). Also there is essential dependence of the structure of asymptotically minimax tests on the parameter κ for the case of Gaussian asymptotics .

In this paper we consider alternative $H_{\varepsilon,\Gamma} : v \in V_\varepsilon(\Gamma)$ for sets $V_\varepsilon(\Gamma) = \bigcup_{\tau\in\Gamma} V_\varepsilon(\tau, \rho_\varepsilon(\tau))$. This corresponds to adaptive setting: τ is unknown, $\tau \in \Gamma$ for a compact $\Gamma = K \times \Delta$; $\Delta = [c,\ C] \subset R^1_+$, $K \subset \Xi_{G_1} \cup \Xi_{G_2}$ where Ξ_{G_2} and Ξ_{G_2} are regions of main tapes of Gaussian asymptotics . First the problems of such types were studied by Spokoiny [16, 17].

For ellipsoidal case we study sharp asymptotics of minimax second kind errors $\beta_\varepsilon(\alpha, \Gamma) = \beta(\alpha, V_\varepsilon(\Gamma))$ and construct asymptotically minimax tests. These asymptotics are analogous to degenerate type. For Besov bodies case we obtain exact rates and construct minimax consistent tests. Analogous exact rates are obtained in a signal detection problem for continuous variant of white Gaussian noise model: alternatives correspond to Besov or Sobolev balls with Sobolev or Besov balls removed.

The study is based on results [11] and on an extension of methods of this paper for degenerate case.

1 Introduction

1.1 Minimax setting

Let an infinitely-dimensional Gaussian random vector $x = \xi + v$ be observed where ξ is a sequence of standard independent Gaussian random variables with zero mean and unit variance, $v \in l_2$ is an unknown mean sequence.

We consider the problem of testing null hypothesis $H_0 : v = 0$ on a sequence v and consider families of alternatives $H_\varepsilon : v \in V_\varepsilon$. Here $V_\varepsilon = \{v\}$ is a given family of sets in the sequence space l_2, $\varepsilon \to 0$ is an asymptotical parameter.

These problems are studied in asymptotically minimax setting (as $\varepsilon \to 0$). A family of (randomized) tests $\psi_\varepsilon = \psi_\varepsilon(x)$, $\psi_\varepsilon(x) \in [0, 1]$ is characterized by the families of the first kind errors $\alpha(\psi_\varepsilon) = E_0(\psi_\varepsilon)$ and by the supremum of the second kind errors

$$\beta(\psi_\varepsilon, V_\varepsilon) = \sup_{v\in V_\varepsilon} \beta(\psi_\varepsilon, v)\ \beta(\psi_\varepsilon, v) = E_v(1-\psi_\varepsilon).$$

Here and later E_v stands for the mean value with respect to the measure P_v which corresponds to an observation $x = \xi + v,\ v \in l_2$. For fixed $\alpha \in (0,1)$ minimax distinguishability is characterized by asymptotics of values

$$\beta(\alpha, V_\varepsilon) = \inf_{\psi\in\Psi_\alpha} \beta(\psi, V_\varepsilon), \quad \Psi_\alpha = \{\psi:\ \alpha(\psi) \le \alpha\}.$$

It is clear that

$$0 \le \beta(\alpha, V_\varepsilon) \le 1-\alpha.$$

The problem is called *trivial*, if $\beta(\alpha, V_\varepsilon) = 1-\alpha$ for any $\alpha \in (0,1)$.

The problem of sharp asymptotics is to investigate asymptotics of values $\beta(\alpha, V_\varepsilon)$ (up to vanishing term, as $\varepsilon \to 0$) and to construct *asymptotically minimax families of tests* $\psi_{\varepsilon,\alpha}$, such that, as $\varepsilon \to 0$,

$$\alpha(\psi_\varepsilon, \alpha) = \alpha + o(1), \quad \beta(\psi_{\varepsilon,\alpha}, V_\varepsilon) = \beta(\alpha, V_\varepsilon) + o(1).$$

The problem of the rates is to obtain conditions of *distinguishability*:

$$\beta(\alpha, V_\varepsilon) \to 0$$

and to construct *minimax consistent families of tests* $\psi_{\varepsilon,\alpha}$:

$$\alpha(\psi_\varepsilon, \alpha) = \alpha + o(1), \quad \beta(\psi_{\varepsilon,\alpha}, V_\varepsilon) = o(1),$$

or to obtain conditions of *indistinguishability (asymptotical triviality)*:

$$\beta(\alpha, V_\varepsilon) \to 1-\alpha.$$

1.2 Alternatives type of ellipsoids and Sobolev or Besov balls and bodies

Certainly, the main point for this setting is a family of alternatives V_ε. It is clear that sets V_ε must not contain any points which are close enough to zero. Also often sets V_ε must not be "wide" enough (see [9] for discussion). Simple enough and important class of such sets $V_\varepsilon = V_\varepsilon(\tau, \rho_\varepsilon)$ are ellipsoids with "small" ellipsoids removed:

$$V_\varepsilon(\tau, \rho_\varepsilon) = E_{q,s}(R/\varepsilon) \setminus E_{p,r}(\rho_\varepsilon/\varepsilon);\ \tau = (\kappa, R),\ \kappa = (p,q,r,s) \in \Xi; \tag{1.1}$$

here $E_{p,r}(R/\varepsilon)$ is l_p-ellipsoid in sequence space of of power sequence of semi-axes $a_i = i^{-r}R/\varepsilon$:

$$E_{p,r}(R) = \{v \in l_2 : \sum_{i=1}^{\infty} i^{rp}|v_i|^p < (R/\varepsilon)^p\}$$

with evident modification for $p = \infty$. The factor ε^{-1} corresponds to normalization in white Gaussian noise model (1.10) (see later). We denote

$$\Xi = \{(p, q, r, s): \ 0 < p, q \le \infty, \ -\infty < r, s < \infty\};$$

a family $\rho_\varepsilon > 0, \ \rho_\varepsilon \to 0$ is given. The case $r = 0$ corresponds to l_p-balls removed. (later in this section we explain the reasons that we consider cases $r \ne 0$.)

For ellipsoidal case with particular $0 < p, q \le \infty, s > 0$ and l_p-balls removed problems of sharp asymptotics had been studied by Ermakov [4] (the case p=q=2), by Ingster [8, 9] ($0 < p = q \le \infty$) , by Suslina [18, 19] ($0 < p, q < \infty, p \ne q$). The case of Besov body with L_2-ball removed have been studied by Ingster and Suslina [12].

The results of these papers show that different types of asymptotics arise in these problems. Full description of sharp asymptotics for ellipsoidal case ($\kappa \in \Xi$) had been obtained by Ingster and Suslina [11]. It was described a partition of the set Ξ onto regions with different types of asymptotics: Ξ_C (*classical*), Ξ_T (*trivial*), Ξ_D (*degenerate*) and $\Xi_{G_l}, l = 1, ..., 5$ (*Gaussian* of two main: $l = 1, 2$ and some "boundary" types: l=3, 4, 5). Asymptotically minimax families of tests for the regions Ξ_D and Ξ_{G_l} (minimax consistent for the region Ξ_C) were constructed as well.

Remind main results [11]. If $p, q < \infty$, then put :

$$\lambda = \lambda(\kappa) = qs - pr, \ \mu = \mu(\kappa) = pq(s - r), \ I = I(\kappa) = 2q(p-2)s - 2p(q-2)r + p - q;$$

if $q = \infty$, then $I = I(\kappa) = 2s(p-2) - 2rp - 1$;

$$r_p = \begin{cases} -1/4 + 1/p, & \text{if } p \le 2 \ , \\ -1/2p, & \text{if } 2 < p < \infty \ , \\ 0, & \text{if } p = \infty \ . \end{cases}$$

Trivial type corresponds to equality: $\beta(\alpha, V_\varepsilon) = 1 - \alpha$ for small enough R and $\rho_\varepsilon/\varepsilon$. The set Ξ_T is defined by the inequality $r \ge r_p$ and joint with following inequalities. If $p, q < \infty$, then

$$\begin{cases} \mu \le 0 \ \& \ \lambda \le 0 \ \& \ I \le 0, & \text{if } 2 > p > q \ , \\ \mu \le q - p \ \& \ I \le 0, & \text{if } 2 < p < q \ , \\ \mu \le 0 \ \& \ \lambda \le 0, & \text{if } p \ge 2, \ p > q \ , \\ \mu \le q - p, & \text{if } p \le 2, \ p \le q \text{ or } p = q > 2. \end{cases}$$

If $q = \infty, \ p < \infty$, then

$$\begin{cases} s - r \le 1/p, & \text{if } p < 2 \ , \\ s - r \le 1/p \ \& \ I \le 0, & \text{if } p \ge 2 \ , \end{cases}$$

and if $p = \infty, \ q \le \infty$, then $s \le r$ and $r \ge 0$.

Classical type of the rates is characterized by relations:

$$\beta(\alpha, V_\varepsilon) \to 1 - \alpha \text{ if and only if } \rho_\varepsilon/\varepsilon \to 0; \quad \beta(\alpha, V_\varepsilon) \to 0 \text{ if and only if } \rho_\varepsilon/\varepsilon \to \infty.$$

The region Ξ_C is defined by the inequality: $r < r_p$.

Degenerate type of sharp asymptotics is defined by the relation

$$\beta(\alpha, V_\varepsilon) = (1-\alpha)\Phi\left(\sqrt{\frac{2\log(R/\rho_\varepsilon)}{s-r}} - \rho_\varepsilon^{s/(s-r)} R^{-r/(s-r)} \varepsilon^{-1}\right) + o(1). \qquad (1.2)$$

Here and later Φ stands for distribution function of a standard Gaussian distribution. Degenerate type arises in the region $\kappa \in \Xi_D$ where

$$\Xi_D = \{\kappa \in \Xi^T : \ s > r > 0, \ \lambda \le 0\}$$

and Ξ^T is the complement of Ξ_T.

The relations (1.2) implies the following rates in the region Ξ_D. Introduce *critical radiuses* of removing sets (rates in Lepski and Spokoiny [14], Spokoiny [17]) :

$$\rho_\varepsilon^*(\tau) = R\left((\varepsilon/R)^2 \log \varepsilon^{-1}\right)^{(s-r)/2s}.$$

Put $\Lambda(\kappa) = (2/s)^{(s-r)/2s}$. Then for any $\alpha \in (0,1)$ one has

$$\beta(\alpha, V_\varepsilon) \to 0 \ \text{ if } \ \liminf \rho_\varepsilon/\rho_\varepsilon^*(\tau) > \Lambda(\kappa); \qquad (1.3)$$

and

$$\beta(\alpha, V_\varepsilon) \to 1-\alpha \ \text{ if } \ \limsup \rho_\varepsilon/\rho_\varepsilon^*(\tau) < \Lambda(\kappa). \qquad (1.4)$$

Gaussian types are described by the relations

$$\beta(\alpha, V_\varepsilon) = \Phi(T_\alpha - u_\varepsilon(\tau, \rho_\varepsilon)) + o(1), \qquad (1.5)$$

Here T_α stands for $(1-\alpha)$-quantile of distribution function of a standard Gaussian distribution : $\Phi(T_\alpha) = 1-\alpha$. The function $u_\varepsilon(\tau, \rho_\varepsilon) = u_\varepsilon$ is characterized minimax distinguishability in the problem. For two main types one has:

$$u_\varepsilon^2(\tau, \rho_\varepsilon) \sim d(\kappa)(\rho_\varepsilon/R)^{A_k(\kappa)}(\varepsilon/R)^{-B_k(\kappa)}, \ k = 1, 2; \qquad (1.6)$$

where $d(\kappa)$ is a positive bounded function. For the type G_1 one has:

$$B_1(\kappa) = 4, \quad A_1(\kappa) = \begin{cases} \frac{p(4-q+4sq)}{pq(s-r)+p-q}, & \text{if } q < \infty \\ \frac{p(4s-1)}{p(s-r)-1}, & \text{if } q = \infty \end{cases}$$

and for the type G_2 one has:

$$\begin{aligned} A_2(\kappa) &= \begin{cases} \frac{p(1+2sq)}{qs-pr}, & \text{if } q < \infty \\ 2p, & \text{if } q = \infty \end{cases} \\ B_2(\kappa) &= \begin{cases} \frac{2pq(s-r)+p-q}{qs-pr}, & \text{if } q < \infty \\ \frac{2p(s-r)-1}{s}, & \text{if } q = \infty \end{cases}. \end{aligned}$$

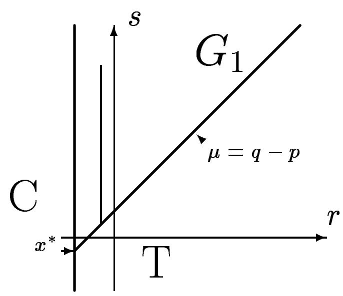

Fig. 1 $\quad p \le 2,\ p \le q \le \infty$

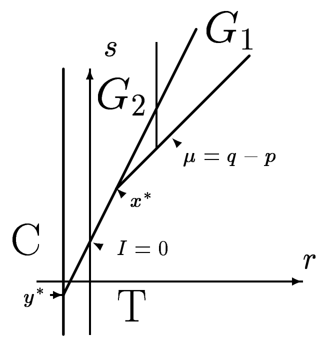

Fig. 2 $\quad 2 < p < q \le \infty$

The region $\Xi_G = \cup_{l=1}^5 \Xi_{G_l}$ is the complement of $\Xi_T \cup \Xi_C \cup \Xi_D$. Boundary types $\Xi_{G_l}, l = 3, 4, 5$ correspond to equalities $I = 0$ and $r = r_p$; the main regions $\Xi_{G_l}, l = 1, 2$ correspond to inequalities $I < 0$ and $I > 0$.

The relations (1.5), (1.6) imply exact rates for the regions $\Xi_{G_l}, l = 1, 2$:

$$\beta(\alpha, V_\varepsilon) \to 0 \ \text{ if and only if } \ \rho_\varepsilon/\rho_\varepsilon^*(\kappa) \to \infty; \tag{1.7}$$

$$\beta(\alpha, V_\varepsilon) \to 1 - \alpha \ \text{ if and only if } \ \rho_\varepsilon/\rho_\varepsilon^*(\kappa) \to 0 \tag{1.8}$$

where *critical radiuses* $\rho_\varepsilon^*(\kappa)$ are are defined by the relations

$$\rho_\varepsilon^*(\kappa) = \varepsilon^{B_k(\kappa)/A_k(\kappa)},\ k = 1, 2. \tag{1.9}$$

The partition of the planes of parameters $\{r, s\}$ onto regions with the asymptotics of different types for fixed values $0 < p < \infty$, $0 < q \le \infty$ is presented on the fig. 1-4. Here

$$x^* = (1/4 - 1/p,\ 1/4 - 1/q),\ y^* = (-1/2p,\ -1/2q)$$

with evident modification for $q = \infty$.

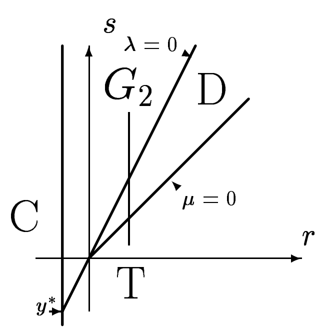

Fig. 3 $\quad p > q,\ p \ge 2$

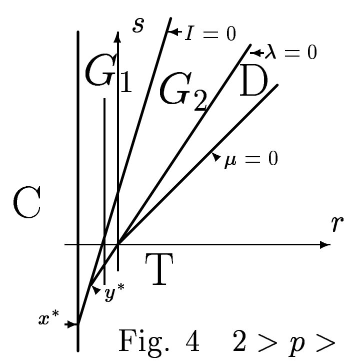

Fig. 4 $\quad 2 > p > q$

Certainly, the problem under consideration is equivalent to well known problem of testing $H_0 : s = 0$ versus a family of alternatives $H_\varepsilon : s \in S_\varepsilon \subset L_2(0, 1)$ in Gaussian white noise model:

$$dX_\varepsilon(t) = s(t)dt + \varepsilon dW(t),\ t \in [0, 1],\ s \in L_2(0, 1),\ \varepsilon > 0; \tag{1.10}$$

for a fixed orthonormal basis $\{\zeta_i\}$. It suffices to consider sequences of normalized empirical Fourier coefficients x_i and sets $V_\varepsilon = \{v_\varepsilon(s),\ s \in S_\varepsilon\}$ of normalized Fourier coefficients:

$$x_i = \varepsilon^{-1} \int_0^1 \zeta_i(t) dX_\varepsilon(t), \quad v_{i,\varepsilon}(s) = \varepsilon^{-1} \int_0^1 \zeta_i(t) s(t) dt.$$

Minimax hypothesis testing problem for Gaussian white noise model is defined by a family of sets $S_\varepsilon \subset L_2(0,1)$. Analogously to above, Sobolev or Besov balls with "small" Sobolev or Besov balls removed are most interesting and important functional sets S_ε in this problem:

$$S_\varepsilon(\tilde{\tau}, \rho_\varepsilon) = S_q^\sigma(R) \setminus S_p^\eta(\rho_\varepsilon); \quad \tilde{\tau} = (\tilde{\kappa}, h, t, R), \tilde{\kappa} = (p, q, \eta, \sigma)$$

or

$$S_\varepsilon(\tilde{\tau}, \rho_\varepsilon) = B_{q,t}^\sigma(R) \setminus E_{p,h}^\eta(\rho_\varepsilon); \quad \tilde{\tau} = (\tilde{\kappa}, h, t, R).$$

Here $S_q^\sigma(R)$ is subset of Sobolev ball of a radius R of $\sigma > 0$-smooth functions in L_q-norm, $1 \le q \le \infty$, (or subset of L_q-ball for $\sigma = 0$) which is contained in $L_2(0,1)$, $B_{q,t}^\sigma(R)$ is similar subset of Besov ball (see Triebel [20] for definitions).

For the case of Sobolev balls with L_p-balls removed ($\eta = 0$) and for particular $p, q, \sigma > 0$ exact rates have been studied by Ingster [6] ($p = q = 2$) and [7] ($p \le 2,\ p \le q$ or $2 \le p = q \le \infty$, by Lepski and Spokoiny [14] and by Ingster and Suslina [12] ($p = 2, q < 2$). Critical radiuses were calculated in these papers. If $p < \infty$, then these critical radiuses imply relations analogous to (1.7); if $p = \infty$, then relations analogous to (1.3), (1.4) hold but with different $\Lambda_1(\kappa),\ \Lambda_2(\kappa)$ in these relations. It was shown by Lepski [13] and by Lepski and Tsybakov [15] that $\Lambda_1(\kappa) = \Lambda_2(\kappa)$ for the case $p = q = \infty$.

The case of Besov balls with L_p-balls removed have been studied by Spokoiny [17]. It was described 3 types of rates (which are analogous to the types G_1, G_2 and D); critical radiuses (up to $\log\log$-factor for the types analogous to G_1, G_2) have been obtained in this paper. For Besov bodies case almost full description of the rates have been obtained by Ingster and Suslina [11].

These studies were based on the wavelet transformation $s \to v = \{v_{\iota,j}\}, v_{\iota,j} = (s, \zeta_{\iota,j})/\varepsilon$ where $\zeta_i,\ i = (\iota, j) \in J$, is some orthonormal pyramidal sequence of wavelet functions; here $J = \{i = (\iota, j) :\ \iota = 1, \dots, \max(1, 2^j),\ -j^{(0)} \le j < \infty$ (see Cohen et al. [2]; Donoho et al. [3]; Spokoiny [17] for example). To simplify notations, we assume $j^{(0)} = 0$ later.

Using well known embedding theorems [2, 3] one can reduce the problem of the rates for alternatives $S_\varepsilon = S_\varepsilon(\tilde{\tau}, \rho_\varepsilon)$ in functional space to similar problem for alternatives $V_\varepsilon = V_\varepsilon(\tilde{\tau}, \rho_\varepsilon)$ in the space of sequences of normalized wavelet coefficients where $V_\varepsilon(\tau, \rho_\varepsilon)$ is Besov body with "small" Besov body removed:

$$V_\varepsilon(\tau, \rho_\varepsilon) = B_{q,t,s}(R/\varepsilon) \setminus B_{p,h,r}(\rho_\varepsilon/\varepsilon), \quad \tau = (\kappa, h, t, R)$$

and the relations between parameters $\tilde{\kappa}$ and κ are defined by equalities

$$r = \eta + 1/2 - 1/p,\ s = \sigma + 1/2 - 1/q. \tag{1.11}$$

Remind that $B_{p,h,r}(R) = \{v =\in l_2 : f_{r,p,h}(v) \le R\}$ is Besov ball of the radius $R > 0$ where if $p, h < \infty$, then

$$f_{r,p,h}(v) = \left(\sum_j \left(2^{jr} \left(\sum_{\iota=1}^{2^j} |v_{\iota,j}|^p \right)^{1/p} \right)^h \right)^{1/h},$$

if $p < h = \infty$, then

$$f_{r,p,h}(v) = \sup_j \left(2^{jr} \left(\sum_{\iota=1}^{2^j} |v_{\iota,j}|^p \right)^{1/p} \right),$$

if $h \le p = \infty$, then we have analogous modifications.

There are some reasons that we consider cases of removing ellipsoids, Besov bodies and balls with $r \ne 0$ and $\eta \ne 0$ in [11] and here.

First, if $p \ne 2$, then L_p-ball in the functional space ($\eta = 0$) roughly corresponds not to l_p-ball in sequense space but to ellipsoid or Besov body with $r = 1/2 - 1/p$.

Next, the cases $\eta \ne 0$ correspond to hypothesis testing on derivatives or on integrals of a signal which is of interest in many problems.

Particularly, the case $\eta = -1$ in the model of unknown distribution density corresponds to hypothesis testing problem on distribution function where classical asymptotics hold. It is of interest to describe the "boundary" between classical and nonclassical asymptotics (the results of [11] give the answer on this question, see above).

It was shown in [11] that (with an exception of "boundary" types) the rates do not depend on parameters t, h. The same partition of the set $\Xi = \{\kappa\}$ onto regions with different types of the rates holds for Besov bodies case. The relations (1.7) and (1.3), (1.4) (may be, with different $\Lambda_1(\kappa)$, $\Lambda_2(\kappa)$) hold in this regions. By the relation (1.11) it implies similar partition of the set $\tilde{\Xi} = \{\tilde{\kappa}\}$ for the cases of Besov and Sobolev balls which extends results of previous papers. For example, vertical half-lines on the fig. 1 – 4 correspond to Sobolev or Besov balls with parameters q, $\sigma = s - 1/2 + 1/q$ and L_p-balls removed: $\eta = r - 1/2 + 1/p = 0$.

Minimax consistent families of tests were constructed for Besov bodies case. By applying inverse wavelet transformation, this yields minimax consistent families of tests for the cases of Besov and Sobolev balls.

1.3 Adaptive setting

Important point in results [16] and [11] is that *asymptotically minimax families of tests for ellipsoidal case (or minimax consistent families of tests for Besov bodies case) do not depend on parameters* $\kappa, R, \rho_\varepsilon$ *for* $\kappa \in \Xi_D$. It means that there exists *common* family of tests which is asymptotically minimax (or consistent) for *all* $\kappa \in \Xi_D$ (and uniformly on any compact $\Gamma = K \times\ D$; $D = [c, C] \subset R^1_+$, $K \subset Int(\Xi_D)$; analogously one can propose common family of minimax consistent tests for $K \subset$

$Int(\Xi_C)$. Certainly it is necessary to consider alternatives $V_\varepsilon(\tau) = V_\varepsilon(\tau, \rho_\varepsilon(\tau))$ with

$$\inf_\tau \rho_\varepsilon(\tau)/\rho^*_\varepsilon(\tau) \geq \Lambda(\kappa)$$

only. It means that these tests are asymptotically minimax (or minimax consistent) for the "union" alternatives

$$V_\varepsilon(\Gamma) = \bigcup_{\tau \in \Gamma} V_\varepsilon(\tau, \rho_\varepsilon(\tau)).$$

However it does not hold for $K \subset \Xi_G$: there exists essential dependence of the structure of asymptotically minimax (or consistent) families of tests on τ or κ; $\kappa \in \Xi_G$. This implies the problem: to construct *common* family of tests which has good minimax or consistent properties for all $\tau,\ \kappa \in \Xi_G$ or for $\kappa \in K \subset \Xi_G$ with wide enough subsets K. Of cause, radiuses of removing sets must depend on τ and be large enough to obtain the minimax consistent tests:

$$\inf_\tau \rho_\varepsilon(\tau)/\rho^*_\varepsilon(\kappa) \to \infty; \tag{1.12}$$

here $\rho^*_\varepsilon(\kappa)$ are defined by (1.12) (we assume R be bounded away from 0 and ∞). One has the same question for the cases of Besov bodies, for Sobolev and Besov balls (critical radiuses are defined by (1.12), (1.11)). This problem is of importance from practical point of view by, as usual, an statistician has not information on a degree and (or) on a norm to measure a smoothness of alternatives and a distance from null-hypothesis.

Following to Spokoiny [16, 17] who starts considerations of this problem, we call this setting as *adaptive*. Asymptotics of minimax second kind errors for "union" alternatives $V_\varepsilon(\Gamma)$ we call as *adaptive* as well (sharp or rate).

The case of the known $p = 2,\ q \geq p,\ \eta = 0$ (L_p-balls removed) and unknown smoothness σ from any interval (σ_0, σ_1) have been considered by Spokoiny [16]. It was shown in [16] that it is not possible to construct any tests with the property (1.12) : it is necessary to increase critical radiuses up to any power of $\log\log\varepsilon^{-1}$. For these increased critical radiuses "adaptive" families of tests $\psi_\varepsilon = \psi_{\varepsilon,p}$ have been constructed by Spokoiny [17] in Besov bodies case with L_p-balls removed. Under analogous to (1.3) assumptions with increased critical radiuses these tests are minimax consistent uniformly on any compact $K \in (\Xi_{G_1} \cup \Xi_{G_2}) \cap \{\eta = 0,\ p = \text{const},\ I \neq 0\}$).

2 Main results

The goal of this paper is to obtain sharp adaptive asymptotics for ellipsoidal case and exact adaptive rates for the case of Besov with Besov bodies removed for the main regions Ξ_G of Gaussian asymptotics. To simplicity we do not consider the boundary cases and some sub-manifolds in these regions. Certainly these results give adaptive rates for white Gaussian noise model (Sobolev or Besov balls with "small" balls removed).

Let us describe sharp adaptive asymptotics for these regions in ellipsoidal case. Assume $\Gamma \subset K \times D$ where $D \subset [c, C] \subset R^1_+$ and K is a compact, $K \subset \Xi_{G_1} \cup \Xi_{G_2}$. Let a family of the radiuses $\rho_\varepsilon(\tau), \tau = (\kappa, R) \in \Gamma$ is given. Put

$$V_\varepsilon(\Gamma) = \bigcup_{\tau \in \Gamma} V_\varepsilon(\tau, \rho_\varepsilon(\tau)).$$

Let us consider the family of functions $u_\varepsilon(\tau, \rho_\varepsilon(\tau))$ in (1.5), (1.6) and put

$$u_\varepsilon(\Gamma) = \inf_{\tau \in \Gamma} u_\varepsilon(\tau, \rho_\varepsilon(\tau)), \;\; H_\varepsilon = \sqrt{2 \log \log \varepsilon^{-1}} - u_\varepsilon(\Gamma).$$

Theorem 1 *Assume K has not intersection with 3-dimensional sub-manifolds $\{I = 0\}$, $\{p = q\}$, $\{p = 2\}$ and $\{\Delta = 0\}$ where $\Delta = sq(4 - p) - rp(4 - q)$. Then:*

1. Following upper bounds hold:

$$\beta(\alpha, V_\varepsilon(\Gamma)) \leq (1 - \alpha)\Phi(H_\varepsilon) + o(1).$$

2. Assume that for any $\delta > 0$ there exists an open set $\Delta = \Delta(\delta) \subset K$ and a function $R(\kappa)$, $\kappa \in \Delta$ such that $\tau = (\kappa, R(\kappa)) \in \Gamma$ and $u_\varepsilon(\tau) < u_\varepsilon(\Gamma) + \delta$ for $\kappa \in \Delta$. Then following lower bounds hold:

$$\beta(\alpha, V_\varepsilon(\Gamma)) \geq (1 - \alpha)\Phi(H_\varepsilon) + o(1).$$

The Theorem 1 shows that under assumption n.2

$$\beta(\alpha, V_\varepsilon(\Gamma)) = (1 - \alpha)\Phi(H_\varepsilon) + o(1).$$

Remark 2.1. The assumption of n.2 means that infimum $u_\varepsilon(\Gamma)$ is "essential". We can use weaker assumption: there exists an interval in K which is not "tangent" to the hyper-surfaces $\{\phi(\kappa) = \text{const}\}$, the function $\phi(\kappa)$ is defined by (3.35), (3.41) later. More exactly, it is enough to assume that the set $\phi(K)$ contains nontrivial interval. However it is possible to show that for any finite number τ we can provide better lower bounds.

Remark 2.2. The asympotics in Theorem 1 are close to degenerate type. In fact, it is shown in the next section that there is close connection between adaptive problem and hypothesis testing problems for degenerate type.

Let us define *adaptive critical radius functions* $\rho^*_{\varepsilon,ad}(\tau)$ by the relation

$$u_\varepsilon(\tau, \rho^*_{\varepsilon,ad}(\tau)) = \sqrt{2 \log \log \varepsilon^{-1}} + O(1).$$

Using (1.6) one has for $\kappa \in \Xi_{G_k}, \; k = 1, 2$

$$\rho^*_{\varepsilon,ad}(\tau) \sim R(\varepsilon/R)^{-B_k(\kappa)/A_k(\kappa)}((2 \log \log \varepsilon^{-1})/d_k(\kappa))^{1/A_k(\kappa)}. \tag{2.1}$$

Note the difference on the rates with non-adaptive critical radiuses (1.7) in the factor $(\log \log \varepsilon^{-1})^{1/A_k(\kappa)}$. These adaptive critical radiuses for $\eta = r - 1/2 + 1/p = 0, \; \sigma = s - 1/2 + 1/q$ correspond to adaptive rates in Spokoiny [17].

Using the Theorem 1 one has for $\Lambda_1 = \Lambda_2 = 1$ that

$$\text{if} \ \liminf_{\varepsilon\to 0} \inf_{\tau\in\Gamma} \rho_\varepsilon(\tau)/\rho^*_{\varepsilon,ad}(\tau) > \Lambda_1, \quad \text{then} \quad \beta(\alpha, V_\varepsilon(\Gamma)) \to 0 \tag{2.2}$$

and

$$\text{if} \quad \limsup_{\varepsilon\to 0} \sup_{\tau\in\Gamma_0} \rho_\varepsilon(\tau)/\rho^*_{\varepsilon,ad}(\tau) < \Lambda_2, \quad \text{then} \quad \beta(\alpha, V_\varepsilon(\Gamma)) \to 1-\alpha \tag{2.3}$$

where $\Gamma_0 = \{(\kappa, R(\kappa));\ \kappa \in \Delta\} \subset \Gamma,\ \Delta$ is an open subset $Int(K)$ and $R(\kappa),\ \kappa\in\Delta$ is any positive function.

Let us consider the Besov bodies case. Let $\tau = (\kappa, h, t, R) \in \Gamma \subset K \times D^{(3)}$ where $D^{(3)} = D_1 \times D_2 \times D_3,\ D_l \subset [c_l, \infty],\ c_l > 0,\ l = 1, 2;\ D_3 \subset [c, C] \subset R^1_+$ and $K \subset \Xi_{G_1} \cup \Xi_{G_2}$ be such compact that K has not intersection with 3-dimensional subset $\{I = 0\}$. Let us consider the family of functions $u_\varepsilon(\tau, \rho_\varepsilon(\tau))$ from sec. 1.3 and put

$$u_\varepsilon(\Gamma) = \inf_{\tau\in\Gamma} u_\varepsilon(\tau, \rho_\varepsilon(\tau)).$$

Theorem 2 *There exist such constants $C = C(\Gamma),\ c = c(\Gamma),\ C > c > 0$ that:*

1. Let

$$\liminf_{\varepsilon\to 0} \inf u^2_\varepsilon(\Gamma)/\log\log\varepsilon^{-1} > C.$$

Then there exist such family of tests ψ_ε that $\alpha(\psi_\varepsilon) = o(1)$ and $\beta(\psi_\varepsilon, V_\varepsilon(\Gamma)) = o(1)$. These relations imply $\beta(\alpha, V_\varepsilon(\Gamma)) \to 0$ for all $\alpha \in (0, 1)$.

2. Assume K has nonempty set $Int(K)$ of interior points, there exist an open set $\Delta \subset Int(K)$ and functions $h : \Delta \to D_1,\ t : \Delta \to D_2,\ R : \Delta \to D_3$, such that

$$\Gamma_0 = \{\tau(\kappa) = (\kappa, h(\kappa), t(\kappa), R(\kappa)),\ \kappa \in \Delta\} \subset \Gamma$$

and

$$\limsup_{\varepsilon\to 0} \sup_{\tau\in\Gamma_0} u^2_\varepsilon(\tau, \rho_\varepsilon(\tau))/\log\log\varepsilon^{-1} < c.$$

Then $\beta(\alpha, V_\varepsilon(\Gamma)) \to (1-\alpha)$ for all $\alpha \in (0, 1)$.

Using the Theorem 2 we obtain

Corollary 2.1 *For Besov bodies case the rates (2.2), (2.3) hold with the adaptive critical radiuses (2.1) and some constants $\Lambda_1 = \Lambda_1(\Gamma),\ \Lambda_2 = \Lambda_2(\Gamma);\ \Lambda_1 > \Lambda_2 > 0$.*

This result for $\eta = 0,\ p = \text{const}$ corresponds to Spokoiny [17].

Next part of the paper contains the proofs.

In the sec. 3 we obtain lower bounds. In the sec. 3.1 we describe the idea of the lower bounds. It follows to Burnashev [1] and to Ingster [8, 9] and is based on the consideration of collections of orthogonal signals. We extend this onto the collections of asymptotically orthogonal product priors in the sense of scalar product which have been introduced in [10, 11]. In the sec. 3.2 we give general estimations for mixtures of product priors. In the sec. 3.3 we present constructions of asymptotically orthogonal collections for ellipsoidal case This constructions are based on the results [11]. In the sec. 3.4 we construct orthogonal collections for Besov body case which are analogous to [16].

In the sec. 4 we construct asymptotically minimax families of adaptive tests for ellipsoidal case. These tests are based on the partition of the set Γ onto small enough sells, on constructions of tests for all sells (this construction is based on the results [11]) and on the "union" of these tests. Of cause, we need to increase test thresholds which implies the loss of efficiency.

In the sec. 5 we propose minimax consistent families of adaptive tests for Besov body case. Test procedure is presented in the sec. 5.1. This construction is close to [17] but differ. The main point of the study of tests is based in the results [11] as well.

3 Lower bounds

3.1 Idea of lower bounds: asymptotically orthogonal signals

Idea of lower bounds in the problem under consideration corresponds to Burnashev [1] and to Ingster [8, 9].

To obtain lower bounds we will use Bayesian approach. Let π^ε be a sequence of probability measures (priors) on the space l_2 (we will consider measures which are supported on finite or on denumerable sets and there are not problems of measurability). Let P_{π^ε} be a sequence of mixtures:

$$P_{\pi^\varepsilon}(A) = \int P_v(A)\pi^\varepsilon(dv).$$

Let x be infinitely dimensional vector of random observations with unknown probability measure P. Consider Bayesian hypothesis testing problem on observations x:

$$H_0: \ P = P_0; \quad H_\varepsilon : P = P_{\pi^\varepsilon}.$$

Let $\beta(\alpha, \pi^\varepsilon)$ be minimum of the second kind errors in this problem for tests of a level α:

$$\beta(\alpha, \pi^\varepsilon) = \inf_{\psi\in\Psi_\alpha} E_{P_{\pi^\varepsilon}}(1-\psi).$$

Assume

$$P_{\pi^\varepsilon}(V_\varepsilon) \to 1. \tag{3.1}$$

It is well known (see [9, sec. 4.1] for example) that under (3.1)

$$\beta(\alpha, V_\varepsilon) \geq \beta(\alpha, \pi^\varepsilon) + o(1). \tag{3.2}$$

Assume that L_1-distance between P_0 and P_π^ε tends to 0:

$$E_{P_0}|dP_{\pi^\varepsilon}/dP_0 - 1| \to 0. \tag{3.3}$$

Then Bayesian problem if asymptotically trivial:

$$\beta(\alpha, \pi^\varepsilon) \to 1 - \alpha. \tag{3.4}$$

The relation (3.3) follows from stronger relation on L_2-distance:

$$E_{P_0}\left(\frac{dP_{\pi^\varepsilon}}{dP_0} - 1\right)^2 \to 0. \tag{3.5}$$

Assume that P_0-distributions of Bayesian likelihood ratios are asymptotically degenerate: for some (nonrandom) sequence C_ε under P_0-probability

$$dP_\pi^\varepsilon / dP_0 = C_\varepsilon + o(1). \tag{3.6}$$

The relation (3.6) yields (see [9, sec. 4.4])

$$\beta(\alpha, \pi^\varepsilon) = (1-\alpha)\ C_\varepsilon + o(1). \tag{3.7}$$

We will consider priors of the type

$$\pi^\varepsilon = M^{-1}\sum_{l=1}^{M} \pi_l^\varepsilon,\ \ M = M_\varepsilon \to \infty. \tag{3.8}$$

Assume for a moment that $\pi_l^\varepsilon = \delta_{v_{\varepsilon,l}}$, where $v_{\varepsilon,1}, ...v_{\varepsilon,M},\ v_{\varepsilon,l} \in l_2$ is an orthogonal collection in l_2:

$$(v_{\varepsilon,l}, v_{\varepsilon,k}) = 0, \|v_l\| = u_\varepsilon;\ \ 1 \le l < k \le M$$

and δ_b is Dirac mass at a point $b \in l_2$. It was shown by Burnashev [1] that if $\exp u_\varepsilon^2 = o(M)$, then $\beta(\alpha, \pi^\varepsilon) = (1-\alpha) + o(1)$.

Also put $R_\varepsilon = \sqrt{2\log M_\varepsilon} - u_\varepsilon$. Then (see [8, 9])

$$\beta(\alpha, \pi^\varepsilon) = (1-\alpha)\Phi(R_\varepsilon) + o(1).$$

3.2 Asymptotically orthogonal priors

3.2.1 Indistinguishability conditions

We consider priors of product type:

$$\pi_l^\varepsilon = \pi_{\varepsilon,l,1} \times ... \times \pi_{\varepsilon,l,n} \times ... \tag{3.9}$$

which correspond to sequences $\bar\pi_{\varepsilon,l} = (\pi_{\varepsilon,l,1}, ..., \pi_{\varepsilon,l,n}, ...,)$ of probability measures on the real line R^1.

Following to [10, 11] introduce scalar product for sequences $\bar\pi = \{\pi_i\} : \bar r = \{r_i\}$

$$(\bar\pi, \bar r) = \sum_i (\pi_i, r_i) = \sum_i \int_{R^1}\int_{R^1} (e^{uv} - 1)\, \pi(du) r(dv),\ \ |\bar\pi\|^2 = \sum_i |\bar\pi_i\|^2 = \sum_i (\pi_i, \pi_i). \tag{3.10}$$

Here i is either an integer for ellipsoidal case or pyramidal index $i = (\iota, j) \in J$ for Besov body case. It is clear that

$$(\pi_i, r_i) = E_{P_{0,1}} \left(\frac{dP_{\pi_i,1}}{dP_{0,1}} \frac{dP_{r_i,1}}{dP_{0,1}} - 1 \right) = E_{P_{0,1}} \frac{dP_{\pi_i,1}}{dP_{0,1}} \frac{dP_{r_i,1}}{dP_{0,1}} - 1$$

where $P_{\pi,1}$ is a mixture of one-dimensional Gaussian measures $P_{v,1} = N(v, 1)$. By this equality

$$\begin{aligned} E_{P_0} \left(\frac{dP_{\hat{\pi}}}{dP_0} - 1 \right) \left(\frac{dP_{\hat{r}}}{dP_0} - 1 \right) &= \prod_i E_{P_{0,1}} \left(\frac{dP_{\pi_i,1}}{dP_{0,1}} \frac{dP_{r_i,1}}{dP_{0,1}} \right) - 1 \\ = \prod_i \left(1 + \sum_i (\pi_i, r_i) \right) - 1 &\leq \exp (\bar{\pi}, \bar{r}) - 1, \end{aligned}$$

where

$$\hat{\pi} = \prod_i \pi_i, \ \hat{r} = \prod_i r_i.$$

By (3.8) this implies

$$\begin{aligned} E_{P_0} \left(\frac{dP_{\pi^\varepsilon}}{dP_0} - 1 \right)^2 &= M^{-2} \sum_{l,k=1}^{M} E_{P_0} \left(\left(\frac{dP_{\pi_l^\varepsilon}}{dP_0} - 1 \right) \left(\frac{dP_{\pi_k^\varepsilon}}{dP_0} - 1 \right) \right) \\ &\leq M^{-2} \sum_{l,k=1}^{M} (\exp(\bar{\pi}_{\varepsilon,l}, \bar{\pi}_{\varepsilon,k}) - 1). \end{aligned}$$

The estimations above yield following indistinguishability conditions analogous to Burnashev [1] :

Lemma 3.1 *Assume $M = M_\varepsilon \to \infty$ and*

$$\sup_{1 \leq l < k \leq M} (\bar{\pi}_{\varepsilon,l}, \bar{\pi}_{\varepsilon,k}) = o(1); \quad \sup_{1 \leq l \leq M} \exp(\|\bar{\pi}_{\varepsilon,l}\|^2) = o(M).$$

Then (3.5) and (3.4) hold. If also $\inf_{1 \leq l \leq M} \pi_l^\varepsilon(V_\varepsilon) = 1 - o(1)$, then $\beta_\varepsilon(\alpha, V_\varepsilon) \to 1 - \alpha$ for any $\alpha \in (0, 1)$.

3.2.2 Asymptotically sharp lower bounds

Let us consider statistics

$$L_\varepsilon = dP_{\pi^\varepsilon}/dP_0 = M^{-1} \sum_{l=1}^{M} L_{\varepsilon,l}; \ \ L_{\varepsilon,l} = dP_{\pi_l^\varepsilon}/dP_0, \ l_{\varepsilon,l} = \log \left(dP_{\pi_l^\varepsilon}/dP_0 \right).$$

Denote

$$t_\varepsilon = \sqrt{2 \log M_\varepsilon}, \ u_{\varepsilon,l} = \|\bar{\pi}_{\varepsilon,l}\|, \ \rho_{\varepsilon;l,k} = (\bar{\pi}_{\varepsilon,l}, \bar{\pi}_{\varepsilon,k})/u_{\varepsilon,l} u_{\varepsilon,k}, \quad \lambda_{\varepsilon,l} = l_{\varepsilon,l}/u_{\varepsilon,l} + u_{\varepsilon,l}/2.$$

Thus $L_{\varepsilon,l} = \exp(\lambda_{-u_{\varepsilon,l}^2/2+\varepsilon,l} u_{\varepsilon,l})$. Let $\Phi_\rho(x, y)$ be distribution function of Gaussian random variables X, Y with zero means, unit variance and $E(XY) = \rho$.

Put the assumptions

A1.

$$\sup_{1\le l\le M_\varepsilon} |u_{\varepsilon,l} - t_\varepsilon| = O(1), \qquad \sup_{1\le l<k\le M_\varepsilon} \rho_{\varepsilon;l,k} = o(t_\varepsilon^{-2}).$$

Assume also for some $\delta > 0$

$$\sup_{1\le l\le M_\varepsilon} \sup_{x\in R^1} |P_0(\lambda_{\varepsilon,l} < x) - \Phi(x)| = O(M_\varepsilon^{2(1+\delta)}) \tag{3.11}$$

and

$$\sup_{1\le l<k\le M_\varepsilon} \sup_{x\in R^1} |P_0(\lambda_{\varepsilon,l} < x, \lambda_{\varepsilon,k} < y) - \Phi_{\rho_{\varepsilon;l,k}}(x,y)| = O(M_\varepsilon^{2(1+\delta)}). \tag{3.12}$$

The assumptions (3.11) and (3.12) are small stronger that asymptotical normality of log-likelihood ratios $l_{\varepsilon,l}$.

Let us consider truncated statistics $L_{\varepsilon,l}$:

$$\hat{L}_{\varepsilon,l}(x) = L_{\varepsilon,l}(x)\mathbf{1}_{\{\lambda_{\varepsilon,l}<t_\varepsilon)\}}(x).$$

Put $X_\varepsilon = \{x :\ \hat{L}_{\varepsilon,l}(x) = L_{\varepsilon,l}(x)\}$ and $\overline{X}_\varepsilon$ be the complement of X_ε. By (3.11)

$$P_0(\overline{X}_\varepsilon) \le \sum_{l=1}^{M} P_0(\lambda_{\varepsilon,l} \ge t_\varepsilon) \le o(1) + M_\varepsilon \Phi(-t_\varepsilon) = o(1). \tag{3.13}$$

By (3.13) one can obtain (3.4) from the relation: under P_0-probability

$$\hat{L}_\varepsilon = M^{-1}\sum_{l=1}^{M} \hat{L}_{\varepsilon,l} = C_\varepsilon + o(1); \quad C_\varepsilon = M^{-1}\sum_{l=1}^{M} \Phi(t_\varepsilon - u_{\varepsilon,l}). \tag{3.14}$$

By Chebyshev inequality to obtain (3.14) it is enough to check that uniformly on $l,k = 1,...,M,\ l \ne k$

$$E_0\hat{L}_{\varepsilon,l} = \Phi(t_\varepsilon - u_{\varepsilon,l}),\ E_0\hat{L}_{\varepsilon,l}^2 = o(M),\ Cov_0(\hat{L}_{\varepsilon,l}, \hat{L}_{\varepsilon,k}) = o(1). \tag{3.15}$$

To check (3.15) we use the equalities for $[0,T]$-truncated moments of random variables (X,Y) with distribution functions $F(x) = P(X < x),\ F(x,y) = P(X < x, Y < y)$:

$$\int_0^T x dF(x) = \int_0^T (1 - F(x))dx; \quad \int_0^T\int_0^T xy dF(x,y) = \int_0^T\int_0^T (1 - F(x,y))dxdy$$

which imply inequalities for differences of moments of bounded random variables $0 \le X_l, Y_l \le T$ with distribution functions $F_l(x),\ F_l(x,y),\ l = 1,2$:

$$|EX_1^k - EX_2^k| \le T^k \sup_x |F_1(x) - F_2(x)|,\ k = 1,2; \tag{3.16}$$

$$|EX_1Y_1 - EX_2Y_2| \le T^2 \sup_{x,y} |F_1(x,y) - F_2(x,y)|. \tag{3.17}$$

Introduce Gaussian random variables $\nu_1, ..., \nu_M$ with zero means, unit variances and $E\nu_l\nu_k = \rho_{\varepsilon;l,k},\ 1 \le l < k \le M$. Put

$$X_l = \exp(-u_{\varepsilon,l}^2/2 + u_{\varepsilon,l}\nu_l),\ \hat{X}_l = X_l \mathbf{1}_{\{\nu_l < t_\varepsilon\}},\ T_l = \exp(-u_{\varepsilon,l}^2/2 + u_{\varepsilon,l}t_\varepsilon).$$

By (3.18), (3.11) and (3.12) one has:

$$E_0\hat{L}_{\varepsilon,l} = E\hat{X}_l + o(T_l/M^{2(1+\delta)});\quad E_0\hat{L}_{\varepsilon,l}^2 = E\hat{X}_l^2 + o(T_l^2/M^{2(1+\delta)});$$
$$Cov_0(\hat{L}_{\varepsilon,l}, \hat{L}_{\varepsilon,k}) = Cov(\hat{X}_l, \hat{X}_k) + o(T_l^2/M^{2(1+\delta)}).$$

For Gaussian variables $\nu_1, ..., \nu_M$ one has

$$\begin{aligned} E\hat{X}_l &= P(\nu_l + u_{\varepsilon,l} < t_\varepsilon) = \Phi(t_\varepsilon - u_{\varepsilon,l}), \\ E\hat{X}_l^2 &= \exp(u_{\varepsilon,l}^2)\Phi(t_\varepsilon - 2u_{\varepsilon,l}) \asymp t_\varepsilon^{-1}\exp((t_\varepsilon^2 + c_{\varepsilon,l}^2)/2) = O(M/t_\varepsilon) \end{aligned}$$

by $c_{\varepsilon,l} = u_{\varepsilon,l} - t_\varepsilon = O(1),\ \exp(t_\varepsilon^2/2) = M$. Note the inequality:

$$\sup_{x,y} |\Phi_\rho(x+a, y+b) - \Phi(x)\Phi(y)| \le B(|\rho| + |a| + |b|) \tag{3.18}$$

which holds for some $B > 0$ and small enough $|\rho|,\ |a|,\ |b|$. One can easily obtain (3.18) by calculation of Hellinger distance. Using A1 and (3.18) one has

$$\begin{aligned} E(\hat{X}_l\hat{X}_k) &= \exp(\rho_{\varepsilon,l,k}u_{\varepsilon,l}u_{\varepsilon,k})\Phi(t_\varepsilon - u_{\varepsilon,l} - \rho_{\varepsilon,l,k}u_{\varepsilon,k},\ t_\varepsilon - u_{\varepsilon,k} - \rho_{\varepsilon,l,k}u_{\varepsilon,l}) \\ &= \Phi(t_\varepsilon - u_{\varepsilon,l})\Phi(t_\varepsilon - u_{\varepsilon,k}) + o(1). \end{aligned}$$

The estimations above and (3.16), (3.17) imply (3.15) and following lower bounds.

Lemma 3.2 *Assume $M = M_\varepsilon \to \infty$ and the assumptions A1, (3.11) – (3.12) hold. Then (3.7) holds with C_ε defined by (3.14). If $\pi_l^\varepsilon(V_\varepsilon) \ge 1 - \delta_\varepsilon,\ l = 1, ..., M$ also, then (3.8) holds for any $\alpha \in (0, 1)$.*

3.2.3 Sequences of symmetrical three-point measures

We need to control the assumptions of asymptotical normality (3.11) – (3.12). To our arms it is enough to consider the case of product priors (3.9) where $\pi_{\varepsilon,l,i} = \pi(z_{\varepsilon,l,i}, h_{\varepsilon,l,i})$ are symmetrical three-point measures at the points 0 , $z_{\varepsilon,l,i}$ and $-z_{\varepsilon,l,i}$:

$$\pi(z, h) = (1-h)\delta_0 + \frac{h}{2}(\delta_z + \delta_{-z}) \tag{3.19}$$

(or two-point measures, if $h_{\varepsilon,l,i} = 1$) and $\bar{\pi}_{\varepsilon,l} = \{\pi(h_{\varepsilon,,l,i}, z_{\varepsilon,l,i})\}$ correspond to two sequences $\bar{h}_{\varepsilon,l},\ \bar{z}_{\varepsilon,l}$ with $h_{\varepsilon,l,i} \in [0, 1],\quad z_{\varepsilon,l,i} \ge 0$. Here and later δ_b is Dirac mass at a point $b \in R^1$.

For these sequences one has:

$$(\bar{\pi}_{\varepsilon,l}, \bar{\pi}_{\varepsilon,k}) = \sum_i (\pi_{\varepsilon,l,i}, \pi_{\varepsilon,k,i}) = 2\sum_i h_{\varepsilon,l,i}h_{\varepsilon,k,i}\sinh^2(z_{\varepsilon,l,i}z_{\varepsilon,k,i}/2), \tag{3.20}$$

$$\|\bar{\pi}_{\varepsilon,l}\|^2 = \sum_i \|\pi_{\varepsilon,l,i}\|^2 = 2\sum_i h^2_{\varepsilon,l,i} \sinh^2 \frac{z^2_{\varepsilon,l,i}}{2}. \tag{3.21}$$

The log-likelihood ratio $l_{\varepsilon,l} = \log\left(dP_{\pi^\varepsilon_l}/dP_0\right)$ is of the form

$$l_{\varepsilon,l} = \sum_i \log(1 + h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i})); \quad \xi(x,z) = e^{-z^2/2}\cosh zx - 1.$$

If x is a standard Gaussian variable, then

$$E\xi(x,z) = 0,\ E\xi^2(x,z) = 2\sinh^2\frac{z^2}{2},\ E(\xi(x,z_1)\xi(x,z_2)) = 2\sinh^2\frac{z_1z_2}{2}, \tag{3.22}$$

and for any $v \in R^1$

$$E\xi(x+v,z) = 2\sinh^2\frac{zv}{2},\ Var\xi(x+v,z) = 2\sinh^2\frac{z^2}{2} + (e^{z^2}-1)\sinh^2 zv. \tag{3.23}$$

Note that

$$1 + h\xi(x,z) \geq 1 - h(e^{-z^2/2} - 1) \geq 1 - \|\pi(h,z)\|/\sqrt{2}. \tag{3.24}$$

Also for an integer $k > 1$ one has

$$E\xi^{2k}(x,z) \leq C_1(k)\exp(C_2(k)z^2)(E\xi^2(x,z))^k \tag{3.25}$$

where $C_1(k) > 0,\ C_2(k) > 0$ are constants (see the Lemma 1 in [10]).

Put the assumptions:

A2. *Uniformly on $l,\ 1 \leq l \leq M_\varepsilon$ for a family $\bar{\pi}_\varepsilon = \{\pi_{\varepsilon,i}\} = \bar{\pi}_{\varepsilon,l}$ one has:*

$$\|\bar{\pi}_\varepsilon\|^2 \asymp \log M_\varepsilon \asymp \log\log(\varepsilon^{-1}).$$

A3. *For some small enough $\delta > 0$ uniformly on $l,\ 1 \leq l \leq M_\varepsilon$ for a family $\bar{\pi}_\varepsilon = \{\pi_{\varepsilon,i}\} = \bar{\pi}_{\varepsilon,l}$ one has:*

$$\sup_i \|\pi_{\varepsilon,i}\|/\|\bar{\pi}_\varepsilon\| = O(\varepsilon^{\delta_0}). \tag{3.26}$$

A4.1. *Uniformly on $l,\ 1 \leq l \leq M$ for a family $\bar{\pi}_\varepsilon = \{\pi_{\varepsilon,i}\} = \bar{\pi}_{\varepsilon,l}$ one has:* $\sup_i z_{\varepsilon,i} = O(1)$.

A4.2. *For all $\eta \in (0,1)$ uniformly on $l,\ 1 \leq l \leq M$ for a family $\bar{\pi}_\varepsilon = \{\pi_{\varepsilon,i}\} = \bar{\pi}_{\varepsilon,l},\ \pi_{\varepsilon,i} = \pi(h_{\varepsilon,i}, z_{\varepsilon,i})$ one has:*

$$\sum_i \exp(\eta z^2_{\varepsilon,i})\|\pi_{\varepsilon,i}\|^2 = O(\|\bar{\pi}_\varepsilon\|^2). \tag{3.27}$$

Put $I_{\varepsilon,c} = \{i : z^2_{\varepsilon,i} > c\log\varepsilon^{-1}\}$. Note that under assumptions A2, A3 and A4.1 or A4.2 for any $\eta_1 \in (0,\eta),\ c > 0$ one has

$$\sum_{i \in I_{\varepsilon,c}} \|\pi_{\varepsilon,i}\|^2 = O(\varepsilon^{c\eta}\|\bar{\pi}_\varepsilon\|^2) = O(\varepsilon^{c\eta_1}). \tag{3.28}$$

Lemma 3.3 *Assume A1, A2, A3 and A4.1 or A4.2. Then for small enough $\delta > 0$*

$$\Delta_{1,\varepsilon} = \sup_{1\le l\le M_\varepsilon} \sup_{x\in R^1} |P_0(\lambda_{\varepsilon,l} < x) - \Phi(x)| = O(\varepsilon^\delta) \tag{3.29}$$

and

$$\Delta_{2,\varepsilon} = \sup_{1\le l<k\le M_\varepsilon} \sup_{x\in R^1} |P_0(\lambda_{\varepsilon,l} < x, \lambda_{\varepsilon,k} < y) - \Phi_{\rho_{\varepsilon;l,k}}(x,y)| = O(\varepsilon^\delta) \tag{3.30}$$

where $\rho_{\varepsilon,l,k} = (\bar\pi_{\varepsilon,l}, \bar\pi_{\varepsilon,k})/\|\bar\pi_{\varepsilon,l}\|\|\bar\pi_{\varepsilon,k}\|$.

Proof. First for any $c > 0$ we can assume that $z^2_{\varepsilon,i} \le c\log\varepsilon^{-1}$. In fact, by (3.28)

$$E_0|\sum_{i\in I_{\varepsilon,c}} h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i})|^2 = \sum_{i\in I_{\varepsilon,c}} E_0(h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i}))^2 = \sum_{i\in I_{\varepsilon,c}} \|\pi_{\varepsilon,i}\|^2 = O(\varepsilon^{c\eta_1}).$$

Note the relation: for any $b \in (0,1)$ one can find such $B > 0$ that $|\log(1+x) - x| \le Bx^2$ for all $x \ge -b$. Using this relation we easily get:

$$\begin{aligned}
&E_0|\sum_{i\in I_{\varepsilon,c}} \log(1 + h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i}))| \\
&\le E_0|\sum_{i\in I_{\varepsilon,c}} h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i})| + B\sum_{i\in I_{\varepsilon,c}} \|\pi_{\varepsilon,i}\|^2 = O(\varepsilon^{c\eta_1/2}).
\end{aligned}$$

These relations and Chebyshev inequality imply the possibility of rejection of "tails". Put

$$\begin{aligned}
l^{(1)}_{\varepsilon,l} &= \sum_i \left(h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i}) - h^2_{\varepsilon,l,i}\xi(x_i, z^2_{\varepsilon,l,i})/2\right), \\
l^{(2)}_{\varepsilon,l} &= \sum_i \left(h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i}) - \|\pi_{\varepsilon,l,i}\|^2/2)\right), \\
\lambda^{(1)}_{\varepsilon,l} &= \|\bar\pi_{\varepsilon,l}\|^{-1}(l^{(2)}_{\varepsilon,l} + |\bar\pi_{\varepsilon,l}\|^2/2) = \|\bar\pi_{\varepsilon,l}\|^{-1}\sum_i h_{\varepsilon,l,i}\xi(x_i, z_{\varepsilon,l,i}).
\end{aligned}$$

Note that for any $b \in (0,1)$ one can find such $B > 0$ that for any $x \ge -b$

$$|\log(1+x) - x + x^2/2| \le B|x|^3.$$

By this inequality and by (3.21), (3.22) – (3.25) one has for some $B_1 > 0$ and small enough $c > 0$, $0 < \delta < \delta_0 - B_1 c$:

$$\begin{aligned}
E_0|l_{\varepsilon,l} - l^{(1)}_{\varepsilon,l}| &\le BE_0\sum_i h^3_{\varepsilon,l,i}|\xi(x_i, z_{\varepsilon,l,i})|^3 \le B_1\sum_i \|\pi_{\varepsilon,l,i}\|^3 \exp(B_1 z^2_{\varepsilon,l,i}) \\
&\le B_1(\sup_i \|\pi_{\varepsilon,l,i}\|)\, \varepsilon^{B_1 c}\sum_i \|\pi_{\varepsilon,l,i}\|^2 o(\varepsilon^\delta).
\end{aligned}$$

Also by analogous estimation for small enough $c > 0$

$$E_0(l^{(1)}_{\varepsilon,l} - l^{(2)}_{\varepsilon,l})^2 \le BE_0\sum_i h^4_{\varepsilon,l,i}\xi^4(x_i, z_{\varepsilon,l,i}) = o(\varepsilon^\delta).$$

By $E_0\lambda^{(1)}_{\varepsilon,l} = 0$, $E_0(\lambda^{(1)}_{\varepsilon,l})^2 = 1$, $E_0(\lambda^{(1)}_{\varepsilon,l}\lambda^{(1)}_{\varepsilon,k}) = \rho_{\varepsilon,l,k}$ the relations (3.29), (3.30) follow from analogous relations for $\lambda^{(1)}_{\varepsilon,l}$ and from one- and two-dimensional von Bahr-Essen inequalities:

$$\Delta^{(1)}_{1,\varepsilon} = \ \leq \ A(1)\frac{\sum_i h^3_{\varepsilon,l,i} E_0|\xi(x_i, z_{\varepsilon,l,i})|^3}{\|\pi_{\varepsilon,l}\|^3} = o(\varepsilon^\delta),$$

$$\Delta^{(1)}_{2,\varepsilon} = \ \leq \ A(2)\frac{\sum_i \left(h^3_{\varepsilon,l,i} E_0|\xi(x_i, z_{\varepsilon,l,i})|^3 \ + \ h^3_{\varepsilon,k,i} E_0|\xi(x_i, z_{\varepsilon,k,i})|^3\right)}{\left((1-|\rho_{\varepsilon,l,i}|)(\|\pi_{\varepsilon,l}\|^2 \ + \ \|\pi_{\varepsilon,k}\|^2)\right)^{3/2}} = o(\varepsilon^\delta),$$

where $A(l)$, $l = 1, 2$ are absolute constants. The Lemma 3.3 is proved.

3.3 Lower bounds for ellipsoids

To obtain lower bounds of the Theorem 1 we can assume that

$$u_\varepsilon(\Gamma) = \sqrt{2\log\log\varepsilon^{-1}} + O(1).$$

Fix small $\delta > 0$ and let $\Delta \subset K$ be such set that for any $\tau = (\kappa,\ R(\kappa))$, $\kappa \in \Delta$ one has

$$u_\varepsilon(\Gamma) \leq u_\varepsilon(\tau) < u_\varepsilon(\Gamma) + \delta. \tag{3.31}$$

At first assume $\Delta \subset \Xi_{G_1}$. It was shown in [11, sec. 3.1, 6.4, 6.5] that for any compact $K \subset \Xi_{G_1}$, $D \subset R^1_+$, for any $\Gamma \subset K \times D$, for any $b > 0$, for any $\tau = (\kappa, R) \in \Gamma$, $\rho_\varepsilon > 0$ such that $b < u_\varepsilon((\tau, \rho_\varepsilon) < \varepsilon^{-\delta}$ for small enough $\delta = \delta(K, c, C) > 0$ one can find such values $m = m_\varepsilon(\tau, \rho_\varepsilon)$, $z_0 = z_{0,\varepsilon}(\tau, \rho_\varepsilon)$ and such sequences $\bar\pi_\varepsilon = \bar\pi_\varepsilon(\tau, \rho_\varepsilon)$ of three-point measures $\pi_{i,\varepsilon} = \pi(h_{i,\varepsilon}(\tau, \rho_\varepsilon),\ z_{i,\varepsilon}(\tau, \rho_\varepsilon))$ that the following properties hold:

P1. Uniformly on $\tau \in \Gamma$ the relations (3.26), (3.27) are fulfilled.

P2. Uniformly on $\tau \in \Gamma$

$$\|\bar\pi_\varepsilon(\tau, \rho_\varepsilon)\| = u_\varepsilon(\tau, \rho_\varepsilon) + o(1);\ \pi^\varepsilon(V_\varepsilon(\tau, \rho_\varepsilon)) \to 1. \tag{3.32}$$

P3. Uniformly on $\tau \in \Gamma$ for some $\delta = \delta(\Gamma) > 0$

$$\begin{aligned} u^2_\varepsilon(\tau, \rho_\varepsilon) &= c_0(\kappa) m z_0^4 (1 + o(\varepsilon^\delta)); \\ c_1(\kappa) m^{1+pr} z_0^p &= (\rho_\varepsilon/\varepsilon)^p (1 + o(\varepsilon^\delta)), \\ c_2(\kappa) m^{1+qs} z_0^q &= (\varepsilon/R)^{-q} (1 + o(\varepsilon^\delta)). \end{aligned}$$

Here $c_l(\kappa)$, $l = 0, 1, 2$ are positive functions which are continuous and bounded away from 0 and ∞ on any compact $K \subset \Xi_{G_1}$.

These relations imply that uniformly on $\tau \in \Gamma$ one has:

$$m_\varepsilon(\tau, \rho_\varepsilon) = d_0(\kappa)\left(u_\varepsilon^{1/2}(\tau, \rho_\varepsilon) R/\varepsilon\right)^{\phi(\kappa)} (1 + o(\varepsilon^\delta)), \tag{3.33}$$

$$z_{0,\varepsilon}(\tau, \rho_\varepsilon) = d_1(\kappa) u_\varepsilon^{1/2}(\tau, \rho_\varepsilon)\left(u_\varepsilon^{1/2}(\tau, \rho_\varepsilon) R/\varepsilon\right)^{-\phi(\kappa)/4} (1 + o(\varepsilon^\delta)) \tag{3.34}$$

where

$$\phi(\kappa) = (s + 1/q - 1/4)^{-1} > 0, \ \kappa \in \Xi_{G_1}, \tag{3.35}$$

$d_l(\kappa), \ l = 0, 1$ are positive functions which are continuous and bounded away from 0 and ∞ on any compact $K \subset \Xi_{G_1}$.

The sequences $\bar{\pi}_\varepsilon(\tau, \rho_\varepsilon)$ correspond to solutions of special extreme problems (see[11, sec. 5.4] and the relation (5.60) in [11]; also sec. 4.2 later). The sequences $h_{i,\varepsilon}(\tau, \rho_\varepsilon) = h_i(m, z_0, \kappa), \ z_{i,\varepsilon}(\tau, \rho_\varepsilon) = z_i(m, z_0, \kappa)$ correspond to solutions of special systems of equations (see [11, sec. 6.2] and the relations (6.83), (6.84) in [11]) with, possibly, small decreasing R, increasing ρ_ε and with "rejection of tails" (replacing of $h_{i,\varepsilon}$ or $z_{i,\varepsilon}$ onto 0 for $i > n_\varepsilon^+$ and $i < n_\varepsilon^-$) to obtain the second relation of the property P2 (see [11, sec. 6.3.2 and 6.5.4]).

Let us consider two families of values $\tau_\varepsilon^{(l)} = \tau^{(l)} = (\kappa^{(l)}, R^{(l)}), \ , \rho_\varepsilon^{(l)}), \ l = 1, 2$ and corresponding families $m_\varepsilon^{(l)} = m_\varepsilon(\tau^{(l)}, \ \rho_\varepsilon^{(l)})$, $\bar{\pi}_\varepsilon^{(l)} = \bar{\pi}_\varepsilon(\tau^{(l)}, \rho_\varepsilon^{(l)})$. It was shown in [11, sec. 6.7.2, Proposition 6.4] that

P4. One can find such constants $\delta_l = \delta_l(\Gamma) > 0, \ l = 0, 1, 2, \ L_0 > 0, \ B > 0, \ b > 0$ that for $0 < \delta < \delta_0, \ L < L_0$, such $\tau_l \in \Gamma$ that $b < \bar{\pi}_\varepsilon^{(l)} < \varepsilon^{-\delta}, \ \|\kappa_1 - \kappa_2\| < L, \ m_\varepsilon^{(1)} < m_\varepsilon^{(2)}$ and for small enough $\varepsilon > 0$ the following inequality holds

$$\frac{(\bar{\pi}_\varepsilon^{(1)}, \bar{\pi}_\varepsilon^{(2)})}{\|\bar{\pi}_\varepsilon^{(1)}\| \|\bar{\pi}_\varepsilon^{(2)}\|} \leq B \left(\left(\frac{m_\varepsilon^{(1)}}{m_\varepsilon^{(2)}} \right)^{\delta_1} + \varepsilon^{\delta_2} \right). \tag{3.36}$$

Put

$$M_\varepsilon \asymp \log \varepsilon^{-1} / (\log \log \varepsilon^{-1}), \ \delta_\varepsilon = (\log \log \varepsilon^{-1})^b / \log \varepsilon^{-1}, \ b \in (0, 1).$$

By $M_\varepsilon \delta_\varepsilon = o(1)$, one can construct such collections

$$\Delta_\varepsilon = \{\tau_{1,\varepsilon}, ..., \tau_{M,\varepsilon}\} \subset \Gamma, \ \tau_{l,\varepsilon} = (\kappa_{l,\varepsilon}, R(\kappa_{l,\varepsilon})), \ \kappa_{l,\varepsilon} \in \Delta, \ l = 1, ..., M_\varepsilon$$

that

$$|\phi(\kappa_{l,\varepsilon}) - \phi(\kappa_{k,\varepsilon})| > \delta_\varepsilon, \ 1 \leq l < k \leq M_\varepsilon.$$

By choose M_ε

$$t_\varepsilon = \sqrt{2 \log M_\varepsilon} = \sqrt{2 \log \log \varepsilon^{-1}} + o(1).$$

Thus the assumption A2 and first part of the assumption A1 hold.

Denote $\bar{\pi}_{\varepsilon,l} = \bar{\pi}_\varepsilon(\tau_{l,\varepsilon}, \rho_\varepsilon(\tau_{l,\varepsilon})), \quad m_{l,\varepsilon} = m_\varepsilon(\tau_{l,\varepsilon}, \rho_\varepsilon(\tau_{l,\varepsilon}))$. Let $l \neq k, \ m_{l,\varepsilon} < m_{k,\varepsilon}$. It follows from (3.31) and (3.33) that for some $B > 0, \ B_1 > 0$ one has

$$m_{l,\varepsilon} / m_{k,\varepsilon} \leq B \varepsilon^{|\phi(\kappa_{l,\varepsilon}) - \phi(\kappa_{k,\varepsilon})|} \leq B \varepsilon^{\delta_\varepsilon} \leq B (\log \varepsilon^{-1})^{-B_1}. \tag{3.37}$$

Put

$$\rho_{\varepsilon,l,k} = \frac{(\bar{\pi}_{\varepsilon,l}, \bar{\pi}_{\varepsilon,k})}{\|\bar{\pi}_{\varepsilon,l}\| \|\bar{\pi}_{\varepsilon,k}\|}, \ 1 \leq l < k \leq M_\varepsilon.$$

Using (3.36), (3.37) we have for some $B_2 > 0$:

$$\sup_{1 \leq l < k \leq M_\varepsilon} \rho_{\varepsilon,l,k} \leq B (\log \varepsilon^{-1})^{-B_2} = o((\log M_\varepsilon)^{-1}) \tag{3.38}$$

which implies the second part of the assumption A1. Using Lemmas 3.2, 3.3 and the properties P1 – P2 of constructed sequences $\bar{\pi}_{\varepsilon,l}$ we obtain the lower bounds of the Theorem 1 for the case $\Delta \subset \Xi_{G_1}$.

Assume $\Delta \subset \Xi_{G_2}$. Also using results [11, sec. 3.1, 6.4, 6.5], as above, for any compact $K \subset \Xi_{G_2}$, $D \subset R^1_+$, for any $\Gamma \subset K \times D$, for any $b > 0$, any $\tau = (\kappa, R) \in \Gamma$, $\rho_\varepsilon > 0$ such that $b < u_\varepsilon((\tau, \rho_\varepsilon) < \varepsilon^{-\delta}$ for small enough $\delta = \delta(\Gamma) > 0$ one can find such values $n = n_\varepsilon(\tau, \rho_\varepsilon)$, $h_0 = h_{0,\varepsilon}(\tau, \rho_\varepsilon)$ and such sequences $\bar{\pi}_\varepsilon = \bar{\pi}_\varepsilon(\tau, \rho_\varepsilon)$ of three-point measures $\pi_{i,\varepsilon} = \pi(h_{i,\varepsilon}(\tau, \rho_\varepsilon),\ z_{i,\varepsilon}(\tau, \rho_\varepsilon))$ that uniformly on $\tau \in \Gamma$ the properties P1, P2 hold. The property P3 is replaced onto

P3a. Uniformly on $\tau \in \Gamma$ for some $\delta = \delta(\Gamma) > 0$

$$\begin{aligned} u^2_\varepsilon(\tau, \rho_\varepsilon) &= c_0(\kappa) n h_0^2 (1 + o(\varepsilon^\delta)); \\ c_1(\kappa) n^{1+pr} h_0 &= (\rho_\varepsilon/\varepsilon)^p (1 + o(\varepsilon^\delta)), \\ c_2(\kappa) n^{1+qs} h_0 &= (\varepsilon/R)^{-q} (1 + o(\varepsilon^\delta)). \end{aligned}$$

Here $c_l(\kappa)$, $l = 0, 1, 2$ are positive functions which are continuous and bounded away from 0 and ∞ on any compact in Ξ_{G_2}.

These relations imply that for some $\delta = \delta(\Gamma) > 0$

$$n_\varepsilon(\tau, \rho_\varepsilon) = d_0(\kappa) \left(u_\varepsilon^{1/q}(\tau, \rho_\varepsilon) R/\varepsilon \right)^{\phi(\kappa)} (1 + o(\varepsilon^\delta)), \tag{3.39}$$

$$h_{0,\varepsilon}(\tau, \rho_\varepsilon) = d_1(\kappa) u_\varepsilon(\tau, \rho_\varepsilon) \left(u_\varepsilon^{1/q}(\tau, \rho_\varepsilon) R/\varepsilon \right)^{-\phi(\kappa)/2} (1 + o(\varepsilon^\delta)) \tag{3.40}$$

where

$$\phi(\kappa) = (s + 1/2q)^{-1} > 0,\ \kappa \in \Xi_{G_2}, \tag{3.41}$$

$d_l(\kappa)$, $l = 0, 1$ are positive functions which are continuous and bounded away from 0 and ∞ on any compact $K \subset \Xi_{G_2}$.

Let $\bar{\pi}^{(l)}_\varepsilon = \bar{\pi}_\varepsilon(\kappa^{(l)}, R^{(l)}, \rho^{(l)}_\varepsilon)$, $m^{(l)}_\varepsilon = m_\varepsilon(\kappa^{(l)}, R^{(l)}, \rho^{(l)}_\varepsilon)$ $l = 1, 2$. It was shown in [11, sec. 6.7.2, Proposition 6.4] that

P4a. One can find such constants $\delta_l = \delta_l(\Gamma) > 0$, $l = 0, 1, 2$, $L_0 > 0$, $B > 0$, $b > 0$ that for $0 < \delta < \delta_0$, $L < L_0$, such $\tau_l \in \Gamma$ that $b < \bar{\pi}^{(l)}_\varepsilon < \varepsilon^{-\delta}$, $\|\kappa_1 - \kappa_2\| < L$, $m^{(1)}_\varepsilon < m^{(2)}_\varepsilon$ and for small enough $\varepsilon > 0$ the following inequality holds

$$\frac{(\bar{\pi}^{(1)}_\varepsilon, \bar{\pi}^{(2)}_\varepsilon)}{\|\bar{\pi}^{(1)}_\varepsilon\| \|\bar{\pi}^{(2)}_\varepsilon\|} \le B \left(\left(\frac{n^{(1)}_\varepsilon}{n^{(2)}_\varepsilon} \right)^{\delta_1} + \varepsilon^{\delta_2} \right). \tag{3.42}$$

Other considerations repeat the case $\Delta \subset \Xi_{G_1}$.

The n.2 of the Theorem 1 is proved.

3.4 Indistinguashability conditions for Besov bodies

To obtain the lower bounds of the Theorem 2 we can assume that

$$c_0 \le u^2_\varepsilon(\tau, \rho_\varepsilon(\tau)) \le c \log\log \varepsilon^{-1} \tag{3.43}$$

for small enough $c > 0$, $c_0 > 0$ and any $(\tau) \in \Delta$. We consider the cases $\Delta \subset \Xi_{G_1} \times D^3$ or $\Delta \subset \Xi_{G_2} \times D^3$.

Following to [11, sec. 7.3] for a value $B > 1$ let us consider families of pyramidal sequences of the three-point measures $\bar{\pi}_\varepsilon = \bar{\pi}_\varepsilon(\tau, \rho_\varepsilon, B) = \{\pi_{\varepsilon,i},\ i = (\iota, j) \in J\}$, $\tau = (\kappa, t, h, R)$:

$$\pi_{\varepsilon,\iota,j} = \begin{cases} \delta_0, & \text{if } j \neq j^* \\ (1-h_j)\delta_0 + h_j(\delta_{z_j} + \delta_{-z_j})/2, & \text{if } j = j^* \end{cases},\quad 1 \le \iota \le 2^j$$

where

$$\begin{cases} j^* = j_0,\ h_{j^*} = h_0,\ z_{j^*} = 1, & \text{if } \kappa \in \Xi_{G_2}; \\ j^* = j_1,\ h_{j^*} = 1,\ z_{j^*} = z_0, & \text{if } \kappa \in \Xi_{G_1}; \end{cases}$$

integers $j_0 = j_0(\tau, \rho_\varepsilon, B) \to \infty$, $j_1 = j_1(\tau, \rho_\varepsilon, B) \to \infty$ and the positive values $h_0 = h_0(\tau, \rho_\varepsilon, B) = o(1)$, $z_0 = h_0(\tau, \rho_\varepsilon, B) = o(1)$ are defined by the relations (later in this section asymptotic relations are uniform on any compact in Ξ_{G_1} or Ξ_{G_2})

$$\begin{cases} 2^{j_1(rp+1)} z_0^p \sim (B\rho_\varepsilon/\varepsilon)^p,\ 2^{j_1(sq+1)} z_0^q \sim (R/B\varepsilon)^q, & \text{if } \kappa \in \Xi_{G_1}; \\ 2^{j_1(rp+1)} h_0 \sim (B\rho_\varepsilon/\varepsilon)^p,\ 2^{(sq+1)} h_0 \sim (R/B\varepsilon)^q, & \text{if } \kappa \in \Xi_{G_2}. \end{cases}$$

In these cases

$$\|\bar{\pi}_\varepsilon(\tau, \rho_\varepsilon, B)\|^2 \asymp u_\varepsilon^2(\tau, \rho_\varepsilon) \asymp \begin{cases} 2^{j_0} z_0^4, & \text{if } \kappa \in \Xi_{G_1}, \\ 2^{j_1} h_0^2, & \text{if } \kappa \in \Xi_{G_2} \end{cases}$$

which imply

$$\begin{cases} j_0 \sim \phi_1(\kappa) \log_2 \varepsilon^{-1}, & \text{if } \kappa \in \Xi_{G_1}, \\ j_1 \sim \phi_2(\kappa) \log_2 \varepsilon^{-1}, & \text{if } \kappa \in \Xi_{G_2}. \end{cases} \tag{3.44}$$

These relations are analogous the properties P3, (3.33) and P3a, (3.39) with $m = 2^{j_1}$ or $n = 2^{j_0}$.

Also it follows from [11, sec. 7.3] that under assumption (3.43) for any $B > 1$

$$\pi^\varepsilon(V_\varepsilon(\tau, \rho_\varepsilon(\tau))) \to 1. \tag{3.45}$$

Put

$$M_\varepsilon \sim \log \varepsilon^{-1} / \log\log \varepsilon^{-1},\ \delta_\varepsilon = (\log\log \varepsilon^{-1})^b / \log \varepsilon^{-1}.$$

By $M_\varepsilon \delta_\varepsilon = o(1)$, for any $b \in (0,1)$ one can construct such collections

$$\Gamma_\varepsilon = \{\tau_{1,\varepsilon}, ..., \tau_{M,\varepsilon}\} \subset \Gamma,\ \tau_{l,\varepsilon} = (\kappa_{l,\varepsilon}, h(\kappa_{l,\varepsilon}), t(\kappa_{l,\varepsilon}), R(\kappa_{l,\varepsilon})),\ \kappa_{l,\varepsilon} \in \Delta,\ l = 1, ..., M_\varepsilon$$

that

$$|\phi(\kappa_{l,\varepsilon}) - \phi(\kappa_{k,\varepsilon})| > \delta_\varepsilon,\ 1 \le l < k \le M_\varepsilon$$

Put $\bar{\pi}_l = \bar{\pi}_\varepsilon(\tau_{l,\varepsilon}, \rho_\varepsilon(\tau_{l,\varepsilon}), B)$ with

$$j_l = \begin{cases} j_1(\tau_{l,\varepsilon}, \rho_\varepsilon(\tau_{l,\varepsilon})), & \text{if } \kappa \in \Xi_{G_1} \\ j_2((\tau_{l,\varepsilon}, \rho_\varepsilon(\tau_{l,\varepsilon})), & \text{if } \kappa \in \Xi_{G_2} \end{cases}$$

By choose $M_\varepsilon,\ t_\varepsilon$ and (3.44)

$$\min_{1\le l<k\le M_\varepsilon} |j_l - j_k| \asymp (\log\log\varepsilon^{-1})^b \to \infty.$$

By measures in $\bar\pi_l$ are supported on one level j_l, this relation implies

$$(\bar\pi_l, \bar\pi_k) = 0,\ \ 1 \le l < k \le M_\varepsilon$$

and one can choose such $c > 0$ in (3.43) that for small enough $\delta > 0$

$$\|\bar\pi_l\| \le (1-\delta)\log M_\varepsilon,\ \ 1 \le l \le M_\varepsilon.$$

Using (3.45) and the Lemma 3.1 we obtain the indistinguashability conditions of the Theorem 2, n.2.

4 Upper bounds for ellipsoids

4.1 Methods of constructions

We need to provide such families of tests $\psi_\varepsilon = \psi_{\varepsilon,\alpha}$ that

$$\alpha(\psi_{\varepsilon,\alpha}) \le \alpha + o(1);\ \ \beta(\psi_{\varepsilon,\alpha}, V_\varepsilon(\Gamma)) \le (1-\alpha)\Phi(H_\varepsilon) + o(1) \tag{4.1}$$

where $H_\varepsilon = \sqrt{2\log\log\varepsilon^{-1}} - u_\varepsilon(\Gamma)$.

It is enough to find such family ψ_ε that

$$\alpha(\psi_\varepsilon) \to 0;\ \ \beta(\psi_\varepsilon, V_\varepsilon(\Gamma)) \le \Phi(H_\varepsilon) + o(1) \tag{4.2}$$

by the relations (4.2) implies (4.1) for tests $\psi_{\varepsilon,\alpha} = \alpha + (1-\alpha)\psi_\varepsilon$. The constructed families will be unions of the collections of the tests $\{\psi_{\varepsilon,l},\ 1 \le l \le M\}$:

$$\psi_\varepsilon(x) = \max_{1\le l\le M} \psi_{\varepsilon,l}(x),\ \ M = M_\varepsilon \to \infty. \tag{4.3}$$

For the tests (4.3) one has

$$\alpha(\psi_\varepsilon) \le \sum_{1\le l\le M} \alpha(\psi_{\varepsilon,l});\ \ \beta(\psi_\varepsilon, v) \le \min_{1\le l\le M} \beta(\psi_{\varepsilon,l}, v),\ \ v \in l_2 \tag{4.4}$$

and to obtain (4.2) it is enough to construct such collections of the tests $\psi_{\varepsilon,l}$ and of the sets $\Gamma_l \subset \Gamma,\ \cup_{1\le l\le M}\Gamma_l = \Gamma$ that uniformly on $l,\ 1 \le l \le M$

$$\alpha(\psi_{\varepsilon,l}) = o(M^{-1}); \tag{4.5}$$

$$\beta(\psi_{\varepsilon,l}, V_\varepsilon(\Gamma_l)) \le \Phi(H_\varepsilon) + o(1). \tag{4.6}$$

We will consider (see [11, sec. 5.3]) the tests $\psi_{\varepsilon,l} = \psi(\bar h_{\varepsilon,l}, \bar z_{\varepsilon,l})$ of the type

$$\psi(\bar h, \bar z) = \mathbf{1}_{\{L_{\bar h,\bar z} > t\} \cup X_{\bar h,\bar z}},\quad t = t_\varepsilon = \sqrt{2\log M} \tag{4.7}$$

which are based on the statistics

$$L_{\bar h,\bar z}(x)=\|\bar\pi\|^{-1}\sum_i h_i\xi(x_i,z_i).$$

Here $\bar\pi=\bar\pi(\bar h,\bar z)=\{\pi_i\}$ is a sequence of three-point measures (3.15). The set $X_{\bar h,\bar z}=\{x:\ \sup_i|x_i|/T_i>1\}$ corresponds to the threshold procedure : for small enough $\delta>0$

$$\begin{cases} T_i=\infty & \text{if } \|\pi_i\|=0\ ,\\ T_i=\sqrt{(2+\delta)\Delta_i}\ ,\ \ \Delta_i=\log(\|\pi_i\|^{-2})-z_i^2(1-\delta) & \text{if } \|\pi_i\|>0 \end{cases} \tag{4.8}$$

(we assume $T_i=0$, if $\sup_i\|\pi_i\|\geq 1$ or $\Delta\leq 0$). These tests are defined by collections

$$\bar h_{\varepsilon,l}=\{h_{\varepsilon,i,l}\}, h_{\varepsilon,i,l}\in[0,1],\ \ \bar z_{\varepsilon,l}=\{z_{\varepsilon,i,l}\},\ \ z_{\varepsilon,i,l}\geq 0;\ \ 1\leq l\leq M. \tag{4.9}$$

Introduce families of sets

$$\Re_{\varepsilon,l}=\{i:\Delta_{\varepsilon,i,l}/9\leq z^2_{\varepsilon,i,l}\leq 9\Delta_{\varepsilon,i,l}\}$$

and put the assumption (see the assumption B4 in [11]):

A5. Uniformly on $l,\ 1\leq l\leq M$ for some families $n_{\varepsilon,l}\to\infty,\quad N_{\varepsilon,l}\to\infty$, $\log n_{\varepsilon,l}\asymp\log N_{\varepsilon,l}$, for some values $\delta\in(0,\delta_0)$ in (4.8), where $\delta_0>0$ is an (absolute) constant, any $i\in\Re_{\varepsilon,l}$, any $r=r(\tau),\ p=p(\tau),\ \tau\in\Gamma_l$ and small enough $\delta^{'}>0$ one has:

$$|\Delta_{\varepsilon,i,l}-\log N_{\varepsilon,l}|<\delta\Delta_{\varepsilon,i,l},\ \ N_{\varepsilon,l}^{-\delta^{'}}<i/n_{\varepsilon,l}<N_{\varepsilon,l}^{\delta^{'}},\ \ n^{rp}_{\varepsilon,l}N_\varepsilon^{1/2}=O((\rho_\varepsilon(\tau)/\varepsilon)^p)N_{\varepsilon,l}^{\delta^{'}})$$

(we denote $r=r(\tau),\ p=p(\tau)$ the components $r,\ p$ of the vector $\tau\in\Gamma$).

It follows from [11], the Corollary 5.2 that under assumptions A2, A3 and A4.1 or A4.2 joint with A5 on the collections (4.9) one has for small enough $\delta_1>0,\ \delta_2>0$

$$\alpha(\psi_{\varepsilon,l})\ \ =\ \ \Phi(-t_\varepsilon)+o(\varepsilon^{\delta_1})=o(M^{-1}), \tag{4.10}$$

$$\beta(\psi_{\varepsilon,l},V_\varepsilon(\Gamma_l))\ \ \leq\ \ \Phi(t_\varepsilon-\inf_{v\in V_\varepsilon^{'}(\Gamma_l)}(\bar\pi_{\varepsilon,l},\ \bar\delta_v)/\|\bar\pi_{\varepsilon,l}\|)+o(\varepsilon^{\delta_1}). \tag{4.11}$$

Here

$$V_\varepsilon^{'}(\Gamma_l)=\bigcup_{\tau\in\Gamma_l}V_\varepsilon(\tau,\rho_\varepsilon^{'}(\tau)),\ \ \rho_\varepsilon^{'}(\tau)=(1-n_\varepsilon^{-\delta_2})\rho_\varepsilon(\tau),\ \ \bar\delta_v=\{\delta_{v_i}\}. \tag{4.12}$$

Thus it is enough to construct such family of collections (4.9) that the assumptions A2, A3 and A4.1 or A4.2 joint with A5 hold and such family of partitions $\Gamma_l\subset\Gamma,\ \cup_{1\leq l\leq M}\Gamma_l=\Gamma$ that

$$\min_{1\leq l\leq M}\inf_{v\in V_\varepsilon^{'}(\Gamma_l)}(\bar\pi_{\varepsilon,l},\ \bar\delta_v)/\|\bar\pi_{\varepsilon,l}\|)\geq u_\varepsilon(\Gamma)+o(1). \tag{4.13}$$

4.2 The construction of collections of tests

To obtain the upper bounds we can assume

$$u_\varepsilon(\Gamma) = \inf_{\tau\in\Gamma} u_\varepsilon(\tau,\rho_\varepsilon(\tau)) \le \sup_{\tau\in\Gamma} u_\varepsilon(\tau,\rho_\varepsilon(\tau)) \le u_\varepsilon(\Gamma) + o(1) \sim \sqrt{2\log\log\varepsilon^{-1}}. \quad (4.14)$$

In fact, let $u_\varepsilon(\tau,\rho_\varepsilon(\tau)) > u_\varepsilon(\Gamma) + b,\ b > 0$. Then, by making $\rho_\varepsilon(\tau)$ smaller and by $u_\varepsilon(\tau,\rho_\varepsilon)$ is monotone and asymptotic continuous on ρ_ε (see [11, sec. 5, the Remarks 5.3 – 5.5]) we get the case $u_\varepsilon(\tau,\rho_\varepsilon(\tau)) = u_\varepsilon(\Gamma) + o(1)$ and obtain wider alternatives $V_\varepsilon(\tau,\rho_\varepsilon(\tau))$.

Let us consider partitions of Γ onto $M = M_\varepsilon$ sells Γ_l of following type. Let $\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)}$ where $\Gamma^{(k)} = \{\tau \in \Gamma : \ \kappa(\tau) \in K^{(k)}\}$ corresponds to subsets $K^{(k)} = K \cap \Xi_{G_k},\ k = 1, 2$. The partitions of $\Gamma^{(k)}$ correspond to partitions of $K^{(k)}$ by levels of the function $\phi(\kappa)$ with the step

$$\Delta^*_\varepsilon \asymp \frac{1}{(\log\log\varepsilon^{-1})^b \log\varepsilon^{-1}}$$

and to the partitions of the other parameters (p, q, r, R) with the step $\Delta_\varepsilon \asymp (\log\log\varepsilon^{-1})^{-b}$. These yield

$$M_\varepsilon \asymp \log\varepsilon^{-1}(\log\log\varepsilon^{-1})^B,\ B = 5b,$$

which implies the Assumption A2.

For $\tau \in \Gamma$ let us consider the radiuses $\rho'_\varepsilon(\tau)$ which are defined by (4.10) and the sequences $\{h_{\varepsilon,i}(\tau,\rho'_\varepsilon(\tau)),\ z_{\varepsilon,i}(\tau,\rho'_\varepsilon(\tau))\}$ which were constructed in [11, sec. 6] (also see sec. 3.2 above. The sequences $\bar\pi_\varepsilon(\tau,\rho_\varepsilon) = \{\pi(h_{\varepsilon,i}(\tau,\rho_\varepsilon),\ z_{\varepsilon,i}(\tau,\rho_\varepsilon))\}$ of the three-point measures (3.19) provide the minimum in extreme problem

$$u_\varepsilon(\tau,\rho_\varepsilon) = \|\bar\pi_\varepsilon(\tau,\rho_\varepsilon)\| = \inf_{\bar\pi\in\Pi_\varepsilon(\tau,\rho_\varepsilon)} \|\bar\pi\|$$

where

$$\Pi_\varepsilon(\tau,\rho_\varepsilon) = \{\bar\pi :\ \sum_i E_{\pi_i} i^{rp}|v|^p \ge (\rho_\varepsilon/\varepsilon)^p,\ \sum_i E_{\pi_i} i^{sq}|v|^q \le (R/\varepsilon)^q\};\ 0 < p, q < \infty$$

with evident modification for $q = \infty$. By the relation (see the Lemma 5.1 in [11])

$$\inf_{\bar\pi\in\Pi_\varepsilon(\tau,\rho_\varepsilon)} (\bar\pi_\varepsilon(\tau,\rho_\varepsilon),\bar\pi)/\|\bar\pi_\varepsilon(\tau,\rho_\varepsilon)\| = u_\varepsilon(\tau,\rho_\varepsilon)$$

and by the embedding

$$\{\bar\delta_v,\ v \in V_\varepsilon(\tau,\rho_\varepsilon)\} \subset \Pi_\varepsilon(\tau,\rho_\varepsilon)$$

these relations imply the inequality

$$\inf_{v\in V_\varepsilon(\tau,\rho_\varepsilon)} (\bar\pi_\varepsilon(\tau,\rho_\varepsilon),\ \bar\delta_v)/\|\bar\pi_\varepsilon(\tau,\rho_\varepsilon)\| \ge u_\varepsilon(\tau,\rho_\varepsilon). \quad (4.15)$$

Put

$$\bar{h}_{l,\varepsilon} = \{h_{\varepsilon,l,i}\},\ \bar{z}_{l\varepsilon} = \{z_{\varepsilon,l,i}\},\ \bar{\pi}_{\varepsilon,l} = \{\pi(h_{\varepsilon,l,i}, z_{\varepsilon,l,i})\}$$

where

$$h_{\varepsilon,l,i} = \sup_{\tau\in\Gamma_l} h_{\varepsilon,i}(\tau, \rho_\varepsilon^{'}(\tau)),\ z_{\varepsilon,l,i} = \sup_{\tau\in\Gamma_l} z_{\varepsilon,i}(\tau, \rho_\varepsilon^{'}(\tau)).$$

It is clear that for any $\tau \in \Gamma_l,\ v \in l_2$

$$(\bar{\pi}_{\varepsilon,l},\ \bar{\delta}_v) \geq (\bar{\pi}_\varepsilon(\tau, \rho_\varepsilon^{'}),\ \bar{\delta}_v) \geq u_\varepsilon(\tau, \rho_\varepsilon^{'})$$

and $u_\varepsilon(\tau, \rho_\varepsilon^{'}) = u_\varepsilon(\Gamma) + o(1)$ by continuous properties of the functions $u_\varepsilon(\tau, \rho_\varepsilon)$.

Thus it is enough to show that

$$u^*_{\varepsilon,l} = \|\bar{\pi}_{\varepsilon,l}\| \leq u_\varepsilon(\Gamma) + o(1) \tag{4.16}$$

and the collections of sequences $\bar{\pi}^*_{\varepsilon,l},\ 1 \leq l \leq M$ satisfy to the Assumptions A3 – A5.

Remind that the families $\bar{h}_\varepsilon(\tau, \rho_\varepsilon),\ \bar{z}_\varepsilon(\tau, \rho_\varepsilon)$ are defined by families

$$m_\varepsilon(\tau, \rho_\varepsilon),\ z_{0,\varepsilon}(\tau, \rho_\varepsilon);\ \kappa \in \Xi_{G_1}$$

or by families

$$n_\varepsilon(\tau, \rho_\varepsilon),\ z_{0,\varepsilon}(\tau, \rho_\varepsilon),\ \kappa \in \Xi_{G_2}$$

and by functions $c_l(\kappa),\ l = 0, 1, 2$ with the properties P3 and P3a (see sec. 3.3). Also it follows from the representation of these functions (see [11], the relations (6.100), (6.127)) that $c_l(\kappa),\ l = 0, 1, 2$ are uniformly continuous and smooth on any compact K which has not intersection with sub-manifolds $\{I = 0\},\ \{p = q\},\ \{p = 2\}$ and $\{\Delta = 0\}$. This implies Lipchisian properties of the functions $d_l(\kappa),\ l = 0, 1$ in the relations (3.33) - (3.35), (3.39) - (3.41).

By these relations one has the relations for the values

$$m_\varepsilon^{(k)} = m_\varepsilon(\tau_k, \rho_\varepsilon^{'}(\tau_k)),\ z_{0,\varepsilon}^{(k)} = z_{0,\varepsilon}(\tau_k, \rho_\varepsilon^{'}(\tau_k)),\ k = 1, 2:$$

uniformly on $\tau_1,\ \tau_2 \in \Gamma^{(1)}$ for some $\delta > 0$

$$\begin{aligned}
\frac{m_\varepsilon^{(1)}}{m_\varepsilon^{(2)}} &= \frac{d_0(\kappa_1)}{d_0(\kappa_2)} \frac{(u_{\varepsilon,1}^{1/2} R_1)^{\phi(\kappa_1)}}{(u_{\varepsilon,2}^{1/2} R_2)^{\phi(\kappa_2)}} \varepsilon^{\phi(\kappa_2)-\phi(\kappa_1)}(1 + o(\varepsilon^\delta)),\\
\frac{z_{0,\varepsilon}^{(1)}}{z_{0,\varepsilon}^{(2)}} &= \frac{d_1(\kappa_1)}{d_1(\kappa_2)} \frac{u_{\varepsilon,1}^{1/2}(u_{\varepsilon,1}^{1/2} R_1)^{-\phi(\kappa_1)/4}}{u_{\varepsilon,2}^{1/2}(u_{\varepsilon,2}^{1/2} R_2)^{-\phi(\kappa_2)/4}} \varepsilon^{(\phi(\kappa_1)-\phi(\kappa_2))/4}(1 + o(\varepsilon^\delta));
\end{aligned}$$

or for the values $n_\varepsilon^{(k)} = n_\varepsilon(\tau_k, \rho_\varepsilon^{'}(\tau_k)),\ h_{0,\varepsilon}^{(k)} = h_{0,\varepsilon}(\tau_k, \rho_\varepsilon^{'}(\tau_k)),\ k = 1, 2$: uniformly on $\tau_1,\ \tau_2 \in \Gamma^{(2)}$ for some $\delta > 0$

$$\begin{aligned}
\frac{n_\varepsilon^{(1)}}{n_\varepsilon^{(2)}} &= \frac{d_0(\kappa_1)}{d_0(\kappa_2)} \frac{(u_{\varepsilon,1}^{1/q_1} R_1)^{\phi(\kappa_1)}}{(u_{\varepsilon,2}^{1/q_2} R_2)^{\phi(\kappa_2)}} \varepsilon^{\phi(\kappa_2)-\phi(\kappa_1)}(1 + o(\varepsilon^\delta)),\\
\frac{h_{0,\varepsilon}^{(1)}}{h_{0,\varepsilon}^{(2)}} &= \frac{d_1(\kappa_1)}{d_1(\kappa_2)} \frac{u_{\varepsilon,1}(u_{\varepsilon,1}^{1/q_1} R_1)^{-\phi(\kappa_1)/2}}{u_{\varepsilon,2}(u_{\varepsilon,2}^{1/q_2} R_2)^{-\phi(\kappa_2)/2}} \varepsilon^{(\phi_2(\kappa_1)-\phi_2(\kappa_2))/2}(1 + o(\varepsilon^\delta));
\end{aligned}$$

here $u_{\varepsilon,k} = u_{\varepsilon,1}(\tau_k, \rho_\varepsilon'(\tau_k))$, the function $\phi(\kappa)$ is defined by (3.35) and (3.41).

By the choose of partitions one has

$$\sup_{\tau_k \in \Gamma_l,\ k=1,2} |r_1 - r_2| + |s_1 - s_2| + |p_1 - p_2| + |q_1^{-1} - q_2^{-1}| + |R_1 - R_2| = O(\Delta_\varepsilon) \quad (4.17)$$

Also by the relations above, by (4.14) and by Lipchisian properties of the functions $d_l(\kappa),\ l = 0, 1$ one has for $\Gamma_l \subset \Gamma^{(1)}$

$$\sup_{\tau_k \in \Gamma_l,\ k=1,2} |m_\varepsilon^{(1)}/m_\varepsilon^{(2)} - 1| + |\tilde{z}_{0,\varepsilon}^{(1)}/z_{0,\varepsilon}^{(2)} - 1| = O((\log\log\varepsilon^{-1})^{-b}) \quad (4.18)$$

and for $\Gamma_l \subset \Gamma^{(2)}$

$$\sup_{\tau_k \in \Gamma_l,\ k=1,2} |n_\varepsilon^{(1)}/n_\varepsilon^{(2)} - 1| + |\tilde{h}_{0,\varepsilon}^{(1)}/h_{0,\varepsilon}^{(2)} - 1| = O((\log\log\varepsilon^{-1})^{-b}). \quad (4.19)$$

It follows from [11, Proposition 6.3] that if K has not intersection with submanifolds $\{I = 0\}$, $\{p = q\}$, $\{p = 2\}$ and $\{\Delta = 0\}$, then the relations (4.17) – (4.19) imply for $b > 1/2$

$$u_{\varepsilon,l}^* = u_\varepsilon(V_l)(1 + O((\log\log\varepsilon^{-1})^{-b}) = u_\varepsilon(\Gamma) + o(1)$$

which imply (4.16). Also using the Propositions 6.1, 6.2 and by the Corollaries 6.1, 6.2 in [11] (where the asymptotics of the sequences $\bar{h}_\varepsilon(\tau),\ \bar{h}_\varepsilon(\tau)$ are described) analogously to the estimations in [11, sec. 6.3.2, sec. 6.5.4] one can check that the Assumptions A2, A3 and A4.1 (for $p \le q$) or A4.2 joint with A5 (for $p > q$) hold.

Thus the upper bounds of the Theorem 1, n. 1 are proved.

5 Upper bounds for Besov bodies

5.1 Test procedure

For Besov bodies case we can assume $u_\varepsilon(\Gamma)/\sqrt{\log\log\varepsilon^{-1}} > C + o(1)$ for large enough $C = C(\Gamma) > 0$. Under this assumption wee need to construct such family of tests ψ_ε that

$$\alpha(\psi_\varepsilon) \to 0;\ \beta(\psi_\varepsilon, V_\varepsilon(\Gamma)) \to 0. \quad (5.1)$$

We assume later without loss of generality that uniformly on $\tau \in \Gamma$

$$u_\varepsilon(\tau, \rho_\varepsilon(\tau)) \sim C\sqrt{\log\log\varepsilon^{-1}}. \quad (5.2)$$

Using standard embedding properties we can assume that $t = \infty$ and $h \le p$. For this case the asymptotics of the values $u_\varepsilon(\tau, \rho_\varepsilon(\tau))$ have been studied in [11, sec. 7]. We may use methods analogous to previous section which are based on small enough partitions of Γ onto sells $\{\Gamma_l\}$ and on using of unions of tests for all sells. However we prefer to give direct constructions.

First, put

$$\psi_{\varepsilon,0} = \mathbf{1}_{X_{\varepsilon,0}},\ X_{\varepsilon,0} = \{\max_j \max_{1\le i\le 2^j} \{|x_{ij}| > T_{\varepsilon,j}\} \tag{5.3}$$

where

$$T_{\varepsilon,j} = \begin{cases} \sqrt{2(\log 2) J_{\varepsilon,0}} & \text{if } j < J_{\varepsilon,0}, \\ \sqrt{2(\log 2) j + 2\log j}, & \text{if } j \ge J_{\varepsilon,0} \end{cases},\ \ J_{\varepsilon,0} \asymp \log\log \varepsilon^{-1}.$$

Note that these tests provide distinguishability for the region D of degenerate type (see sec. 1.2 and [11, Theorem 7]).

Next, for $j \ge J_{\varepsilon,0}$ let us consider statistics

$$l_j = 2^{-(j+1)/2} \sum_{\iota=1}^{2^j} (x_{\iota j}^2 - 1)$$

and note that

$$E_0 l_j \;=\; 0,\ E_0 l_j^2 = 1,\ E_0 l_j^4 = O(1); \tag{5.4}$$

$$E_v l_j \;=\; 2^{-(j+1)/2} \sum_{\iota=1}^{2^j} v_{\iota j}^2,\ Var_v l_j \le 1 + 2^{-(j-1)} \sum_{\iota=1}^{2^j} v_{\iota j}^2. \tag{5.5}$$

Put

$$\psi_{\varepsilon,1} = \mathbf{1}_{X_{\varepsilon,1}},\ X_{\varepsilon,1} = \{\max_{J_{\varepsilon,0}\le j} l_j/T_j > 1\},\ T_j = 2\sqrt{\log j}. \tag{5.6}$$

At last, fix a value $c \in (0, \log_2 2/C_2(2))$ where $C_2(2)$ is the constant from (3.25) and put

$$K = K(j) = (\log\log j)/2,\ K(c,j) = K + cj^{1/2}.$$

For levels

$$j:\ \ J_{\varepsilon,0} \le j \le J_{\varepsilon,1} \asymp (\log\log \varepsilon^{-1}) \log \varepsilon^{-1}$$

let us consider collections of statistics

$$l_{j,k} = \sum_{\iota=1}^{2^j} l_{\iota;j,k} = (2^{j-1} \sinh^2(z_{j,k}^2/2))^{-1/2} \sum_{\iota=1}^{2^j} \xi(x_{\iota j}, z_{j,k}),\ 1 \le k \le K(c,j)$$

where

$$z_{j,k} = \begin{cases} e^{k-1}/\sqrt{\log j}, & 1 \le k \le K \\ \sqrt{k-K}, & K < k \le K(c,j) \end{cases}$$

which correspond to normalized sequences of measures

$$\bar\pi_{j,k} = \bar\pi_j(z_{j,k}) = \{\pi_{\iota j}(z_{j,k}), 1 \le \iota \le 2^j\},$$

supported on the level j:

$$\pi_{\iota j}(z) = \pi(h(z,j), z),\ h(z,j) = (2^{j+1} \sinh^2(z^2/2))^{-1/2};\ \|\bar\pi_j(z)\| = 1. \tag{5.7}$$

Put

$$\psi_{\varepsilon,j,k} = \mathbf{1}_{l_{j,k}>t_j}, \ t_j = \sqrt{2\log j}.$$

We consider the tests

$$\psi_\varepsilon = \max\{\psi_{\varepsilon,0}, \ \psi_{\varepsilon,1}, \ \max_{J_{\varepsilon,0}\le j\le J_{\varepsilon,1}} \max_{1\le k\le K(c,j)} \psi_{\varepsilon;j,k}\}. \tag{5.8}$$

Remark 5.1. The structure of the tests (5.8) is close to adaptive tests which have been proposed by Spokoiny [17]. Particularly, the tests $\psi_{\varepsilon,0}, \ \psi_{\varepsilon,1}$ are analogous to the tests $\phi_{n,\infty}, \ \phi_{n,2}$; $n^{-1/2} = \varepsilon$ in [17]. The main difference is that we use the tests $\max_{1\le k\le K(c,j)} \psi_{\varepsilon;j,k}$ in place of the tests $\phi_{n,p,j}$ and $\max_k \phi_{n,p,\lambda_k,j}$ which are based on the statistics $\sum_\iota |x_{\iota j}|^p$ and $\sum_\iota(|x_{\iota j}|^p \mathbf{1}(|x_{\iota j}| > \lambda_k)$ for fixed $p > 2$.

Remark 5.2. It follows from the considerations later that we can use the tests $\psi_{\varepsilon,0}, \ \psi_{\varepsilon,1}$ only, if it is known that $\kappa \in \Xi_{G_1}$. Also, it follows from the Remark 5.3 to the Proposition 5.1 later that if it is known that $s > r$ for $p > q$ or $s+1/q > r+1/p$ for $p \le q$, then we can consider only finite number $j \le J_{\varepsilon,1}$ in the tests $\psi_{\varepsilon,0}, \ \psi_{\varepsilon,1}$ also.

5.2 First kind errors

To estimate first kind errors note that

$$\alpha(\psi_\varepsilon) \le \alpha(\psi_{\varepsilon,0}) + \alpha(\psi_{\varepsilon,1}) + \sum_{J_{\varepsilon,0}\le j\le J_{\varepsilon,1}} \sum_{1\le k\le K(c,j)} \alpha(\psi_{\varepsilon;j,k}). \tag{5.9}$$

First, one can see that

$$\alpha(\psi_{\varepsilon,0}) \le \sum_{j\ge J_{\varepsilon,0}} 2^{j+1}\Phi(-T_j) \asymp \sum_{j\ge J_{\varepsilon,0}} \frac{1}{j(\log j)^{3/2}} = o(1). \tag{5.10}$$

Next, using Gaussian approximation, Bahr-Essen inequality and relations (5.5), one can easily check that

$$\alpha(\psi_{\varepsilon,1}) \le \sum_{j\ge J_{\varepsilon,0}} \left(2\Phi(-2\sqrt{\log j}) + O(2^{-j/2})\right) = o(1). \tag{5.11}$$

At last, note that $E_0 l_{j,k} = 0, \ Var_0 l_{j,k} = \|\bar{\pi}_{j,k}\|^2 = 1$ by (3.22), (3.25) and by $d = 1 - Bc/\log 2 > 0, \ B = C_2(2)$, we get:

$$R_{j,k} = \sum_\iota E_0 l^4_{\iota;j,k} = O\left(2^{-j}\exp(Bcj)\right) = O(2^{-jd})$$

which yield the asymptotic normality of statistics $l_{j,k}$ and estimations of the accuracy in Bahr-Essen inequality. Using this inequality, we get:

$$\sum_{J_{\varepsilon,0}\le j\le J_{\varepsilon,1}} \sum_{1\le k\le K(c,j)} P_0(l_{j,k} > t_j) \le \sum_{J_{\varepsilon,0}\le j\le J_{\varepsilon,1}} \sum_{1\le k\le K(c,j)} (2\Phi(-t_j) + O(R_{j,k})) = o(1).$$

These relations and (5.9) imply $\alpha(\psi_\varepsilon) = o(1)$.

5.3 Second kind errors for particular tests

To obtain the second relation in (5.1) under assumption (5.2), for any family $\tau = \tau_\varepsilon \in \Gamma,\ v = v_\varepsilon \in V_\varepsilon = V_\varepsilon(\tau_\varepsilon, \rho_\varepsilon(\tau_\varepsilon))$ one needs to find such $l = l_\varepsilon(v_\varepsilon),\ l = 0,1$ or $l = (j,k),\ j = j_\varepsilon,\ k = k_\varepsilon,\ J_{\varepsilon,0} \le j \le J_{\varepsilon,1},\ 1 \le k \le K(c,j)$ that

$$\beta(\psi_{\varepsilon,l}, v_\varepsilon) \to 0. \tag{5.12}$$

First, let us consider such families $v_\varepsilon \in l_2\ \limsup_{\iota,j} |v_{\varepsilon,\iota,j}|/T_{\varepsilon,j} > 1$ (we assume later without loss of generality that there exist all limits under consideration as $\varepsilon \to \infty$.) One can easily see that for all such families that

$$\beta(\psi_{\varepsilon,0}(v_\varepsilon)) \le \inf_{\iota,j} \Phi(T_{\varepsilon,j} - v_{\varepsilon,\iota,j}) = o(1).$$

Next, let us consider such families $v_\varepsilon \in l_2$ that

$$\lim \sup_{j \ge J_{\varepsilon,0}} H_j(v_\varepsilon)/T_j > 1;\ H_j(v) = E_v l_j = 2^{-(j+1)/2} \sum_{\iota=1}^{2^j} v_{\iota j}^2.$$

Using Chebyshev inequality and relations (5.5) one can easily see that for all such families

$$\beta(\psi_{\varepsilon,1}(v_\varepsilon)) \le \inf_j \frac{1 + 2^{-(j-3)/2} H_j(v_\varepsilon)}{(H_j(v_\varepsilon) - T_j)^2} = o(1).$$

At last, for any $(j,k):\ J_{\varepsilon,0} \le j \le J_{\varepsilon,0},\ 1 \le k \le K(c,j)$, let us consider families

$$v_\varepsilon \in l_2 :\ \lim\sup_{\iota,j} |v_{\varepsilon,\iota,j}|/T_{\varepsilon,j} \le 1,\ \lim H_{j,k}(v_\varepsilon)/t_j > 1$$

where

$$H_{j,k}(v) = E_v l_{j,k} = (\bar\pi_{j,k}, \bar\delta_v) = 2h_{j,k} \sum_\iota \sinh^2(v_{\iota j} z_{j,k}/2).$$

Using the relations (3.23), the equality

$$e^{x^2} - 1 = 2e^{x^2} \sinh^2(x^2/2)$$

and inequality

$$\sinh^2 x \le e^x \sinh^2(x/2)$$

one can see that for all such families

$$\begin{aligned} Var_v l_{j,k} &\le 1 + h_{j,k} \sinh^2(z_{j,k}^2/2) \exp(z_{j,k} T_{\varepsilon,j,0} + z_{j,k}^2) H_{j,k}(v) \\ &\le 1 + 2^{-j/2} \exp(z_{j,k} T_{\varepsilon,j,0} + z_{j,k}^2/2) H_{j,k}(v). \end{aligned}$$

For small enough $c > 0$ by the inequality

$$z_{j,k} T_{\varepsilon,j,0} + z_{j,k}^2/2 \le ((2\log 2)^{1/2} c + c^2/2) j < (\log 2/2) j$$

and using Chebyshev inequality, we get:

$$\beta(\psi_{\varepsilon;j,k}(v_\varepsilon)) \le Var_v l_{j,k}/(H_{j,k}(v_\varepsilon) - t_j)^2 = o(1)$$

for all families v_ε under consideration.

Note that the set $X_\varepsilon = \{x : \psi_\varepsilon(x) = 0\}$ is convex and symmetrical on all coordinates $x_{\iota j}$. By Anderson's lemma (see Ibragimov and Khasminskii [5]) this implies the inequality

$$\beta(\psi_\varepsilon(v)) \le \beta(\psi_\varepsilon(\tilde{v})),$$

where

$$\tilde{v} = \tilde{v}(J_\varepsilon);\ \tilde{v}_{\iota j} = \begin{cases} v_{\iota j}, & \text{if } (\iota j) \in J_\varepsilon \\ 0, & \text{if } (\iota j) \notin J_\varepsilon \end{cases},$$

for any fixed family of subsets $J_\varepsilon \subset J = \{(\iota j)\}$.

The estimation above show that to obtain (5.12) it is enough to establish the following

Lemma 5.1 *Under assumptions above one can find such constant $C = C(\Gamma) > 0$ in (5.2) that for all families $\tau_\varepsilon \in \Gamma$, all families v_ε:*

$$v_\varepsilon \in V_\varepsilon = V_\varepsilon(\tau_\varepsilon, \rho_\varepsilon(\tau_\varepsilon)) : \limsup_{\iota j} |v_{\iota j}|/T_{\varepsilon,j} \le 1 \tag{5.13}$$

there exist values $\delta_\varepsilon = o(1)$ and sets $J_\varepsilon \subset \{J_{\varepsilon,0} \le j \le J_{\varepsilon,1}\}$ such that $\tilde{v}_\varepsilon = \tilde{v}_\varepsilon(J_\varepsilon) \in \tilde{V}_\varepsilon = V_\varepsilon(\tau_\varepsilon, \tilde{\rho}_\varepsilon(\tau_\varepsilon))$ with $\tilde{\rho}_\varepsilon = (1-\delta_\varepsilon)\rho_\varepsilon$ and either

$$\limsup_{j} H_j(\tilde{v}_\varepsilon)/T_j = \limsup_{j \in J_\varepsilon} H_j(v_\varepsilon)/T_j > 1; \quad J_\varepsilon \subset \{j \ge J_{\varepsilon,0}\} \tag{5.14}$$

or

$$\limsup_{j} \sup_{1 \le k \le K(c,j)} H_{j,k}(\tilde{v}_\varepsilon)/t_j = \limsup_{j \in J_\varepsilon} \sup_{1 \le k \le K(c,j)} H_{j,k}(v_\varepsilon)/t_j > 1. \tag{5.15}$$

5.4 Proof of the Lemma 5.1

Let $\tilde{u}_\varepsilon$ be the values $u_\varepsilon(\tau_\varepsilon, \tilde{\rho}_\varepsilon(\tau_\varepsilon))$ corresponding to $\tilde{\rho}_\varepsilon(\tau) = (1-\delta_\varepsilon)\rho_\varepsilon(\tau)$; values $\delta_\varepsilon = o(1)$ will be defined concretely below. It follows from investigation of extreme problem in [11, sec. 7] [1] that under assumptions (5.2) and for $t = \infty,\ 0 < h \le p$ one has:

$$u_\varepsilon(\tau, \tilde{\rho}_\varepsilon) \sim u_\varepsilon(\tau, \rho_\varepsilon)$$

uniformly on $\tau \in \Gamma$ which imply

$$\tilde{u}_\varepsilon \sim u_\varepsilon = u_\varepsilon(\tau_\varepsilon, \rho_\varepsilon(\tau_\varepsilon)).$$

Remind that for $t = \infty,\ h \le p$

$$u_\varepsilon(\tau, \rho_\varepsilon) \ = \ \|\bar{\pi}_\varepsilon(\tau, \rho_\varepsilon)\| \ = \ \inf_{\bar{\pi} \in |\Pi_\varepsilon(\tau, \rho_\varepsilon)} \|\bar{\pi}\|$$

[1] To simplicity the only the case $p \ne q$ is considered in [11, sec 7.1]. For the case $p = q$ one can easily see that all necessary hold true also.

where

$$\Pi_\varepsilon(\tau,\rho_\varepsilon)=\{\bar\pi:\sum_j(2^{rhj}\sum_\iota E_{\pi_{\iota j}}|v|^p)^{h/p}\ge(\rho_\varepsilon/\varepsilon)^h,\ \sup_j 2^{sqj}\sum_\iota E_{\pi_{\iota j}}|v|^q\le(R/\varepsilon)^q\}.$$

Also (see [11, sec. 7])

$$\pi_{\iota j;\varepsilon}(\tau,\rho_\varepsilon)=\pi(h_j((\tau,\rho_\varepsilon),z_j((\tau,\rho_\varepsilon))$$

are three-point measures and there exist such values

$$z_0=z_{0,\varepsilon}(\tau,\rho_\varepsilon),j_1=j_{1,\varepsilon}(\tau,\rho_\varepsilon),\ \text{if}\ \kappa\in\Xi_{G_1}$$

or

$$h_0=h_{0,\varepsilon}(\tau,\rho_\varepsilon),j_0=j_{0,\varepsilon}(\tau,\rho_\varepsilon),\ \text{if}\ \kappa\in\Xi_{G_2}$$

and functions $d_l(\tau),\ l=0,1,2$ which are bounded away from 0 and ∞ that for $\kappa\in\Xi_{G_1}$

$$u^2_\varepsilon(\tau,\rho_\varepsilon)\sim d_0(\tau)z_0^4 2^{j_1},\left(\frac{\rho_\varepsilon}{\varepsilon}\right)^p\sim d_1(\tau)z_0^p2^{(1+rp)j_1},\left(\frac{R}{\varepsilon}\right)^q\sim d_2(\tau)z_0^q2^{(1+sq)j_1}\quad(5.16)$$

or for $\kappa\in\Xi_{G_2}$

$$u^2_\varepsilon(\tau,\rho_\varepsilon)\sim d_0(\tau)h_0^2 2^{j_0},\left(\frac{\rho_\varepsilon}{\varepsilon}\right)^p\sim d_1(\tau)h_02^{(1+rp)j_0},\left(\frac{R}{\varepsilon}\right)^q\sim d_2(\tau)h_02^{(1+sq)j_0}.\quad(5.17)$$

Note that the relations (5.16), (5.17) and the assumption (5.2) imply that uniformly on $\tau\in\Gamma$

$$j^*_\varepsilon\asymp\log\varepsilon^{-1},\ j^*_\varepsilon=\begin{cases}j_1, & \text{if}\ \kappa_\varepsilon\in\Xi_{G_1}\\ j_0, & \text{if}\ \kappa_\varepsilon\in\Xi_{G_2}\end{cases}.\quad(5.18)$$

Also if $\kappa_\varepsilon\in\Xi_{G_1}$, then $r+1/p-1/4>0,\ s+1/q-1/4>0$ and

$$(\rho_\varepsilon/\varepsilon)\asymp u_\varepsilon^{1/2}2^{j_1(r+1/p-1/4)},\ (R/\varepsilon)\asymp u_\varepsilon^{1/2}2^{j_1(s+1/q-1/4)};\quad(5.19)$$

if $\kappa_\varepsilon\in\Xi_{G_2}$, then $r+1/2p>0,\ s+1/2q>0$ and

$$(\rho_\varepsilon/\varepsilon)\asymp u_\varepsilon^{1/p}2^{j_1(r+1/2p)},\ (R/\varepsilon)\asymp u_\varepsilon^{1/q}2^{j_1(s+1/2q)}.\quad(5.20)$$

Let $\tilde\pi_\varepsilon=\{\tilde\pi_{\varepsilon,j}\}=\bar\pi_\varepsilon(\tau_\varepsilon,\tilde\rho_\varepsilon)$ be the family of sequences which corresponds to $\tilde\rho_\varepsilon$:

$$\|\tilde\pi_\varepsilon\|\sim u_\varepsilon(\tau_\varepsilon,\rho_\varepsilon)\sim C\sqrt{\log\log\varepsilon^{-1}}.\quad(5.21)$$

One can easily see (compare with Lemma 5.1 in [11] and sec. 4.2 above)

$$\|\bar\pi_\varepsilon(\tau,\rho_\varepsilon)\|^{-1}\inf_{v\in V_\varepsilon(\tau,\rho_\varepsilon)}(\bar\pi_\varepsilon(\tau,\rho_\varepsilon),\bar\delta_v)\ge\|\bar\pi_\varepsilon(\tau,\rho_\varepsilon)\|\quad(5.22)$$

which implies that for all $\tilde v=\tilde v_\varepsilon\in\tilde V_\varepsilon$

$$\|\tilde\pi_\varepsilon\|^{-1}\sum_j 2h_{\varepsilon,j}\sum_\iota\sinh^2(z_{\varepsilon,j}\tilde v_{\iota j}/2)=\sum_j r_{\varepsilon,j}(\bar\pi_{\varepsilon,j},\bar\delta_{\tilde v_\varepsilon})\ge C\sqrt{\log\log\varepsilon^{-1}}(1+o(1))\quad(5.23)$$

where

$$r_{\varepsilon,j}^2 = \frac{2^j \|\bar{\pi}_{\varepsilon,j}\|^2}{\|\tilde{\pi}_\varepsilon\|^2} = \frac{2^j h_{\varepsilon,j}^2 \sinh^2(z_{\varepsilon,j}^2/2)}{\sum_j 2^j h_{\varepsilon,j}^2 \sinh^2(z_{\varepsilon,j}^2/2)};\ \sum_j r_{\varepsilon,j}^2 = 1$$

and $\bar{\pi}_{\varepsilon,j} = \bar{\pi}_j(z_{\varepsilon,j}),\ \|\bar{\pi}_{\varepsilon,j}\| = 1$ are normalized sequences defined by (5.7).

It follows from the study of extreme problem in [11] (see [11], Proposition 7.1, where asymptotics of the sequences $h_{\varepsilon,j} = h_{\varepsilon,j}(\tau, \rho_\varepsilon),\ z_{\varepsilon,j} = z_{\varepsilon,j}(\tau, \rho_\varepsilon)$ have been described, and [11, sec. 7.2] where asymptotics (5.16) and (5.16) are established) that for all $\tau_\varepsilon \in \Gamma$ there exist values $a = a(\Gamma) > 0$ and $b = b(\Gamma) > 0$ such that for small enough ε

$$r_{\varepsilon,j} \leq a 2^{-b|j-j_\varepsilon^*|} \tag{5.24}$$

where j_ε^* is defined by (5.18). The relation (5.24) implies that

$$\sum_j r_{\varepsilon,j}(1 + |j - j_\varepsilon^*|) \leq B = B(\Gamma)$$

and using (5.23) one has:

$$\sup_j(\bar{\pi}_{\varepsilon,j}, \bar{\delta}_{\bar{v}_\varepsilon}))/(1 + |j - j_\varepsilon^*|) \geq C_1\sqrt{\log\log\varepsilon^{-1}}(1 + o(1));\ C_1 = C/B. \tag{5.25}$$

We can assume later without loss of generality that $\kappa = \kappa(\tau_\varepsilon) \in \Xi_{G_1}$ or $\kappa = \kappa(\tau_\varepsilon) \in \Xi_{G_2}$ and consider differently these cases. Also without loss of generality we can assume that various necessary relations later between parameters p, q, r, s hold true for all small enough $\varepsilon > 0$.

The following simple proposition will be used later.

Proposition 5.1 *For a set $J_\varepsilon \subset J$ denote*

$$a_j^p = 2^{rpj} \sum_\iota |v_{\iota j}|^p,\ D_\varepsilon(J_\varepsilon) = \sum_{j \in J_\varepsilon} a_j^h/(\rho_\varepsilon/\varepsilon)^h.$$

Assume (5.2) and $v = v_\varepsilon$ satisfy to (5.13). For any $\eta > 0$ put $J_\varepsilon^- = \{j < J_{\varepsilon,0}\}$. Then $D_\varepsilon(J_\varepsilon^-) = o(1)$.

Proof of the Proposition 5.1. Note that under assumption (5.13) we have:

$$a_j^p = O(J_{\varepsilon,0}^{p/2} 2^{(rp+1)j}),\ j < J_{\varepsilon,0}$$

which implies

$$\sum_{j<J_{\varepsilon,0}} a_j^h = O(J_{\varepsilon,0}^{h/2} 2^{(rh+h/p)J_{\varepsilon,0}}) = O((\log\varepsilon^{-1})^B)$$

for some $B > 0$. From the other hand, under assumptions (5.2) the relations (5.19), (5.20), (5.18) imply: $(\rho_\varepsilon/\varepsilon) = o(\varepsilon^{-b})$ for some $b > 0$. These relations imply $D_\varepsilon(J_\varepsilon^-) = o(1)$.

Remark 5.3. Assume $p > q,\ \delta = s - r > 0$ or $p \leq q,\ \delta = s + -r + 1/q - 1/p > 0$ and put $(J_\varepsilon^+) = \{j > (J_{\varepsilon,1}\}$. Then analogous estimations show that $D_\varepsilon(J_\varepsilon^+) = o(1)$.

In fact, note the inequalities:

$$2^{sqj}\sum_{\iota}|v_{\iota j}|^q \le (R/\varepsilon)^q$$

and

$$(\sum_{i=1}^n |x_i|^p)^{1/p} \le (\sum_{i=1}^n |x_i|^q)^{1/q}, \text{ if } p>q; \ \ (n^{-1}\sum_{i=1}^n |x_i|^p)^{1/p} \le (n^{-1}\sum_{i=1}^n |x_i|^q)^{1/q}, \text{ if } p\le q.$$

Using these inequalities, we get: $a_j \le 2^{-\delta j}$. It follows from (5.19), (5.20), (5.18) that $(R/\rho_\varepsilon) = O(\varepsilon^{-B})$ for some $B = B(\Gamma) > 0$ which imply

$$D_\varepsilon(J_\varepsilon^+) = O((R/\rho_\varepsilon)^h 2^{-\delta h J_{\varepsilon,1}}) = o(1).$$

5.4.1 The case $\kappa \in \Xi_{G_1}$

Note that this case corresponds to all $p \le q,\ p \le 2$ and it is possible for $q < p \le 2$ and for $2 < p < q$ (see Fig. 1 – 4 above).

First, assume $p \le q,\ p \le 2$ (we omit index ε). These relations imply (see [11, Proposition 7.1]) [2] then there exists $\delta = \delta(\Gamma) > 0$ such that if $\tau \in \Xi_G^1 \cap \Gamma$, then

$$\sup_j z_{\varepsilon,j}(\tau_\varepsilon) 2^{\delta \max(j,j_1)} = o(1) \tag{5.26}$$

Fix small enough $\eta > 0$ and put

$$J_\varepsilon = \{j \notin J_\varepsilon^-\},\ \delta_\varepsilon = D_\varepsilon(J_\varepsilon).$$

Using the Proposition 5.1 we have: $\tilde v_\varepsilon \in \tilde V_\varepsilon$ with $\delta_\varepsilon = o(1)$.

Using (5.26) and (5.13) one has:

$$\sup_{\iota j} |v_{\iota j}| z_{\varepsilon,j} = o(1),$$

and by $\sinh x \sim x$ as $x = o(1)$, we get:

$$(\bar\pi_{\varepsilon,j}, \bar\delta_v) \sim H_j(v). \tag{5.27}$$

Using (5.25), we obtain :

$$\sup_j H_j(\tilde v_\varepsilon)/(1+|j-j_\varepsilon^*|) \ge C_1\sqrt{\log\log\varepsilon^{-1}}(1+o(1)),\ C_1 = C/B(\Gamma) \tag{5.28}$$

and if $C_1 > 2$, this relation and (5.18) imply (5.14).

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[2] The case $p \ne q$ is considered in [11, Proposition 7.1] only. For the case $p = q$ one can easily see that if $p \le 2$, then $h_{\varepsilon,j} = 1$, $\max_{j<j_1} z_{\varepsilon,j} = o(\varepsilon^\delta)$ and $z_{\varepsilon,j} = 0$ for $j > j_1$; if $p > 2$, then $z_{\varepsilon,j} = z(p)$ for all (ι, j); $h_{\varepsilon,j} = 0$ for $j > j_0$. Thus we can assume $z_{\varepsilon,j} = 0$ for $j > j_\varepsilon^*$ in the case $p = q$.

Next, assume $q < p \le 2$ or $2 < p < q$. Remind that inclusion $\kappa \in \Xi_{G_1}$ implies the relation: $I = I(\kappa) < 0$ where $I = 2qs(p-2) - 2pr(q-2) + p - q$.

Denote

$$b_{\varepsilon,j} = 2^{jrp} \sum_{\iota} |v_{\iota,j}|^p / (\rho_\varepsilon/\varepsilon)^p. \tag{5.29}$$

Fix $\eta > 0$ and let $J^* = J^*_\varepsilon(v_\varepsilon) = J^1_\varepsilon(v_\varepsilon) \cap J^0_\varepsilon$ where

$$\begin{array}{rcl} J^1_\varepsilon(v_\varepsilon) & = & \{j \ge J_{\varepsilon,0} : \ H_j(v_\varepsilon)/T_j < 2\}, \\ J^0_\varepsilon & = & \begin{cases} \{j > j_1(1+\eta)\}, & \text{if } \ q < p \le 2, \\ \{J_{\varepsilon,0} \le j < j_1(1-\eta)\}, & \text{if } \ q > p > 2 \end{cases} . \end{array}$$

Proposition 5.2 *Under assumptions above there exist* $\tau = \tau(\eta, \Gamma) > 0$ *such that*

$$\sup_{j \in J^*} b_{\varepsilon,j} 2^{\tau|j-j_1|} = o(1).$$

Proof of the Proposition 5.2. Using Holder inequality, for $q < p \le 2$ or $2 < p < q$ one can obtain that

$$\sum_{\iota} |v_{\iota,j}|^p \le \left(\sum_{\iota} |v_{\iota,j}|^2 \right)^{(q-p)/(q-2)} \left(\sum_{\iota} |v_{\iota,j}|^q \right)^{(p-2)/(q-2)} .$$

Because

$$\sum_{\iota} |v_{\iota,j}|^q \le (R/\varepsilon)^p, \ \sum_{\iota} |v_{\iota,j}|^2 \le 4(\log j)^{1/2} 2^{j/2},$$

using relations (5.19), (5.2), after simple arithmetical calculations, we get:

$$b_{\varepsilon,j} \le B u_\varepsilon^{(q-p)/2} (\log j)^{(q-p)/2(q-2)} 2^{-I(j-j_1)/2(q-2)}$$

which implies necessary relation.

Put

$$J_\varepsilon = \{j \notin (J^-_\varepsilon \cap J^*\}, \ \delta_\varepsilon = D_\varepsilon(J_\varepsilon).$$

Using the Propositions 5.1, 5.2 we easily get : $\tilde{v}_\varepsilon \in \tilde{V}_\varepsilon$ with $\delta_\varepsilon = o(1)$.

It follows from [11, Proposition 7.1] that there exists $\delta = \delta(\Gamma) > 0$ such that if $\kappa \in \Xi_{G_1} \cap \Gamma$, $q < p \le 2$ or $2 < p < q$, then

$$\sup_{j \notin J^0_\varepsilon} z_{\varepsilon,j}(\tau_\varepsilon) 2^{\delta \max(j,j_1)} = o(1) \tag{5.30}$$

which imply $(\bar{\pi}_{\varepsilon,j}, \bar{\delta}_v) \sim H_j(v)$ uniformly on $j \notin J^0_\varepsilon$. Using (5.25), we obtain (5.28) which imply (5.14).

5.4.2 The case $\kappa \in \Xi_{G_2}$

This case corresponds to all $p > q,\ p > 2$ with $\lambda = sq - rp > 0$ and it is possible for $2 < p \le q$ and for $2 \ge p > q$, if $I > 0$ and $\lambda > 0$ (see Fig. 1 – 4 above).

First, assume $2 < p \le q$. Put

$$J_\varepsilon = \{j \notin J_\varepsilon^-\},\ \delta_\varepsilon = D_\varepsilon(J_\varepsilon).$$

Using the Proposition 5.1 we have: $\tilde{v}_\varepsilon \in \tilde{V}_\varepsilon$ with $\delta_\varepsilon = o(1)$.

Fix small enough $\eta > 0$. It follows from [11, Proposition 7.1] that there exists $A = A(\Gamma) > 0,\ \delta = \delta(\Gamma) > 0$ such that if $\kappa \in \Xi_{G_2} \cap \Gamma,\ 2 < p \le q$, then

$$z_{\varepsilon,j} \le \begin{cases} A, & \text{if } j \le j_0(1+\eta), \\ 2^{-\delta j}, & \text{if } j > j_0(1+\eta) \end{cases} \tag{5.31}$$

which imply $(\bar{\pi}_{\varepsilon,j}, \bar{\delta}_v) \sim H_j(v)$ uniformly on $j > j_0(1+\eta)$.

For $j \le j_0(1+\eta)$ put

$$k = k_{\varepsilon,j} = \min\{k:\ z_{j,k} \ge z_{\varepsilon,j}\}.$$

Note that $1 \le k \le O(K(j))$ because (5.31). If $z_{\varepsilon,j} < z_{j,1} = 1/\sqrt{\log j}$, then $z_{\varepsilon,j}|v_{\iota j}| \le 2$, and by choosing $z_{j,k}$ we have:

$$\sinh^2(z_{\varepsilon,j}v_{\iota j}/2)/\sinh(z_{\varepsilon,j}^2/2) \ge c_1 \sinh^2(z_{k,j}v_{\iota j}/2)/\sinh(z_{k,j}^2/2)$$

for some (absolute) constant c_1. This relation implies

$$H_{j,k_\varepsilon}(\tilde{v}_\varepsilon) = (\bar{\pi}_{j,k_\varepsilon}, \bar{\delta}_{\tilde{v}_\varepsilon}) \ge c_1(\bar{\pi}_{\varepsilon,j}, \bar{\delta}_{\tilde{v}_\varepsilon}). \tag{5.32}$$

Thus, using (5.25) we get:

$$\begin{aligned} &\max\{\sup_{j \le j_0(1+\eta)} H_{j,k_{\varepsilon,j}}(\tilde{v}_\varepsilon)/(1+|j-j_\varepsilon^1|),\ \sup_{j \le j_0(1+\eta)} H_j(1+|j-j_\varepsilon^1|)\} \\ &\ge C_1\sqrt{\log\log\varepsilon^{-1}}(1+o(1)),\ C_1 = c_1 C/B(\Gamma). \end{aligned}$$

This relation and (5.18) imply (5.14) or (5.15).

Next, assume $2 < p,\ q < p$ or $2 \ge p > q$. It follows from [11, Proposition 7.1] that there exists $A = A(\Gamma) > 0,\ \delta = \delta(\Gamma) > 0$ such that if $\kappa \in \Xi_{G_2} \cap \Gamma,\ q < p \le 2$ or $2 < p, p > q$, then

$$z_{\varepsilon,j} \le \begin{cases} A, & \text{if } j \le j_0, \\ A\sqrt{1+j-j_0}, & \text{if } j > j_0 \end{cases}. \tag{5.33}$$

Fix small enough $\eta > 0$, such that $z_{\varepsilon,j} < K(c,j)/2$ for $j \le j_0(1+\eta)$. Denote $J^1 = \{j > j_0(1+\eta)\}$.

Proposition 5.3 *Under assumptions above there exist $\tau = \tau(\eta, \Gamma) > 0$ such that*

$$\sup_{j \in J^1} b_{\varepsilon,j} 2^{\tau(j-j_0)} = o(1).$$

Proof of the Proposition 5.3. By $p \geq q$, using the inequality

$$\sum_\iota |v_{\iota,j}|^p \leq \max_\iota |v_{\iota,j}|^{p-q} \sum_\iota |v_{\iota,j}|^q$$

and because

$$\sum_\iota |v_{\iota,j}|^q \leq (R/\varepsilon)^p, \ \max_\iota |v_{\iota,j}| \leq 2(\log j)^{1/2},$$

using also relations (5.19), (5.2), after simple arithmetical calculations, we get:

$$b_{\varepsilon,j} \leq B(\log j)^{(p-q)/2} 2^{-\lambda(j-j_0)}, \ \lambda = \lambda(\kappa) > 0$$

which implies necessary relation.

For $j \leq j_0(1+\eta)$ put

$$k = k_{\varepsilon,j} = \min\{k : \ z_{j,k} = k/j^{1/2} \geq z_{\varepsilon,j}\}.$$

Note that $1 \leq k \leq K(c,j)$ because (5.33). Then, analogously to (5.32), we can find such (absolute) constant c_1 that

$$H_{j,k_\varepsilon}(\tilde{v}_\varepsilon) = (\bar{\pi}_{j,k_\varepsilon}, \bar{\delta}_{\tilde{v}_\varepsilon}) \geq c_1(\bar{\pi}_{\varepsilon,j}, \bar{\delta}_{\tilde{v}_\varepsilon}).$$

Thus, using (5.25) we get:

$$\sup_j H_{j,k_{\varepsilon,j}}(\tilde{v}_\varepsilon)/(1+|j-j^*_\varepsilon|) \geq C_2\sqrt{\log\log\varepsilon^{-1}}(1+o(1)), \ C_2 = c_1 C/B(\Gamma).$$

This relation and (5.18) imply (5.15).

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